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On ε-nets in a complex

by

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§ 1. Let F be a compact metric space or a closed subset of such a space. A finite subset A of F is called an ε-net in F if $\varrho(x, A) < \varepsilon$ for any point x of F. [1]

The point x of F is said to be of order λ with respect to the net A if there are exactly λ points $x_1, x_2, \ldots, x_\lambda$ of A for which $\varrho(x, x_i) = \varrho(x, A)$, $(i = 1, 2, \ldots, \lambda)$. [1]

Alexandroff has then stated the problem [2]:

Is it possible, for any λ-dimensional closed set F and for any ε, to find an ε-net so that no point of F has an order $> \lambda + 1$ with respect to the net?

The object of this paper is to prove that to any n-dimensional complex K there is a homeomorphic metric space K' for which the answer to Alexandroff's question is in the affirmative.

Let K be a finite, connected, n-dimensional complex which we imagine to be topologically immersed in the Euclidean R_{2n+1} so that the metric in K may be taken as the metric of R_{2n+1} i.e. the distance between two points of K is their distance in R_{2n+1}.

Consider the infinite sequence of complexes $K_0, K_1, K_2, \ldots$ in which K_0 is the complex K and K_{i+1} is a regular subdivision [3]) of K_i such that the new vertices introduced are centres [4]) of simplexes of K_i. Let the vertices of K_i be $x_{i,1}, x_{i,2}, \ldots, x_{i\,\alpha_i}$ and those of K_{i+1} be $x_{i+1,1}, x_{i+1,2}, \ldots, x_{i+1,\alpha_i}, x_{i+1,\alpha_i+1}, \ldots, x_{i+1,\alpha_{i+1}}$ where $x_{i+1,j} = x_{i,j}$ $(j = 1, 2, \ldots, \alpha_i)$. The set $\{x_{i,j}\}$ of all vertices $x_{i,j}$ $(i = 0, 1, 2, \ldots; j = 1, 2, \ldots, \alpha_i)$ is dense in K.

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[1]) P. Alexandroff, Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension [Annals of Math. (2) **30** (1928), 123].

[2]) P. Alexandroff l.c. [1]) 125.

[3]) O. Veblen, Colloquium Lectures on Analysis Situs (1922).

[4]) By the centre of a simplex is understood that point whose barycentric coordinates with respect to the vertices are all equal.

§ 2. Introduction of the new metric in K. By a path in K_i joining vertices x and y of K_i is understood a 1-chain $x_{i,1}x_{i,2} + x_{i,2}x_{i,3} + \ldots + x_{i,l}x_{i,l+1}$ where $x_{i,1} = x$ and $x_{i,l+1} = y$; then (2.1) the length of this path is defined to be $\frac{l}{2^i}$. Of the finite number of paths in K_i which join vertices $x_{i,s}$ and $x_{i,t}$ of K_i there are one or more whose lengths as above defined have the minimum possible value — such a path is called a minimum path in K_i. We now define:

(2.2) The distance $\varrho(x_{i,s}, x_{i,t})$ is the length of a minimum path in K_i joining $x_{i,s}$ and $x_{i,t}$, $(i = 0, 1, 2, \ldots)$. We then have

(2.3) $\varrho(x_{i,r}, x_{i,s}) + \varrho(x_{i,s}, x_{i,t}) \geqq \varrho(x_{i,r}, x_{i,t})$, $(i = 0, 1, 2, \ldots)$; for otherwise a path in K_i from $x_{i,r}$ to $x_{i,t}$ via $x_{i,s}$ would have a length $< \varrho(x_{i,r}, x_{i,t})$ contrary to the definition (2.2).

Let a minimum path L_i in K_i joining $x_{i,s}$ and $x_{i,t}$ consist of l_i 1-cells of K_i, $(i = 0, 1, 2, \ldots)$.

(2.4) No two 1-cells of L_i belong to the same simplex E of K_i, $(i = 0, 1, \ldots)$; for otherwise two or more 1-cells of L_i could be replaced by a single 1-cell of E, thus replacing L_i by a shorter path in K_i contrary to hypothesis.

Let the upper index α indicate that the vertex x^α of K_{i+1} is the centre of an α-simplex of K_i. We then have:

(2.5) A minimum path L_{i+1} of K_{i+1} joining $x_{i,s} = x_{i+1,s}$ and $x_{i,t} = x_{i+1,t}$ has the form

$$x_1^{\alpha_1} x_2^{\alpha_2} + x_2^{\alpha_2} x_3^{\alpha_3} + \ldots + x_{l_{i+1}}^{\alpha_{l_{i+1}}} x_{l_{i+1}+1}^{\alpha_{l_{i+1}+1}},$$

where $\alpha_1 = \alpha_{l_{i+1}+1} = 0$, $\alpha_{2m-1} < \alpha_{2m} > \alpha_{2m+1}$, $l_{i+1} = 2h$ and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} + x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ is in the subdivision of a simplex E_m of K_i, $(m = 1, 2, \ldots, h)$.

(a) $\alpha_1 = \alpha_{l_{i+1}+1} = 0$, since $x_1^{\alpha_1}$ and $x_{l_{i+1}+1}^{\alpha_{l_{i+1}+1}}$ are the vertices $x_{i,s}$ and $x_{i,t}$ of K_i.

(b) Assume $\alpha_{2m-1} < \alpha_{2m}$; then $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}}$ is in the subdivision of an α_{2m}-simplex E_m of K_i of centre $x_{2m}^{\alpha_{2m}}$; if $\alpha_{2m} < \alpha_{2m+1}$, then $x_{2m+1}^{\alpha_{2m+1}}$ would be the centre of an α_{2m+1}-simplex of K_i having E_m in its boundary and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ would be a 2-simplex of K_{i+1} contrary to (2.4), hence $\alpha_{2m} > \alpha_{2m+1}$, hence $x_{2m+1}^{\alpha_{2m+1}}$ is the centre of a face of E_m and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} + x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ is in the subdivision of E_m. Similarly if $\alpha_{2m+1} > \alpha_{2m+2}$, $x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}} x_{2m+2}^{\alpha_{2m+2}}$ would be a 2-simplex of K_{i+1} contrary to (2.4), hence $\alpha_{2m+1} < \alpha_{2m+2}$ i.e. $\alpha_{2(m+1)-1} < \alpha_{2(m+1)}$. From (a) and (b), (2.5) follows by induction (to prove that $l_{i+1} = 2h$ we merely note that when $\alpha_j > \alpha_{j+1}$, j is even, and since $\alpha_{l_{i+1}} > \alpha_{l_{i+1}+1} = 0$, $\alpha_{l_{i+1}}$ is even).

From (2.5) $x_{i,s}$ and $x_{i,t}$ can be joined by the path $L'_i = E'_1 + E'_2 + \ldots + E'_h$ in K_i where E'_m is a 1-simplex of the simplex E_m of K_i and $h = \frac{1}{2} l_{i+1}$.

Since L_i is a minimum path in K_i joining $x_{i,s}$ and $x_{i,t}$ we have (c) length $L_i \leqq$ length $L'_i = \frac{\frac{1}{2} l_{i+1}}{2^i} = \frac{l_{i+1}}{2^{i+1}} =$ length L_{i+1}. But by a regular subdivision of L_i we obtain a path L'_{i+1} in K_{i+1} joining $x_{i+1,s} = x_{i,s}$ and $x_{i+1,t} = x_{i,t}$ and composed of $2l_i$ 1-simplexes of K_{i+1}; hence length $L'_{i+1} = \frac{2l_i}{2^{i+1}} = \frac{l_i}{2^i} =$ length L_i. Since L_{i+1} is a minimum path in K_{i+1}, length $L'_{i+1} \geqq$ length L_{i+1}, hence (d) length $L_i \geqq$ length L_{i+1}. From (c) and (d) we have length L_i $=$ length L_{i+1}, hence

(2.6) $$\varrho(x_{i,s}, x_{i,t}) = \varrho(x_{i+1,s}, x_{i+1,t}), \quad (i = 0, 1, 2, \ldots)$$

Let $x_{i,r}$, $x_{j,s}$ and $x_{k,t}$ be any three vertices of $\{x_{i,j}\}$ $(i = 0, 1, 2, \ldots; j = 1, 2, \ldots, \alpha_i)$, then

(2.7) $$\varrho(x_{i,r}, x_{j,s}) + \varrho(x_{j,s}, x_{k,t}) \geqq \varrho(x_{i,r}, x_{k,t});$$

for let m be an integer greater than i, j and k, such that $x_{i,r} = x_{m,r}$, $x_{j,s} = x_{m,s}$ and $x_{k,t} = x_{m,t}$, then by (2.3) we have

$$\varrho(x_{m,r}, x_{m,s}) + \varrho(x_{m,s}, x_{m,t}) \geqq \varrho(x_{m,r}, x_{m,t})$$

from which, using (2.6), we obtain (2.7).

Let now x and y be any points of K and x_{i,r_i} and x_{i,s_i} vertices of K_i such that the sequences $x_{1,r_1}, x_{2,r_2}, \ldots$ and $x_{1,s_1}, x_{2,s_2}, \ldots$ converge to x and y respectively in R_{2n+1}; we then make the definition

(2.8) $$\varrho(x, y) = \lim_{i \to \infty} \varrho(x_{i,r_i}, x_{i,s_i}).$$

(2.9) From (2.2) and (2.7) it follows that the metric thus introduced satisfies the usual axioms

$$\begin{cases} \varrho(x, x) = 0, \\ \varrho(x, y) = \varrho(y, x), \\ \varrho(x, y) + \varrho(y, z) \geqq \varrho(x, z). \end{cases}$$

The points of K with the new metric thus constitute a metric space K'.

§ 3. (3.1) The distance in K' from a vertex of a simplex of K_i to a point of the opposite face is $\leqq \frac{1}{2^i}$.

Let x_0 be a vertex of an h-dimensional simplex

$$x_0 E_0 = x_0 x_1 \ldots x_h \text{ of } K_i, \ (h = 1, 2, \ldots, n), \ E_0$$

being the face opposite x_0; let y be any point of E_0, E_{m+1} that simplex of K_{i+m+1} in the subdivision of E_m which contains y, $(m = 0, 1, \ldots)$, and x_{m+1} the centre of $x_m E_m$; then

$$\varrho(x_0, y) \leqq \sum_{m=0,1,\ldots,\infty} \varrho(x_m, x_{m+1}) = \sum_{j=1,2,\ldots,\infty} \frac{1}{2^{i+j}} = \frac{1}{2^i}.$$

A similar proof gives:

(3.2) The distance in K' from a vertex of a simplex of K_i to any point of the simplex or its boundary is $\leqq \frac{1}{2^i}$.

(3.3) The distance in K' from a vertex of a simplex of K_i to a point of the opposite face is $\frac{1}{2^i}$.

Let x_1 and x_2 be vertices of a simplex E of K_{i-1}, $x_1 E'$ and $x_2 E'$ simplexes of K_i in the subdivision of E having a common face E', and y any point of E'; then by (3.1)

$$\varrho(x_1, y) \leqq \frac{1}{2^i} \text{ and } \varrho(x_2, y) \leqq \frac{1}{2^i},$$

hence if $\varrho(x_1, y) < \frac{1}{2^i}$ we should have

$$\varrho(x_1, x_2) \leqq \varrho(x_1, y) + \varrho(y, x_2) < \frac{1}{2^{i-1}};$$

but by (2.1), $\varrho(x_1, x_2) = \frac{1}{2^{i-1}}$ since x_1 and x_2 are vertices of the simplex E of K_{i-1}; from this contradiction we have $\varrho(x_1, y) = \frac{1}{2^i}$; the theorem (3.3) is thus true for the vertex x_1 and face E' of $x_1 E'$; but from definitions (2.1) and (2.2) the distance from a vertex of a simplex of K_i to a point of the opposite face is a function of i only $(i = 0, 1, \ldots)$, so that (3.3) holds for all simplexes of K_i.

From (3.3) we have:

(3.4) The distance in K' from the centre of a simplex star of K_i to a point of its boundary [5]) is $\frac{1}{2^i}$, $(i = 0, 1, \ldots,)$.

From (2.2), (2.8) and (3.4) we have:

(3.5) The distance in K' from the centre of a simplex star of K_i to a point of K_i neither in the interior [5]) nor boundary of the star is $> \frac{1}{2^i}$, $(i = 0, 1, \dots)$.

From (3.2) we have:

(3.6) The vertices of K_i constitute a $\frac{1}{2^{i+1}}$-net in K', $(i = 0, 1, \dots)$.

§ 4. Let $E = x_1 x_2 \dots x_{h+1}$ be any simplex of K_i of centre c, then from (3.4)

$$\varrho(c, x_j) = \frac{1}{2^{i+1}}, \quad (j = 1, 2, \dots, h+1;\ i = 0, 1, \dots),$$

hence:

(4.1) The order of c with respect to the net of the vertices of E is $h + 1$.

Let x be an inner point of E other than c, then there are vertices x_j and x_k of E such that x is in the interior or boundary of the star of K_{i+1} of centre x_j but neither in the interior nor boundary of the star of K_{i+1} of centre x_k; hence from (3.2) and (3.5) we have:

(4.2) $\varrho(x, x_j) \leqq \frac{1}{2^{i+1}}$, $\varrho(x, x_k) > \frac{1}{2^{i+1}}$, thus $\varrho(x, x_j) < \varrho(x, x_k)$;

hence:

(4.3) The order of x with respect to the net of the vertices of E is $< h + 1$.

(4.4) The order of an inner point x of E with respect to the net of all vertices of K_i is equal to its order with respect to the net of vertices of E.

For let y be any vertex of K_i other than a vertex of E; if $y x_1 x_2 \dots x_{h+1}$ is a simplex of K_i then by (3.1) $\varrho(x, y) = \frac{1}{2^i}$; if $y x_1 x_2 \dots x_{h+1}$ is not a simplex of K_i then $\varrho(x, y) > \frac{1}{2^i}$, thus in all cases $\varrho(x, y) \geqq \frac{1}{2^i}$; but by (3.4) if $x = c$ or by (4.2) if $x \neq c$,

[5]) Those simplexes of K_i having a common vertex constitute a simplex star whose centre is this vertex; the points of those simplexes of the star of which the centre is not a vertex constitute the boundary of the star and the remaining points of the star constitute its interior.

there is a vertex x_j of E such that $\varrho(x, x_j) \leqq \frac{1}{2^{i+1}} < \varrho(x, y)$, so that the order of x with respect to the net $y, x_1, x_2, \ldots, x_{h+1}$ is equal to its order with respect to the net $x_1, x_2, \ldots, x_{h+1}$.

From (4.1), (4.3) and (4.4):

(4.5) An inner point of an h-dimensional simplex E of K_i, ($h = 1, 2, \ldots, n$; $i = 0, 1, \ldots$), is of order $h+1$ or $< h+1$ with respect to the vertices of K_i according as it is or is not, respectively, the centre of E.

An immediate consequence is:

(4.6) The order of any point of K' with respect to the net of vertices of K_i is $\leqq n+1$.

§ **5.** Let D_i ($i = 0, 1, \ldots$) be the maximum of the diameters, in K, of the simplexes of K_i, then:

(5.1) $D_i \leqq \left(\frac{n}{n+1}\right)^i D_0$, hence $\lim\limits_{i \to \infty} D_i = 0$.

From (3.3) we have:

(5.2) The diameter in K' of a simplex star Δ_i of K_i is $\frac{2}{2^i}$, thus $\lim\limits_{i \to \infty}$ (diam. Δ_i in K') $= 0$.

(5.3) If x be an arbitrary point of K there is a sequence Δ_i, ($i = 0, 1, \ldots$), of simplex stars such that Δ_i is a star of K_i containing x in its interior, $\Delta_{i+1} \subset \Delta_i$ and, K being closed and compact, $x = \prod\limits_{i=0,1,\ldots,\infty} \Delta_i$.

The proof is sufficiently obvious to be omitted.

(5.4) Let $\overline{\Delta}_i$ represent Δ_i together with those simplexes of K_i having vertices in common with Δ_i, then:

(a) Diam. $\overline{\Delta}_i$ in $K' = \frac{4}{2^i}$, by (3.3) and (5.2);

(b) Diam. $\overline{\Delta}_i$ in $K \leqq 4\left(\frac{n}{n+1}\right)^i D_0$, by (5.1);

(c) $\overline{\Delta}_i \supset \Delta_i \supset x$ and

(d) $\varrho(x, K - \overline{\Delta}_i) \geqq \frac{1}{2^i}$.

Let $S(x, r)$ be the spherical region of R_{2n+1} of centre x and radius r, then:

(5.5) We can choose r so small that for arbitrary i $S(x, r)K \subset \Delta_i$; further, from (5.4b) and (5.4c), for arbitrary r we can choose i so great that $S(x, r) \supset \overline{\Delta}_i$.

Let $x_1, x_2, \ldots$ converge to x in K; then each $S(x, r)$ contains almost all the x_j, hence by (5.5) each Δ_i contains almost all the x_j, hence by (5.2) and (5.3), $x_1, x_2, \ldots$ converges to x in K'.

Let $x_i, x_2, \ldots$ converge to x in K'; then by (5.4d) each $\overline{\Delta}_i$ contains almost all the x_j, hence by (5.5) each $S(x, r)$ contains almost all the x_j, hence $x_1, x_2, \ldots$ converges to x in K.

Thus K and K' are continuous images of each other and since each point corresponds to itself we have:

(5.6) K and K' are homeomorphic.

From (4.6) and (5.6) we have:

(5.7) To any finite n-dimensional complex K there can be constructed a homeomorphic metric space K' in which, for arbitrary ε, an ε-net can be constructed such that the order of any point of K' with respect to the net is $\leqq n + 1$.

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