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# Analytical methods in hypercomplex systems

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A hypercomplex system over the field of real numbers, with $n$ base-elements, is a topological ring with respect to the topology of the $n$-dimensional Euclidean vector space; i.e. addition and multiplication are continuous functions with respect to the topology of the vector space. Hypercomplex systems over the field of real numbers can be characterized entirely topologically among other rings as locally compact connected separable topological rings whose additive group does not involve any compact subgroup besides $\{0\}$ [1]. A single hypercomplex system is usually characterized by algebraic properties. In what follows several algebraic properties of hypercomplex systems over the field of real numbers will be expressed by means of convergence of infinite sequences.

In 3 the case of hypercomplex systems with division (fields) is discussed by means of continuous groups.

## 1.

Let $\mathfrak{S}$ be a hypercomplex system with $n$ base-elements with respect to the field of real numbers. $\mathfrak{S}$ can be considered as an $n$-dimensional Euclidean vector space. In what follows we shall use the term „neighbourhood" in the sense which is commonly used for $n$-dimensional vector spaces.

A sequence $\{\alpha_\nu\}$ of elements of $\mathfrak{S}$ will be called convergent to $\alpha$, ($\alpha_\nu \rightarrow \alpha$ or $\alpha = \lim \alpha_\nu$), if every neighbourhood of $\alpha$ contains almost all $\alpha_\nu$. A sequence will be called divergent if it does not contain any convergent subsequence. In this sense a divergent sequence is unbounded. Convergence and divergence are, of course, independent of the base of $\mathfrak{S}$.

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[1]) N. Jacobson and O. Taussky, Locally compact rings [Proc. Nat. Acad. Sc. U.S.A. 21 (1935), 106—108].

If $\mathfrak{S}$ is a field (i.e. the system of real or complex numbers or of quaternions) the product of two divergent sequences is divergent again. This does not hold if $\mathfrak{S}$ contains divisors of zero besides zero. For instance, from the divergence of a sequence $\{\alpha_\nu\}$ of regular elements (i.e. elements which are not divisors of zero) the convergence of $\{\alpha_\nu^{-1}\}$ does not necessarily follow. However the following converse always holds.

Lemma 1. If $\{\alpha_\nu\}$ converges to a divisor of zero [2]) and $\alpha_\nu^{-1}$ exists for every $\nu$, then $\{\alpha_\nu^{-1}\}$ is divergent.

Proof. If $\{\alpha_\nu^{-1}\}$ were not divergent it would involve a convergent subsequence $\{\alpha_{m_\nu}^{-1}\}$. The product $\alpha_{m_\nu} \cdot \alpha_{m_\nu}^{-1} = 1$ would be converging to a divisor of zero, which is a contradiction.

## 2.

Another essential difference between fields and systems which are not fields is the following: When $\mathfrak{S}$ is a field one can choose a base $\omega_1, \ldots, \omega_n$ for $\mathfrak{S}$ such that the elements $\nu = a_1\omega_1 + \ldots + a_n\omega_n$ with $\sum a_i^2 = 1$ form a group, while all elements $\lambda$ with $\sum a_i^2 < 1$ have the property that $\lambda^n \to 0$. For the system of real numbers this special base is $\omega_1 = 1$; for the complex numbers $\omega_1 = 1$, $\omega_2 = i$ $(i^2 = -1)$; for the quaternions $\omega_1 = 1$, $\omega_2 = i$, $\omega_3 = j$, $\omega_4 = k$ $(i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j)$. Geometrically interpreted this means that for a special base the elements on the surface of the $n$-dimensional unit sphere form a group, while the powers of its inner points converge to 0. However, it is always possible to find a sphere with 0 as centre, whose points all have the property that their powers converge to 0. We shall prove a more general fact in

Lemma 2. The set $\Lambda$ formed by all the elements $\lambda$ such that $\lambda^n \to 0$ is open.

At first we prove

Lemma 3. There exists a real number $\varrho$ such that the open sphere $S_\varrho$ with radius $\varrho$ and centre 0 contains the product of any two elements of its boundary.

Proof. The set of products $\alpha\beta$, where $\alpha$, $\beta$ are elements of the closed sphere of radius 1 and centre 0 is a compact set. It is therefore contained in a certain sphere of radius P with centre 0. Lemma 3 holds for $\varrho = \dfrac{1}{2\mathrm{P}}$.

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[2]) In this paper the expression „divisor of zero" will be used for „divisor of zero or zero".

Proof of Lemma 2. Let $S$ be the open sphere of radius $\frac{1}{2}\varrho$ and centre 0, where $\varrho$ is defined in Lemma 3. $S$ has the property that the powers of all its elements converge to zero. Thus the existence of a sphere with centre 0 and which belongs to $\Lambda$ is proved. Let $\lambda$ be any element of $\Lambda$ which is different from 0. Since $\lambda \epsilon \Lambda$ and $S$ is a neighbourhood of 0, there exists an exponent $m$ such that $\lambda^m \epsilon S$. From the continuity of multiplication in $\mathfrak{S}$ it follows that $\lambda'^m \epsilon S$ for all elements $\lambda'$ of a certain sphere $S(\lambda)$ with centre $\lambda$. For otherwise there would exist a sequence $\lambda'_\nu$ of elements with $\lim \lambda'_\nu = \lambda$, such that $\lambda'^m_\nu \notin S$. Since $S$ is open this contradicts $\lim \lambda'^m_\nu \epsilon S$. Hence: if $\lambda \epsilon \Lambda$, there exists a sphere $S(\lambda)$ with centre $\lambda$ which also belongs to $\Lambda$, i.e. $\Lambda$ is open.

Lemma 4. Let $\alpha$ be an arbitrary element of $\mathfrak{S}$. There exists a real number $r$ such that $\{(r\,\alpha)\}^n \to 0$.

Proof. Let $S$ denote the same sphere as in Lemma 2. There exists a real number $r$ such that $r\alpha \epsilon S$. Hence $r\alpha \epsilon \Lambda$.

Lemma 5. Let $\alpha$ be a regular element such that $\{\alpha^n\}$ does not diverge. There exists a real number $r$ such that $\{(r\alpha)^n\}$ is divergent.

Proof. Let $S$ denote the same sphere as in Lemma 2. We multiply $\alpha^{-1}$ by a real number $r'$ such that $\alpha^{-1}r'$ is contained in $S$. Then $\{(\alpha^{-1}r')^n\} \to 0$. Hence by Lemma 1 the sequence $\{(\alpha r'^{-1})^n\}$ diverges.

As an application of Lemma 5 we prove

Lemma 6. The group formed by the regular elements of a hypercomplex system over the field of real numbers cannot involve any real commutator except $\pm 1$ [3]).

Proof. Let $\alpha^{-1}\beta^{-1}\alpha\beta = a$, where $a$ is real and $|a| \neq 1$. Then

$$\beta^{-1}\alpha\beta = a\alpha. \tag{1}$$

We may assume that the sequence $\{\alpha^n\} \to 0$. For otherwise we could consider $r\alpha$ instead of $\alpha$ for a real number $r$ such that $(r\alpha)^n \to 0$. By Lemma 2 such a number $r$ always exists. We may further assume that $|a| > 1$. For otherwise we could deal with the equation $\beta^{-1}a^{-1}\alpha\beta = \alpha$.

From (1) follows

$$\beta^{-1}\alpha^n\beta = (a\alpha)^n, \quad n = 1, 2, \ldots \tag{2}$$

Since $\{\beta^{-1}\alpha^n\beta\}$ is a continuous transformation of the sequence

[3]) This theorem follows easily also by means of the representation of $\mathfrak{S}$ by matrices.

$\{\alpha^n\}$, the element $a\alpha$ belongs to $\Lambda$ again. Since

$$\beta^{-1}a\alpha\beta = a^2\alpha,$$

we can deduce in the same way that $a^2\alpha\epsilon\Lambda$, and generally: $a^m\alpha\epsilon\Lambda$ for $m = 1, 2, \ldots$ This contradicts Lemma 5.

We shall now investigate the structure of the set $\Lambda$. $\Lambda$ is always a subset of the elements $\alpha$ of norm $\alpha < 1$, the norm being defined by the regular representation of $\mathfrak{S}$. For the elements $\alpha$ of norm $\alpha = 1$ form a multiplicative group. Hence: if $\alpha$ is any element of norm $\alpha = 1$, all its powers have this property too, therefore they cannot converge to zero. If an element is of norm $> 1$, it is a product of an element of norm $= 1$ and a real number $> 1$, therefore its powers diverge. Hence the elements of $\Lambda$ have norm $< 1$. When $\mathfrak{S}$ is a field, i.e. for real or complex numbers or quaternions, the group formed by the elements of norm $= 1$ is a compact group. Therefore in this case $\Lambda$ coincides with the set of elements of norm $< 1$. This is, for a particularly chosen base, the set of inner points of the sphere. However, the compactness of the group of elements of norm $= 1$ is not a necessary condition for the coincidence of $\Lambda$ with the set of elements of norm $< 1$.

This is shown by the following .

Example 1. Let $\mathfrak{S}$ be the hypercomplex system with two base-elements $1$, $\epsilon$, where $\epsilon^2 = 0$. In this case $\Lambda$ is the set of all numbers $a + b\epsilon$, where $a$, $b$ are real and $|a| < 1$. The boundary of $\Lambda$ in $\mathfrak{S}$ consists of the two lines $\pm 1 + b\varepsilon$. Since norm $(a+b\varepsilon) = a^2$, the set $\Lambda$ coincides with the set of elements whose norm $< 1$, and the two lines $\pm 1 + b\varepsilon$ are the group of elements of norm $= 1$.

We shall mention now a case where $\Lambda$ is a proper subset of the elements of norm $< 1$.

Example 2. Let $\mathfrak{S}$ be the hypercomplex system with two base-elements $1$, $\gamma$, where $\gamma^2 = 1$. Another base for this ring is given by the elements $e_1 = \dfrac{1+\gamma}{2}$, $e_2 = \dfrac{1-\gamma}{2}$, which satisfy the relations $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = 0$. An arbitrary element of this ring is given by $a_1e_1 + a_2e_2$, where $a_1$, $a_2$ are real numbers. As

$$(a_1e_1+a_2e_2)^n = a_1^ne_1 + a_2^ne_2\,,$$

the set $\Lambda$ consists of all elements $a_1e_1 + a_2e_2$ with $|a_1|$, $|a_2| < 1$. The boundary $\mathbf{N}$ of $\Lambda$ does not form a group, but we can easily prove that for every element $\nu\epsilon\mathbf{N}$ holds: $\nu^m\epsilon\mathbf{N}$ for $m = 1, 2, \ldots$.

Since norm $(a+b\gamma) = |a^2 - b^2|$, the set $\Lambda$ is different from the set of elements of norm $< 1$, as the latter is not compact.

In what follows we shall denote by $\mathfrak{R}$ the radical of $\mathfrak{S}$, by $\mathfrak{S}/\mathfrak{R}$ the ring of residue-classes mod $\mathfrak{R}$, and by $\mathfrak{L}$ the set of elements of $\mathfrak{S}$ with norm $< 1$, the norm being defined by the regular representation of $\mathfrak{S}/\mathfrak{R}$[4]).

THEOREM I 1. Let $\mathfrak{S}/\mathfrak{R}$ be a field. Then the set $\Lambda$ of $\mathfrak{S}$ coincides with $\mathfrak{L}$.

THEOREM I 2. If $\mathfrak{S}/\mathfrak{R}$ is not a field, then $\Lambda$ is a proper subset of $\mathfrak{L}$.

We shall prove this with the help of:

THEOREM II. Let $\bar{\Lambda}$ be the subset of $\mathfrak{S}/\mathfrak{R}$ consisting of all elements $\bar{\lambda} \epsilon \mathfrak{S}/\mathfrak{R}$ such that $\bar{\lambda}^n \to \mathfrak{R}$. $\bar{\Lambda}$ coincides with the set of residue-classes of $\mathfrak{S}$ mod $\mathfrak{R}$ defined by the elements of $\Lambda$.

First we shall prove a further Lemma.

Let us choose a fixed base $\omega_1, \ldots, \omega_n$ for $\mathfrak{S}$.

If $\alpha = a_1\omega_1 + \ldots + a_n\omega_n$, we shall denote $|\sqrt{\Sigma a_i^2}|$ by $||\alpha||$. If $r$ is a real number, $||r\alpha|| = r||\alpha||$. But only if $\mathfrak{S}$ is a field, there exists a base such that the relation $||\alpha \cdot \beta|| = ||\alpha|| \cdot ||\beta||$ holds for every pair of elements $\alpha, \beta$. In this case $||\alpha|| = |\text{norm}\,\alpha|^{\frac{1}{n}}$ $(n = 1, 2, 4)$.

LEMMA 7. Let $\alpha_1, \ldots, \alpha_n$ be any elements of $\mathfrak{S}$ and $i_1, \ldots, i_n$ an arbitrary permutation of $1, \ldots, n$; then

$$||\alpha_1 \cdots \alpha_n - \alpha_{i_1} \cdots \alpha_{i_n}|| \leqq 2||\alpha_1|| \cdots ||\alpha_n|| \cdot \varrho^{1-n},$$

where $\varrho$ is a fixed real positive number $< 1$ which satisfies the conditions required in Lemma 3.

PROOF. Obviously

$$||\alpha_1 \cdots \alpha_n - \alpha_{i_1} \cdots \alpha_{i_n}|| =$$
$$= ||\alpha_1|| \cdots ||\alpha_n|| \cdot \frac{1}{\varrho^n} \cdot \left\| \frac{\alpha_1\varrho}{||\alpha_1||} \cdots \frac{\alpha_n\varrho}{||\alpha_n||} - \frac{\alpha_{i_1}\varrho}{||\alpha_{i_1}||} \cdots \frac{\alpha_{i_n}\varrho}{||\alpha_{i_n}||} \right\|.$$

As every $\dfrac{\alpha_k \cdot \varrho}{||\alpha_k||}$ is a point contained in the boundary of the sphere of radius $\varrho$ and centre 0, the two products $\Pi \dfrac{\alpha_k \cdot \varrho^n}{||\alpha_k||}$ and

[4]) For the definition of the radical and its main properties see for instance B. L. VAN DER WAERDEN, Moderne Algebra II [Berlin, 1931].

$\Pi \frac{\alpha_{i_k} \cdot \varrho^n}{||\alpha_{i_k}||}$ are contained in the same sphere and therefore their difference has an absolute value $\leqq 2\varrho$. From this Lemma 7 follows.

PROOF of THEOREM II. If $\lambda \epsilon \Lambda$, the residue-class mod $\mathfrak{R}$ generated by $\lambda$ obviously belongs to $\overline{\Lambda}$. We shall prove the converse: Let $\lambda'$ be an element such that the residue-class mod $\mathfrak{R}$ generated by $\lambda'$ belongs to $\overline{\Lambda}$. Then we can prove: $\lambda'^n \to 0$,

i.e. $\lambda'^n \to 0$ mod $\mathfrak{R}$ implies $\lambda'^n \to 0$.

$$\lambda'^n \to 0 \text{ mod } \mathfrak{R} \text{ means}$$

$$\lambda'^n = r_n + \varepsilon_n$$

where $r_n \epsilon \mathfrak{R}$ and $\lim \varepsilon_n = 0$. We choose $n = \overline{n}$ so large that $\|\varepsilon_{\overline{n}}\| < \frac{1}{2}\varrho$, where $\varrho$ has the same meaning as in Lemma 3, and prove that the subsequence $\{\lambda'^{\overline{n}n}\} \to 0$. Let us denote $\varepsilon_{\overline{n}}$ by $\varepsilon$ and $r_{\overline{n}}$ by $r$. Since $r \epsilon \mathfrak{R}$, there exists an integer $z$ such that the ideal $(r)^z = (0)$. We shall now prove

$$(\varepsilon + r)^n \to 0.$$

We can write

$$(\varepsilon + r)^n = \sum_{\substack{\sum \nu_i + \mu_i = n \\ \nu_i, \mu_i \geqq 0}} \varepsilon^{\nu_1} r^{\mu_1} \varepsilon^{\nu_2} r^{\mu_2} \cdots$$

$$= \varepsilon^n + \binom{n}{1} \varepsilon^{n-1} r + \cdots + \binom{n}{z-1} \varepsilon^{n-z+1} r^{z-1} + R,$$

where $z$ was defined such that $(r)^z = 0$ and

$$\begin{aligned} R = & \sum_{\substack{\sum \nu_i = n-1,\ \sum \mu_i = 1 \\ \nu_i, \mu_i \geqq 0,\ \nu_1 < n-1}} (\varepsilon^{\nu_1} r^{\mu_1} \varepsilon^{\nu_2} r^{\mu_2} \cdots - \varepsilon^{n-1} r) + \\ & + \sum_{\substack{\sum \nu_i = n-2,\ \sum \mu_i = 2 \\ \nu_i, \mu_i \geqq 0,\ \nu_1 < n-2}} (\varepsilon^{\nu_1} r^{\mu_1} \varepsilon^{\nu_2} r^{\mu_2} \cdots - \varepsilon^{n-2} r^2) \\ & + \cdots + \sum_{\substack{\sum \nu_i = n-z+1,\ \sum \mu_i = z-1 \\ \nu_i, \mu_i \geqq 0,\ \nu_1 < n-z+1}} (\varepsilon^{\nu_1} r^{\mu_1} \varepsilon^{\nu_2} r^{\mu_2} \cdots - \varepsilon^{n-z+1} r^{z-1}). \end{aligned}$$

According to Lemma 7

$$\begin{aligned} ||R|| \leqq 2 \Bigg[ & \left(\binom{n}{1} - 1\right) \frac{||\varepsilon||^{n-1}}{\varrho^{n-1}} ||r|| + \left(\binom{n}{2} - 1\right) \frac{||\varepsilon||^{n-2}}{\varrho^{n-2}} \frac{||r||^2}{\varrho} + \\ & + \ldots + \left(\binom{n}{z-1} - 1\right) \frac{||\varepsilon||^{n-z+1}}{\varrho^{n-z+1}} \cdot \frac{||r||^{z-1}}{\varrho^{z-2}} \Bigg] = \frac{||\varepsilon||^{n-z+1}}{\varrho^{n-z+1}} \cdot P(n), \end{aligned}$$

where $P(n)$ is a polynomial in $n$. Since $||\varepsilon|| < \varrho$,

$$\lim_{n \to \infty} R = 0.$$

In order to prove that

$$(\varepsilon + r)^n - R = \varepsilon^n + \binom{n}{1}\varepsilon^{n-1}r + \cdots + \binom{n}{z-1}\varepsilon^{n-z+1}r^{z-1} \to 0,$$

we use the fact that $||2\varepsilon|| < \varrho$. Therefore $||(2\varepsilon)^m|| < \varrho$, or $||\varepsilon^m|| < \frac{\varrho}{2^m}$. Hence every term of this finite sum tends to zero.

Proof of Theorem I 1. Let $\mathfrak{S}/\mathfrak{R}$ be a field. $\mathfrak{S}/\mathfrak{R}$ is also a hypercomplex system over the field of real numbers and — as already mentioned above — for fields $\overline{A} = \overline{\mathfrak{L}}$, where $\overline{\mathfrak{L}}$ is the set of elements of $\mathfrak{S}/\mathfrak{R}$ generated by the elements of $\mathfrak{L}$[5]). By theorem II it follows that $A = \mathfrak{L}$.

Proof of theorem I 2. $\mathfrak{S}/\mathfrak{R}$ has no radical. Since it is not a field, it must therefore involve an idempotent element $e \neq \overline{1}$, where $\overline{1}$ is the unit element of $\mathfrak{S}/\mathfrak{R}$. $e$ cannot be an element of $\overline{A}$, and no real multiple $r e$, where $|r| > 1$, can belong to $\overline{A}$. However, since $e$ is a divisor of zero, its norm $= 0$ and hence every real multiple of $e$ belongs to $\overline{\mathfrak{L}}$. Hence $\overline{A} \subset \overline{\mathfrak{L}}$. By Theorem II follows:

$$A \subset \mathfrak{L}.$$

The two special rings mentionned above are examples for Theorem I 1 and Theorem I 2. The first is a ring which is a field modulo its radical. The radical consists namely of all real multiples of $\varepsilon$, and the ring of residue-classes is isomorphic with the field of real numbers. The open interval $(-1, 1)$ generates the classes of $\overline{A}$. Therefore $A$ consists of the open strip $(-1 + b\varepsilon, 1 + b\varepsilon)$, where $b$ is arbitrary real. — The second ring has no radical and two idempotent elements besides 1.

Theorem III. Every element $\nu$ of the boundary $\mathbf{N}$ of $A$ has the property that $\nu^k \in \mathbf{N}$ for every $k > 0$. If $\nu$ has an inverse $\nu^{-1}$, then either $\nu^{-1} \in \mathbf{N}$ or $\nu^{-k}$ is divergent. If an element $\nu'$ has the property that its powers $\{\nu'^k\}$ are neither divergent nor convergent to 0, it must be contained in $\mathbf{N}$.

Proof. Let $\nu$ be an element of $\mathbf{N}$ and $k$ a positive integer. We shall prove: $\nu^k$ is a limit of elements of $A$, but does not belong

[5]) As $\mathfrak{L}$ is defined by the representation of $\mathfrak{S}/\mathfrak{R}$, it is obvious that every element which is contained in a class of $\overline{\mathfrak{L}}$ belongs to $\mathfrak{L}$.

to $A$, whence it will follow: $\nu^k \epsilon \mathbf{N}$. Since $\nu = \lim \lambda_i$, where $\lambda_i \epsilon A$, it follows that $\nu^k = \lim_{i \to \infty} \lambda_i^k$. $\lambda_i^k$ is contained in $A$ again, but not $\lim_{i \to \infty} \lambda_i^k = (\lim_{i \to \infty} \lambda_i)^k$. For otherwise $(\lim_{i \to \infty} \lambda_i^k)^n \to 0$ $(n=1, 2, \ldots)$. Hence $(\lim \lambda_i)^{kn} \to 0$ and $\lim \lambda_i \epsilon A$, which contradicts the assumption.

If $\nu^{-1}$ exists we consider $\nu^{-1}$ as $\lim r_n \nu^{-1}$, where $r_n$ are real numbers $< 1$, $\lim r_n = 1$. If $\{\nu^{-k}\}$ does not diverge, the sequence $r_n \nu^{-1}$ is contained in $A$, and its limit, namely $\nu^{-1}$ is not contained in $A$, for otherwise by Lemma 1 the powers of $\nu$ would diverge. — The third part of Theorem III is nearly obvious. If the sequence $\{\nu^k\}$ is bounded, the element $r\nu$ belongs to $A$ for every real number $|r| < 1$. Hence $\nu = \lim r_n \nu$, where $0 < r_n < 1$ and $r_n \to 1$ and $r_n \nu \epsilon A$.

As a special result of Theorem III every compact subgroup which is contained in the multiplicative group of regular elements of $\mathfrak{S}$, is contained in $\mathbf{N}$.

## 3.

This part deals with hypercomplex systems which are fields. The classical result in this case is the theorem of Frobenius: The real and complex numbers and quaternions are the only fields which are hypercomplex systems over the field of real numbers. This theorem can be deduced from the following theorem on continuous groups which has been proved by E. Cartan [6]: *The n-dimensional Euclidean sphere is a group space only for* $n = 0, 1, 3$. From this it follows that: if a hypercomplex system $\mathfrak{S}$ with $n$ base-elements with respect to the field of real numbers is a field, $n$ must be one of the numbers 1, 2, 4. For let $\mathfrak{S}$ be a field. Its multiplicative group $M$ contains the positive real numbers as a self conjugate subgroup (this follows from the definition of hypercomplex systems). The quotientgroup with respect to this subgroup is a topological group and on the other side a topological image of the $(n-1)$-dimensional unit sphere, since the elements of the latter are in a continuous one-one-correspondance with the half-lines from the origin. Hence the sphere is a topological group, although its elements need not form a group in $M$. Hence $n - 1 = 0, 1, 3$. From this the theorem of Frobenius follows easily. In these three cases the multiplicative group $M$ is furthermore the direct product of the

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[6]) This result follows from E. Cartan [Annales Soc. Polonaise Math. 8 (1930), 181—225].

group formed by the elements of the unit sphere and the group of positive real numbers. The latter is a one-dimensional vector group. Since $M$ is locally compact, connected (for $n \geqq 2$), separable and the direct product of a compact group and a vector group, it is a simple example of a type of groups which has been investigated recently [7]).

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[7]) H. FREUDENTHAL, Topologische Gruppen mit genügend vielen fastperiodischen Funktionen [Annals of Math. (2) 37 (1936), 57—77].