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A Remark on Fourier Transforms and Functions Analytic in a Half-Plane

by

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Let \mathfrak{L}_p be the class of functions $f(x)$ measurable over $(-\infty, \infty)$ such that the integral

$$(1) \quad \int_{-\infty}^{\infty} |f(x)|^p dx, \quad p \text{ fixed, } p \geq 1,$$

is finite. It is well known that \mathfrak{L}_p becomes a complete linear metric vector space if we define its metric by

$$(2) \quad \|f(x)\|_p = \|f\|_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p}.$$

The value $p = \infty$ will also be admitted, with the agreement that \mathfrak{L}_∞ is the class of functions $f(x)$ continuous and bounded over $(-\infty, \infty)$ with the metric

$$(3) \quad \|f\|_\infty = \sup_{-\infty < x < \infty} |f(x)|.$$

Let $g(x) \in \mathfrak{L}_p$, $1 \leq p < \infty$. Put

$$(4) \quad G(u; a) = (2\pi)^{-1/2} \int_{-a}^a g(x) e^{-iux} dx.$$

If there exists a function $G(u) \in \mathfrak{L}_q$, $1 \leq q \leq \infty$, such that

$$(5) \quad \|G(u; a) - G(u)\|_q \rightarrow 0 \text{ as } a \rightarrow \infty,$$

then we say that $G(u)$ is the Fourier transform of $g(x)$ in \mathfrak{L}_q and write

$$(6) \quad G(u) = L^q G(u; a).$$

Such a function is known to exist when $1 \leq p \leq 2$ and $q = p' = \frac{p}{p-1}$.

Let $f(z)$ be a function of the complex variable $z = x + iy$, analytic in the upper half-plane $y > 0$. If, for almost all x ,

$f(z)$ tends to a definite limit, $f(x)$, as $z \rightarrow x$ along all non-tangential paths, the function $f(x)$ so defined is said to be the limit-function of $f(z)$. We shall assume that $f(x) \in \mathfrak{L}_p$, $1 \leq p < \infty$.

In the present note we are concerned with the class \mathfrak{A}_p of functions $f(z)$, analytic in the half-plane $y > 0$, each possessing a limit-function $f(x) \in \mathfrak{C}_p$, $1 \leq p < \infty$, and representable by its Cauchy integral,

$$(7) \quad f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{\xi - z} \equiv I(z; f),$$

or, what is equivalent ¹⁾, by its Poisson integral,

$$(8) \quad f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} \equiv Q(z; f).$$

The following problem presents itself naturally: find necessary and sufficient conditions which must be satisfied by a given function $f(x) \in \mathfrak{L}_p$ in order that $f(x)$ be the limit-function of a function $f(z) \in \mathfrak{A}_p$. In a recent note ²⁾ we gave a solution of this problem under the assumption that $f(x)$ possesses a Fourier transform (in a certain generalized sense). It turns out that this transform must vanish for negative values of its argument. The purpose of the present note is to investigate the same problem under a different assumption, that $f(x)$ itself is the Fourier transform in \mathfrak{L}_p of a function $g(u) \in \mathfrak{L}_q$, $1 \leq q \leq \infty$.

Theorem. Let $f(x)$ be the Fourier transform in \mathfrak{L}_p of a function $\varphi(u) \in \mathfrak{C}_q$, $1 \leq q \leq \infty$. In order that $f(x)$ be the limit-function of a function $f(z) \in \mathfrak{A}_p$ it is necessary and sufficient that $\varphi(u)$ vanish for $u > 0$.

Proof. For convenience we replace $\varphi(u)$ by $g(-u)$ and set

$$(9) \quad f(\xi) = L^p F(\xi; a),$$

$a \rightarrow \infty$

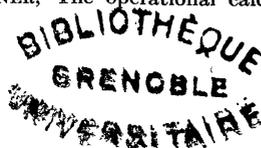
where

$$(10) \quad F(\xi; a) = \int_{-a}^a g(u) e^{i\xi u} du,$$

and $g(u) \in \mathfrak{L}_q$. We first assume that $1 < q < \infty$, and proceed to the computation of the Poisson integral of $f(\xi)$,

¹⁾ Cf. HILLE and TAMARKIN, On a theorem of Paley and Wiener [Annals of Math. (2) 34 (1933), 606—614].

²⁾ Loc. cit. ¹⁾ For a special case see N. WIENER, The operational calculus [Math. Ann. 95 (1926), 557—584; (580)].



$$(11) \quad Q(z; f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2} = \lim_{N \rightarrow \infty} \frac{1}{\pi} \int_{-N}^N \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2},$$

which obviously converges absolutely. In view of (9) we have

$$(12) \quad \int_{-N}^N \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2} = \int_{-\infty}^{\infty} g(u)du \int_{-N}^N \frac{ye^{i\xi u}d\xi}{(\xi-x)^2+y^2}.$$

On the other hand, by a direct computation,

$$(13) \quad \frac{y}{(\xi-x)^2+y^2} = \frac{1}{2} \int_{-\infty}^{\infty} e^{i(x-\xi)t} e^{-y|t|} dt.$$

Hence

$$(14) \quad \int_{-N}^N \frac{ye^{i\xi u}d\xi}{(\xi-x)^2+y^2} = \frac{1}{2} \int_{-\infty}^{\infty} e^{ixt-y|t|} dt \int_{-N}^N e^{i\xi(u-t)} d\xi = \\ = \int_{-\infty}^{\infty} e^{ixt-y|t|} \frac{\sin N(u-t)}{u-t} dt.$$

On substituting into (12) and interchanging the order of integration, which is clearly permissible, we get

$$(15) \quad \frac{1}{\pi} \int_{-N}^N \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2} = \int_{-\infty}^{\infty} e^{ixt-y|t|} dt \int_{-\infty}^{\infty} g(u) \frac{1}{\pi} \frac{\sin N(u-t)}{u-t} du = \\ = \int_{-\infty}^{\infty} e^{ixt-y|t|} dt \int_{-\infty}^{\infty} g(u) \mathfrak{D}_N(u-t) du,$$

where

$$\mathfrak{D}_N(u) = \frac{1}{\pi} \frac{\sin Nu}{u}$$

is the classical Dirichlet kernel. Now put

$$g_N(t) = \int_{-\infty}^{\infty} g(u) \mathfrak{D}_N(u-t) du.$$

It is known ¹⁾ that

$$\|g_N(t) - g(t)\|_q \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Consequently

$$(16) \quad Q(z; f) = \int_{-\infty}^{\infty} e^{ixt-y|t|} g(t) dt = \\ = \int_0^{\infty} e^{ixt} g(t) dt + \int_{-\infty}^0 e^{i\bar{z}t} g(t) dt \equiv \Omega_1(z) + \Omega_2(\bar{z}),$$

¹⁾ See, for instance, HILLE and TAMARKIN, On the theory of Fourier transforms [Bulletin of the Amer. Math. Soc. 39 (1933)].

where $\bar{z} = x - iy$. Here $\Omega_1(z)$ is analytic in z and $\Omega_2(\bar{z})$ is analytic in \bar{z} , while

$$\Omega_1(z) \rightarrow 0, \quad \Omega_2(\bar{z}) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Hence the vanishing of $\Omega_2(\bar{z})$ is a necessary and sufficient condition for the analyticity of $Q(z; f)$. Now, if $f(z) \in \mathfrak{A}_p$, then $f(z)$ is represented by its Poisson integral $Q(z; f)$, and $\Omega_2(\bar{z}) \equiv 0$; conversely, if $\Omega_2(\bar{z}) \equiv 0$, $Q(z; f)$ is analytic and represents a function $f(z) \in \mathfrak{A}_p$ whose limit-function is precisely $f(x)$. We see therefore that the condition $\Omega_2(\bar{z}) \equiv 0$ is necessary and sufficient in order that $f(x)$ be the limit-function of a function $f(z) \in \mathfrak{A}_p$. In view of the uniqueness theorem for Fourier integrals, however, the condition $\Omega_2(\bar{z}) \equiv 0$ is equivalent to the condition $g(t) = 0$ for $t < 0$; which is the desired result.

The treatment of the cases $q = 1$, $q = \infty$ is slightly more complicated, but formula (16) and all the subsequent conclusions will still be valid. When $q = 1$ or $q = \infty$, we apply the method of arithmetic means to evaluate the integral (11). Thus

$$\begin{aligned} & \frac{1}{N} \int_0^N dn \frac{1}{\pi} \int_{-n}^n \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2} = \lim_{a \rightarrow \infty} \int_{-a}^a g(u) du \frac{1}{N} \int_0^N dn \frac{1}{\pi} \int_{-n}^n \frac{ye^{i\xi u} d\xi}{(\xi-x)^2+y^2} = \\ & = \lim_{a \rightarrow \infty} \int_{-a}^a g(u) du \frac{1}{N} \int_0^N dn \int_{-\infty}^{\infty} e^{ixt-y|t|} \frac{1}{\pi} \frac{\sin n(u-t)}{u-t} dt = \\ (17) \quad & = \lim_{a \rightarrow \infty} \int_{-a}^a g(u) du \int_{-\infty}^{\infty} e^{ixt-y|t|} \frac{1}{2N\pi} \left[\frac{\sin \frac{N(u-t)}{2}}{\frac{u-t}{2}} \right]^2 dt = \\ & = \lim_{a \rightarrow \infty} \int_{-\infty}^{\infty} e^{ixt-y|t|} dt \int_{-a}^a g(u) \mathfrak{F}_N(u-t) du = \int_{-\infty}^{\infty} e^{ixt-y|t|} dt \int_{-\infty}^{\infty} g(u) \mathfrak{F}_N(u-t) du \end{aligned}$$

where

$$\mathfrak{F}_N(u) = \frac{1}{2N\pi} \left[\frac{\sin \frac{Nu}{2}}{\frac{u}{2}} \right]^2$$

is the Fejér kernel. Now put

$$g'_N(t) = \int_{-\infty}^{\infty} g(u) \mathfrak{F}_N(u-t) du.$$

When $N \rightarrow \infty$, the left-hand member of (17) tends to $Q(z; f)$

since the integral (11) converges. On the other hand, when $q = 1$,

$$\|g'_N(t) - g(t)\|_1 \rightarrow 0$$

while, when $q = \infty$, $g'_N(t) \rightarrow g(t)$ boundedly, and in fact uniformly over every finite range. Hence allowing $N \rightarrow \infty$, we obtain (16) again, and finish the proof as above. This method of course might have been used in the case $1 < q < \infty$ as well.

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