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PULL-BACK OF CURRENTS BY MEROMORPHIC MAPS

BY TUYEN TRUNG TRUONG

ABSTRACT. — Let X and Y be compact Kähler manifolds, and let $f : X \to Y$ be a dominant meromorphic map. Based upon a regularization theorem of Dinh and Sibony for DSH currents, we define a pullback operator $f^\sharp$ for currents of bidegrees (p,p) of finite order on Y (and thus for *any* current, since Y is compact). This operator has good properties as may be expected.

Our definition and results are compatible to those of various previous works of Meo, Russakovskii and Shiffman, Alessandrini and Bassanelli, Dinh and Sibony, and can be readily extended to the case of meromorphic correspondences.

We give an example of a meromorphic map f and two nonzero positive closed currents T_1, T_2 for which $f^\sharp(T_1) = -T_2$. We use Siu's decomposition to help further study on pulling back positive closed currents. Many applications on finding invariant currents are given.

RÉSUMÉ (*Pull-back de courants par des applications méromorphes*)

Soient X et Y des variétés kählériennes compactes, et $f: X \to Y$ une application méromorphe dominante. En nous basant sur un théorème de régularisation de Dinh et Sibony pour des courants DSH, nous définissons un opérateur pullback $f^\sharp$ pour les courants de bidegré (p,p) d'ordre fini sur Y (et donc pour *tout* courant, puisque Y est compact. Cet opérateur a des bonnes propriétés, comme attendu.

Notre définition et nos résultats sont compatibles avec ceux des travaux précédents de Meo, Russakovskii et Shiffman, Alessandrini et Bassanelli, Dinh et Sibony, et peut être facilement étendu au cas des correspondances méromorphes.

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Nous donnons un exemple d'application méromorphe f et deux courants fermés positifs non-nuls T_1, T_2 pour lesquels $f^\sharp(T_1) = -T_2$. Nous utilisons la décomposition de Siu pour faciliter l'étude des courants fermés positifs pullback. Nous donnons une multitude d'applications autour de la recherche de courants invariants.

1. Introduction

Let X and Y be two compact Kähler manifolds, and let $f : X \to Y$ be a dominant meromorphic map. For a (p,p)-current T on Y, we seek to define a pullback $f^\sharp(T)$ which has good properties. Such a pullback operator will be helpful in complex dynamics, in particular in the problem of finding invariant closed currents for a selfmap.

We let $\pi_X, \pi_Y : X \times Y \to X, Y$ be the two projections (When $X = Y$ we denote these maps by π_1 and π_2). Let $\Gamma_f \subset X \times Y$ be the graph of f, and let $\mathscr{C}_f \subset \Gamma_f$ be the critical set of π_Y, i.e., the smallest analytic subvariety of Γ_f so that the restriction of π_Y to $\Gamma_f - \mathscr{C}_f$ has fibers of dimension $dim(X) - dim(Y)$. For a set $B \subset Y$, we define $f^{-1}(B) = \pi_X(\pi_Y^{-1}(B) \cap \Gamma_f)$, and for a set $A \subset X$ we define $f(A) = \pi_Y(\pi_X^{-1}(A) \cap \Gamma_f)$.

If T is a smooth form on Y, then it is standard to define $f^*(T)$ as a current on X by the formula $f^*(T) = (\pi_X)_*(\pi_Y^*(T) \wedge [\Gamma_f])$. This definition descends to cohomology classes: If T_1 and T_2 are two closed smooth forms on Y having the same cohomology classes, then $f^*(T_1)$ and $f^*(T_2)$ have the same cohomology class in X. This allows us to define a pullback operator on cohomology classes. These considerations apply equally to continuous forms. When T is an arbitrary current on Y, we can still define $\pi_Y^*(T)$ as a current on $X \times Y$. However, in general it is not known how to define the wedge product of the two currents $\pi_Y^*(T)$ and $[\Gamma_f]$. This is the source of difficulty for defining pullback for a general current.

For some important classes of currents (positive closed and positive dd^c-closed currents, DSH currents, for definitions see the next subsection), there have been works on this topic by Meo [15], Russakovskii and Shiffman [16], Alessandrini and Bassanelli [1], Dinh and Sibony [10],[11]. We will give more details on these works later, but here will discuss only some general ideas used in these papers. Roughly speaking, in the works cited above, to define pullback of a (p,p) current T, the authors use approximations of T by sequences of smooth (p,p) forms T_n satisfying certain properties, and then define $f^\sharp(T) = \lim_{n\to\infty} f^*(T_n)$ if the limit exists and is the same for all such sequences. In order to have such approximations then T must have some positive property. In these definitions, the resulting pullback of a positive current is again positive.

Our idea for pulling back a general (p,p) current T is as follows. Assume that we have a well-define pullback $f^{\sharp}(T)$. Then for any smooth form of complement bidegree α we should have

$$\int_X f^{\sharp}(T)\wedge\alpha=\int_Y T\wedge f_*(\alpha).$$

The wedge product in the integral of the RHS is not well-defined in general. To define it we adapt the above idea, that is to use smooth approximations of either T or $f_*(\alpha)$. Fortunately, since Y is compact, any current T is of a finite order s. Moreover since $f_*(\alpha)$ is a DSH current, we can use the regularization theorem in [11] to produce approximation by C^s forms $K_n(f_*(\alpha))$ with desired properties. Then we define

$$\int_X f^{\sharp}(T)\wedge\alpha=\lim_{n\to\infty}\int_Y T\wedge K_n(f_*(\alpha)),$$

if the limit exists and is the same for such good approximations. The details of this definition will be given in the next subsection. We conclude this subsection commenting on the main results of this paper:

-Our pullback operator is compatible with the standard definition for continuous form and with the definitions in the works cited above.

-There are examples of losing positivity for currents of higher bidegrees when pulled back by meromorphic maps.

-We obtain a natural criterion on pulling back analytic varieties which, combined with Siu's decomposition, can be used to help further study pullback of general positive closed currents.

-We can apply the definition to examples having invariant positive closed currents of higher bidegrees whose supports are contained in pluripolar sets.

1.1. Definitions. — For convenience, let us first recall some facts about currents. The notations of positive and strongly positive currents in this paper follow the book [6]. For a current T on Y, let $supp(T)$ denote the support of T. Given $s\geq 0$, a current T is of order s if it acts continuously on the space of C^s forms on Y equipped with the usual C^s norm. A positive (p,p) current T is of order 0. If T is a positive (p,p) current then its mass is defined as $||T||=<T,\omega_Y^{dim(Y)-p}>$, where ω_Y is a given Kähler $(1,1)$ form of Y. If T is a closed current on Y, we denote by $\{T\}$ its cohomology class. If V is a subvariety in Y, we denote by $[V]$ the current of integration on V, which is a strongly positive closed current. We use $\rightharpoonup$ for weak convergence of currents.

For any p, we define $DSH^p(Y)$ (see Dinh and Sibony [8]) to be the space of (p,p) currents $T=T_1-T_2$, where T_i are positive currents, such that $dd^cT_i=\Omega_i^+-\Omega_i^-$ with $\Omega_i^{\pm}$ positive closed. Observe that $||\Omega_i^+||=||\Omega_i^-||$ since they are cohomologous to each other because $dd^c(T_i)$ is an exact current. Define the DSH-norm of T as

$$||T||_{DSH} := \min\{||T_1|| + ||T_2|| + ||\Omega_1^+|| + ||\Omega_2^+||,\ T_i,\ \Omega_i,\ \text{as above}\}.$$

Using compactness of positive currents, it can be seen that we can find T_i, $\Omega_i^\pm$ which realize $||T||_{DSH}$, hence the minimum on the RHS of the definition of DSH norm. We say that $T_n \rightharpoonup T$ in $DSH^p(Z)$ if T_n weakly converges to T and $||T_n||_{DSH}$ is bounded.

Our definition is modelled on the smooth approximations given by Dinh and Sibony [8]. However, some restrictions should be imposed on the approximations when we deal with the case of general maps:

1) Since any definition using local approximations will give a positive current as the resulting pullback of positive currents, in general we need to use global approximations in order to deal with the cases like the map J_X in Section 4.

2) For a general compact Kähler manifold, it is not always possible to approximate a positive closed current by positive closed smooth forms (see Proposition 2 for an example where even the negative parts of the approximation are not bounded).

3) The more flexible we allow in approximating currents, the more restrictive the maps and currents we can define pullback. For example, we have the following observation

Lemma 1. — *Assume that for any positive closed smooth (p,p) form T and for every sequence of positive closed smooth forms $T_n^\pm$ whose masses $||T_n^\pm||$ are uniformly bounded and $T_n^+ - T_n^- \rightharpoonup T$, then $f^*(T_n^+ - T_n^-) \rightharpoonup f^*(T)$. Then the same property holds for any positive closed (p,p) current T.*

Proof. — In fact, let $T_n^+ - T_n^-$ and $S_n^+ - S_n^-$ be two sequences weakly converging to a positive closed (p,p) current T, where $T_n^\pm$ and $S_n^\pm$ are positive closed smooth (p,p) forms having uniformly bounded masses. Then $(T_n^+ + S_n^-) - (T_n^- + S_n^+)$ is a sequence weakly converges to 0 with the same property, and because 0 is a smooth form, we must have $f^*(T_n^+ + S_n^-) - f^*(T_n^- + S_n^+)$ weakly converges to 0 by assumption. Hence $f^*(T_n^+ - T_n^-)$ and $f^*(S_n^+ - S_n^-)$ converges to the same limit. □

Roughly speaking, under the conditions of Lemma 1 then all positive closed currents can be pulled back. However, this is not true in general (see Example 2). We will restrict to use only good approximation schemes, defined as follows

Definition 1. — *Let Y be a compact Kähler manifold. Let $s \geq 0$ be an integer. We define a good approximation scheme by C^s forms for DSH currents on Y to be an assignment that for a DSH current T gives two sequences $K_n^\pm(T)$ (here $n = 1, 2, \ldots$) where $K_n^\pm(T)$ are C^s forms of the same bidegrees as T, so* *that $K_n(T) = K_n^+(T) - K_n^-(T)$ weakly converges to T, and moreover the following properties are satisfied:*

1) *Boundedness: The DSH norms of $K_n^{\pm}(T)$ are uniformly bounded.*

2) *Positivity: If T is positive then $K_n^{\pm}(T)$ are positive, and $||K_n^{\pm}(T)||$ is uniformly bounded with respect to n.*

3) *Closedness: If T is positive closed then $K_n^{\pm}(T)$ are positive closed.*

4) *Continuity: If $U \subset Y$ is an open set so that $T|_U$ is a continuous form then $K_n^{\pm}(T)$ converges locally uniformly on U.*

5) *Additivity: If T_1 and T_2 are two DSH^p currents, then $K_n^{\pm}(T_1 + T_2) = K_n^{\pm}(T_1) + K_n^{\pm}(T_2)$.*

6) *Commutativity: If T and S are DSH currents with complements bidegrees then*

$$\lim_{n\to\infty}[\int_Y K_n(T) \wedge S - \int_Y T \wedge K_n(S)] = 0.$$

7) *Compatibility with the differentials: $dd^c K_n^{\pm}(T) = K_n^{\pm}(dd^c T)$.*

8) *Condition on support: The support of $K_n(T)$ converges to the support of T. By this we mean that if U is an open neighborhood of supp(T), then there is n_0 so that when $n \geq n_0$ then supp$(K_n(T))$ is contained in U. Moreover, the number n_0 can be chosen so that it depends only on supp(T) and U but not on the current T.*

Now we give the definition of pullback operator on $DSH^p(Y)$ currents

DEFINITION 2. — *Let T be a $DSH^p(Y)$ current on Y. We say that $f^\sharp(T)$ is well-defined if there is a number $s \geq 0$ and a current S on X so that*

$$\lim_{n\to\infty} f^*(K_n(T)) = S,$$

for any good approximation scheme by C^{s+2} forms $K_n^{\pm}$. Then we write $f^\sharp(T) = S$.

By the commutativity property of good approximation schemes by C^s forms, if T is DSH so that $f^\sharp(T) = S$ is well-defined then for any smooth form α we have

$$\int_X f^\sharp(T) \wedge \alpha = \lim_{n\to\infty} \int_Y T \wedge K_n(f_*(\alpha)).$$

This equality helps to extend Definition 2 to any (p,p) current T. Recall that since Y is a compact manifold, any current on Y is of finite order.

DEFINITION 3. — *Let T be a (p,p) current of order s_0. We say that $f^\sharp(T)$ is well-defined if there is a number $s \geq s_0$ and a current S on X so that*

$$\lim_{n\to}\int_Y T\wedge K_n(f_*(\alpha) = \int_X S\wedge\alpha,$$

for any smooth form α on X and any good approximation scheme by C^{s+2} forms. Then we write $f^\sharp(T)=S$.

1.2. Results. — The operator $f^\sharp$ in Definitions 2 and 3 has the following properties:

LEMMA 2. — i) *If T is a continuous (p,p) form (not necessarily DSH) then $f^\sharp(T)$ is well-defined and coincides with the standard definition $f^*(T) := (\pi_1)_*(\pi_2^*(T)\wedge[\Gamma_f])$.*

ii) *$f^\sharp$ is closed under linear combinations: If $f^\sharp(T_1)$ and $f^\sharp(T_2)$ are well-defined, then so is $f^\sharp(a_1T_1+a_2T_2)$ for any complex numbers a_1 and a_2. Moreover $f^\sharp(a_1T_1+a_2T_2)=a_1f^\sharp(T_1)+a_2f^\sharp(T_2)$.*

iii) *If T is DSH and $f^\sharp(T)$ is well-defined, then the support of $f^\sharp(T)$ is contained in $f^{-1}(supp(T))$.*

iv) *If T is closed then $f^\sharp(T)$ is also closed, and in cohomology $\{f^\sharp(T)\} = f^*\{T\}$.*

For a smooth form, we can also define its pullback by using any desingularization of the graph of the map. We have an analog result

THEOREM 4. — *Let $\widetilde{\Gamma_f}$ be a desingularization of Γ_f, and let $\pi:\widetilde{\Gamma_f}\to X$ and $g:\widetilde{\Gamma_f}\to Y$ be the induced maps of $\pi_X|\Gamma_f$ and $\pi_Y|\Gamma_f$. Thus $\widetilde{\Gamma_f}$ is a compact Kähler manifold, π is a modification, and g is a surjective holomorphic map so that $f=g\circ\pi^{-1}$. Let T be a (p,p) current on Y. If $g^\sharp(T)$ is well-defined, then $f^\sharp(T)$ is also well-defined. Moreover $f^\sharp(T)=\pi_*(g^\sharp(T))$.*

The following result is a restatement of a result of Dinh and Sibony (Section 5 in [10]):

THEOREM 5. — *Let θ be a smooth function on $X\times Y$ so that $supp(\theta)\cap\Gamma_f\subset \Gamma_f-\mathscr{C}_f$. Then for any DSH^p current T on Y, $(\pi_X)_*(\theta[\Gamma_f]\wedge\pi_Y^*(T))$ is well-defined (see also [15]).*

The following result is a generalization of a result proved by Dinh and Sibony in the case of projective spaces (see Proposition 5.2.4 in [11])

THEOREM 6. — *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Assume that $\pi_X(\mathscr{C}_f)$ is of codimension $\geq p$. Then $f^\sharp(T)$ is well-defined for any positive closed (p,p) current T on Y. Moreover the following continuity holds: if T_j are positive closed (p,p) currents weakly converging to T then $f^\sharp(T_j)$ weakly converges to $f^\sharp(T)$.*

Example 1: In [3], Bedford and Kim studied the linear quasi-automorphisms. These are birational selfmaps f of rational 3-manifolds X so that both f and f^{-1} have no exceptional hypersurfaces. Hence we can apply Theorem 6 to pullback and pushforward any positive closed $(2,2)$ current on X. The map J_X in Section 4 is also a quasi-automorphism.

Below is a more general result, dealing with the case when the current T is good (say continuous) outside a closed set A whose preimage is not big.

THEOREM 7. — *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let $A \subset Y$ be a closed subset so that $f^{-1}(A) \cap \pi_X(\mathscr{C}_f) \subset V$ where V is an analytic subvariety of X having codim $\geq p$. If T is a positive closed (p,p)-current on Y which is continuous on $Y - A$, then $f^\sharp(T)$ is well-defined. Moreover, the following continuity holds: If $T_n^\pm$ are positive closed continuous (p,p) forms so that $||T_n^\pm||$ are uniformly bounded, $T_n^+ - T_n^- \rightharpoonup T$, and $T_n^\pm$ locally uniformly converges on $Y - A$, then $f^*(T_n^+ - T_n^-) \rightharpoonup f^\sharp(T)$.*

When $\pi_X(\mathscr{C}_f)$ has codimension $\geq p$, then we can choose $A = Y$ in Theorem 7, and thus recover Theorem 6.

As a consequence, we have the following result on pulling back of varieties:

COROLLARY 1. — *Let f, X, Y be as in Theorem 7. Let V be an analytic variety of Y of codim p. Assume that $f^{-1}(V)$ has codim $\geq p$. Then $f^\sharp[V]$ is well-defined.*

The assumptions in Corollary 1 are optimal, as can be seen from

Example 2: Let Y = a compact Kähler 3-fold, and let L_0 be an irreducible smooth curve in Y. Let $\pi : X \to Y$ be the blowup of Y along L_0. If L is an irreducible curve in Y which does not coincide with L_0 then $\pi^{-1}(L)$ has dimension 1, hence $\pi^\sharp[L]$ is well-defined. In contrast, it is expected that $\pi^\sharp[L_0]$ is not well-defined. One explanation (which is communicated to us by Professor Tien Cuong Dinh, see also the introduction in [1]) is that if $\pi^\sharp[L_0]$ was to be defined, then it should be a special $(2,2)$ current on the hypersurface $\pi^{-1}(L_0)$. However, we have too many $(2,2)$ currents on that hypersurface to point out a special one.

We have the following example of losing positivity

COROLLARY 2. — *Let X be the blowup of $\mathbb{P}^3$ along 4 points $e_0 = [1:0:0:0], e_1 = [0:1:0:0], e_2 = [0:0:1:0], e_3 = [0:0:0:1]$. Let $J : \mathbb{P}^3 \to \mathbb{P}^3$ be the Cremona map $J[x_0 : x_1 : x_2 : x_3] = [1/x_0 : 1/x_1 : 1/x_2 : 1/x_3]$, and let J_X be the lifting of J to X.*

For $0 \leq i \neq j \leq 3$, let $\Sigma_{i,j}$ be the line in $\mathbb{P}^3$ consisting of points $[x_0 : x_1 : x_2 : x_3]$ where $x_i = x_j = 0$. Let $\widetilde{\Sigma_{i,j}}$ be the strict transform of $\Sigma_{i,j}$ in X.

For any positive closed $(2,2)$ current T, $J_X^\sharp(T)$ is well-defined. Moreover, $J_X^\sharp([\widetilde{\Sigma_{0,1}}]) = -[\widetilde{\Sigma_{2,3}}]$ and $J_X^\sharp([\widetilde{\Sigma_{2,3}}]) = -[\widetilde{\Sigma_{0,1}}]$.

REMARK 1. — *The map J_X was given in Example 2.5 page 33 in [14] where the author showed that the map $J_X^* : H^{2,2}(X) \to H^{2,2}(X)$ does not preserve the cone of cohomology classes generated by positive closed $(2,2)$ currents.*

In Lemma 10, it will be shown that $(J_X^\sharp)^2(T) = T$ for any positive closed $(2,2)$ current T. Thus this example gives positive support to an open question posed in Section 6.

We conclude this subsection discussing pullback of a positive closed (p,p) current T in general. For $c > 0$ define $E_c(T) = \{y \in Y : \nu(T,y) \geq c\}$, where $\nu(T,y)$ is the Lelong number of T at y (see [6] for definition). Then by the semi-continuity theorem of Siu (see [18], and also [6]), $E_c(T)$ is an analytic subvariety of Y of codimension $\geq p$. Moreover, we have a decomposition

$$T = R + \sum_{j=1}^{\infty} \lambda_j [V_j],$$

where $\lambda_j \geq 0$, V_j is an irreducible analytic variety of codimension p and is contained in $E(T) = \cup_{c>0} E_c(T)$, and R is a positive closed current such that $E_c(R)$ has codimension $> p$ for all $c > 0$. Note that $E(T) =$ the union of $E_c(T)$'s for rational numbers $c > 0$, hence is a (at most) countable union of analytic varieties.

THEOREM 8. — *Notations are as above. Assume that for any irreducible variety V of codimension p contained in $E(T)$, then $f^{-1}(V)$ has codimension $\geq p$. Then $f^\sharp(\sum_{j=1}^\infty \lambda_j [V_j])$ is well-defined and is equal to $\sum_{j=1}^\infty \lambda_j f^\sharp[V_j]$. Hence $f^\sharp(T)$ is well-defined iff $f^\sharp(R)$ is well-defined.*

1.3. Compatibility with previous works. — In this subsection we compare our results with the results in previous papers.

The pullback of positive closed $(1,1)$ currents was defined by Meo [15] for finite holomorphic maps between complex manifolds (not necessarily compact or Kähler). Our definition coincides with his in the case of compact Kähler manifolds

COROLLARY 3. — *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive closed $(1,1)$-current on Y. Then $f^\sharp(T)$ is well-defined, and coincides with the usual definition.*

Proof. — Since $\pi_X(\mathscr{C}_f)$ is a proper analytic subvariety of X, it has codimension ≥ 1, thus we can apply Theorem 6. □

The pullback of positive dd^c closed $(1,1)$ currents were defined by Alessandrini - Bassanel [1] and Dinh -Sibony [10] under several contexts. Our definition coincides with theirs in the case of compact Kähler manifolds

COROLLARY 4. — *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive dd^c-closed $(1,1)$-current on Y. Then $f^\sharp(T)$ is well-defined, and coincides with the usual definition.*

Proof. — Consider a desingulariztion $\widetilde{\Gamma_f}$ and $\pi : \widetilde{\Gamma_f} \to X$ and $g : \widetilde{\Gamma_f} \to Y$ as in Theorem 4. Then it suffices to show that $g^\sharp(T)$ is well-defined. This later follows from the proof of Theorem 5.5 in [10]. □

For a map $f : \mathbb{P}^k \to \mathbb{P}^k$, Russakovskii and Shiffman [16] defined the pullback of a linear subspace V of codimension p in $\mathbb{P}^k$ for which $\pi_2^{-1}(V) \cap \Gamma_f$ has codimension $\geq p$ in Γ_f. It can be easily seen that this is a special case of Corollary 1. In the same paper, we also find a definition for pullback of a measure having no mass on $\pi_Y(\mathscr{C}_f)$. Our definition coincides with theirs

THEOREM 9. — *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive measure having no mass on $\pi_Y(\mathscr{C}_f)$. Then $f^\sharp(T)$ is well-defined, and coincides with the usual definition. Moreover, if T has no mass on proper analytic subvarieties of Y, then $f^\sharp(T)$ has no mass on proper analytic subvarieties of X.*

1.4. Applications. — We now discuss the problem of finding an invariant current of a dominant meromorphic self-map f. Let $f : X \to X$ be a dominant meromorphic selfmap of a compact Kähler manifold X of dimension k. Define by $r_p(f)$ the spectral radius of $f^* : H^{p,p}(X) \to H^{p,p}(X)$. Then the p-th dynamical degree of f is defined as follows:

$$\delta_p(f) = \lim_{n\to\infty} (r_p(f^n))^{1/n},$$

where $f^n = f \circ f \circ \cdots \circ f$ is the n-th iteration of f. When $p = dim(X)$ then $\delta_p(f)$ is the topological degree of f.

The map f is called p-algebraic stable (see, for example [11]) if $(f^*)^n = (f^n)^*$ as linear maps on $H^{p,p}(X)$ for all $n = 1, 2, \ldots$. When this condition is satisfied, it follows that $\delta_p(f) = r_p(f)$, thus helps in determining the p-th dynamical degree of f.

There is also the related condition of p-analytic stable (see [11]) which requires that

1) $(f^n)^\sharp(T)$ is well-defined for any positive closed (p,p) current T and any $n\geq 1$.

2) Moreover, $(f^n)^\sharp(T) = (f^\sharp)^n(T)$ for any positive closed (p,p) current T and any $n\geq 2$.

Since $H^{p,p}(X)$ is generated by classes of positive closed smooth (p,p) forms, p-analytic stability implies p-algebraic stability. In fact, if $\pi_1(\mathscr{C}_f)$ has codimension $\geq p$, then f is p-analytic stable iff it is p-algebraic stable and satisfies condition 1) above so that $(f^\sharp)^n(\alpha)$ is positive closed for any positive closed smooth (p,p) form and for any $n\geq 1$. Hence 1-algebraic stability is the same as 1-analytic stability.

For any map f then f is k-algebraic stable where k =dimension of X. If f is holomorphic then it is p-algebraic stable for any p. We have the following result

Lemma 3. — *Let X be a compact Kahler manifold with a Kahler form ω_X and $f: X\rightarrow X$ be a dominant meromorphic map. Assume that $\pi_1(\mathscr{C}_f)$ has codimension $\geq p$ and f is p-analytic stable. Let $0\neq\theta$ be an eigenvector with respect to the eigenvalue $\lambda = r_p(f)$ the spectral radius of the linear map $f^*: H^{p,p}(X)\rightarrow H^{p,p}(X)$. Assume moreover that $||(f^n)^*(\omega_X^p)||\sim\lambda^n$ as $n\rightarrow\infty$. Then there is a closed (p,p) current T which is a difference of two positive closed (p,p) currents satisfying $\{T\}=\theta$ and $f^\sharp(T)=\lambda T$.*

Since f is p-analytic stable, the condition on $||(f^n)^*(\omega_X^p)||$ can be easily checked by looking at the Jordan form for f^* (see e.g., [14]). Variants of this condition are also available. Lemma 3 generalizes the results for the standard case $p=1$ and for the case $X=\mathbb{P}^k$ in Dinh and Sibony [11]. We suspect that the pseudo-automorphism in [3] are 2-analytic stable, the latter may probably be checked using the method of the proof of Lemma 10. If so, Lemma 3 can be applied to these maps to produce invariant closed $(2,2)$ currents. However, these invariant currents may not be unique, since for the maps in [3] the first and second dynamical degrees are the same. The map J_X in Section 4 has invariant $(2,2)$ current $\widetilde{\Sigma_{0,1}}-\widetilde{\Sigma_{2,3}}$ which is not positive. The relation between p-algebraic and p-analytic stabilities to the problem of finding invariant currents will be discussed more in Sections 5 and 6.

Let us continue with an application concerning invariant positive closed currents whose supports are contained in pluripolar sets.

Corollary 5. — *Let $f_1:\mathbb{P}^{k_1}\rightarrow\mathbb{P}^{k_1}$ and $f_2:\mathbb{P}^{k_2}\rightarrow\mathbb{P}^{k_2}$ be dominant rational maps not 1-algebraic stable, of degrees d_1 and d_2 respectively. Then there* *is a nonzero positive closed* $(2,2)$ *current* T *on* $\mathbb{P}^{k_1}\times\mathbb{P}^{k_2}$ *with the following properties:*

1) $f^{\sharp}(T)$ *is well-defined and moreover* $f^{\sharp}(T)=d_1d_2T$*, here* $f=f_1\times f_2$.
2) *The support of* T *is pluripolar.*

The existence of Green currents T_1 and T_2 for f_1 and f_2 were proved by Sibony [17] (see also [5]). The current T is in fact the product $T_1\times T_2$. Its support is contained in a countable union of analytic varieties of codimension 2 in $\mathbb{P}^{k_1}\times\mathbb{P}^{k_2}$. The subtlety in proving the Conclusion 1) of Corollary 5 lies in the fact that for general choices of f_1 and f_2 it is not clear that we can pullback every positive closed $(2,2)$ currents, and even if we can do so, we may not have the continuity on pullback like in the case of $(1,1)$ currents.

Corollary 6. — *Let* X *be a compact Kähler manifold of dimension* k*, and let* $f:X\to X$ *be a dominant meromorphic map. Assume that* f *has large topological degree, i.e.,* $\delta_k(f)>\delta_{k-1}(f)$*. Then* f *has an invariant positive measure* μ*, i.e.,* $f^*(\mu)=\delta_k(f)\mu$.

The result of Corollary 6 belongs to Guedj [13]. Our proof here is slightly different from his proof in that we don't need to show that the measure μ has no mass on proper analytic subvarieties.

Corollary 7. — *Let* X *be a compact Kähler manifold, and let* $f:X\to X$ *be a surjective holomorphic map. Let* λ *be a real eigenvalue of* $f^*:H^{p,p}(X)\to H^{p,p}(X)$*, and let* $0\neq\theta_\lambda\in H^{p,p}(X)$ *be an eigenvector with eigenvalue* λ*. Assume moreover that* $|\lambda|>\delta_{p-1}(f)$*. Then there is a closed current* T *of order* 2 *with* $\{T\}=\theta_\lambda$ *so that* $f^{\sharp}(T)$ *is well-defined, and moreover* $f^{\sharp}(T)=\lambda T$.

Example 3: Let $X=\mathbb{P}^2_{w_1}\times\mathbb{P}^2_{w_2}\times\mathbb{P}^2_{w_3}$, and let $f:X\to X$ to be $f(w_1,w_2,w_3)=(P_2(w_2),P_3(w_3),P_1(w_1))$ where $P_1,P_2,P_3:\mathbb{P}^2\to\mathbb{P}^2$ are surjective holomorphic maps of degrees ≥ 2, and not all of them are submersions (For example, we can choose one of them to be $P[z_0:z_1:z_2]=[z_0^d:z_1^d:z_2^d]$ for some integer $d\geq 2$). Theorem 7 can be applied to find invariant currents for f.

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The rest of this paper is organized as follows: In Section 2 we collect some simple but helpful properties of positive currents. Then we consider the pull-back operator in Section 3. In Section 4 we explore the properties of the map J_X. We will also give results concerning the operator f^o on positive closed currents defined by Dinh-Nguyen [7] (see Proposition 1), and concerning the regularization results of Dinh-Sibony [8] (see Proposition 2). In Section 5 we consider invariant currents. We give examples of good approximation schemes and discuss some open questions in the last section.

2. Some preliminary results

In this section, we collect some simple but useful facts about positive currents. All the results presented are well known, but we include the proofs for the convenience of the readers. Through out this section, let Z be a compact Kähler manifold of dimension k, with a Kähler $(1,1)$ form ω_Z. Let $\pi_1, \pi_2 : Z \times Z \to Z$ be the projections, and let $\Delta_Z \subset Z \times Z$ be the diagonal.

Lemma 4. — *Let T be a continuous real (p,p) form on Z. Then there exists a constant $A > 0$ independent of T so that*

$$A||T||_{L^\infty}\omega_Z^p \pm T$$

are both strongly positive forms.

Proof. — Since Z is a compact Kähler manifold, there is a finite covering of Z by open sets U's each of them is biholomorphic to a ball in $\mathbb{C}^k$. Using a partition of unity for this covering, we reduce the problem to the case where T is a continuous real (p,p) form compactly supported in a ball in $\mathbb{C}^k$. Since T is a real form, we can write

$$T = \sum_{|I|=|J|=p} (f_{I,J} dz_I \wedge d\overline{z_J} + \overline{f_{I,J}} d\overline{z_I} \wedge dz_J),$$

where $f_{I,J}$ are bounded continuous complex-valued functions. By Lemma 1.4 page 130 in [6], $dz_I \wedge d\overline{z_J}$ can be represented as a linear combination of strongly positive forms with complex coefficients. Let us write

$$dz_I \wedge d\overline{z_J} = \sum_{i \in \mathscr{C}} \alpha_{I,J,i} \varphi_i,$$

where $\mathscr{C}$ is a finite set independent of I and J, φ_i are fixed strongly positive (p,p) forms, and $\alpha_{I,J,i}$ are complex numbers. Then

$$d\overline{z_I} \wedge dz_J = \sum_{i \in \mathscr{C}} \overline{\alpha_{I,J,i}} \varphi_i.$$

Hence T can be represented in the form

$$T = \sum_{|I|=|J|=p} \sum_{i \in \mathscr{C}} f_{I,J,i} \varphi_i,$$

where $f_{I,J,i} = \alpha_{I,J,i} f_{I,J} + \overline{\alpha_{I,J,i} f_{I,J}}$ are bounded continuous real-valued functions satisfying $||f_{I,J,i}||_{L^\infty} \leq A||T||_{L^\infty}$ for some constant $A > 0$ independent of T. Each of the forms φ_i can be bound by a multiplicity of ω_Z^p, hence we can find a constant $A > 0$ independent of T so that $A||T||_{L^\infty} \omega_Z^p \pm T$ are strongly positive forms. □

Lemma 5. — *Let S be a strongly positive current on Z, and let T be a continuous positive (p,p) form. Then $S \wedge T$ is well-defined and is a positive current.*

Similarly, if S is a positive current on Z, and T is a continuous strongly positive (p,p) form then $S \wedge T$ is well-defined and is a positive current.

Proof. — Since S is a strongly positive current on Z, it is of order zero, hence can be wedged with a continuous form. Thus $S \wedge T$ is well-defined. Now we show that $S \wedge T$ is a positive current.

We can approximate T uniformly by smooth (p,p) forms T_n. Then use Lemma 4, there is a constant $A > 0$ independent of n so that $A||T - T_n||_{L^\infty} \omega_Z^p \pm (T - T_n)$ are strongly positive. Since T is a positive form, this implies that $T_n + A||T - T_n||_{L^\infty} \omega_Z^p$ are positive for all n. Since the current S acts continuously on C^0 forms, and we chose T_n to converge uniformly to T, we have that

$$S \wedge T = \lim_{n \to \infty} S \wedge T_n = \lim_{n \to \infty} S \wedge (T_n + A||T - T_n||_{L^\infty} \omega_Z^p).$$

Since S is strongly positive and $T_n + A||T - T_n||_{L^\infty} \omega_Z^p$ are positive smooth forms, $S \wedge (T_n + A||T - T_n||_{L^\infty} \omega_Z^p)$ are positive currents. Thus $S \wedge T$ is the weak limit of a sequence of positive currents, hence itself a positive current. □

LEMMA 6. — *Let T be a positive closed (p,p) current on Z. Then there is a closed smooth (p,p) form θ on Z so that $\{\theta\}=\{T\}$ in cohomology, and moreover*

$$-A||T||\omega_Z^p\leq \theta\leq A||T||\omega_Z^p.$$

Here $A>0$ is independent of T.

Proof. — Let $\pi_1,\pi_2 : Z\times Z\rightarrow Z$ be the two projections, and let Δ_Z be the diagonal of Z. Let Δ be a closed smooth form on $Z\times Z$ representing the cohomology class of $[\Delta_Z]$. If we define

$$\theta=(\pi_1)_*(\pi_2^*(T)\wedge \Delta),$$

it is a smooth (p,p) current on Z having the same cohomology class as T. Since Z is compact, so is $Z\times Z$, and by Lemma 4 there is a constant $A>0$ so that $A(\pi_1^*\omega_Z+\pi_2^*\omega_Z)^{dim(Z)}\pm\Delta$ are strongly positive forms. Since T is a positive current, by Lemma 5 it follows that

$$\theta=(\pi_1)_*(\pi_2^*(T)\wedge \Delta)\leq A(\pi_1)_*((\pi_1^*\omega_Z+\pi_2^*\omega_Z)^{dim(Z)}\wedge\pi_2^*(T))=A||T||\omega_Z^p.$$

Similarly, we have also $\theta\geq -A||T||\omega_Z^p$. □

LEMMA 7. — *Let T_j be a sequence of $DSH^p(Z)$ currents converging in DSH to a current T. Then for any continuous $(k-p,k-p)$ form S we have*

$$\lim_{j\rightarrow\infty}\int_Z T_j\wedge S=\int_Z T\wedge S.$$

Proof. — By assumption, T_j weakly converges to T in the sense of currents, and moreover we can write $T_j=T_j^+-T_j^-$ and $T=T^+-T^-$ where $T_j^{\pm}$ and $T^{\pm}$ are positive currents, whose norms are uniformly bounded. Since S is a continuous form, we can find a sequence of smooth forms S_n uniformly converging to S, i.e., we can choose S_n smooth forms so that

$$-\frac{1}{n}\omega_Z^{k-p}\leq S-S_n\leq \frac{1}{n}\omega_Z^{k-p}.$$

Hence by Lemma 5, for any j and n

$$-\frac{1}{n}(||T_j^+||+||T_j^-||)+\int_Z T_j\wedge S_n\leq\int_Z T_j\wedge S\leq\frac{1}{n}(||T_j^+||+||T_j^-||)+\int_Z T_j\wedge S_n.$$

Hence given a number n, letting $j\rightarrow\infty$, using the fact that $T_j\rightharpoonup T$, S_n is smooth, and $||T_j||_{DSH}$ is uniformly bounded

$$-\frac{A}{n}+\int_Z T\wedge S_n\leq\liminf_{j\rightarrow\infty}\int_Z T_j\wedge S\leq\limsup_{j\rightarrow\infty}\int_Z T_j\wedge S\leq\frac{A}{n}+\int_Z T\wedge S_n,$$

where $A > 0$ is independent of n. Since T is a difference of two positive currents, it is a current of order zero, hence acting continuously on the space of continuous forms equipped with the sup norm. Since S_n converges uniformly to S, we have

$$\lim_{n\to\infty}\int_Z T\wedge S_n = \int_Z T\wedge S.$$

Combining this and the previous inequalities, letting $n\to\infty$, we obtain

$$\lim_{j\to\infty}\int_Z T_j\wedge S = \int_Z T\wedge S,$$

as wanted. □

3. Pull-back of DSH currents

First, we show the good properties of the operator $f^\sharp$

Proof. — (Of Lemma 2) Let $K_n = K_n^+ - K_n^-$ be a good approximation scheme by C^2 forms.

i) If T is a continuous form, then $K_n^\pm(T)$ uniformly converges on Y. Hence there are continuous forms T^+, T^- and constants ϵ_n decreasing to 0, so that $T = T^+ - T^-$ and $-\epsilon_n\omega_Y^p \leq K_n^\pm(T) - T^\pm \leq \epsilon_n\omega_Y^p$. Then

$$-\epsilon_n f^*(\omega_Y^p) \leq f^*(K_n^\pm(T)) - f^*(T^\pm) \leq \epsilon_n f^*(\omega_Y^p),$$

and thus $f^*(K_n^\pm(T))$ weakly converges to $f^*(T^\pm)$. Therefore, $f^*(K_n^+(T) - K_n^-(T))$ weakly converges to $f^*(T^+) - f^*(T^-) = f^*(T)$. This shows that $f^\sharp(T)$ is well-defined and coincides with the usual definition.

ii) Follows easily from the definition.

iii) If T is DSH, the result follows from the definition and the fact that support of $K_n(T)$ converges to support of T.

iv) First we show that if $T = T_1 + dd^cT_2$ is closed, where T_1 is a (p,p) current and T_2 is a $(p-1,p-1)$ current both of order 0, and $f^\sharp(T)$ is well-defined, then $f^\sharp(T)$ is closed.

From the assumption, it follows that T_1 is closed. To show that $f^\sharp(T)$ is closed, it suffices to show that if α is a d-exact $(dim(X)-p, dim(X)-p)$ smooth form, then

$$\int_X f^\sharp(T)\wedge\alpha = 0.$$

In fact, by definition

$$\int_X f^\sharp(T)\wedge\alpha = \lim_{n\to\infty}\int_Y T_1\wedge K_n(f_*(\alpha)) + T_2\wedge dd^cK_n(f_*(\alpha)).$$

By the dd^c lemma, there is a smooth form β so that $\alpha = dd^c(\beta)$. Then by the compatibility with differentials of good approximation schemes, we have

$K_n(f_*(\alpha)) = K_n(f_*(dd^c\beta)) = dd^cK_n(f_*(\beta))$ is d-exact. Thus each of the two integrals in the RHS of the above equality is 0, independent of n. Hence the limit is 0 as well.

Now we show that $\{f^\sharp(T)\} = f^*\{T\}$. Let θ be a smooth closed form so that $\{T\} = \{\theta\}$. Then there is a current R so that $T - \theta = dd^c(R)$. If α is a closed smooth form then

$$\begin{aligned}\int_X (f^\sharp(T) - f^*(\theta)) \wedge \alpha &= \lim_{n\to\infty} \int_Y (T-\theta)\wedge K_n(f_*(\alpha))\\ &= \lim_{n\to\infty} \int_Y dd^c(R)\wedge K_n(f_*(\alpha))\\ &= \lim_{n\to\infty} \int_Y R\wedge K_n(f_*(dd^c\alpha)) = 0,\end{aligned}$$

since $dd^c(\alpha) = 0$. This shows that $\{f^\sharp(T)\} = \{f^*(\theta)\}$, and the latter is $f^*\{T\}$ by definition. □

Proof of Theorem 4. — Assume that $g^\sharp(T)$ is well-defined with respect to number s in Definition 3. Let α be a smooth form on X and K_n a good approximation scheme by C^{s+2} forms on Y. Then $f_*(\alpha) = g_*(\pi^*\alpha)$. Since $\pi^*(\alpha)$ is smooth on $\widetilde{\Gamma_f}$ and $g^\sharp(T)$ is well-defined, we have

$$\begin{aligned}\lim_{n\to\infty} \int_Y T\wedge K_n(f_*\alpha) &= \lim_{n\to\infty} \int_Y T\wedge K_n(g_*\pi^*\alpha)\\ &= \int_{\widetilde{\Gamma_f}} g^\sharp(T)\wedge\pi^*\alpha = \int_X \pi_*g^\sharp(T)\wedge\alpha,\end{aligned}$$

as wanted. □

Now we give the proofs of Theorems 5, 6, 7, 8 and 9.

Proof of Theorem 5. — In this proof we use the value $s = 0$ in Definitions 2 and 3. The proof is the same as the proof of Lemma 3.3 in [10] using the following observations:

i) Lemma 3.1 in [10] applies for C^2 forms T_n. Hence Lemma 3.3 in [10] applies to C^2 forms T_n.

ii) Let us choose two difference good approximation schemes by C^2 forms $K_n = K_n^+ - K_n^-$ and $H_n = H_n^+ - H_n^-$. Then the sequences $K_n^+(T) + H_n^-(T)$ and $K_n^-(T) + H_n^+(T)$ converges in DSH to a same positive current.

iii) Apply Lemma 3.3 in [10] to the sequences $K_n^+(T)+H_n^-(T)$ and $K_n^-(T)+H_n^+(T)$, we conclude that in $\Gamma_f - \mathscr{C}_f$, the sequences $f^*(K_n^+(T)) + f^*(H_n^-(T))$ and $f^*(K_n^-(T)) + f^*(H_n^+(T))$ converges to a same current. Thus we have that the sequences $f^*(K_n^+(T) - K_n^-(T))$ and $f^*(H_n^+(T) - H_n^-(T))$ converges in $\Gamma_f - \mathscr{C}_f$ to a same current. □

Proof of Theorem 6. — We follow the proof of Proposition 5.2.4 in [11] with some appropriate modifications. Let $K_n = K_n^+ - K_n^-$ be a good approximation scheme by C^2 forms.

a) First we show that $f^\sharp(T)$ is well-defined for any positive closed (p,p) current T.

Let θ be a smooth closed (p,p) form so that $\{\theta\} = \{T\}$ in cohomology classes. Since $T = (T-\theta)+\theta$, by Lemma 2, to show that $f^\sharp(T)$ is well-defined, it is enough to show that $f^\sharp(T-\theta)$ is well-defined. By dd^c lemma (see also [9]), there is a DSH current R so that $T-\theta = dd^c(R)$. Hence to show that $f^\sharp(T-\theta)$ is well-defined, it is enough to show that $f^\sharp(R)$ is well-defined.

We can write $K_n(R) = R_{1,n} - R_{2,n}$, where $R_{i,n}$ are positive $(p-1,p-1)$ forms of class C^2, and $dd^c(R_{i,n}) = \Omega_{i,n}^+ - \Omega_{i,n}^-$, where $\Omega_{i,n}^\pm$ are positive closed C^2 (p,p) forms. Moreover, $||R_{i,n}||$ and $||\Omega_{i,n}^\pm||$ are uniformly bounded.

i) First we show that $||f^*(R_{i,n})||$ are uniformly bounded. Theorem 5 implies that $f^*(R_{i,n})$ converges in $X - \pi_X(\mathscr{C}_f)$ to a current. Since the codimension of $\pi_X(\mathscr{C}_f)$ is $\geq p$, it is weakly p-pseudoconvex (see Lemma 5.2.2 in [11]). Hence there exists a smooth $(dim(X)-p, dim(X)-p)$ form Θ defined on X so that $dd^c\Theta \geq 2\omega_X^{dim(X)-p+1}$ on $\pi_X(\mathscr{C}_f)$. We can choose a small neighborhood V of $\pi_X(\mathscr{C}_f)$ so that $dd^c\Theta \geq \omega_X^{dim(X)-p+1}$ on V. Since $R_{i,n}$ is a positive C^2 form, $f^*(R_{i,n})$ is well defined and is a positive current. Since $f^*(R_{i,n})$ converges in $X-\pi_X(\mathscr{C}_f)$ to a current, it follows that $||f^*(R_{i,n})||_{X-V}$ is bounded. Because

$$||f^*(R_{i,n})||_X = ||f^*(R_{i,n})||_{X-V} + ||f^*(R_{i,n})||_V,$$

to show that $||f^*(R_{i,n})||_X$ is bounded, it is enough to estimate $||f^*(R_{i,n})||_V$. We have

$$\begin{aligned}||f^*(R_{i,n})||_V &= \int_V f^*(R_{i,n}) \wedge \omega_X^{dim(X)-p+1} \leq \int_V f^*(R_{i,n}) \wedge dd^c(\Theta)\\ &= \int_X f^*(R_{i,n}) \wedge dd^c(\Theta) - \int_{X-V} f^*(R_{i,n}) \wedge dd^c(\Theta).\end{aligned}$$

The term

$$|\int_{X-V} f^*(R_{i,n}) \wedge dd^c(\Theta)|$$

can be bound by $||f^*(R_{i,n})||_{X-V}$, and thus is bounded. We estimate the other term: Since X is compact

$$\begin{aligned}|\int_X f^*(R_{i,n}) \wedge dd^c(\Theta)| &= |\int_X dd^c f^*(R_{i,n}) \wedge \Theta| = |\int_X f^*(dd^c R_{i,n}) \wedge \Theta|\\ &= |\int_X f^*(\Omega_{i,n}^+ - \Omega_{i,n}^-) \wedge \Theta|.\end{aligned}$$

Since $\Omega^{\pm}_{i,n}$ are positive closed C^2 forms, $f^*(\Omega^{\pm}_{i,n})$ are well-defined and are positive closed currents. Choose a constant $A>0$ so that $A\omega_X^{dim(X)-p}\pm\Theta$ are strictly positive forms, we have

$$\begin{aligned}
&|\int_X f^*(\Omega^{+}_{i,n}-\Omega^{-}_{i,n})\wedge\Theta|\\
&\leq |\int_X f^*(\Omega^{+}_{i,n})\wedge\Theta|+|\int_X f^*(\Omega^{-}_{i,n})\wedge\Theta|\\
&\leq A\int_X f^*(\Omega^{+}_{i,n})\wedge\omega_X^{dim(X)-p}+A\int_X f^*(\Omega^{-}_{i,n})\wedge\omega_X^{dim(X)-p}.
\end{aligned}$$

Since $\Omega^{\pm}_{i,n}$ are positive closed currents with uniformly bounded norms, the last integrals are uniformly bounded as well.

ii) From i) we see that for any good approximation scheme by C^2 forms K_n, the sequence $f^*(R_{1,n})-f^*(R_{2,n})$ has a convergent sequence. We now show that the limit is unique, hence complete the proof of Theorem 6. So let τ be the limit of the sequence $f^*(R_{1,n})-f^*(R_{2,n})$. Such a τ is a DSH^{p-1} current by the consideration in i). Let $H_n=H_n^+-H_n^-$ be another good approximation scheme by C^2 forms, and let τ' be the corresponding limit, which is in DSH^{p-1}. We want to show that $\tau=\tau'$. or equivalently, to show that $\tau-\tau'=0$.

By Theorem 5, $\tau-\tau'=0$ in $X-\pi_X(\mathscr{C}_f)$. Hence support of $\tau-\tau'$ is contained in $\pi_X(\mathscr{C}_f)$. Since $\tau-\tau'$ is in DSH^{p-1}, it is a $\mathbb{C}$-flat $(p-1,p-1)$ current (see Bassanelli [2]). Because the codimension of $\pi_X(\mathscr{C}_f)$ is $\geq p$, it follows by Federer-type support theorem for $\mathbb{C}$-flat currents (see Theorem 1.13 in [2]) that $\tau-\tau'=0$ identically.

b) Finally, we show that if T_j are positive closed (p,p) currents converging in DSH to T then $f^{\sharp}(T_j)$ weakly converges to $f^{\sharp}(T)$.

We let $\pi_1,\pi_2:Y\times Y\to Y$ be the projections, and let Δ_Y be the diagonal. As in the proof of Lemma 6, we choose Δ to be a smooth closed $(dim(Y),dim(Y))$ on Y having the same cohomology class with $[\Delta_Y]$. We write $\Delta=\Delta^+-\Delta^-$, where $\Delta^{\pm}$ are strongly positive smooth closed $(dim(Y),dim(Y))$ forms. If we define $\phi_j^{\pm}=(\pi_1)_*(\pi_2^*(T_j)\wedge\Delta^{\pm})$ and $\phi^{\pm}=(\pi_1)_*(\pi_2^*(T)\wedge\Delta^{\pm})$, then $\{T_j\}=\{\phi_j^+-\phi_j^-\}$ and $\{T\}=\{\phi^+-\phi^-\}$. Moreover, $\phi_j^{\pm}$ are positive closed smooth forms converging uniformly to $\phi^{\pm}$. Hence $f^*(\phi_j^{\pm})$ weakly converges to $f^*(\phi^{\pm})$. Thus to show that $f^{\sharp}(T_j)$ weakly converges to $f^{\sharp}(T)$, it is enough to show that $f^{\sharp}(T_j-\phi_j)$ weakly converges to $f^{\sharp}(T-\phi)$, where we define $\phi_j=\phi_j^+-\phi_j^-$ and $\phi=\phi^+-\phi^-$.

By Proposition 2.1 in [9], there are positive $(p-1,p-1)$ currents $R_j^{\pm}$ and $R^{\pm}$ so that $T_j-\phi_j=dd^c(R_j^+-R_j^-)$, $T-\phi=dd^c(R^+-R^-)$. Moreover, we can choose these in such a way that $R_j^{\pm}$ converges in DSH to $R^{\pm}$. From the proof of a), $f^\sharp$ is well-defined on the set of DSH^{p-1} currents. Thus to prove b) we need to show only that $f^\sharp(R_j^\pm)$ weakly converges to $f^\sharp(R^\pm)$.

By Theorem 5, on $X - \pi_X(\mathscr{C}_f)$ the currents $f^\sharp(R_j^\pm)$ and $f^\sharp(R^\pm)$ are the same as the currents $f^o(R_j^\pm)$ and $f^o(R^\pm)$ defined in [10]. Hence by the results in [10], it follows that $f^\sharp(R_j^\pm)$ weakly converges in $X - \pi_X(\mathscr{C}_f)$ to $f^\sharp(R^\pm)$. Thus as in the proof of a), to show that $f^\sharp(R_j^\pm)$ weakly converges to $f^\sharp(R^\pm)$, it suffices to show that $||f^\sharp(R_j)||_{DSH}$ is uniformly bounded.

The current $f^\sharp(R_j)$ is the limit of $f^*(\mathscr{K}_n(R_j))$. As in a), we write $K_n(R_j) = R^+_{j,n} - R^-_{j,n}$ where $R^\pm_{j,n}$ are positive $DSH^{p-1}(Y)$ forms of class C^2. Moreover, by Theorem 13, there is a constant $A > 0$ independent of j and n so that $||R^\pm_{j,n}||_{DSH} \leq A||R^\pm_j||_{DSH}$. It can be seen from the proof of a) that $f^\sharp(R_j)$ is a DSH^{p-1} current. Moreover $||f^\sharp(dd^cR_j)||_{DSH}$, which can be bound using intersections of cohomology classes, is $\leq A||R_j||_{DSH}$, where $A > 0$ is independent of j.

We choose an open neighborhood V of $\pi_X(\mathscr{C}_f)$ and a form Φ as in the proof of a). Then we can see from a) that

$$||f^\sharp(R_j)||_{DSH} \leq A||f^\sharp(R_j)||_{X-V,DSH} + A||f^\sharp(dd^cR_j)||_{DSH},$$

where $A > 0$ is a constant independent of j, and $||f^\sharp(R_j)||_{X-V,DSH}$ means the DSH norm of $f^\sharp(R_j)$ computed on the set $X - V$. From the results in [10], $||f^\sharp(R_j)||_{X-V,DSH}$ is uniformly bounded. The term $||f^\sharp(dd^cR_j)||_{DSH}$ was shown above to be uniformly bounded as well. Thus $||f^\sharp(R_j)||_{DSH}$ is uniformly bounded as desired. □

Proof of Theorem 7. — Let θ be a closed smooth form on Y having the same cohomology class as T. Since T is continuous on $U = X - A$, there are DSH^{p-1} currents $R^\pm$ so that $T - \theta = dd^c(R^+) - dd^c(R^-)$, where $R^\pm|_U$ are continuous (see Proposition 2.1 in [9]). As in the proof of the Theorem 6, we will show that $f^\sharp(R^\pm)$ are well-defined. Since $f^{-1}(A) \cap \pi_X(\mathscr{C}_f) \subset V$, where V is of codimension $\geq p$, it is enough as before to show that $f^*(K_n^\pm(R^\pm))$ have bounded masses outside a small neighborhood of $f^{-1}(A)\cap\pi_X(\mathscr{C}_f)$. First, by the proof of Theorem 6, $f^*(K_n^\pm(R^\pm))$ have bounded masses outside a small neighborhood of $\pi_X(\mathscr{C}_f)$. Hence it remains to show that $f^*(K_n^\pm(R^\pm))$ have bounded masses outside a small neighborhood of $f^{-1}(A)$.

Let B be a small neighborhood of $f^{-1}(A)$. Then there is a cutoff function χ for A, so that $f^{-1}(supp(\chi)) \subset B$. We write

$$f^*(K_n^\pm(R^\pm)) = f^*(\chi K_n^\pm(R^\pm)) + f^*((1-\chi)K_n^\pm(R^\pm)).$$

The first current has support in B, and hence has no contribution for the mass of $f^*(K_n^\pm(R^\pm))$ outside B. By properties of good approximation schemes by C^2 forms, $(1-\chi)K_n^\pm(R^\pm)$ uniformly converges to a continuous form on Y, and hence $f^*((1-\chi)K_n^{\pm}(R^{\pm}))$ has uniformly bounded masses on X, which is what wanted to prove.

To complete the proof, we need to show the continuity stated in the theorem. This continuity can be proved using the arguments from the first part of the proof, and from part b) of the proof of Theorem 6 and the proof of Proposition 3. □

Proof of Theorem 8. — By assumption and Corollary 1, if V is an analytic variety of codimension p contained in $E(T)$, then $f^{\sharp}[V]$ is well-defined with the number $s=0$ in Definition 3. Hence the currents

$$W_N=\sum_{j=1}^{n}\lambda_j[V_j]$$

can be pulled back with the same number $s=0$ in Definition 3, here N is a positive integer. Since $0\leq\sum_j\lambda_j[V_j]-W_N=S_N$ where $S_N\rightharpoonup 0$ as $N\rightarrow\infty$, by Theorem 12 it follows that $f^{\sharp}(\sum_j\lambda_j[V_j])=\sum_j\lambda_j f^{\sharp}[V_j]$ is well-defined. □

Proof of Theorem 9. — Let T be a positive measure on Y having no mass on $\pi_Y(\mathscr{C}_f)$. Let K_n be a good approximation scheme by C^2 forms. Then we will show that as n converges to ∞, any limit point of $[\Gamma_f]\wedge\pi_Y^*(K_n(T))$ has no mass on $\mathscr{C}_f$. Thus $\lim_{n\rightarrow\infty}[\Gamma_f]\wedge\pi_Y^*(K_n(T))=(\pi_Y|_{\Gamma_f})^*(T)$ where the RHS is defined in [10]. Then $f^{\sharp}(T)$ is well-defined, and moreover equals to the current $f^o(T)$ defined in [10], thus satisfies all the conclusions of Theorem 9.

Now we proceed to prove that any limit point τ of $[\Gamma_f]\wedge\pi_Y^*(K_n(T))$ has no mass on $\mathscr{C}_f$. This is equivalent to showing that for a smooth $(dim(X),dim(X))$ form α on $X\times Y$, and for a sequence θ_j of smooth functions on $X\times Y$ having the properties: $0\leq\theta_j\leq 1$, $\theta_j=1$ on a neighborhood of $\mathscr{C}_f$, and support of θ_j converges to $\mathscr{C}_f$ then:

$$\lim_{j\rightarrow\infty}\lim_{n\rightarrow\infty}\int_{X\times Y}\theta_j\alpha\wedge[\Gamma_f]\wedge\pi_Y^*(K_n(T))=0.$$

By properties of good approximation schemes by C^2 forms, we can write the above equality as

$$\lim_{j\rightarrow\infty}\lim_{n\rightarrow\infty}\int_{X\times Y}T\wedge K_n((\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f]))=0. \tag{3.1}$$

Writing α as the difference of two positive smooth forms, we may assume that α is positive. Now α is a positive smooth form, since $0\leq\theta_j\leq 1$ for all j, we can bound the function $(\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f])$ by a multiplicity of $(\pi_Y)_*(\omega_{X\times Y}^{dim(X)}\wedge[\Gamma_f])$ independently of j. The later is a constant, thus $(\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f])$ is a positive bounded function. Then $K_n((\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f]))$ are C^2 functions uniformly bounded w.r.t. j and n. Moreover, the support of $K_n((\pi_Y)_*(\theta_j\alpha\wedge$

$[\Gamma_f]))$ converges to $\pi_Y(\mathscr{C}_f)$ as $j\to\infty$, independent of n. Because T has no mass on $\pi_Y(\mathscr{C}_f)$, we can then apply Lebesgue's dominated convergence theorem to obtain (3.1). □

4. The map J_X

Through out this section, let X be the blowup of $\mathbb{P}^3$ along 4 points $e_0=[1:0:0:0], e_1=[0:1:0:0], e_2=[0:0:1:0], e_3=[0:0:0:1]$; $J:\mathbb{P}^3\to\mathbb{P}^3$ is the Cremona map $J[x_0:x_1:x_2:x_3]=[1/x_0:1/x_1:1/x_2:1/x_3]$, and let J_X be the lifting of J to X. For $0\leq i\neq j\leq 3$, $\Sigma_{i,j}$ is the line in $\mathbb{P}^3$ consisting of points $[x_0:x_1:x_2:x_3]$ where $x_i=x_j=0$, and $\widetilde{\Sigma_{i,j}}$ is the strict transform of $\Sigma_{i,j}$ in X.

Let E_0,E_1,E_2,E_3 be the corresponding exceptional divisors of the blowup $X\to\mathbb{P}^3$, and let L_0,L_1,L_2,L_3 be any lines in E_0,E_1,E_2,E_3 correspondingly. Let H be a generic hyperplane in $\mathbb{P}^3$, and let H^2 be a generic line in $\mathbb{P}^3$. Then H,E_0,E_1,E_2,E_3 are a basis for $H^{1,1}(X)$, and H^2,L_0,L_1,L_2,L_3 are a basis for $H^{2,2}(X)$. Intersection products in complementary dimensions are (see for Example Chapter 4 in [12]):

$$\begin{aligned}
&H.H^2=1,\ H.L_0=0,\ H.L_1=0,\ H.L_2=0,\ H.L_3=0,\\
&E_0.H^2=0,\ E_0.L_0=-1,\ E_0.L_1=0,\ E_0.L_2=0,\ E_0.L_3=0,\\
&E_1.H^2=0,\ E_1.L_0=0,\ E_1.L_1=-1,\ E_1.L_2=0,\ E_1.L_3=0,\\
&E_2.H^2=0,\ E_2.L_0=0,\ E_2.L_1=0,\ E_2.L_2=-1,\ E_1.L_3=0,\\
&E_3.H^2=0,\ E_3.L_0=0,\ E_3.L_1=0,\ E_3.L_2=0,\ E_3.L_3=-1.
\end{aligned}$$

The map $J_X^*:H^{1,1}(X)\to H^{1,1}(X)$ is not hard to compute (see for example the computations in Example 2.5 in [14]):

$$\begin{aligned}
J_X^*(H)&=3H-2E_0-2E_1-2E_2-2E_3,\\
J_X^*(E_0)&=H-E_1-E_2-E_3,\\
J_X^*(E_1)&=H-E_0-E_2-E_3,\\
J_X^*(E_2)&=H-E_0-E_1-E_3,\\
J_X^*(E_3)&=H-E_0-E_1-E_2.
\end{aligned}$$

If $x\in H^{1,1}(X)$ and $y\in H^{2,2}(X)$, since $J_X^2=$the identity map on X, we have the duality $(J_X^*y).x=y.(J_X^*x)$. Thus from the above data, we can write down the map $J_X^*: H^{2,2}(X) \to H^{2,2}(X)$:

$$\begin{aligned} J_X^*(H^2) &= 3H^2 - L_0 - L_1 - L_2 - L_3, \\ J_X^*(L_0) &= 2H^2 - L_1 - L_2 - L_3, \\ J_X^*(L_1) &= 2H^2 - L_0 - L_2 - L_3, \\ J_X^*(L_2) &= 2H^2 - L_0 - L_1 - L_3, \\ J_X^*(L_3) &= 2H^2 - L_0 - L_1 - L_2. \end{aligned}$$

Now we are ready to prove Corollary 2.

Proof of Corollary 2. — The restriction $J_X : X - \bigcup \widetilde{\Sigma_{i,j}} \to X - \bigcup \widetilde{\Sigma_{i,j}}$ is a biholomorphic map, as can be seen by using local coordinate projections for the blowup π near the exceptional divisors E_i's. Moreover it can be shown that $J_X(\widetilde{\Sigma_{i,j}}) = \widetilde{\Sigma_{3-i,3-j}}$, and every point on $\widetilde{\Sigma_{i,j}}$ blows up to $\widetilde{\Sigma_{3-i,3-j}}$. Hence $\pi_1(\mathscr{C}_{J_X}) = \bigcup \widetilde{\Sigma_{i,j}}$. Therefore the map J_X satisfies Theorem 6 for $p = 2$. Thus if T is a positive closed $(2,2)$ current on X then $J_X^\sharp(T)$ is well-defined. For an alternative proof of this fact, see Lemma 8 below.

It remains to show that $J_X^\sharp[\widetilde{\Sigma_{0,1}}] = -[\widetilde{\Sigma_{2,3}}]$. Since $J_X^{-1}(\widetilde{\Sigma_{0,1}}) = \widetilde{\Sigma_{2,3}}$, by Theorem 7 there is a number λ so that $J_X^\sharp[\widetilde{\Sigma_{0,1}}] = \lambda[\widetilde{\Sigma_{2,3}}]$. To determine λ, we need to know $J_X^*\{\widetilde{\Sigma_{0,1}}\}$. We have $\{\widetilde{\Sigma_{0,1}}\} = \{H^2 - L_2 - L_3\}$, hence from the above data we have

$$J_X^*\{\widetilde{\Sigma_{0,1}}\} = J_X^*\{H^2\} - J_X^*\{L_2\} - J_X^*\{L_3\} = \{-H^2 + L_0 + L_1\} = -\{\widetilde{\Sigma_{2,3}}\},$$

thus $\lambda = -1$, and $J_X^\sharp[\widetilde{\Sigma_{0,1}}] = -[\widetilde{\Sigma_{2,3}}]$. □

The following result gives an alternative proof to the conclusions of Corollary 2. In its proof we will make use of the space Y defined in the statement of Proposition 1 below. Here $\pi : Y \to X$ is the blowup of X along all submanifolds $\widetilde{\Sigma_{i,j}}$ $(1 \le i < j \le 3)$. Then the lifting map J_Y of J to Y is an involutive automorphism. Moreover, if we let $S_{i,j}$ denote the exceptional divisor of Y over $\widetilde{\Sigma_{i,j}}$, then $J_Y(S_{0,1}) = S_{2,3}$, $J_Y(S_{0,2}) = S_{1,3}$, and $J_Y(S_{0,3}) = S_{1,2}$.

Lemma 8. — *Let T_n^+ and T_n^- be positive closed smooth $(2,2)$ forms on X, so that*

i) $||T_n^+||$, $||T_n^-||$ *are uniformly bounded,*

and

ii) $T_n^+ - T_n^- \rightharpoonup [\widetilde{\Sigma_{0,1}}]$.

Then $J_X^*(T_n^+ - T_n^-) \rightharpoonup -[\widetilde{\Sigma_{2,3}}]$.

As a consequence, if we replace $[\widetilde{\Sigma_{0,1}}]$ in i) and ii) above by any positive closed $(2,2)$ current T then $J_X^*(T_n^+ - T_n^-)$ converges to $J_X^\sharp(T)$.

Proof. — Let $\tau_n^+ = \pi^*(T_n^+)$ and $\tau_n^- = \pi^*(T_n^-)$, which are positive closed currents on Y. By assumption i), $||\tau_n^+||$ and $||\tau_n^-||$ are uniformly bounded. Thus we may assume that $\tau_n^+ \rightharpoonup \tau^+$ and $\tau_n^- \rightharpoonup \tau^-$, where τ^+ and τ^- are positive closed currents on Y.

Since J_Y is a biholomorphic map, we can pull-back any current on Y by J_Y. It is not hard to see that

$$J_X^*(T_n^+) = \pi_*(J_Y^*\tau_n^+),$$
$$J_X^*(T_n^-) = \pi_*(J_Y^*\tau_n^-).$$

Hence

$$J_X^*(T_n^+ - T_n^-) \rightharpoonup \pi_*(J_Y^*(\tau^+ - \tau^-)).$$

We need to show that the latter current is $-[\widetilde{\Sigma_{2,3}}]$. To this end, it suffices to show that support of $\pi_*(J_Y^*(\tau^+ - \tau^-))$ is in $\widetilde{\Sigma_{2,3}}$. In fact, then we will have $\pi_*(J_Y^*(\tau^+ - \tau^-)) = \lambda[\widetilde{\Sigma_{2,3}}]$, and the computation on cohomology shows that $\lambda = -1$.

It is not hard to see that support of $\tau^+ - \tau^-$ is contained in the union of $S_{i,j}$'s $(1 \leq i < j \leq 3)$. Let $\tau_{i,j} = \tau^+|_{S_{i,j}} - \tau^-|_{S_{i,j}}$ with support in $S_{i,j}$ so that $\tau = \sum_{1\leq i<j\leq 3} \tau_{i,j}$. In $H^{2,2}(Y)$ we have:

$$\pi^*\{\widetilde{\Sigma_{0,1}}\} = \{\tau^+ - \tau^-\} = \sum_{i,j}\{\tau_{i,j}\},$$

here $\pi^*\{\widetilde{\Sigma_{0,1}}\}$ can be represented by currents with support in $S_{0,1}$. Moreover, by considering the push-forwards $\pi_*(\tau_n^+ - \tau_n^-)$, it follows that $\pi_*(\tau_{i,j}) = 0$ where $(i,j) \neq (0,1)$. It can be checked that each fiber $S_{i,j}$ is a product $S_{i,j} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, hence by Kuneth's theorem $H^{2,2}(S_{i,j})$ is generated by a "horizontal curve" $\alpha_{i,j}$ and a "vertical curve" (or fiber) $\beta_{i,j}$. Here the properties of "horizontal curve" and "vertical curve" that we use are that $\pi_*(\alpha_{i,j}) = \widetilde{\Sigma}_{i,j}$ and $\pi_*(\beta_{i,j}) = 0$. Hence there are numbers $a_{i,j}$ and $b_{i,j}$ so that the cohomology class of $\tau_{i,j} - a_{i,j}\alpha_{i,j} - b_{i,j}\beta_{i,j}$ is zero. For $(i,j) \neq (0,1)$, since $\pi_*(\tau_{i,j}) = 0$, it follows that

$$a_{i,j}\{\widetilde{\Sigma}_{i,j}\} = \pi_*\{a_{i,j}\alpha_{i,j} + b_{i,j}\beta_{i,j}\} = \pi_*\{\tau_{i,j}\} = \{\pi_*(\tau_{i,j})\} = 0.$$

Hence $a_{i,j} = 0$ for $(i,j) \neq (0,1)$.

Note that a non-zero $(2,2)$-cohomology class in $H^{2,2}(Y)$ represented by currents with supports in $S_{0,1}$ can not be represented by a linear combinations of "vertical curves" with support in $\bigcup_{(i,j)\neq(0,1)} S_{i,j}$: Assume that

$$\{a_{0,1}\alpha_{0,1} + b_{0,1}\beta_{0,1} + \sum_{(i,j)\neq(0,1)} b_{i,j}\beta_{i,j}\} = 0$$

in $H^{2,2}(Y)$. Push-forward by the map π implies that $a_{0,1}\{\widetilde{\Sigma}_{0,1}\} = 0$ in $H^{2,2}(X)$, and hence $a_{0,1} = 0$. Thus $\{\sum b_{i,j}\beta_{i,j}\} = 0$ in $H^{2,2}(Y)$. Use the fact that

$\{S_{i,j}\}.\{\beta_{k,l}\} = -1$ if $(k,l) = (i,j)$, and $= 0$ otherwise (see for Example Chapter 4 in [12]), we imply that $b_{i,j} = 0$ for all (i,j) as claimed.

Hence it follows that $\{\tau_{i,j}\} = 0$ in $H^{2,2}(Y)$ for $(i,j) \neq (0,1)$.

We have

$$\pi_*(J_Y^*(\tau^+ - \tau^-)) = \sum_{i,j} \pi_*(J_Y^* \tau_{i,j}),$$

where support of $\pi_*(J_Y^*\tau_{i,j})$ is contained in $\widetilde{\Sigma_{3-i,3-j}}$. Here we use the convention that $\widetilde{\Sigma_{k,l}} := \widetilde{\Sigma_{l,k}}$ if $k > l$. Since $\pi_*(J_Y^*\tau_{i,j})$ is a normal $(2,2)$ current, it follows from the structure theorem for normal currents that there is $\lambda_{i,j} \in \mathbb{R}$ so that $\pi_*(J_Y^*\tau_{i,j}) = \lambda_{i,j}[\widetilde{\Sigma_{3-i,3-j}}]$. If $(i,j) \neq (0,1)$ then $\{\tau_{i,j}\} = 0$ in $H^{2,2}(Y)$, thus $\{\pi_*(J_Y^*\tau_{i,j})\} = 0$ in $H^{2,2}(X)$, which implies $\lambda_{i,j} = 0$ for such (i,j)'s. Hence

$$\pi_*(J_Y^*(\tau^+ - \tau^-)) = \pi_*(J_Y^*\tau_{0,1})$$

has support in $\widetilde{\Sigma_{2,3}}$ as wanted. □

PROPOSITION 1. — *Let X be the space constructed in Corollary 2. Let $\pi : Y \to X$ be the blowup of X along all submanifolds $\widetilde{\Sigma_{i,j}}$ $(1 \leq i < j \leq 3)$. Then there is a positive closed $(2,2)$-current T on X with L^1 coefficients so that: in $H^{2,2}(Y)$,*

$$\{\pi^o(T)\} \neq \pi^*\{T\}.$$

Here the operator π^o is defined in Dinh and Nguyen [7]. In this case, in fact $\pi^o(T)$ is also the operator defined in Dinh and Sibony [10].

Proof. — We assume in order to reach a contradiction that for any positive closed $(2,2)$ current T on X with L^1-coefficients then $\{\pi^o(T)\} = \pi^*\{T\}$ in $H^{2,2}(Y)$.

By regularization theorem of Dinh and Sibony, there is a sequence T_n^+ and T^- of positive closed $(2,2)$ currents with L^1-coefficients such that $||T_n^+||$ are uniformly bounded and $T_n^+ \rightharpoonup T^- + [\widetilde{\Sigma_{0,1}}]$. By the assumption we have $\{\pi^o(T_n^+)\} = \pi^*\{T_n^+\}$ for any n, and $\{\pi^o(T^-)\} = \pi^*\{T^-\}$. Now for the maps J_X and J_Y considered above, it is not hard to see that $J_X^o = \pi_* J_Y^* \pi^o$. Thus, we also have $\{J_X^o(T_n^+)\} = J_X^*\{T_n^+\}$ and $\{J_X^o(T^-)\} = J_X^*\{T^-\}$.

Let τ^+ be a cluster point of $J_X^o(T_n^+)$. Then it is easy to see that

$$\tau^+ \geq J_X^o(T^- + [\widetilde{\Sigma_{0,1}}]) = J_X^o(T^-) + J_X^o([\widetilde{\Sigma_{0,1}}]) = J_X^o(T^-).$$

But then this contradicts the fact that in $H^{2,2}(X)$:

$$\begin{aligned}\{\tau^+\} &= \lim_{n\to\infty}\{J_X^o(T_n^+)\} = \lim_{n\to\infty} J_X^*\{(T_n^+)\}\\ &= J_X^*\{T^-\} + J_X^*\{\widetilde{\Sigma_{0,1}}\} = \{J_X^o(T^-)\} - \{[\widetilde{\Sigma_{2,3}}]\},\end{aligned}$$

here we used the assumption that $J_X^*\{(T_n^+)\} = \{J_X^o(T_n^+)\}$ and $J_X^*\{(T^-)\} = \{J_X^o(T^-)\}$. □

PROPOSITION 2. — *Let X be the space constructed in Corollary 2. There is no sequence T_n^+ and T^- of positive closed smooth $(2,2)$ forms on X such that*

i) $||T_n^+||$ *are uniformly bounded*

ii) $T_n^+ - T^- \rightharpoonup [\widetilde{\Sigma_{0,1}}]$.

REMARK 2. — *In Example 6.3 of the paper* [4] *of Bost, Gillet, and Soule, a related result was given.*

Proof. — Use the same argument as that in the proof of Proposition 1, but now use that if $T_n^\pm$ are positive closed smooth forms then $J_X^*(T_n^\pm) = J_X^o(T_n^\pm)$, and hence $\{J_X^o(T_n^\pm)\} = J_X^*\{T_n^\pm\}$. □

5. Invariant currents

Throughout this section, we let X be a compact Kähler manifold of dimension k, and let $f : X \to X$ be a dominant meromorphic map.

We introduce in the below a condition, called dd^c-p stability. This condition seems to be natural for the problem of finding invariant (p,p) currents for a self-map f (see the discussions and the results after the definition).

DEFINITION 10. — *We say that f satisfies the dd^c-p stability condition if the following holds: For any smooth $(p-1,p-1)$ form α and for any n, $f^\sharp((f^n)^*dd^c\alpha)$ is well-defined, and moreover $f^\sharp((f^n)^*dd^c\alpha) = (f^{n+1})^*(dd^c\alpha)$.*

In general, condition of dd^c-p stability has no relation with condition of p-algebraic stability. On the one hand, the dd^c-p stability condition requires no constraints on the action of f^* on $H^{p,p}(X)$, because the cohomology class of $dd^c(\alpha)$ is zero. On the other hand, it asks for the possibility of iterated pull-back $dd^c(\alpha)$ by f. Any map f is dd^c-1 stable, whether being or not 1-algebraic stable. If f is p-analytic stable then f is dd^c-p stable. Using the method in Step 1 of the proof of Lemma 10, it can be shown that the linear pseudo-automorphisms in [3] are dd^c-2 stable. We suspect that these pseudo-automorphisms are also 2-analytic stable even though it seems not be easily checked.

We first introduce an abstract result on invariant (p,p) currents.

THEOREM 11. — *Assume that $f : X \to X$ satisfies the dd^c-p condition. Let λ be a real eigenvalue of $f^* : H^{p,p}(X) \to H^{p,p}(X)$, and let $0 \neq \theta_\lambda \in H^{p,p}(X)$ be an eigenvector with eigenvalue λ. Assume moreover that $|\lambda| > \delta_{p-1}(f)$ and let $s \geq 2$ be an integer. Then any of the following statements is equivalent to each other:*

1) *There is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and moreover $f^\sharp(T) = \lambda T$.*

2) *There are a smooth $(p-1,p-1)$ form α and a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and moreover $f^\sharp(T) = \lambda T + \lambda dd^c(\alpha)$.*

3) *For any smooth $(p-1,p-1)$ form α, there is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and moreover $f^\sharp(T) = \lambda T + \lambda dd^c(\alpha)$.*

4) *There is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and moreover $f^\sharp(T) - \lambda T$ is a smooth form.*

Note that for the current T in Theorem 11, we do not know whether $(f^n)^\sharp(T)$ (for $n \geq 2$) is well-defined or not. The proof of Theorem 11 makes use of the following result, which is interesting in itself.

THEOREM 12. — *Let T_j and T be (p,p) currents of order s_0. Assume that $-S_j \leq T - T_j \leq S_j$ for any j, where S_j are positive closed (p,p) currents with $||S_j|| \to 0$ as $j \to \infty$.*

1) *If $f^\sharp(T_j)$ is well-defined for any j with the same number s in Definition 3, then $f^\sharp(T)$ is well-defined. Moreover $f^\sharp(T_j)$ weakly converges to $f^\sharp(T)$.*

2) *If $f^\sharp(dd^cT_j)$ is well-defined for any j with the same number s in Definition 3, then $f^\sharp(dd^cT)$ is well-defined. Moreover $f^\sharp(dd^cT_j)$ weakly converges to $f^\sharp(dd^cT)$.*

Note that when $p = 0$, a closed $(0,0)$ current on X is a constant, hence the S_j in Theorem 12 are positive constants converging to zero.

Proof of Theorem 12. — i) Let $K_n = K_n^+ - K_n^-$ be a good approximation scheme by C^{s+2} forms. Let α be a strongly positive smooth $(k-p,k-p)$ form on X. then $f_*(\alpha)$ is a strongly positive form. Therefore $K_n^\pm f_*(\alpha)$ are strongly positive forms of class C^2. Since $-S_j \leq T_j - T \leq S_j$, by Lemma 5 we obtain

$$-\int_X S_j \wedge K_n^\pm f_*(\alpha) \leq \int_X (T_j - T) \wedge K_n^\pm f_*(\alpha) \leq \int_X S_j \wedge K_n^\pm f_*(\alpha).$$

From Lemma 4, there is a constant $A > 0$ independent of α so that $A||\alpha||_{L^\infty}\omega_X^{k-p} \pm \alpha$ are strongly positive forms. Then $A||\alpha||_{L^\infty} f_*(\omega_X^{k-p}) \pm f_*(\alpha)$ are strongly positive forms on X. Hence we have

$$\int_X S_j \wedge K_n^{\pm} f_*(\alpha) \leq A||\alpha||_{L^\infty} \int_X S_j \wedge K_n^{\pm} f_*(\omega_X^{k-p}).$$

The latter integral can be computed cohomologously, hence can be bound as

$$\begin{aligned} A||\alpha||_{L^\infty} \int_X S_j \wedge K_n^{\pm} f_*(\omega_X^{k-p}) &\leq A||\alpha||_{L^\infty} ||S_j|| \times ||K_n^{\pm} f_*(\omega_X^{k-p})|| \\ &\leq A||\alpha||_{L^\infty} ||S_j|| \times ||f_*(\omega_X^{k-p})||. \end{aligned}$$

The latter inequality comes from Theorem 13. Hence,

$$-A||\alpha||_{L^\infty}||S_j|| \leq \int_X (T_j - T) \wedge K_n^{\pm} f_*(\alpha) \leq A||\alpha||_{L^\infty}||S_j||. \tag{5.1}$$

Since $f^\sharp(T_j)$ are well-defined for all j, if we take limit as $n \to \infty$ in (5.1), we get

$$\begin{aligned} -A||\alpha||_{L^\infty}||S_j|| &\leq \int_X f^\sharp(T_j) \wedge \alpha - \limsup_{n\to\infty} \int_X T \wedge K_n f_*(\alpha) \\ &\leq \int_X f^\sharp(T_j) \wedge \alpha - \liminf_{n\to\infty} \int_X T \wedge K_n f_*(\alpha) \\ &\leq A||\alpha||_{L^\infty}||S_j||. \end{aligned}$$

Since $||S_j|| \to 0$, taking limit as $j \to \infty$ shows that

$$L(\alpha) := \lim_{n\to\infty} \int_X T \wedge K_n f_*(\alpha)$$

exists, and moreover it satisfies

$$-A||\alpha||_{L^\infty}||S_j|| \leq \int_X f^\sharp(T_j) \wedge \alpha - L(\alpha) \leq A||\alpha||_{L^\infty}||S_j||, \tag{5.2}$$

for all j, and all strongly positive smooth $(dim(X) - p, dim(X) - p)$ form α. Since any smooth $(dim(X) - p, dim(X) - p)$ form α is the difference of two strongly positive smooth $(dim(X) - p, dim(X) - p)$ forms α_1 and α_2 whose L^∞ norms are uniformly bounded (up to a multiplicative constant) by $||\alpha||_{L^\infty}$ by Lemma 4, it follows that (5.2) holds for any smooth form α. From this, it follows easily that the assignment $\alpha \mapsto L(\alpha)$ is a well-defined functional on smooth forms α. Now we show that it is a current on X. For this end, it suffices to show that if α_n are smooth forms so that $||\alpha_n||_{C^s} \to 0$ for any fixed $s \geq 0$ then $L(\alpha_n) \to 0$. This follows easily from (5.2) by first taking limit when

$n \to \infty$ and then taking limit when $j \to \infty$, using the assumptions that $f^\sharp(T_j)$ are currents, hence

$$\lim_{n\to\infty} \int_X f^\sharp(T_j) \wedge \alpha_n = 0,$$

for any j.

ii) The proof is similar to the proof of i), with a small change: The estimate (5.1) is modified to

$$-A||dd^c\alpha||_{L^\infty}||S_j|| \leq \int_X (T_j - T) \wedge K_n^\pm f_*(dd^c\alpha) \leq A||dd^c\alpha||_{L^\infty}||S_j||. \qquad \square$$

The proof of Theorem 11 also uses the following result:

Lemma 9. — *Assume that f satisfies the dd^c-p stability condition. Let λ be a positive real number. If $|\lambda| > \delta_{p-1}(f)$, then for any smooth $(p-1,p-1)$ form α, there is a current R_α of order 0, so that $f^\sharp(dd^cR_\alpha)$ is well-defined, and moreover*

$$f^\sharp(dd^cR_\alpha) - \lambda dd^cR_\alpha = \lambda dd^c\alpha.$$

Proof. — Define $\beta = -\alpha$, and consider

$$R_n = \sum_{j=0}^{n} \frac{(f^j)^*(\beta)}{\lambda^j}.$$

Since β is a smooth $(p-1,p-1)$ form, there is a constant $A > 0$ so that $-A\omega_X^{p-1} \leq \beta \leq A\omega_X^{p-1}$. It follows that

$$R_\alpha = \sum_{j=0}^{\infty} \frac{(f^j)^*(\beta)}{\lambda^j}$$

is a well-defined current which is a difference of two positive currents, hence of order 0. Moreover $-S_n \leq R_n - R \leq S_n$, where

$$S_n = A \sum_{j=n+1}^{\infty} \frac{(f^j)^*(\omega_X^{p-1})}{|\lambda|^j}.$$

The S_n are well-defined positive closed $(p-1,p-1)$ currents, because it is well-known (see for Example Chapter 2 in [14]) that

$$\lim_{n\to\infty} ||(f^n)^*(\omega_X^{p-1})||^{1/n} = \delta_{p-1}(f),$$

and the latter is $< |\lambda|$ by assumption. The above inequality also shows that $||S_n|| \to 0$ as $n \to \infty$. The dd^c-p stability condition shows that $f^\sharp(dd^cR_n)$

is well-defined for any n, and moreover $f^\sharp(dd^c R_n) - \lambda dd^c R_{n+1} = -\lambda dd^c\beta = \lambda dd^c\alpha$. Applying Theorem 12, using that R_n weakly converges to R_α, we have

$$f^\sharp(dd^c R_\alpha) - \lambda dd^c R_\alpha = \lambda dd^c\alpha. \qquad \square$$

Proof of Theorem 11. — All of the equivalences follow easily from Lemma 9.

1) $\Rightarrow$ 3): Let T_0 be a closed (p,p) current of order s with $\{T_0\} = \theta_\lambda$ so that $f^\sharp(T_0)$ is well-defined, and $f^\sharp(T_0) - \lambda T_0 = 0$. For any smooth $(p-1,p-1)$ form α on X, let R_α be the current constructed in Lemma 9. Then $T = T_0 + dd^c(R_\alpha)$ is a closed (p,p) current of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and $f^\sharp(T) - \lambda T = dd^c(R_\alpha)$.

3) $\Rightarrow$ 2: Obviously.

2) $\Rightarrow$ 1): Let α_0 be a smooth $(p-1,p-1)$ form, and let T_0 be a closed (p,p) current of order s with $\{T_0\} = \theta_\lambda$ so that $f^\sharp(T_0)$ is well-defined, and $f^\sharp(T_0) - \lambda T_0 = dd^c(\alpha_0)$. Let R_α be the current constructed in Lemma 9. Then $T = T_0 - dd^c(R_\alpha)$ is a closed (p,p) current of order s with $\{T\} = \theta_\lambda$ so that $f^\sharp(T)$ is well-defined, and $f^\sharp(T) - \lambda T = 0$.

Finally, that 2) and 4) are equivalent follows from the dd^c lemma, since the current $f^\sharp(T) - \lambda T$ is a smooth form cohomologous to 0. $\square$

Now we give the proofs of Lemma 3 and Corollaries 5, 6 and 7.

Proof of Lemma 3. — Since $\pi_1(\mathscr{C}_f)$ has codimension $\geq p$, it follows from Theorem 6 any positive closed (p,p) current can be pulled back, and the pullback operator is continuous with respect to the weak topology on positive closed (p,p) currents. We can represent θ by a difference $\alpha = \alpha^+ - \alpha^-$ of two positive closed smooth (p,p) forms $\alpha^\pm$. Since f is p-analytic stable, it follows that $(f^n)^*(\alpha^\pm) = (f^\sharp)^n(\alpha^\pm)$ are positive closed (p,p) currents for any $n \geq 1$. Moreover there is a constant $C_1 > 0$ so that $||(f^\sharp)^n(\alpha^\pm)|| = ||(f^n)^*(\alpha^\pm)|| \leq C_1 r_p(f)^n = C_1\lambda^n$ (see e.g [14]). We follow the standard construction of an invariant current under these assumptions (see [17] and [5]). Consider the currents $T_N = T_N^+ - T_N^-$, where

$$T_N^\pm = \frac{1}{N}\sum_{j=0}^{N-1}\frac{(f^\sharp)^j(\alpha^\pm)}{\lambda^j}.$$

Then $T_N^\pm$ are positive closed (p,p) currents with uniformly bounded masses, thus after passing to a subsequence, we may assume that they converge to $T^\pm$. We define $T = T^+ - T^-$. Since $\{T_N\} = \{\alpha\}$ for any N, we also have $\{T\} = \{\alpha\}$. Since $f^\sharp(T_N^\pm) - \lambda T_N^\pm$ converges to 0, it follows that $f^\sharp(T) = \lambda T$. $\square$

Proof of Corollary 5. — Let T_1 and T_2 be the Green $(1,1)$ currents for the maps f_1 and f_2 as constructed in Sibony [17], respectively. Then we can write

$$T_i = \sum_j \lambda_{j,i}[V_{j,i}]$$

for $i = 1, 2$, where $\lambda_{j,i} > 0$ and $V_{j,i}$ are irreducible hypersurfaces in $\mathbb{P}^{k_i}$. Moreover $f^*(T_1) = d_1 T_1$ and $f^*(T_2) = d_2 T_2$. We choose $T = T_1 \times T_2$. Consider the finite summands

$$S_{N,i} = \sum_{j=0}^{N} \lambda_{j,i}[V_{j,i}].$$

Then $f^{-1}(S_{N,1} \times S_{N,2}) = f_1^{-1}(S_{N,1}) \times f_2^{-1}(S_{N,2})$ has codimension 2 in $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2}$, thus $f^\sharp(S_{N,1} \times S_{N,2})$ are well-defined by Corollary 1. Since $T_1 \times T_2 - S_{N,1} \times S_{N,2}$ are positive closed currents decreasing to 0, it follows by Theorem 12 that $f^\sharp(T_1 \times T_2)$ is well-defined and moreover

$$f^\sharp(T_1 \times T_2) = \lim_{N \to \infty} f^\sharp(S_{N,1} \times S_{N,2}).$$

It remains to show that $f^\sharp(T_1 \times T_2) = d_1 d_2 T_1 \times T_2$. To this end, first we show that $f^\sharp(S_{N,1} \times S_{N,2}) = f_1^*(S_{N,1}) \times f_2^*(S_{N,2})$ for any N. By the results in [11] (see also the last section), there are positive closed $(1,1)$ currents $W_{j,N,1}$ on $\mathbb{P}^{k_1}$ and $W_{j,N,2}$ on $\mathbb{P}^{k_2}$ with uniformly bounded norms so that $S_{N,1} = \lim_{j\to\infty} W_{j,N,1}$ and $S_{N,2} = \lim_{j\to\infty} W_{j,N,2}$. Moreover, we can choose these approximations in such a way that support of $W_{j,N,1}$ converges to $S_{N,1}$ and support of $W_{j,N,2}$ converges to $S_{N,2}$. Then $\lim_{j\to\infty} W_{j,N,1} \times W_{j,N,2} = S_{N,1} \times S_{N,2}$, and $W_{j,N,1} \times W_{j,N,2}$ has uniformly bounded mass and locally uniformly converges to 0 on $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2} - S_{N,1} \times S_{N,2}$. Hence we can apply Theorem 7 to obtain that

$$\begin{aligned} f^\sharp(S_{N,1} \times S_{N,2}) &= \lim_{j\to\infty} f^*(W_{j,N,1} \times W_{j,N,2}) = \lim_{j\to\infty} f_1^*(W_{j,N,1}) \times f_2^*(W_{j,N,2}) \\ &= f_1^*(S_{N,1}) \times f_2^*(S_{N,2}). \end{aligned}$$

Having this, it follows from the continuity of pullback on positive closed $(1,1)$ currents and the definitions of T_1 and T_2 that

$$\begin{aligned} f^\sharp(T_1 \times T_2) &= \lim_{N\to\infty} f^\sharp(S_{N,1} \times S_{N,2}) = \lim_{N\to\infty} f_1^*(S_{N,1}) \times f_2^*(S_{N,2}) \\ &= f^*(T_1) \times f^*(T_2) = d_1 d_2 T_1 \times T_2. \end{aligned}$$ □

Proof of Corollary 6. — It is well-known that for any smooth (k,k) form θ then $(f^n)^\sharp(\theta) = (f^\sharp)^n(\theta)$ (see for example [13]). Hence f satisfies dd^c-k stability condition. As in [13], we can find a smooth probability measure θ so that $f^*(\theta)$ is again a smooth probability measure. Hence $f^*(\theta) - \delta_k(f)\theta = dd^c(\varphi)$, where φ is a smooth $(p-1, p-1)$ form. Hence we can apply Theorem 11. □

Proof of Corollary 7. — Let θ be a smooth form then $f^*(\theta)$ is again a smooth form since f is holomorphic. Then we can use the same arguments as that in the proof of Corollary 6. $\square$

6. Examples of good approximation schemes, and open questions

We give some examples of good approximation schemes in Definition 1 in the first two subsections, and then discuss some open problems in the last subsection.

6.1. The case of general Kähler manifolds. — Let Z be a compact Kähler manifold of dimension k. Let $\pi_1, \pi_2 : Z \times Z \to Z$ be the two projections, and let $\Delta_Z \subset Z \times Z$ be the diagonal. Our construction of examples use the following regularization theorem of DSH currents in [8].

THEOREM 13. — *There is a sequence of strongly positive closed (k,k) forms $K_n^{\pm}$ on $Z \times Z$ of L^1 coefficients with the following properties:*

i) $K_n^+ - K_n^- \rightharpoonup [\Delta_Z]$, *and $||K_n^{\pm}||$ are uniformly bounded. The singularities of $K_n^{\pm}$ are the same as that of the Bochner-Martinelli kernel.*

ii) *Support of $K_n^+ - K_n^-$ converges to Δ_Z. By this we mean, for any open neighborhood U of Δ_Z, there exists n_0 so that if $n \geq n_0$ then support of $K_n^+ - K_n^-$ is contained in U.*

iii) *If T is a DSH^p current then $(K_n^+ - K_n^-) \wedge \pi_2^*(T) \rightharpoonup [\Delta_Z] \wedge \pi_2^*(T)$. Moreover, if T_j converges to T in $DSH^p(Z)$, then for a given number n: $(\pi_1)_*(K_n^{\pm} \wedge \pi_2^*(T_j)) \rightharpoonup (\pi_1)_*(K_n^{\pm} \wedge \pi_2^*(T))$ when $j \to \infty$.*

Define $K_n^{\pm}(T) = (\pi_1)_*(K_n^{\pm} \wedge \pi_2^*(T))$, and $K_n(T) = K_n^+(T) - K_n^-(T)$. Then $K_n(T) \rightharpoonup T$ in $DSH^p(Z)$ as $n \to \infty$. Moreover, $||K_n^{\pm}(T)||_{DSH} \leq A||T||_{DSH}$, where $A > 0$ is independent of T and n.

iv) *For any $s > 0$, there exists a number $l_0 = l_0(s)$ so that $K_{n_l} \circ K_{n_{l-1}} \circ \cdots \circ K_{n_1}(T)$ is a C^s form for any $l \geq l_0$, any integers $n_1, n_2, \ldots, n_l$, and any DSH^p current T.*

v) *If T is a continuous form then $K_n(T)$ converges uniformly to T.*

Proof. — The definition of $K_n^{\pm}$ is given in Section 3 in [8], and we will recall the construction later in this subsection. All of the references below are from the same paper

i) is given in Lemma 3.1.

ii) is given in Remark 4.5.

iii) is given in Theorems 1.1 and 4.4.

iv) is given in Lemma 2.1.

v) is given in Proposition 4.6. $\square$

Let us mention some notations used later on.

REMARK 3. — *We use the following notations:*

For integers $n_1, \ldots, n_l$ *and a* DSH^p *current or continuous* (p,p) *form* T *on* Y, *we define* $K_{n_l, n_{l-1}, \ldots, n_1}(T) = K_{n_l} \circ K_{n_{l-1}} \circ \cdots \circ K_{n_1}(T)$. *For simplicity, we write* (l) *instead of* $(n_1, \ldots, n_l)$, *and* $\mathcal{K}_{(l)}(T)$ *instead of* $K_{n_l, \ldots, n_1}(T)$.

We write

$$\lim_{(n_1, n_2, \ldots, n_l) \to \infty} T_{n_1, \ldots, n_l} = T$$

if for any sequence $(n_1)_k, \ldots, (n_l)_k \to \infty$ *we have*

$$\lim_{k \to \infty} T_{(n_1)_k, \ldots, (n_l)_k} = T.$$

For simplicity we use

$$\lim_{(l)} T_{(l)} = T$$

for such a limit.

Example 4: By Theorem 14 below, if T is a DSH current then

$$\lim_{(l)} \mathcal{K}_{(l)}(T) = T,$$

for any $l \geq 0$.

The following consequence of Theorem 13 will be used to approximate $DSH^p(Y)$ currents by C^s forms in a linear way

THEOREM 14. — i) *If* T_1 *is a* $DSH^p(Z)$ *current and* T_2 *is a continuous* $(dim(Z) - p, dim(Z) - p)$ *form on* Z *then*

$$\int_Z K_n^{\pm}(T_1) \wedge T_2 = \int_Z T_1 \wedge K_n^{\pm}(T_2).$$

ii) *For any integer* l *and any* $DSH^p(YZ$ *current* T, $\mathcal{K}_{(l)}(T)$ *converges in* $DSH^p(Z)$ *to* T. *Here the convergence is understood in the sense of Remark 3.*

Proof of Theorem 14. — i) By Theorem 13, the LHS of the equality we want to prove is continuous for the DSH convergence w.r.t. T_1. By Lemma 7, the RHS of the equality is also continuous for the DSH convergence w.r.t. T_1. Hence using the approximation theorem for DSH currents of Dinh and Sibony, it suffices to prove the equality when T_1 is a smooth form, in which case it is easy to be verified.

ii) Note that since $||\mathcal{K}_{(l)}(T)||_{DSH} \leq A^l ||T||_{DSH}$ by Theorem 13, to prove ii) it suffices to show that $\mathcal{K}_{(l)}(T)$ converges weakly to T in the sense of currents.

We prove by induction on l. If $l = 1$, ii) is the content of Theorem 13. To illustrate the idea of the proof, we show for example how to prove ii) for the case $l=2$ when knowing ii) for $l=1$. Hence we need to show that: For a smooth $(dim(Z)-p, dim(Z)-p)$ form α

$$\lim_{(2)}\int_Z K_{n_2}\circ K_{n_1}(T)\wedge\alpha=\int_Z T\wedge\alpha.$$

Since α is smooth, by i) we have

$$\lim_{(2)}\int_Z K_{n_2}\circ K_{n_1}(T)\wedge\alpha=\lim_{(2)}\int_Z K_{n_1}(T)\wedge K_{n_2}(\alpha).$$

By the case $l=1$ we know that $K_{n_1}(T)$ converges to T in DSH^p. By Theorem 13, $K_{n_2}(\alpha)$ converges uniformly to α. Hence $\alpha-K_{n_2}(\alpha)$ is bound by $\epsilon_{n_2}\omega_Z^{dim(Z)-p}$, where $\epsilon_{n_2}\to 0$ as $n_2\to\infty$. A similar argument to that of the proof of Lemma 7 shows that

$$|\int_Z K_{n_1}(T)\wedge K_{n_2}(\alpha)-\int_Z K_{n_1}(T)\wedge\alpha|\leq A\epsilon_{n_2},$$

where $A>0$ is independent of n_1 and n_2. Letting limit when n_1,n_2 converges to ∞ and using the induction assumption for $l=1$, we obtain the claim for $l=2$. □

Now we define a good approximation scheme by C^2 forms as follows: Choose $l=l_0(2)$ in Theorem 13, and choose the approximation $\mathscr{K}_{(2l)}$. Most of the requirements for good approximation scheme can be checked directly on $\mathscr{K}_{(2l)}$. The rest of this subsection shows the remaining requirements. The next remark concerns the dd^c of $\mathscr{K}_{(2l)}$.

Remark 4. — *If T is a $DSH^p(Y)$ current $T=T_1-T_2$ with T_1,T_2 positive (p,p) currents, and $dd^c(T_i)=\Omega_i^+-\Omega_i^-$ where $\Omega_i^\pm$ positive closed currents, then we can write:*

i) *$K_n(T)=T_{1,n}-T_{2,n}$ where $T_{1,n}=K_n^+(T_1)+K_n^-(T_2)$ and $T_{2,n}=K_n^-(T_1)+K_n^+(T_2)$ are positive currents with L^1 coefficients.*

ii) *$dd^c(T_{1,n})=\Omega_{1,n}^+-\Omega_{1,n}^-$, where $\Omega_{1,n}^+=K_n^+(\Omega^+)+K_n^-(\Omega_2^+)$ and $\Omega_{1,n}^-=K_n^+(\Omega^-)+K_n^-(\Omega_2^-)$ are positive closed $(p+1,p+1)$ currents with L^1 coefficients. Similarly, we can write $dd^c(T_{2,n})=\Omega_{2,n}^+-\Omega_{2,n}^-$, where $\Omega_{2,n}^+$ and $\Omega_{2,n}^-$ are positive closed $(p+1,p+1)$ currents with L^1 coefficients.*

iii) *$||T_{i,n}||, ||\Omega_{i,n}^\pm||\leq A||T||_{DSH}$, where $A>0$ is independent of T.*

If we repeat this argument and use Theorem 13, we see that for $l=2l_0(2)$ as above, we can write $\mathscr{K}_{(2l_0)}(T)=T_{1,(2l_0)}-T_{2,(2l_0)}$ where

i) *$T_{i,(2l_0)}$ are positive C^2 forms.*

ii) *$dd^c(T_{i,(2l_0)})=\Omega_{i,(2l_0)}^+-\Omega_{i,(2l_0)}^-$, where $\Omega_{i,(2l_0)}^\pm$ are positive closed C^2 forms.*

iii) *$||T_{i,(2l_0)}||, ||\Omega_{i,(2l_0)}^\pm||\leq A||T||_{DSH}$, where $A>0$ is independent of T.*

More explicitly, we can write $\mathcal{K}_{(2l_0)} = \mathcal{K}^+_{(2l_0)} - \mathcal{K}^-_{(2l_0)}$*, where* $\mathcal{K}^{\pm}_{(2l_0)}$ *are convex combinations of compositions of* $K^{\pm}_m$ *(here* m *belongs to the set* $n_1, n_2, \ldots, n_{2l_0}$*), so that if* T *is a positive DSH current, then* $\mathcal{K}^{\pm}_{(2l_0)}(T)$ *are positive currents. For example, if* $l_0 = 1$*, then* $\mathcal{K}^+_{(2)} = K^+_{n_2} \circ K^+_{n_1} + K^-_{n_2} \circ K^-_{n_1}$ *and* $\mathcal{K}^-_{(2)} = K^-_{n_2} \circ K^+_{n_1} + K^+_{n_2} \circ K^-_{n_1}$*. Then we define*

$$\begin{aligned}
T_{1,(2l_0)} &= \mathcal{K}^+_{(2l_0)}(T_1) + \mathcal{K}^-_{(2l_0)}(T_2),\\
T_{2,(2l_0)} &= \mathcal{K}^+_{(2l_0)}(T_2) + \mathcal{K}^-_{(2l_0)}(T_1),\\
\Omega^+_{1,(2l_0)} &= \mathcal{K}^+_{(2l_0)}(\Omega^+_1) + \mathcal{K}^-_{(2l_0)}(\Omega^+_2),\\
\Omega^-_{1,(2l_0)} &= \mathcal{K}^+_{(2l_0)}(\Omega^-_1) + \mathcal{K}^-_{(2l_0)}(\Omega^-_2),
\end{aligned}$$

and similarly for $\Omega^{\pm}_{2,(2l_0)}$*.*

The following refinements of Proposition 4.6 in [8] concern the continuity property of $\mathcal{K}_{(2l)}$. Its proof uses explicitly the properties of the kernels K_n in Theorem 13 from Section 3 in [8], which we recall briefly here. Let $\pi : \widetilde{Z\times Z} \to Z\times Z$ be the blowup along the diagonal Δ_Z, and let $\widetilde{\Delta_Z} = \pi^{-1}(\Delta_Z)$. Choose a strictly positive closed $(k-1,k-1)$ form γ on $\widetilde{Z\times Z}$ so that $\pi_*(\gamma\wedge[\widetilde{\Delta_Z}]) = [\Delta_Z]$. We let Θ' be a smooth closed $(1,1)$ form on $\widetilde{Z\times Z}$ having the same cohomology class with $[\widetilde{\Delta_Z}]$, and let φ be a quasi PSH function so that $dd^c\varphi = [\widetilde{\Delta_Z}] - \Theta'$. Observe that φ is smooth out of $[\widetilde{\Delta_Z}]$, and $\varphi^{-1}(-\infty) = \widetilde{\Delta_Z}$. Let $\chi : \mathbb{R}\cup\{-\infty\} \to \mathbb{R}$ be a smooth increasing convex function such that $\chi(x) = 0$ on $[-\infty,-1]$, $\chi(x) = x$ on $[1,+\infty]$, and $0\leq \chi' \leq 1$. Define $\chi_n(x) = \chi(x+n) - n$, and $\varphi_n = \chi_n\circ\varphi$. The functions φ_n are smooth decreasing to φ, and $dd^c\varphi_n \geq -\Theta$ for every n, where Θ is a strictly positive closed smooth $(1,1)$ form so that $\Theta - \Theta'$ is positive. Then we define $\Theta^+_n = dd^c\varphi_n + \Theta$ and $\Theta^-_n = \Theta^- = \Theta - \Theta'$, and finally $K^{\pm}_n = \pi_*(\gamma\wedge\Theta^{\pm}_n)$, and $K_n = K^+_n - K^-_n$.

Proposition 3. — i) *Let* T_n *be a sequence of* $DSH^p(Z)$ *currents converging in DSH to* T*. Assume that there is an open set* $U\subset Z$ *so that* $T_n|_U$ *are continuous forms, and* T_n *converges locally uniformly on* U *to* T*. Then* $K^{\pm}_n(T_n)|_U$ *are continuous and converges locally uniformly on* U*.*

ii) *Let* T *be a* $DSH^p(Z)$ *current. Assume that there is an open set* $U\subset Z$ *so that* $T|_U$ *is a continuous form. Then for any positive integer* l*,* $\mathcal{K}^{\pm}_{(l)}(T)|_U$ *are continuous forms, and converges locally uniformly on* U*.*

Proof. — i) Let $U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset U$ be a relative compact open sets in U. We will show that $K^{\pm}_n(T_n)$ converges uniformly on U_1. Let $\chi_2 : Z\to[0,1]$ be a cutoff function for U_2 so that χ_2 is smooth, $\chi_2 = 1$ on U_2 and $\chi_2 = 0$ outside of U_3. We write $K^{\pm}_n(T_n) = K^{\pm}_n(\chi_2 T_n) + K^{\pm}_n((1-\chi_2)T_n)$. By assumptions, $\chi_2 T_n$

converges uniformly on Z to $\chi_2 T$, so there are ϵ_n decreasing to 0 as $n \to 0$ so that $-\epsilon_n \omega_Z^p \leq \chi_2 T_n - \chi_2 T \leq \epsilon_n \omega_Z^p$. Then

$$-\epsilon_n K_n^{\pm}(\omega_Z^p) \leq K_n^{\pm}(\chi_2 T_n) - K_n^{\pm}(\chi_2 T) \leq \epsilon_n K_n^{\pm}(\omega_Z^p).$$

Now $K_n^-(\omega_Z^p) = K^-(\omega_Z^p)$ is a smooth form, and hence $K_n^+(\omega_Z^p) = K_n(\omega^p) - K^-(\omega_Z^p)$ is a sequence of smooth forms converging uniformly on Z, by applying Proposition 4.6 in [8] to ω_Z^p. Hence to prove i), it remains to show that $K_n^{\pm}((1-\chi_2)T_n)$ converges uniformly on U_1.

We let $\chi_1 : Z \to [0,1]$ be a cutoff function for U_1 so that χ_1 is smooth, $\chi_1 = 1$ on U_1 and $\chi_1 = 0$ outside of U_2. Then it suffices to show that $\chi_1 K_n^{\pm}((1-\chi_2)T_n)$ uniformly converges on Z. By definition, we have

$$\begin{aligned}\chi_1 K_n^{\pm}((1-\chi_2)T_n)(x) &= \int_Z \chi_1(x) K_n^{\pm}(x,y) \wedge (1-\chi_2(y)) T_n(y) dy \\ &= \int_Z \chi_1(x)(1-\chi_2(y)) K_n^{\pm}(x,y) \wedge T_n(y) dy.\end{aligned}$$

By definition of χ_1 and χ_2, the support of $\chi_1(x)(1-\chi_2(y))K_n^{\pm}(x,y)$ is contained in a fixed compact set of $Z \times Z - \Delta_Z$. Hence by definition of $K_n^{\pm}$, there is an n_0 and smooth forms $k^{\pm}(x,y)$ on $Z \times Z$ so that $\chi_1(x)(1-\chi_2(y))K_n^{\pm}(x,y) = k^{\pm}(x,y)$ for all $n \geq n_0$. Then for $n \geq n_0$ we have

$$\chi_1 K_n^{\pm}((1-\chi_2)T_n)(x) = \int_Z k^{\pm}(x,y) \wedge T_n(y) dy,$$

and the RHS converges uniformly to $\int_Z k^{\pm}(x,y) \wedge T(y) dy$ since $T_n \rightharpoonup T$.

ii) We prove the claim for example for the case $l = 1$ and $l = 2$.

First, consider the case $l = 1$. Then ii) follows by applying i) to the constant sequence $T_n = T$.

Now we consider the case $l = 2$. Then $\mathscr{K}_{(2)}^+(T) = K_{n_2}^+ \circ K_{n_1}^+(T) + K_{n_2}^- \circ K_{n_1}^-(T)$, and $\mathscr{K}_{(2)}^-(T) = K_{n_2}^+ \circ K_{n_1}^-(T) + K_{n_2}^- \circ K_{n_1}^+(T)$. We show for example that $K_{n_2}^+ \circ K_{n_1}^+(T)$ converges uniformly locally on U as both n_1 and n_2 go to ∞. We apply i) to the sequence $T_n = K_n^+(T)$. The two conditions of i) are not hard to check: First, by the case $l = 1$ the sequence T_n converges locally uniformly on U. Second, by Theorem 14, $T_n = K_n(T) + K^-(T) \rightharpoonup T + K^-(T)$. □

6.2. The case of projective spaces. — In this case, Dinh and Sibony [11] used super-potential to define pullback of a positive closed current. We recall their definition in this subsection. The reader is referred to [11] for more detail.

a) Quasi-potentials:

Let ω be the Fubini-Study form on $\mathbb{P}^k$, normalized so that $||\omega|| = 1$. Let $\mathscr{C}_p$ be the convex set of positive closed (p,p) currents T on $\mathbb{P}^k$, normalized so that $||T|| = 1$. If $T \in \mathscr{C}_p$, then there is a $(p-1,p-1)$ current U_T bounded from above so that $T - \omega^p = dd^c(U_T)$, and we call $m = \int_X U_T \wedge \omega^{k-p+1}$ the mean of U_T. We call U_T a quasi-potential of T of mean m. For simplicity we choose $m = 0$.

b) Deformation of currents:

The group $Aut(\mathbb{P}^k)$ of automorphisms of $\mathbb{P}^k$ is the complex Lie group $PGL(k+1, \mathbb{C})$ of dimension $k^2 + 2k$. We choose a local holomorphic coordinate chart y ($y \in \mathbb{C}^{k^2+2k}$, with $|y| < 2$) of $Aut(\mathbb{P}^k)$ near the identity $id \in Aut(\mathbb{P}^k)$, in such a way that $y = 0$ at id. The element in $Aut(\mathbb{P}^k)$ with coordinate y is denoted by τ_y. Assume that the norm $|y|$ is invariant under the involution $\tau \leftrightarrow \tau^{-1}$. Choose a smooth probability ρ with support in $|y| < 1$ so that ρ is radially and decreasing in $|y|$.

Let R be a positive or negative current on $\mathbb{P}^k$. For $\theta \in \mathbb{C}$ with $|\theta| \leq 1$, define

$$R_\theta := \int_{Aut(\mathbb{P}^k)} (\tau_{\theta y})_*(R) d\rho(y) = \int_{Aut(\mathbb{P}^k)} (\tau_{\theta y})^*(R) d\rho(y). \tag{6.1}$$

This has the same positiveness or negativeness as R. Lemma 2.1.5 in [11] shows that as $\theta \to 0$ then R_θ weakly converges to R and $supp(R_\theta)$ converges to $supp(R)$. Moreover, if $U \subset \mathbb{P}^k$ is open and $R|_U$ is continuous, then R_θ converges locally uniformly on U to R.

c) Super-potential:

Let S be a smooth form in $\mathscr{C}_p$, and let R be in $\mathscr{C}_{k-p+1}$. If U_R is a quasi-potential of R (of mean 0), then the number $\int_X S \wedge U_R$ is independent of the choice of U_R, and is denoted by

$$\mathscr{U}_S(R) = \int_X S \wedge U_R,$$

and $\mathscr{U}_S$ is called the superpotential (of mean 0) of S.

For arbitrary $S \in \mathscr{C}_p$ and $R \in \mathscr{C}_{k-p+1}$, define

$$\mathscr{U}_S(R) := \lim_{\theta \to 0} \mathscr{U}_{S_\theta}(R) = \lim_{\theta \to 0} \mathscr{U}_S(R_\theta).$$

Note that this definition is symmetric $\mathscr{U}_S(R) = \mathscr{U}_R(S)$.

d) Pullback of currents:

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominant rational map. A positive closed (p,p) current T is called f^*-admissible if

$$\mathscr{U}_T(f_*(\omega^{k-p+1})) > -\infty.$$

In this case, we define $f^*(T)$ as follows:

$$f^*(T) = \lim_{\theta \to 0} f^*(T_\theta).$$

6.3. Some open questions. — Let X be a compact Kähler manifold, and let $f: X \to X$ be a dominant meromorphic map.

A) Let T be a positive closed (p,p) current on X with Siu's decomposition $T = R + \sum_j \lambda_j [V_j]$. Let $E(T)$ be as in Theorem 8. Assume that for any irreducible analytic V contained in $E(T)$ then $f^{-1}(V)$ has codimension $\geq p$. Is $f^\sharp(T)$ well-defined? If so, is $f^\sharp(R)$ positive? Note that by Corollary 2, $f^\sharp(T)$ may not be positive though.

B) Assume that $\pi_1(\mathscr{C}_f)$ has codimension $\geq p$.

a) When $X = \mathbb{P}^k$, [11] showed that $\pi_1(\mathscr{C}_{f^n})$ has codimension $\geq p$ for all n. Is the same true for a general X?

b) Does f satisfy dd^c-p stability condition? This holds for $p = 1$.

c) Using a) and the fact that when $X = \mathbb{P}^k$ then $f^\sharp$ preserves the convex cone of positive (p,p) currents, [11] showed that if moreover f is p-algebraic stable then $(f^n)^\sharp = (f^\sharp)^n$ for all n. Does the same conclusion hold when X is arbitrary Kähler manifold? We check that the answer to this question is positive when $f = J_X$:

Lemma 10. — *Let J_X be the same map in Section 4. Then J_X is 2-algebraic stable and $(J_X^\sharp)^2 = Id = (J_X^2)^\sharp$ on positive closed $(2,2)$ currents.*

Proof. — Since J_X has no exceptional hypersurface, J_X is 1-algebraic stable. Because $J_X = J_X^{-1}$, it follows by duality that J_X is also 2-algebraic stable. Since $J_X^2 = Id$, it remains to check that $(J_X^\sharp)^2 = Id$. Define $A = \bigcup_{i\neq j} \widetilde{\Sigma}_{i,j}$.

1) First we show that for a DSH^1 current R then:

$$(J_X^\sharp)^2(R) = R. \tag{6.2}$$

For this end, first we show that $(J_X^\sharp)^2(R) = R$ on $X - A$. Since $J_X^\sharp$ is continuous in the DSH^1 topology by Theorem 6, using Theorem 13 it suffices to show (6.2) for a smooth $(1,1)$ form R. In that case it is easy to see, since $(J_X^\sharp)^2(R)$ is determined by its restriction on $X - A$, and on $X - A$ it is not other than the usual pullback of smooth forms $(J_X|_{X-A}^*)^2(R)$.

Having $(J_X^\sharp)^2(R) = R$ on $X - A$, then (6.2) follows by the Federer type of support in [2].

2) It follows from 1) that if T is a positive closed $(2,2)$ current on X, then $(J_X^\sharp)^2(T) - T$ depends only on the cohomology class of T. In fact, if T' is a positive closed $(2,2)$ current having the same cohomology class as T, then $T - T' = dd^c(R)$ for a DSH^1 current R. Then from 1)

$$(J_X^\sharp)^2(T) - (J_X^\sharp)^2(T') = dd^c(J_X^\sharp)^2(R) = dd^c(R) = T - T'.$$

3) From 2), to prove Lemma 10 it suffices to show it for a set of positive closed currents whose cohomology classes generate $H^{2,2}(X)$. For such a set, we can consider the currents of integrations on a generic line in $\mathbb{P}^3$, a generic line

in the exceptional divisors E_0, E_1, E_2, E_3, and the line $\widetilde{\Sigma}_{i,j}$. In these cases, the wanted equality is easy to be checked. □

C) Can the constructions of invariant currents in the Subsection 1.4 be extended to other cases, for example for a map in Question B?

Lemma 3 gives a positive support to this question. More generally, for any meromorphic map f, there are natural candidates μ for an invariant measure of f. These measures can be standardly constructed as in the proof of Lemma 3: Let α be a smooth probability measure. Then μ is a cluster point of the sequence

$$\mu_N = \frac{1}{N} \sum_{j=0}^{N-1} \frac{(f^*)^j(\alpha)}{\delta_k(f)^j}.$$

There are two problems remain to be solved. First, we don't know whether the measure μ constructed this way can be pulled back or not. Second, we don't know whether we have a continuity property to help showing that $f^\sharp(\mu) = \delta_k(f)\mu$. If we can extend Theorem 12 to be applicable to the sequence μ_N then we can solve these two problems altogether.

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