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Jean-Yves Chemin & Fabrice Planchon

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SELF-IMPROVING BOUNDS FOR THE NAVIER-STOKES EQUATIONS

BY JEAN-YVES CHEMIN & FABRICE PLANCHON

ABSTRACT. — We consider regular solutions to the Navier-Stokes equation and provide an extension to the Escauriaza-Seregin-Sverak blow-up criterion in the negative regularity Besov scale, with regularity arbitrarly close to -1. Our results rely on turning a priori bounds for the solution in negative Besov spaces into bounds in the positive regularity scale.

RÉSUMÉ (*Estimations de bootstrap* a priori *pour Navier-Stokes*). — On considère des solutions régulières des équations de Navier-Stokes pour lesquelles on prouve une extension du critère d'explosion d'Escauriaza-Seregin-Sverak dans l'échelle des espaces de Besov de régularité négative, arbitrairement proche de -1. Nos résultats reposent sur l'amélioration d'estimations *a priori* en régularité négative pour devenir à régularité positive.

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JEAN-YVES CHEMIN, Laboratoire Jacques-Louis Lions, UMR CNRS 7598, Université Pierre et Marie Curie, 4 place Jussieu 75005 Paris, France • *E-mail :* `chemin@ann.jussieu.fr`
FABRICE PLANCHON, Laboratoire J. A. Dieudonné, UMR CNRS 7351, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France and Institut universitaire de France • *E-mail :* `fabrice.planchon@unice.fr`

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1. Introduction

We consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

$$\text{(NS)} \qquad \begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\otimes u) - \nabla\pi, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0 \end{cases}$$

for $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$, where $u=(u_i(x,t))_{i=1}^3\in\mathbb{R}^3$ is the velocity vector field, $\pi(x,t)\in\mathbb{R}$ is the associated pressure function and

$$\nabla\cdot(u\otimes u) := \Big(\sum_{j=1}^d \partial_{x_j}(u_i u_j)\Big)_{i=1}^d.$$

In the pioneering work [11], J. Leray proved the existence of global turbulent (weak in the modern terminology) solutions of (NS) for initial data with finite kinetic energy, i.e. initial data in L^2. These solutions need not be unique or preserve regularity of the initial data. In this same work, J. Leray proved that for regular enough initial data (namely H^1 initial data), a local (in time) unique solution exists. He also proved that as long as this solution is regular enough, it is unique among all the possible turbulent solutions, and moreover, if such a turbulent solution satisfies

$$u\in L^p([0,T[;L^q(\mathbb{R}^3)) \text{ with } \frac{2}{p}+\frac{3}{q}=1,\ q>3, \tag{1.1}$$

then the solution remains regular on $[0,T]$ and can be extended beyond time T. This is now known as Serrin's criterion.

On the other hand, there is a long line of works on constructing local in time solutions, from H. Fujita and T. Kato (see [9]) to H. Koch and D. Tataru (see [10]). For these results, the main feature is that the initial data belongs to spaces which are invariant under the scaling of the equations. Between [9] and [10], T. Kato (see [8]) proved wellposedness of (NS) for initial data u_0 in L^3. In this framework of local in time (strong, e.g. unique) solutions, Serrin's criterion may be understood as a non blow-up criterion at time T: e.g. if u is a strong solution with $u_0\in L^3(\mathbb{R}^3)$, that is $u\in C([0,T[;L^3(\mathbb{R}^3))$, and if (1.1) is satisfied, then one may (continuously and uniquely) extend the solution u past time T.

In the recent important work [7], L. Escauriaza, G. Seregin and V. Šverák obtained the endpoint version of Serrin's criterion, using blow-up techniques to construct a special solution vanishing at blow-up time and then backward uniqueness to rule out its existence. Earlier work of Giga and Von Wahl proved this endpoint under a continuity in time assumption in L^3, and such a continuity result was recently improved to match the local in time theory by Cheskidov-Shvydkoy [4].

Our first theorem (Theorem 1 below) may be seen as an extension of the end-point criterion by Escauriaza-Seregin-Šverák, in the negative regularity scale. Before providing an exact statement, we need to introduce a few notations and definitions.

Since we are interested in smooth (or at least strong in the Kato sense) solutions, (NS) is equivalent for our purpose with its integral formulation, where the pressure has been disposed of with the projection operator $\mathbb{P}$ over divergence free vector fields:

$$u = S(t)u_0 - \int_0^t \mathbb{P}S(t-s)\nabla\cdot(u\otimes u)(s)\,ds = u_L + B(u,u) \tag{1.2}$$

where $S(t) = \exp(t\mathbb{P}\Delta) = \mathbb{P}\exp(t\Delta)$ is the Stokes flow (which is nothing but the heat flow in $\mathbb{R}^3$ on divergence free vector fields) and $B(u,u)$ is the Duhamel term which reads, component wise

$$B(f,g) = -\int_0^t R_jR_kR_l|\nabla|S(t-s)(fg)(s)\,ds, \tag{1.3}$$

where the $R_{(\cdot)}$ are the usual Riesz transforms (recall $\mathbb{P}$ is a Fourier multiplier with matrix valued symbol $\mathrm{Id}-|\xi|^{-2}\xi\otimes\xi$). We will denote the Lebesgue norm by

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^3}|f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Let us recall a definition of Besov spaces using the heat flow $S(\sigma)$.

Definition 1.1. — *Let $Q(\sigma) = \sigma\partial_\sigma S(\sigma)$. We define $\dot{B}^{s,q}_p$ as the set of tempered distributions f such that*

- *the integral $\int_{1/N}^N Q(\sigma)f\,d\sigma/\sigma$ converges to f when $N\to+\infty$ as a tempered distribution if $s<\frac{d}{p}$ and after taking the quotient with polynomials if not, and*
- *the function $\sigma^{-s/2}\|Q(\sigma)f\|_p$ is in $L^q(d\sigma/\sigma)$; its norm defines the Besov norm of f:*

$$\|f\|^q_{\dot{B}^{s,q}_p} = \int_0^{+\infty}\sigma^{-sq/2}\|Q(\sigma)f\|^q_p\,\frac{d\sigma}{\sigma}. \tag{1.4}$$

We recall that the usual (homogeneous) Sobolev spaces $\dot{H}^s$, defined through the Fourier transform by $|\xi|^s\widehat{f}(\xi)\in L^2$, may be identified with $\dot{B}^{s,2}_2$, while the critical Sobolev embedding holds as follows: $\dot{B}^{s,q}_p\hookrightarrow\dot{B}^{\rho,\lambda}_r$ provided $s-d/p=\rho-d/r$, $s\geq\rho$ and $q\leq\lambda$, as well as $\dot{B}^{s,q}_p\hookrightarrow L^r_x$ provided $s-d/p=-d/r$, $s\geq 0$ and $q\leq r$.

We are now in a position to state our first result:

THEOREM 1. — *Let u be a local in time solution to (NS) such that $u_0 \in \dot{H}^{1/2}$. Assume that there exist $p \in]3, +\infty[$ and $q < 2p'$ such that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,q}} < +\infty, \tag{1.5}$$

then the solution may be uniquely extended past time T.

We remark that our hypothesis allows for smooth, compactly supported data; actually, one may simply assume that the vorticity $\omega_0 = \nabla \wedge u_0$ belongs to $L^{3/2}$. By Sobolev embedding and the Biot-Savart law, this implies that u_0 belongs to $\dot{H}^{\frac{1}{2}} \subset \dot{B}_p^{3/p-1,2}$. Hence by local Cauchy theory so does u and (1.5) is finite at least for small times.

It is of independent interest to consider the case of L^3 data, without any extra regularity hypothesis:

THEOREM 2. — *Let u be a local in time strong solution to (NS) with data u_0 in $L^3 \cap \dot{B}_p^{3/p-1,q}$, with $3 < p < +\infty$ and $q < 2p'$. Assume that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,q}} < +\infty, \tag{1.6}$$

then the solution may be uniquely extended past time T.

The restriction on q for the data implies that $q < 3$ as $p > 3$. As such, our result does not include the L^3 case, as we are still assuming a subtle decay hypothesis through the q indice. However, the restriction is mostly technical and all is required to lift it is to generalize the results from [6], most specifically the compactness result which is only stated in L^3 rather than in the Besov scale. This will be adressed elsewhere, providing generalizations of the present note and the results of [6]. Our purpose here is to illustrate that these blow up criterions do not require positive regularity on the data; in fact, they will extend to non L^3 data into the negative Besov scale.

Both Theorem 1 and 2 rely crucially on improving the rather weak a priori bound on u from the hypothesis. Such "self-improvements" are of independent interest and we state examples of them below. We start with a (spatial) regularity improvement for negative Besov-valued data (see the forthcoming Remark 2.7 on the p range restriction which is only technical).

THEOREM 3. — *Let u be a local in time strong solution to (NS) with data $u_0 \in \dot{B}_p^{3/p-1,q}$, with $3 < p < 6$ and $q < +\infty$. Assume that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,\infty}} \leq M, \tag{1.7}$$

then we have the following improved uniform bound on $w_3 = u - u_L - B(u_L, u_L)$,

$$\sup_{t\in[0,T[} \|w_3(\cdot,t)\|_{\dot{B}_2^{\frac{1}{2},\infty}} \leq C(M), \tag{1.8}$$

where C is an explicit smooth function of its argument.

For any initial datum $u_0 \in \dot{B}_p^{-(1-3/p),q}$, with $1 \leq p, q < +\infty$, there exists a unique, local in time, strong solution to (NS). Such solutions were obtained in [1] for $3 < p \leq 6$ and for all finite p in [12], and we refer to the appendix of [5] for a proof which is closer in spirit to the present note. One should point out that all these Besov spaces are embedded in VMO^{-1} (limits of smooth, compactly supported functions in BMO^{-1}) and that strong solutions in this endpoint space were obtained in [10].

Strong solutions are known to obey the same space-time estimates as the heat flow on any compact time interval on which they exist: one may take advantage of these estimates to improve regularity on $w = u - u_L$ in this context, as was done in [2] for L^3 data and in [12, 5] for $\dot{B}_p^{3/p-1,q}$ by substracting further iterates of the heat flow. However, to our knowledge, the only known result assuming an a priori bound with no time integrability was proved in [2] where the conclusion of Theorem 3 is obtained assuming a slightly weaker condition than $u \in L^\infty_t L^3_x$ (the Lebesgue space is replaced by its larger weak counterpart).

Finally, we provide a time regularity improvement, whose proof can be used to obtain Theorem 3 in the range $p \leq 4$, but should be of independent interest.

THEOREM 4. — *Let u be a local in time strong solution to (NS) with data $u_0 \in \dot{B}_4^{-1/4,4}$. Assume that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_4^{-1/4,4}} \leq M, \tag{1.9}$$

then u has the following Hölder in time regularity:

$$\forall (t,t') \in [0,T[^2\,,\ \|u(\cdot,t) - u(\cdot,t')\|_{\dot{B}_4^{-3/4,4}} \leq C(M)|t-t'|^{\frac{1}{4}}. \tag{1.10}$$

For notational convenience, set, for any $1 \leq p$,

$$B_p = \dot{B}_p^{-(1-3/p),\infty},\ \ B^s = B_p \ \text{ with } s - 3/p = -1.$$

In other words, indices are tight by scaling and we indifferently use regularity or decay to label spaces with scale -1. In what follows, we shall also need a suitable modification of Besov spaces, taking into account the time variable.

DEFINITION 1.2. — *For $1 \leq \rho \leq +\infty$, we shall say that $u(x,t)$ belongs to $\mathscr{L}^\rho([a,b]; \dot{B}_p^{s,q})$ if $u(t)$ is in $\dot{B}_p^{s,q}$ for all $t \in [a,b]$ and*

$$\int_0^{+\infty} \sigma^{-sq} \|Q(\sigma)u\|^q_{L^\rho([a,b];L^p_x)} \frac{d\sigma}{\sigma} < +\infty\ .$$

The associated norm is defined in the obvious way and $\mathscr{L}^\rho_T(\dot B^{s,q}_p) := \mathscr{L}^\rho([0,T];\dot B^{s,q}_p)$.

As before, we will adopt the following shorthand notation

$$\mathscr{L}^\rho B_p = \mathscr{L}^\rho B^s = \mathscr{L}^\rho_t(\dot B^{s,\infty}_p) \ \text{ with } \ s = -1 + 3/p + 2/\rho,$$

which is consistent with the previous one: $L^\infty_t B_p = \mathscr{L}^\infty B_p$.

We will denote by $\lesssim$ a less or equal sign with a harmless constant, and C any irrelevant constant which may change from line to line.

2. From a priori bounds to a generalized endpoint Serrin's criterion

From Sobolev's embedding, Theorem 1 immediately follows from Theorem 2. In turn, Theorem 2 is a consequence of the following key proposition.

PROPOSITION 2.1. — *Let u be as in Theorem 2. Then there exists a decomposition $u = v + w$ such that*

$$\sup_{t\in[0,T[} \|v(\cdot,t)\|_{L^3\cap \dot B^{3/p-1,q}_p} \le C(M,u_0), \tag{2.1}$$

$$\sup_{t\in[0,T[} \|w(\cdot,t)\|_{B_{1/(1-\varepsilon)}} \le C(M,u_0), \tag{2.2}$$

where ε may be chosen arbitrarly small.

Postponing the proof of this proposition for a moment, we prove Theorem 2: notice that (2.1) provides an a priori bound for the v part in $L^\infty([0,T[;L^3)$; we seek to obtain a similar bound for the w part. As $w = u - v$, we also have a bound on w in $L^\infty([0,T[;\dot B^{3/p-1,q}_p)$, from (2.1) and (1.6). As $q < 2p'$, let us write

$$q = \frac{2p'}{1+\eta} \quad \text{with } \eta \text{ small enough.}$$

Then define

$$r = \frac{3}{1+2\eta}\,, \quad \theta = \frac{1+2\eta-3/p}{3(1/p'-3\varepsilon)} \quad \text{and} \quad b = \frac{q}{1-\theta}\,,$$

and notice that $b \le 3$. We now combine this bound with (2.2), using convexity of norms and Sobolev embedding of Besov spaces into Lebesgue ones. This gives

$$\|w\|_{L^3} \lesssim \|w\|_{\dot B^{(3/r-1),b}_r} \lesssim \|w\|^\theta_{B_{1/(1-\varepsilon)}} \|w\|^{1-\theta}_{\dot B^{(3/p-1),q}_p}.$$

As such, we have obtained control of $u = v + w$ in $L^\infty_t(L^3_x)$, which allows to use the Escauriaza-Seregin-Šverák result to conclude the proof of Theorem 2. ☐

We now prove Proposition 2.1. Note that a local in time solution with data in $\dot{B}_p^{3/p-1,q}$ exists and additional regularity is preserved (see for instance [3] or [5]). Hence we do not worry about existence, but rather focus on improving bounds. It is convenient to present the argument in a rather abstract setting. Recall B was defined in (1.2), and set $w_2 = u - u_L = B(u,u)$, then

$$w_2 = B(u_L, u_L) + 2B(u_L, w_2) + B(w_2, w_2) \tag{2.3}$$

where we are obviously abusing notations (writing $B(u,v) = B(v,u)$). Note that from a priori bound (1.5) and local existence theory, we have $u_L \in L_t^\infty \dot{B}_p^{-(1-3/p),q}$ with a uniform bound $2M$, while obviously $u_L \in C_t(L_x^3)$ with bound $\|u_0\|_{L^3}$.

We start with an easy case which already provides the key features of the general argument without technicalities.

Lemma 2.2. — *Assume in addition to the hypothesis of Theorem 2 that $\omega_0 \in L^{3/2}$; then Proposition 2.1 holds with $v = u_L$, $w = w_2$ and $\varepsilon = 0$.*

We just remarked that, even without additional requirements, (2.1) holds for $v = u_L$. We are left with proving (2.2) for w_2: we will use (2.3). Note that by the Biot-Savart law, ∇u_0 belongs to $L^{3/2}$ and thus ∇u_L to $C_t(L^{3/2})$. By chain rule, $\nabla_x(u_L \otimes u_L)$ is in $L_t^\infty(L^1)$. Using Proposition 4.1 in [5], we infer

$$\|B(u_L, u_L)\|_{\mathscr{L}^\infty B_1} \lesssim \|u_0\|_{L^3} \|\nabla u_0\|_{L^{\frac{3}{2}}}. \tag{2.4}$$

Therefore, we seek an a priori bound for w_2 in $\mathscr{L}^\infty B_1$ from the weaker bound (1.5) on u.

To deal with the remaining terms in (2.3), we use the following lemma:

Lemma 2.3. — *Let $1 \le r < 3 < p < +\infty$, $f, g \in \mathscr{L}^\infty B_r \cap \mathscr{L}^\infty B_p$, $2/3 < 1/r + 1/p \le 1$ and $1/\eta \le 1/r + 1/p$, then*

$$\|B(f,g)\|_{\mathscr{L}^\infty B_\eta} \lesssim \|f\|_{\mathscr{L}^\infty B_r} \|g\|_{\mathscr{L}^\infty B_p} + \|g\|_{\mathscr{L}^\infty B_r} \|f\|_{\mathscr{L}^\infty B_p}. \tag{2.5}$$

If $p = 3$, the same estimate holds with B_3 replaced by L_x^3,

$$\|B(f,g)\|_{\mathscr{L}^\infty B_\eta} \lesssim \|f\|_{\mathscr{L}^\infty B_r} \|g\|_{L_t^\infty(L_x^3)} + \|g\|_{\mathscr{L}^\infty B_r} \|f\|_{L_t^\infty(L_x^3)}. \tag{2.6}$$

The proof of the lemma follows directly from standard product rules in Besov spaces and properties of the operator B defined by (1.3), see e.g. Proposition 4.1 in [5]. $\square$

For the term $B(w_2,w_2)$, (2.5) yields

$$\|B(w_2,w_2)\|_{\mathscr{L}^\infty B_1} \lesssim \|w_2\|_{\mathscr{L}^\infty B_p}\|w_2\|_{\mathscr{L}^\infty B_{p'}}\,, \tag{2.7}$$

and by convexity of Besov norms,

$$\|w_2\|_{\mathscr{L}^\infty B_{p'}} \lesssim \|w_2\|^{\lambda}_{\mathscr{L}^\infty B_1}\|w_2\|^{(1-\lambda)}_{\mathscr{L}^\infty B_p}, \quad \text{with} \quad \lambda = \frac{p-2}{p-1}\,\cdot$$

Therefore,

$$\|B(w_2,w_2)\|_{\mathscr{L}^\infty B_1} \lesssim K^{2-\lambda}\|w_2\|^{\lambda}_{\mathscr{L}^\infty B_1} \quad \text{with} \quad K = \sup_{t\in[0,T[} \|u(\cdot,t)\|_{B_p}\,. \tag{2.8}$$

The crossterm is handled in a similar way: convexity of norms yields again

$$\|w_2\|_{\mathscr{L}^\infty B_{3/2}} \lesssim \|w_2\|^{\eta}_{\mathscr{L}^\infty B_1}\|w_2\|^{(1-\eta)}_{\mathscr{L}^\infty B_p}, \quad \text{with} \quad \eta = \frac{2p-3}{3(p-1)}\,,$$

and by (2.6)

$$\|B(u_L,w_2)\|_{\mathscr{L}^\infty B_1} \lesssim \|u_0\|_{L^3}K^{1-\eta}\|w_2\|^{\eta}_{\mathscr{L}^\infty B_1} + \|\nabla u_0\|_{L^{\frac32}}K^{1-\theta}\|w_2\|^{\theta}_{\mathscr{L}^\infty B_1}. \tag{2.9}$$

Gathering (2.4), (2.9) and (2.8) and using convexity, we obtain the desired control of w_2 in $\mathscr{L}^\infty B_1$, which ends the proof of Lemma 2.2. □

In order to lower the regularity requirement on u_0, we need to deal with the crossterm in a different way: in fact, the part of $B(u_L,w_2)$ which carries high frequencies of u_L has no reason to be any better than $B_{3/(2(1-\varepsilon))}$. Hence, we seek first such an a priori estimate for w_2, and then bootstrap this intermediate estimate to a suitable estimate in $B_{1/(1-\varepsilon)}$ for the next term in the expansion:

Lemma 2.4. — *Under the hypothesis of Theorem 2, Proposition 2.1 holds with*

$$v = u_L + B(u_L,u_L) + 2B(u_L,w_2) \quad \textit{and} \quad w = B(w_2,w_2).$$

Remark that, by standard heat estimates, the bound (2.1) holds for $B(u_L,u_L)$ as it already does for u_L. We now use the following lemma to take care of the crossterm:

Lemma 2.5. — *Let $3<p<+\infty$, $f\in \mathscr{L}^\infty B_p$, then*

$$\|B(u_L,f)\|_{L^\infty_t(L^3)} \lesssim \|f\|_{\mathscr{L}^\infty B_p}\|u_0\|_{L^3}\,, \tag{2.10}$$

and

$$\|B(u_L,f)\|_{L^\infty_t(\dot B_p^{-(1-3/p),q})} \lesssim \|f\|_{\mathscr{L}^\infty B_p}\|u_0\|_{\dot B_p^{-(1-3/p),q}}\,. \tag{2.11}$$

The proof of Lemma 2.5 follows once again from product rules and properties of B (Proposition 4.1, [5]), provided one uses heat estimates on u_L: for (2.10), one uses $u_L \in \mathscr{L}^{2p'}_t(\dot B_3^{1/p',3})$, while for (2.11) one uses $u_L \in \mathscr{L}^1_t(\dot B_p^{3/p+1,q}) \cap \mathscr{L}^\infty_t(\dot B_p^{3/p-1,q})$. □

We now apply the lemma to $f = w_2$ (which was already proved to be in $\mathscr{L}^\infty B_p$) and finally get bound (2.1) on our $v = u_L + B(u_L, u_L) + 2B(u_L, w_2)$. We now turn to the bound on w, with a new product lemma:

LEMMA 2.6. — *Let $\varepsilon > 0$ be small, $f \in \mathscr{L}^\infty B_{3/(1+\varepsilon)}$, then*

$$\|B(u_L, f)\|_{\mathscr{L}^\infty B_{3/(2(1-\varepsilon))}} \lesssim \|f\|_{\mathscr{L}^\infty B_{3/(1+\varepsilon)}} \|u_0\|_{L^3}\,. \tag{2.12}$$

As before, the lemma follows from product rules in Besov spaces, actually requiring only $u_L \in \mathscr{L}^\infty B^{-3\varepsilon}$. □

Going back to (2.3), we have $B(u_L, u_L) \in \mathscr{L}^\infty B^1$ from standard estimates (or suitable tweaking of the previous lemma, or [2]). From Lemmata 2.3 and 2.6,

$$\|w_2\|_{\mathscr{L}^\infty B_{3/(2(1-\varepsilon))}} \lesssim C(u_0) + \|w_2\|_{\mathscr{L}^\infty B_{3/(1+\varepsilon)}} \|u_0\|_{L^3} + \|w_2\|^2_{\mathscr{L}^\infty B_{3/(1+\varepsilon)}}$$

and by convexity of Besov norms,

$$\|w_2\|_{B_{3/(1+\varepsilon)}} \leq \|w_2\|^\lambda_{B_{3/(2(1-\varepsilon))}} \|w_2\|^{1-\lambda}_{B_p}\,,$$

where $\lambda = ((1+\epsilon)p - 3)/(2(1-\epsilon)p - 3) < 1/2$, provided $\varepsilon < 3/(4p)$. Hence, combining the three previous inequalities and convexity, we obtain

$$\|w_2\|_{\mathscr{L}^\infty B_{3/(2(1-\varepsilon))}} \leq C(u_0, M).$$

We can now proceed with $w = B(w_2, w_2)$: another application of Lemma 2.3 yields

$$\|B(w_2, w_2)\|_{\mathscr{L}^\infty B_{1/(1-\varepsilon)}} \lesssim \|w_2\|_{\mathscr{L}^\infty B_{3/(2(1-\varepsilon))}} \|w_2\|_{\mathscr{L}^\infty B_{3/(1-\varepsilon)}}\,,$$

which concludes the proof of Lemma 2.4 and therefore the proof of Proposition 2.1. □

For the remaining part of this section we prove Theorem 3. Recall we then have $3 < p < 6$ and the solution u satisfies a priori bound (1.7).

In order to compensate for the lack of positive regularity on the linear flow u_L, we need one further iteration: set $w_2 = B(u_L, u_L) + w_3$, then

$$\begin{aligned} w_3 = {} & 2B(u_L, B(u_L, u_L)) + B(B(u_L, u_L), B(u_L, u_L)) \\ & + 2B(u_L, w_3) + 2B(B(u_L, u_L), w_3) + B(w_3, w_3). \end{aligned} \tag{2.13}$$

We start with terms involving only the linear flow: standard heat estimates yield (see e.g. Proposition 4.1 of [5])

$$B(u_L, u_L) \in \mathscr{L}^\infty B_{p/2} \cap \mathscr{L}^1 B_{p/2}\,;$$

then, by standard product rules, with $p < q < 6$, where $\kappa = 3/p - 3/q > 0$ is understood to be small,

$$B(u_L, B(u_L, u_L)) \in \mathscr{L}^\infty B_{3q/(q+3)} \cap \mathscr{L}^1 B_{3q/(q+3)},$$

as the worst case is when low frequencies are on $B(u_L, u_L) \in \mathscr{L}^\infty B^{-\kappa}$. Notice that for $q = 6$ we would get $\mathscr{L}^\infty B^{1/2} = \mathscr{L}^\infty B_2$. The quadrilinear term is dealt with in a similar way.

REMARK 2.7. — *This may be iterated again, of course, but we will not do so here. Our restriction on p comes from the balance between regularity $3/q$ (on the trilinear term in u_L) and $3/p - 1$ (our a priori bound), which requires $3/q + 3/p - 1 > 0$.*

Next, we prove the following proposition, which is a slight improvement over the statement from Theorem 3.

PROPOSITION 2.8. — *Assume (1.7) on u for $3 < p < 6$, then, for $p < q < 6$,*

$$\|w_3\|_{\mathscr{L}^\infty B_{3q/(q+3)}} \lesssim C(\|u\|_{\mathscr{L}^\infty B_p}).$$

We already dealt with terms involving only u_L in (2.13). All $B(\cdot, \cdot)$ terms involving w_3 itself are like $B(v, w_3)$, where $v \in \mathscr{L}^\infty B_p$ and $\|v\|_{\mathscr{L}^\infty B_p} \lesssim M = \|u\|_{\mathscr{L}^\infty B_p}$.

LEMMA 2.9. — *Let r be such that $3/r = (q+3)/q - \varepsilon$ and $\varepsilon < 6/q - 1$. Let v be in $\mathscr{L}^\infty B_p$ and w_3 in $\mathscr{L}^\infty B_r$, then*

$$\|B(v, w_3)\|_{\mathscr{L}^\infty B^{3/q}} \lesssim \|v\|_{\mathscr{L}^\infty B_p} \|w_3\|_{\mathscr{L}^\infty B_r}. \tag{2.14}$$

The lemma is again a direct consequence of product rules and properties of B. □

By convexity of Besov norms,

$$\|w_3\|_{B_r} \lesssim \|w_3\|_{B^{3/q}}^{1-\eta} \|w_3\|_{B_p}^{\eta}$$

with $\eta = \varepsilon/(1-\kappa)$, and

$$\|B(v, w_3)\|_{\mathscr{L}^\infty B^{3/q}} \lesssim M^{1+\eta} \|w_3\|_{L^\infty B^{3/q}}^{1-\eta} \lesssim C(\eta) M^{\frac{1+\eta}{\eta}} + \gamma(\eta) \|w_3\|_{\mathscr{L}^\infty B^{3/q}},$$

where we may chose $\gamma \ll 1$. Summing estimates, we close on w_3,

$$\|w_3\|_{\mathscr{L}^\infty B^{3/q}} \lesssim C(\delta, M) + \delta \|w_3\|_{\mathscr{L}^\infty B^{3/q}},$$

with a small suitable δ.

REMARK 2.10. — *Note that we assume that u_0 is actually in $\dot{B}_p^{3/p-1,q}$ with $q < +\infty$; then a local in time strong solution exists, and the a priori bound is valid as long as the strong solution exists, because w_3 is already known to be in $B^{3/p}$ as a byproduct of local existence theory. We are **not** constructing w_3, merely improving a bound.*

3. Hölder regularity in time

3.1. Scaled energy estimates. — Consider a local in time solution u such that $u_0 \in \dot{B}_4^{-1/4,4}$. Assuming it exists past time T, one may prove that $\sup_{0<t<T} t^{\frac{1}{8}}\|u(\cdot,t)\|_{L^4} < +\infty$; from the Duhamel formula, one then obtains that $\sup_{0<t<T}\|u-u_L\|_{L^2_x} \lesssim T^{1/4}$. The next proposition proves that such a bound does not depend on the local Cauchy theory but only on a suitable a priori bound:

PROPOSITION 3.1. — *Let u be a solution of (NS). Then, recalling $u = u_L + w$, we have*

$$\frac{1}{t^{\frac{1}{2}}}\|w(t)\|_{L^2}^2 + \int_0^t \frac{1}{t'^{\frac{1}{2}}}\Big(\|\nabla w(t')\|_{L^2}^2 + \frac{1}{t'}\|w(t')\|_{L^2}^2\Big)\,dt' \lesssim \|u_0\|^4_{\dot{B}_4^{-1/4,4}} \exp\big(C\|u_0\|^5_{\dot{B}_5^{-2/5,5}}\big).$$

The equation on w reads

$$(3.1)\qquad \begin{cases} \partial_t w - \Delta w + w\cdot\nabla w + u_L\cdot\nabla w = -w\cdot\nabla u_L - u_L\cdot\nabla u_L - \nabla p \\ \operatorname{div} w = 0 \quad \text{and} \quad w_{|t=0} = 0. \end{cases}$$

Performing an L^2 energy estimate on (3.1) yields

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \le \|u_L\|_{L^4}^2\|\nabla w(t)\|_{L^2} + \|u_L\|_{L^5}\|w\|_{L^{\frac{10}{3}}}\|\nabla w\|_{L^2}$$

where integration by parts was done on all terms on the right using the divergence free condition, followed by Hölder. As $\|w\|_{10/3} \le \|w\|_{L^2}^{2/5}\|\nabla w\|_{L^2}^{3/5}$, by convexity

$$(3.2)\qquad \frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \le 2\|u_L(t)\|_4^4 + C\|w(t)\|_{L^2}^2\|u_L(t)\|_5^5.$$

Introduce the correct scaling in time, $\psi(t) = t^{-\frac{1}{2}}\|w(t)\|_{L^2}^2$ and let $\phi(t) = \int_0^t \|u_L(t')\|_{L^5}^5\,dt'$,

$$\frac{d}{dt}\Big(\psi(t)e^{-C\phi(t)}\Big) + \frac{1}{t^{\frac{1}{2}}}\left(\psi(t) + \|\nabla w(t)\|_2^2\right)e^{-C\phi(t)} \le \frac{2}{t^{\frac{1}{2}}}\|u_L(t)\|_{L^4}^4 e^{-C\phi(t)}.$$

We now integrate over $[0,t]$,

$$\psi(t) + \int_0^t \frac{1}{t'^{\frac{1}{2}}}\left(\psi(t') + \|\nabla w(t')\|_{L^2}^2\right)dt' \lesssim \int_0^{+\infty} t'^{\frac{1}{2}}\|S(t')u_0\|_4^4\frac{dt'}{t'} \times \exp\Big(C\int_0^{+\infty} t'\|S(t')u_0\|_{L^5}^5\frac{dt'}{t'}\Big).$$

Recalling that in our definition of Besov norms (1.4) we may replace $Q(t)$ by $S(t)$ for negative regularity, we identify equivalent norms for the Besov norms $\dot{B}_4^{-1/4,4}$ and $\dot{B}_5^{-2/5,5}$ in our last inequality, and get the desired result. $\square$

3.2. From scaled energy estimates to regularity improvement. — We now (re)prove a particular case of Theorem 3, namely $p = q = 4$, from the estimates in the previous subsection. Iterating the scaled energy estimate on higher order fluctuations would allow larger p, q.

PROPOSITION 3.2. — *Let u be a strong solution of (NS) on a time interval $[0, T[$, with data u_0 in $\dot{B}_4^{-1/4,4}$ and such that $u \in L^\infty([0, T[; \dot{B}_4^{-1/4,4})$. Then w is in $L^\infty([0, T[; \dot{B}_2^{1/2,\infty})$.*

Proof. — It requires an alternate definition of Besov spaces, using discrete Littlewood-Paley decompositions rather than heat operators.

DEFINITION 3.1. — *Let ϕ be a smooth function in the Schwartz class such that $\widehat{\phi} = 1$ for $|\xi| \leq 1$ and $\widehat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_j(x) := 2^{dj}\phi(2^j x)$, and frequency localization operators $S_j := \phi_j * \cdot$, $\Delta_j := S_{j+1} - S_j$. An equivalent definition of $\dot{B}_p^{s,q}$ is the set of tempered distributions f such that*

- *the partial sum $\sum_{-m}^{m} \Delta_j f$ converges to f as a tempered distribution if $s < \frac{d}{p}$ and after taking the quotient with polynomials if not, and*
- *the sequence $\epsilon_j := 2^{js}\|\Delta_j f\|_p$ is in ℓ^q; its ℓ^q-norm defines the Besov norm of f.*

We proceed with proving Proposition 3.2. From standard heat kernel bounds for frequency localized functions, (1.2) yields the inequality

$$
(3.3) \quad 2^{\frac{j}{2}}\|\Delta_j w(t)\|_{L^2} \lesssim \int_0^t e^{-c2^{2j}(t-t')}2^{\frac{3}{2}j}\|\Delta_j(u_L \otimes u_L(t')\|_{L^2}\,dt'
$$
$$
+\int_0^t e^{-c2^{2j}(t-t')}2^{\frac{3}{2}j}\|\Delta_j(u_L \otimes w)(t')\|_{L^2}\,dt' + \int_0^t e^{-c2^{2j}(t-t')}2^{\frac{3}{2}j}\|\Delta_j(w \otimes w)(t')\|_{L^2}\,dt'.
$$

Let us denote by $K_j(t) + J_j(t) + I_j(t)$ the righthand side. The first term is easy, using standard heat decay: $t^{1/8}\|u_L(t)\|_4 \lesssim \|u_0\|_{\dot{B}_4^{-1/4,\infty}}$, and

$$
K_j(t) \lesssim \int_0^t e^{-c2^{2j}(t-t')}2^{\frac{3}{2}j}t'^{-1/4}\,dt' = \int_0^{2^{2j}t} e^{-c(2^{2j}t-\tau)}\tau^{-1/4}\,d\tau \lesssim 1.
$$

The second term is similar, using $t^{1/2}\|u_L(t)\|_\infty \lesssim \|u_0\|_{\dot{B}_\infty^{-1,\infty}}$, and $\|w(t)\|_{L^2} \lesssim t^{\frac{1}{4}}$,

$$
J_j(t) \lesssim \int_0^t e^{-c2^{2j}(t-t')}2^{\frac{3}{2}j}t'^{1/4}t'^{-1/2}\,dt' = \int_0^{2^{2j}t} e^{-c(2^{2j}t-\tau)}\tau^{-1/4}\,d\tau \lesssim 1.
$$

Let us decompose $I_j(t)$ by introducing $t_{j,\Lambda} \stackrel{\text{def}}{=} t-\Lambda 2^{-2j}$ (where Λ will be chosen later on) and set $I_j(t) = I_{j,1}(t) + I_{j,2}(t)$ with

$$\begin{aligned} I_{j,1}(t) &= \int_0^{t_{j,\Lambda}} e^{-c2^{2j}(t-t')} 2^{\frac{3}{2}j} \|\Delta_j(w\otimes w)(t')\|_{L^2}\, dt' \quad \text{and} \\ I_{j,2}(t) &= \int_{t_{j,\Lambda}}^t e^{-c2^{2j}(t-t')} 2^{\frac{3}{2}j} \|\Delta_j(w\otimes w)(t')\|_{L^2}\, dt' \,. \end{aligned}$$

We have

$$I_{j,1}(t) \leq e^{-\frac{c}{2}\Lambda} \int_0^{t_{j,\Lambda}} e^{-\frac{c}{2}2^{2j}(t_{j,\Lambda}-t')} 2^{\frac{3}{2}j} \|\Delta_j(u\otimes u(t')\|_{L^2}\, dt'.$$

From product rules in Besov spaces,

$$\|\Delta_j(w\otimes w)\|_{L^\infty(0,T;L^2)} \lesssim 2^{\frac{j}{2}} \|w\|_{L^\infty(0,T;\dot B_4^{-1/4,\infty})} \|w\|_{L^\infty(0,T;\dot B_2^{1/2,\infty})}.$$

Choosing Λ such that $Ce^{-\frac{c}{2}\Lambda}\|w\|_{L^\infty(0,T;\dot B_4^{-1/4,\infty})} \leq \frac{1}{2}$, we get

$$\sup_{t\in[0,T[} I_{j,1}(t) \leq \frac{1}{2}\|w\|_{L^\infty(0,T;\dot B_2^{1/2,\infty})}. \tag{3.4}$$

We are left with $I_{j,2}(t)$. We may replace w by u, as this just adds terms which are similar to the K_j and J_j terms. We then split u on the interval $[t_{j,\Lambda}, t]$ in the following way

$$u = u_{L,j} + w_j \quad \text{with} \quad u_{L,j}(t) \stackrel{\text{def}}{=} e^{(t-t_{j,\Lambda})\Delta} u(t_{j,\Lambda}).$$

By the same reasonning that took care of the K_j and J_j terms, the $u_{L,j}\otimes u_{L,j}$ and $u_{L,j}\otimes w_j$ terms in I_j are uniformly bounded. We are left with quadratic terms $w_j\otimes w_j$. Using Bernstein inequality we have

$$\|\Delta_j(w_j\otimes w_j)\|_{L^2} \lesssim 2^{\frac{3}{2}j}\|w_j(t)\|_{L^2}^2.$$

By integration on the interval $[t_{j,\Lambda}, t]$ (the length of which is less than $\Lambda 2^{-2j}$)

$$2^{\frac{3j}{2}} \int_{t_{j,\Lambda}}^t \|\Delta_j(w_j(t')\otimes w_j(t'))\|_{L^2} dt' \lesssim 2^{3j}(\Lambda 2^{-2j})\|w_j\|^2_{L^\infty[t_{j,\Lambda},t];L^2)}.$$

Proposition 3.1 with initial time $t_{j,\Lambda}$ implies

$$2^{\frac{3j}{2}} \int_{t_{j,\Lambda}}^t \|\Delta_j(w_j(t')\otimes w_j(t'))\|_{L^2} dt' \lesssim \Lambda\|u\|^4_{L^\infty(0,T;\dot B_4^{-1/4,4})} \exp\bigl(C\|u\|^5_{L^\infty(0,T;\dot B_5^{-2/5,5})}\bigr).$$

Then plugging all this in (3.3) and (3.4), we get, for any $t<T$,

$$\begin{aligned} \|w\|_{L^\infty([0,t];\dot B_2^{1/2,\infty})} &\lesssim \|u_0\|^2_{\dot B_4^{-1/4,\infty}} + \frac{1}{2}\|w\|_{L^\infty([0,t];\dot B_2^{1/2,\infty})} \\ &\quad + C(1+\Lambda)\|u\|^4_{L^\infty(0,T;\dot B_4^{-1/4,4})} \exp\bigl(C\|u\|^5_{L^\infty(0,T;\dot B_5^{-2/5,5})}\bigr). \end{aligned}$$

The choice of Λ means that $\Lambda \sim \log\bigl(e + \|u\|_{L^\infty(0,T;\dot B_4^{-1/4,4})}\bigr)$. This concludes the proof of Proposition 3.2. $\square$

Let us prove Theorem 4. Let us consider two times t and t_0 in $[0,T[$. We can assume that $t_0 < t$. Then, let us write that

$$u(t) - u(t_0) = u(t) - S(t-t_0)u(t_0) + \bigl(S(t-t_0) - \mathrm{Id}\bigr)u(t_0).$$

Applying Proposition 3.2 at time t_0 and using $L^2 \hookrightarrow \dot B_4^{-3/4,4}$

$$\|u(t) - S(t-t_0)u(t_0)\|_{\dot B_4^{-3/4,4}} \le C(M)|t-t_0|^{\frac14}.$$

Moreover, we have

$$\bigl\|\bigl(S(t-t_0) - \mathrm{Id}\bigr)u(t_0)\bigr\|_{\dot B_4^{-3/4,4}} \le C|t-t_0|^{\frac14}\|u(t_0)\|_{\dot B_4^{-1/4,4}}\,,$$

and Theorem 4 is proved. $\square$

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