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RATIONAL BV-ALGEBRA IN STRING TOPOLOGY

BY YVES FÉLIX & JEAN-CLAUDE THOMAS

To Micheline Vigué-Poirrier on her 60th birthday

ABSTRACT. — Let M be a 1-connected closed manifold of dimension m and LM be the space of free loops on M. M. Chas and D. Sullivan defined a structure of BV-algebra on the singular homology of LM, $H_*(LM;\boldsymbol{k})$. When the ring of coefficients is a field of characteristic zero, we prove that there exists a BV-algebra structure on the Hochschild cohomology $HH^*(C^*(M);C^*(M))$ which extends the canonical structure of Gerstenhaber algebra. We construct then an isomorphism of BV-algebras between $HH^*(C^*(M);C^*(M))$ and the shifted homology $H_{*+m}(LM;\boldsymbol{k})$. We also prove that the Chas-Sullivan product and the BV-operator behave well with a Hodge decomposition of $H_*(LM)$.

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Résumé (*BV-algèbres rationnelles en topologie des lacets libres*)
Soit M une variété simplement connexe compacte sans bord de dimension m. Désignons par LM l'espace des lacets libres sur M. M. Chas et D. Sullivan ont défini une structure de BV-algèbre sur l'homologie singulière $H_*(LM;\boldsymbol{k})$. Lorsque l'anneau des coefficients $\boldsymbol{k}$ est un corps de caractéristique nulle, nous établissons l'existence d'une structure de BV-algèbre sur la cohomologie de Hochschild $HH^*(C^*(M);C^*(M))$ qui étend la structure canonique d'algèbre de Gerstenhaber. De plus nous construisons un isomorphisme de BV-algèbres entre $H_{*+m}(LM;\boldsymbol{k})$ et $HH^*(C^*(M);C^*(M))$. Finalement nous démontrons que le produit de Chas-Sullivan ainsi que le BV-opérateur sont compatibles avec la décomposition de Hodge de $H_*(LM;\boldsymbol{k})$.

1. Introduction

Chas and Sullivan considered in [3] the free loop space $LM = \mathrm{map}(S^1, M)$ for a smooth orientable closed manifold of dimension m. They use geometric methods to show that the shifted homology $\mathbb{H}_*(LM) := H_{*+m}(LM)$ has the structure of a Batalin-Vilkovisky algebra (BV-algebra for short). Later on Cohen and Jones defined in [5] a ring spectrum structure on the Thom spectrum LM^{-TM} which realizes the Chas-Sullivan product in homology. More recently, Gruher and Salvatore proved in [17] that the algebra structure (and thus the BV-algebra structure) on $\mathbb{H}_*(LM)$ is natural with respect to smooth orientation preserving homotopy equivalences.

Assume that the coefficients ring is a field. By a result of Jones, [19, Thm. 4.1] there exists a natural linear isomorphism

$$HH_*\big(C^*(M);C^*(M)\big) \cong H^*(LM),$$

and by duality an isomorphism $H_*(LM) \cong HH^*(C^*(M);C_*(M))$. Here $HH_*(A;Q)$ (respectively $HH^*(A;Q)$) denotes the Hochschild homology (respectively cohomology) of a differential graded algebra A with coefficients in the differential graded A-bimodule Q, $C^*(M)$ denotes the singular cochains algebra and $C_*(M)$ the complex of singular chains. The cap product induces an isomorphism of graded vector spaces (for instance see [11, Appendix]), $HH^*(C^*(M);C_*(M)) \cong HH^{*-m}(C^*(M);C^*(M))$, and therefore an isomorphism of graded vector spaces

$$\mathbb{H}_*(LM) \cong HH^*\big(C^*(M);C^*(M)\big).$$

Since $HH^*(A;A)$ is canonically a Gerstenhaber algebra, for any differential graded algebra A, it is natural to ask:

Question 1. — *Does there exist an isomorphism of Gerstenhaber algebras between $\mathbb{H}_*(LM)$ and $HH^*(C^*(M);C^*(M))$?*

Various isomorphisms of graded algebras have been constructed. The first one has been constructed by Merkulov for real coefficients [24], [13] using iterated integrals. An another isomorphism has been constructed for rational coefficients by M. Vigué and the two authors, [12], using the chain coalgebra of the Quillen minimal model of M.

Although $HH^*(A;A)$ does not have, for any differential graded algebra A, a natural structure of BV-algebra extending the canonical Gerstenhaber algebra, a second natural question is:

QUESTION 2. — *Does there exist on $HH^*(C^*(M);C^*(M))$ a structure of BV-algebra extending the structure of Gerstenhaber algebra and an isomorphism of BV-algebras between $\mathbb{H}_*(LM)$ and $HH^*(C^*(M);C^*(M))$?*

The main result of this paper furnishes a positive answer to Question 2 and thus to Question 1 when the field of coefficients is assumed of characteristic zero.

THEOREM 1. — *If M is 1-connected and the field of coefficients has characteristic zero then*

(i) *Poincaré duality induces a* BV*-structure on $HH^*(C^*(M);C^*(M))$ extending the structure of Gerstenhaber algebra;*
(ii) *there exits an isomophism of* BV*-algebras*

$$\mathbb{H}_*(LM) \cong HH^*\big(C^*(M);C^*(M)\big).$$

BV-algebra structures on the Hochschild cohomology $HH^*(A;A)$ have been constructed by different authors under some conditions on A. First of all, Tradler and Zeinalian [29] did it when A is the dual of an A_∞-coalgebra with ∞-duality (rational coefficients). This is in particular the case when $A = C^*(M)$, see [28]. Menichi [23] constructed also a BV-structure in the case when A is a symmetric algebra (any coefficients). Let us mention that Ginzburg [16, Thm. 3.4.3] has proved that $HH^*(A;A)$ is a BV-algebra for certain algebras A. Using this result Vaintrob [30] constructed an isomorphism of BV-algebras between $\mathbb{H}_*(LM)$ and $HH^*(A;A)$ when A is the group ring with rational coefficients of the fundamental group of an aspherical manifold M. This is coherent with our Theorem 1 because in this case $C_*(\Omega M)$ is quasi-isomorphic to A and using [9, Prop. 3.3] we have isomorphisms of Gerstenhaber algebras

$$HH^*(A;A) \cong HH^*\big(C_*(\Omega M);C_*(\Omega M)\big) \cong HH^*\big(C^*(M);C^*(M)\big).$$

Extending Theorem 1 to finite fields of coefficients would be difficult. For instance Menichi [22] proved that algebras $\mathbb{H}_*(LS^2)$ and $HH^*(H^*(S^2);H^*(S^2))$

are isomorphic as Gerstenhaber algebras but not as BV-algebras for $\mathbb{Z}/2$-coefficients.

In this paper we work over a field of characteristic zero. We use rational homotopy theory for which we refer systematically to [7]. We only recall here that a morphism in some category of complexes is a *quasi-isomorphism* if it induces an isomorphism in homology. Two objects are *quasi-isomorphic* if they are related by a finite sequence of quasi-isomorphisms. We shall use the classical convention $V^i = V_{-i}$ for degrees and $V^\vee$ denotes the graded dual of the graded vector space V.

Let $\boldsymbol{C}_*(A;A) := (A\otimes T(s\bar{A}),\partial)$ be the Hochschild chain complex of a differential graded algebra A with coefficients in A. Here $T(s\bar{A})$ denotes the free coalgebra generated by the graded vector space $s\bar{A}$ with $\bar{A} = \{A^i\}_{i\geq 1}$ and $(s\bar{A})^i = A^{i+1}$. We emphazise that $\boldsymbol{C}_*(A;A) = A\otimes T(s\bar{A})$ is considered as a cochain complex for upper degrees.

Now by a recent result of Lambrechts and Stanley [20] there is a commutative differential graded algebra A satisfying:

1) A is quasi-isomorphic to the differential graded algebra $C^*(M)$.
2) A is connected, finite dimensional and satisfies Poincaré duality in dimension m. This means there exists a A-linear isomorphism $\theta : A \to A^\vee$ of degree $-m$ which commutes with the differentials.

We call A a *Poincaré duality model* for M.

The starting point of the proof is to replace $C^*(M)$ by A because there is an isomorphism of Gerstenhaber algebras, [9, Prop. 3.3],

$$HH^*(A;A) \cong HH^*\big(C^*(M);C^*(M)\big). \tag{1}$$

This will allows us to use Poincaré duality at the chain level.

Denote by μ the multiplication of A. This is a model of the diagonal map. We define then the linear map $\mu_A\colon A \to A\otimes A$ by the commutative diagram

$$\begin{array}{ccc} A^\vee & \xrightarrow{\mu^\vee} & (A\otimes A)^\vee = A^\vee\otimes A^\vee \\ {\scriptstyle\theta}\Big\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\Big\uparrow{\scriptstyle\theta\otimes\theta} \\ A & \xrightarrow{\quad\mu_A\quad} & A\otimes A \end{array} \tag{2}$$

By definition μ_A is a $A\otimes A$-linear map degree m which commutes with the differentials (Here A is a $A\otimes A$-module via μ). This is a representative of the Gysin map associated to the diagonal embedding. With these notation we prove in §4:

PROPOSITION 1. — 1) *The cochain complex $\boldsymbol{C}_*(A;A)$ is quasi-isomorphic to the complex $C^*(LM)$. In particular, there is an isomorphism of graded vector spaces*

$$HH_*(A;A) \cong H^*(LM).$$

2) *If μ denotes the multiplication of A and ϕ denotes the coproduct of the coalgebra $T(s\bar{A})$ then the composite Φ*

$$\begin{array}{ccc} A\otimes T(s\bar{A}) & \xrightarrow{id\otimes\phi} A\otimes T(s\bar{A})\otimes T(s\bar{A}) \cong & A\otimes_{A^{\otimes 2}} (A\otimes T(s\bar{A}))^{\otimes 2} \\ \Phi \downarrow & & \downarrow \mu_A\otimes id \\ (A\otimes T(s\bar{A}))^{\otimes 2} & \xleftarrow{\cong} & A^{\otimes 2}\otimes_{A^{\otimes 2}} (A\otimes T(s\bar{A}))^{\otimes 2} \end{array}$$

is a linear map of degree m which commutes with the differentials.

3) *The isomorphism $HH_*(A;A) \cong H^*(LM)$, considered in 1), transfers the map induced by Φ on $HH_*(A;A)$ to the dual of the Chas-Sullivan product on $H^{*-m}(LM)$.*

4) *The duality isomorphism $HH_*(A;A)^\vee \cong HH^*(A;A^\vee) \overset{(\theta)}{\cong} HH^{*-m}(A;A)$ transfers the map induced by Φ on $HH_*(A;A)^\vee$ to the Gerstenhaber product on $HH^*(A;A)$.*

Denote by $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$ and $\Delta' : \mathbb{H}^*(LM) \to \mathbb{H}^{*-1}(LM)$ the morphisms induced by the canonical action of S^1 on LM. As proved by Chas and Sullivan this operator Δ defines on $\mathbb{H}_*(M)$ a structure of BV-algebra. In section 5 we prove:

PROPOSITION 2. — *The isomorphism $HH_*(A;A) \cong H^*(LM)$, considered in Proposition 1, transfers Connes' boundary $B : HH_*(A;A) \to HH_{*+1}(A;A)$ to the operator Δ'.*

L. Menichi [23] proved that the duality isomorphism

$$HH_*(A;A)^\vee \cong HH^*(A;A^\vee) \overset{(\theta)}{\cong} HH^*(A;A)$$

transfers $B^\vee : (HH_{*+1}(A;A)^\vee \to (HH_*(A;A))^\vee$ to a BV-operator on $HH^*(A;A)$ that defines a BV-structure extending the Gerstenhaber algebra structure. The isomorphisms of Gerstenhaber algebras (1) carries on the right hand term a structure of BV-algebra extending the Gerstenhaber algebra. This fact combined with Proposition 1 and 2 gives Theorem 1.

Since the field of coefficients is of characteristic zero, the homology of LM admits a Hodge decomposition, $\mathbb{H}_*(LM) = \bigoplus_{r\geq 0} \mathbb{H}_*^{[r]}(LM)$ (see [33], [32], [15] and [21, Thm. 4.5.10]). We prove that this decomposition behaves well with respect to the product $\bullet$ and the BV-operator Δ defined by Chas-Sullivan.

THEOREM 2. — *With the above notation, we have*

1) $\mathbb{H}_*^{[r]}(LM)\otimes\mathbb{H}_*^{[s]}(LM)\xrightarrow{\bullet}\mathbb{H}_*^{[\le r+s]}(LM)$,
2) $\Delta:\mathbb{H}_*^{[r]}(LM)\longrightarrow\mathbb{H}_{*+1}^{[r+1]}(LM)$.

By definition $\mathbb{H}_*^{[0]}(LM)$ is the image of $H_{*+m}(M)$ by the homomorphism induced in homology by the canonical section $M\to LM$. It has been proved in [10] that if $\operatorname{aut} M$ denotes the monoid of (unbased) self-equivalences of M then there exists a natural isomorphism of graded algebras

$$\mathbb{H}_*^{[1]}(LM)\cong H_{*+m}(M)\otimes\pi_*(\Omega\operatorname{aut} M).$$

For any $r\ge 0$, a description of $\mathbb{H}_*^{[r]}(LM)$ can be obtained, using a Lie model (L,d) of M, as proved in the last result.

PROPOSITION 3. — *The graded vector space $\mathbb{H}_*^{[r]}(LM)$ is isomorphic to $\operatorname{Tor}^{UL}(\boldsymbol{k},\Gamma^r(L))$ where $\Gamma^r(L)$ is the sub-UL-module of UL for the adjoint representation that is the image of $\bigwedge^r L$ by the classical Poincaré-Birkoff-Witt isomorphism of coalgebras $\wedge L\to UL$.*

The text is organized as follows. Notation and definitions are made precise in sections 2 and 3. Proposition 1 is proved in Sections 4, Proposition 2 is proved in section 5. Theorem 2 and Proposition 3 are proved in the last section.

2. Hochschild homology and cohomology

2.1. Bar construction. — Let A be a differential graded augmented cochain algebra and let P (res. N) be a differential graded right (resp. left) A-module,

$$A=\{A^i\}_{i\ge 0},\quad P=\{P^j\}_{j\in\mathbb{Z}},\quad N=\{N^j\}_{j\in\mathbb{Z}}\quad\text{and}\quad \bar A=\ker(\varepsilon:A\to\boldsymbol{k}).$$

The *two-sided* (*normalized*) *bar construction*,

$$\mathbb{B}(P;A;N)=P\otimes T(s\bar A)\otimes N,\quad \mathbb{B}_k(P;A;N)^\ell=(P\otimes T^k(s\bar A)\otimes N)^\ell,$$

is the cochain complex defined as follows. For $k\ge 1$, a generic element $p[a_1|a_2|\cdots|a_k]n$ in $\mathbb{B}_k(P;A;N)$ has (upper) degree $|p|+|n|+\sum_{i=1}^k(|sa_i|)$. If $k=0$, we write $p[\,]n=p\otimes 1\otimes n\in P\otimes T^0(s\bar A)\otimes N$. The differential $d=d_0+d_1$ is defined by

$$\mathbb{B}_k(P;A;N)^\ell\xrightarrow{d_0}\mathbb{B}_k(P;A;N)^{\ell+1},$$

$$d_0\big(p[a_1|a_2|\cdots|a_k]n\big) = d(p)[a_1|a_2|\cdots|a_k]n - \sum_{i=1}^{k}(-1)^{\epsilon_i}p\big[a_1|a_2|\cdots|d(a_i)|\cdots|a_k\big]n + (-1)^{\epsilon_{k+1}}p[a_1|a_2|\cdots|a_k]d(n),$$

$$\mathbb{B}_k(P;A;N)^\ell \xrightarrow{d_1} \mathbb{B}_{k-1}(P;A;N)^{\ell+1},$$

$$d_1\big(p[a_1|a_2|\cdots|a_k]n\big) = (-1)^{|p|}pa_1[a_2|\cdots|a_k]n + \sum_{i=2}^{k}(-1)^{\epsilon_i}p[a_1|a_2|\cdots|a_{i-1}a_i|\cdots|a_k]n - (-1)^{\epsilon_k}p[a_1|a_2|\cdots|a_{k-1}]a_kn.$$

Here $\epsilon_i = |p| + \sum_{j<i}(|sa_j|)$.

In particular, considering $\boldsymbol{k}$ as a trivial A-bimodule we obtain the complex

$$\mathbb{B}A = \mathbb{B}(\boldsymbol{k};A;\boldsymbol{k})$$

which is a differential graded coalgebra whose comultiplication is defined by

$$\phi([a_1|\cdots|a_r]) = \sum_{i=0}^{r}[a_1|\cdots|a_i]\otimes[a_{i+1}|\cdots||a_r].$$

Recall that a differential A-module N is called *semifree* if N is the union of an increasing sequence of sub-modules $N(i)$, $i\geq 0$, such that each $N(i)/N(i-1)$ is an R-free module on a basis of cycles (see [7]). Then,

LEMMA 1 (see [7, Lemma 4.3]). — *The canonical map $\varphi : \mathbb{B}(A;A;A) \to A$ defined by $\varphi[\ ] = 1$ and $\varphi([a_1|\cdots|a_k]) = 0$ if $k>0$, is a semifree resolution of A as an A-bimodule.*

2.2. Hochschild complexes. — Let us denote by $A^e = A\otimes A^{\mathrm{op}}$ the envelopping algebra of A.

If P is a differential graded right A^e-module then the cochain complex

$$\boldsymbol{C}_*(P;A) := (P\otimes T(s\bar{A}),\partial) \overset{\mathrm{def}}{\cong} P\otimes_{A^e}\mathbb{B}(A;A;A),$$

is called the *Hochschild chain complex of A with coefficients in P*. Its homology is called the *Hochschild homology of A with coefficients in P* and is denoted by $HH_*(A;P)$. When we consider $\boldsymbol{C}_*(A;A)$ as well as $HH_*(A;A)$, A is supposed equipped with its canonical right A^e-module structure.

For sake of completeness, let us recall the definition of the Connes' coboundary:

$$B : \boldsymbol{C}_*(A;A) \longrightarrow \boldsymbol{C}_*(A;A).$$

One has $B\big(a_0\otimes[a_1|\cdots|a_n]\big)=0$ if $|a_0|=0$ and

$$B\big(a_0\otimes[a_1|\cdots|a_n]\big)=\sum_{i=0}^{n}(-1)^{\bar\epsilon_i}1\otimes[a_i|\cdots|a_n|a_0|a_1|\cdots|a_{i-1}]$$

if $|a_0|>0$, where

$$\bar\epsilon_i=\big(|sa_0|+|sa_1|+\cdots+|sa_{i-1}|\big)\big(|sa_i|+\cdots+|sa_n|\big).$$

It is well known that $B^2=0$ and $B\circ\partial+\partial\circ B=0$. We also denote by B the induced operator in Hochschild homology $HH_*(A;A)$.

If N is a (left) differential graded A^e-module then the ($\mathbb{Z}$-graded) complex

$$\boldsymbol{C}^*(A;N):=\big(\operatorname{Hom}(T(s\bar A),N),\delta\big)\overset{\text{def}}{\cong}\operatorname{Hom}_{A^e}\big(\mathbb{B}(A;A;A),N\big),$$

is called the *Hochschild cochain complex* of A with coefficients in the differential graded A-bimodule N. Its homology is called the *Hochschild cohomology of A with coefficients in N* and is denoted by $HH^*(A;N)$. When we consider $\boldsymbol{C}^*(A;A)$ as well as $HH^*(A;A)$, A is supposed equipped with its canonical left A^e-bimodule structure.

Consider the graded dual, $V^\vee$, of the graded vector space $V=\{V^i\}_{i\in\mathbb{Z}}$, i.e. $V^\vee=\{V_i^\vee\}_{i\in\mathbb{Z}}$ with $V_i^\vee:=\operatorname{Hom}(V^i,\boldsymbol{k})$. The canonical isomorphism

$$\operatorname{Hom}\big(A\otimes_{A^e}\mathbb{B}(A;A;A),\boldsymbol{k}\big)\longrightarrow\operatorname{Hom}_{A^e}\big(\mathbb{B}(A;A;A),A^\vee\big)$$

induces the isomorphism of complexes $\boldsymbol{C}_*(A;A)^\vee\to\boldsymbol{C}^*(A;A^\vee)$.

2.3. The Gerstenhaber algebra on $HH^*(A;A)$. — A *Gerstenhaber algebra* is a commutative graded algebra $H=\{H_i\}_{i\in\mathbb{Z}}$ with a bracket

$$H_i\otimes H_j\to H_{i+j+1},\quad x\otimes y\mapsto\{x,y\}$$

such that for $a,a',a''\in H$:

(a) $\{a,a'\}=(-1)^{(|a|-1)(|a'|-1)}\{a',a\}$;

(b) $\{a,\{a',a''\}\}=\{\{a,a'\},a''\}+(-1)^{(|a|-1)(|a'|-1)}\{a',\{a,a''\}\}$.

For instance the Hochschild cohomology $HH^*(A;A)$ is a Gerstenhaber algebra [14]. The bracket can be defined by identifying $\boldsymbol{C}^*(A;A)$ with a differential graded Lie algebra of coderivations (see [26] and [9, 2.4]).

2.4. BV-algebras and differential graded Poincaré duality algebras. — A Batalin-Vilkovisky algebra (BV-algebra for short) is a commutative graded algebra, H together with a linear map (called a BV-operator)

$$\Delta : H^k \longrightarrow H^{k-1}$$

such that:

1) $\Delta \circ \Delta = 0$;

2) H is a Gerstenhaber algebra with the bracket defined by

$$\{a, a'\} := (-1)^{|a|}\big(\Delta(aa') - \Delta(a)a' - (-1)^{|a|}ab\Delta(a')\big).$$

3. The Chas-Sullivan algebra structure on $\mathbb{H}_*(LM)$ and its dual

We assume in this section and in the following ones that $\boldsymbol{k}$ is a field of characteristic zero.

Denote by $p_0 : LM \to M$ the evaluation map at the base point of S^1, and recall that the space LM can be replaced by a smooth manifold ([4], [25]) so that p_0 is a smooth locally trivial fibre bundle ([1], [25]).

The Chas-Sullivan product

$$\bullet : H_*(LM)^{\otimes 2} \longrightarrow H_{*-m}(LM), \quad x \otimes y \longmapsto x \bullet y$$

was first defined in [3] by using "transversal geometric chains". Then

$$\mathbb{H}_*(LM) := H_{*+m}(LM)$$

becomes a commutative graded algebra.

It is convenient for our purpose to introduce the *dual of the loop product* $H^*(LM) \to H^{*+m}(LM^{\times 2})$. Consider the commutative diagram

$$(1) \qquad \begin{array}{ccccc} LM^{\times 2} & \xleftarrow{\ i\ } & LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\ \downarrow p_0^{\times 2} & & \downarrow p_0 & & \downarrow p_0 \\ M^{\times 2} & \xleftarrow{\ \Delta\ } & M & = & M \end{array}$$

where

- Comp denotes composition of free loops,
- the left hand square is a pullback diagram of locally trivial fibrations,
- i is the embedding of the manifold of composable loops into $LM \times LM$.

The embeddings Δ and i have both codimension m. Thus, using the Thom-Pontryagin construction we obtain the Gysin maps

$$\Delta^! : H^k(M) \longrightarrow H^{k+m}(M^{\times 2}), \quad i^! : H^k(LM \times_M LM) \longrightarrow H^{k+m}(LM^{\times 2}).$$

Thus diagram (1) yields the diagram

$$(2)\qquad \begin{array}{ccccc} H^{k+m}(LM^{\times 2}) & \xleftarrow{i^!} & H^k(LM \times_M LM) & \xleftarrow{H^k(\mathrm{Comp})} & H^k(LM) \\ \uparrow{\scriptstyle H^*(p_0)^{\otimes 2}} & & \uparrow{\scriptstyle H^*(p_0)} & & \uparrow{\scriptstyle H^*(p_0)} \\ H^{k+m}(M^{\times 2}) & \xleftarrow{\Delta^!} & H^k(M) & = & H^k(M) \end{array}$$

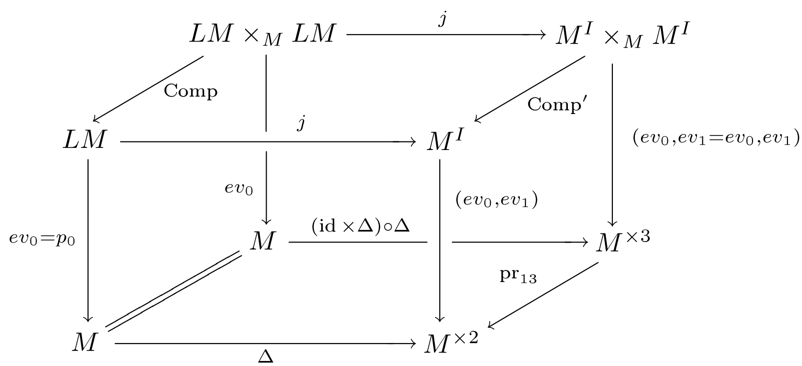
Following [27], [6], the *dual of the loop product* is defined by composition of maps on the upper line :

$$i^! \circ H^*(\mathrm{Comp}) : H^*(LM) \longrightarrow H^{*+m}(LM^{\times 2}).$$

4. Proof of Proposition 1 and the Cohen-Jones-Yan spectral sequence.

The composition of free loops $\mathrm{Comp} : LM \times_M LM \to LM$ is obtained by pullback from the composition of paths $\mathrm{Comp}' : M^I \times_M M^I \to M^I$ in the following commutative diagram.

(Comp)



Here Δ denotes the diagonal embedding, j the obvious inclusions, ev_t denotes the evaluation maps at t, and pr_{13} the map defined by $\mathrm{pr}_{13}(a, b, c) = (a, c)$.

Let (A, d) be a commutative differential graded algebra quasi-isomorphic to the differential graded algebra $C^*(M)$. A cochain model of the right hand square in diagram (Comp) is given by the commutative diagram

$$(\dagger)\qquad \begin{array}{ccc} \mathbb{B}(A; A; A) & \xrightarrow{\Psi} & \mathbb{B}(A; A; A) \otimes_A \mathbb{B}(A; A; A) \\ \uparrow & & \uparrow \\ A^{\otimes 2} & \xrightarrow{\psi} & A^{\otimes 3} \end{array}$$

where Ψ and ψ denote the homorphism of cochain complexes defined by

$$\Psi\big(a\otimes[a_1|\cdots|a_k]\otimes a'\big)=\sum_{i=0}^{k}a\otimes[a_1|\cdots|a_i]\otimes 1\otimes[a_{i+1}|\cdots|a_k]\otimes a',$$
$$\psi(a\otimes a')=a\otimes 1\otimes a'.$$

We consider now the commutative diagram obtained by tensoring diagram (†) by A:

$$(\ddagger)\qquad\begin{array}{ccc} A\otimes_{A^{\otimes 2}}\mathbb{B}(A,A,A) & \xrightarrow{\mathrm{id}\otimes\Psi} & A\otimes_{A^{\otimes 3}}\big(\mathbb{B}(A;A;A)\otimes_A\mathbb{B}(A;A;A)\big)\\ \uparrow & & \uparrow\\ A\otimes_{A^{\otimes 2}}A^{\otimes 2} & \xrightarrow{\mathrm{id}\otimes\psi} & A\otimes_{A^{\otimes 3}}A^{\otimes 3}\end{array}$$

Since $\mathbb{B}(A;A;A)$ is a semifree model of A as A-bimodule, we deduce from [8], p. 78, that diagram (‡) is a cochain model of the left hand square in diagram (Comp). Obviously, we have also the commutative diagram

$$\begin{array}{ccc} A\otimes_{A^{\otimes 2}}\mathbb{B}(A,A,A) & \xrightarrow{\mathrm{id}\otimes\Psi} & A\otimes_{A^{\otimes 3}}\mathbb{B}(A;A;A)\otimes_A\mathbb{B}(A;A;A)\\ \uparrow\cong & & \cong\uparrow\\ A\otimes T(s\bar{A}) & \xrightarrow{\mathrm{id}\otimes\phi} & A\otimes T(s\bar{A})\otimes T(s\bar{A})\end{array}$$

where ϕ denotes the coproduct of the coalgebra $T(s\bar{A})$. Thus we have proved:

Lemma 2. — *The cochain complex $\boldsymbol{C}_*(A;A)$ is a cochain model of LM, (i.e. we have an isomorphism of graded vector spaces $HH_*(A;A)\cong H^*(LM)$.) Moreover, the composite*

$$\begin{array}{ccc} \boldsymbol{C}_*(A;A) & \longrightarrow & \boldsymbol{C}_*(A;A)\otimes_A\boldsymbol{C}_*(A;A)\\ \| & & \uparrow\cong\\ A\otimes T(s\bar{A}) & \xrightarrow{\mathrm{id}\otimes\phi} & A\otimes T(s\bar{A})\otimes T(s\bar{A})\end{array}$$

is model of the composition of free loops.

Recall now that the Gysin map $\Delta^!$ of the diagonal embedding $\Delta: M\to M\times M$ is the Poincaré dual of the homomorphism $H_*(\Delta)$. This means that the following diagram is commutative:

$$\begin{array}{ccc} H_*(M) & \xrightarrow{H_*(\Delta)} & H_*(M\times M)\\ {\scriptstyle -\cap[M]}\uparrow\cong & & \cong\uparrow{\scriptstyle -\cap[M\times M]}\\ H^*(M) & \xrightarrow{\Delta^!} & H^*(M\times M)\end{array}$$

Let A be a Poincaré duality model of M and μ_A as defined by diagram (2) of the introduction. The linear map $\mu_A = A \to A \otimes A$ is a cochain model for $\Delta^!$. Next observe that, [26], we can choose the pullback of a tubular neighborhood of the diagonal embedding Δ as a tubular neighborhood of the embedding $i : LM \times_M LM \to LM \times LM$. Thus the Gysin map $i^!$ is obtained by pullback from $\Delta^!$. Therefore, since A is graded commutative, then $C_*(A;A)$ is a A-semifree and we have proved:

LEMMA 3. — *The linear map of degree m*

$$\boldsymbol{C}_*(A;A) \otimes_A \boldsymbol{C}_*(A;A) \xrightarrow{\cong} A \otimes_{A^{\otimes 2}} \boldsymbol{C}_*(A;A)^{\otimes 2} \xrightarrow{\mu_A \otimes \mathrm{id}} \boldsymbol{C}_*(A;A)^{\otimes 2}$$

commutes with the differential and induces $i^!$ in homology.

Then a combination of Lemmas 2, 3 and Lemma 4 below gives Proposition 1 of the introduction.

LEMMA 4. — *The duality isomorphism $(HH_{*+m}(A;A))^\vee \cong HH^{*+m}(A;A^\vee) \overset{(\theta)}{\cong} HH^*(A;A)$ transfers the map induced by Φ on $HH_*(A;A)$ to the Gerstenhaber product on $HH^*(A;A)$.*

Proof. — Observe that the composite (dotted arrow in the next diagram) induces the Gerstenhaber product in $HH^*(A;A)$.

$$\begin{array}{ccccc}
\mathrm{Hom}(T(s\bar{A}),A)^{\otimes 2} & \xrightarrow{\cong} & \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A),A)^{\otimes 2} & & f \otimes g \\
\vdots & & \downarrow & & \downarrow \\
\vdots & & \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A) \otimes_A \mathbb{B}(A;A;A), A \otimes_A A) & & f \otimes_A g \\
\vdots & & \cong \downarrow \mathrm{Hom}(\mathbb{B}(A;A;A)\otimes_A \mathbb{B}(A;A;A),\mu) & & \\
\vdots & & \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A) \otimes_A \mathbb{B}(A;A;A), A) & & \\
\vdots & & \downarrow \mathrm{Hom}(\Psi,A) & & \\
\mathrm{Hom}(T(s\bar{A}),A) & \xrightarrow{\cong} & \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A),A) & &
\end{array}$$

Then the remaining of the proof follows by considering an obvious commutative diagram. □

Spectral sequence. — By putting $F_p := A \otimes \big(T(s\bar{A})\big)^{\leq p}$, for $p \geq 0$, we define a filtration

$$A \otimes T(s\bar{A}) \supset \cdots \supset F_p \supset F_{p-1} \supset \cdots \supset A = F_0$$

such that $\partial F_p \subset F_p$ and $\Phi(F_p) \subset \bigoplus_{k+\ell=p} F_k \otimes F_\ell$. The resulting spectral sequence

$$E_2^{p,q} = \; H^q(M) \otimes H^p(\Omega M) \Longrightarrow H^{p+q}(LM)$$

is the comultiplicative "regraded" Serre spectral sequence for the fibration $p_0 : LM \to M$. It dualizes into a spectral sequence of algebras

$$H_{q+m}(M) \otimes H_p(\Omega M) \Longrightarrow \mathbb{H}_{p+q}(LM).$$

We recover in this way, for coefficients in a field of characteristic zero, the spectral sequence defined previously by Cohen, Jones and Yan [6].

5. Proof of Proposition 2.

Let $\rho : S^1 \times LM \to LM$ be the canonical action of the circle on the space LM. The action ρ induces an operator $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$. The Chas-Sullivan product together with Δ gives to $\mathbb{H}_*(LM)$ a BV-structure [3].

Denote by $\mathfrak{M}_M = (\bigwedge V, d)$ a (non necessary minimal) Sullivan model for M [8, §12]. We put $sV = \overline{V}$ and denote by S the derivation of $\bigwedge V \otimes \bigwedge \overline{V}$ defined by $S(v) = \bar{v}$ and $S(\bar{v}) = 0$ for $v \in V$ and $\bar{v} \in \overline{V}$. Then a Sullivan model for LM is given by the commutative differential graded algebra $(\bigwedge V \otimes \bigwedge \overline{V}, \bar{d})$ where $\bar{d}(\bar{v}) = -S(dv)$ [34]. Moreover in [33] Burghelea and Vigué prove that a Sullivan model of the action $\rho : S^1 \times LM \to LM$ is given by

$$\mathfrak{M}_\rho : \big(\bigwedge V \otimes \bigwedge \overline{V}, \bar{d}\big) \longrightarrow \big(\bigwedge u, 0\big) \otimes \big(\bigwedge V \otimes \bigwedge \overline{V}, \bar{d}\big), \quad |u| = 1\big),$$

$$\mathfrak{M}_\rho(\alpha) = 1 \otimes \alpha + u \otimes S(\alpha), \quad \alpha \in \bigwedge V \otimes \bigwedge \overline{V}.$$

In particular the map induced in cohomology by the action of S^1 on LM is given by the derivation $S : H^*(\bigwedge V \otimes \bigwedge \overline{V}) \to H^{*-1}(\bigwedge V \otimes \bigwedge \overline{V})$. Denote now by B the Connes' boundary on $\boldsymbol{C}_*(\mathfrak{M}_M; \mathfrak{M}_M) = \bigwedge V \otimes T\big(s\overline{\bigwedge V}\big)$. D. Burghelea and M. Vigué proved the following lemma in [31, Thm. 2.4].

Lemma 5. — *The morphism $f : \boldsymbol{C}_*(\mathfrak{M}_M; \mathfrak{M}_M) \to (\mathfrak{M}_M \otimes \bigwedge \overline{V})$ defined by*

$$f\big(a \otimes [a_1|\cdots|a_n]\big) = \frac{1}{n!} aS(a_1)\cdots S(a_n)$$

is a quasi-isomorphism of complexes and $f \circ B = S \circ f$.

Lemma 5 identifies the Connes boundary, B acting on $HH_*(A;A) \cong H_*(\mathfrak{M}_M; \mathfrak{M}_M)$ with the circle action and thus with the Chas-Sullivan BV-operator on $H^*(LM) \cong HH_*(A;A)$. This is Proposition 2 of the introduction.

6. Hodge decomposition

With the notation of the previous sections, let $(\mathfrak{M}_M \otimes \bigwedge \overline{V}, \bar{d})$ be a Sullivan model for LM. Denote by $G^p = \bigwedge V \otimes \bigwedge^p \overline{V}$ the subvector space generated by the words of length p in $\overline{V}$. The differential $\bar{d}$ satisfies $\bar{d}(G^p) \subset G^p$. Thus we put

$$H^n_{[p]}(LM) := H^n(G^p).$$

This decomposition splits $H^*(LM; \boldsymbol{k})$ into summands given as eigenspaces of the maps $LM \to LM$ induced from the n-power maps of the circle $e^{it} \mapsto e^{int}$ [33]. It defines by duality a Hodge decomposition on $H_*(LM)$. We are now ready to prove Theorem 2 of the introduction.

Proof of Theorem 2. — Recall that the differential ∂ in $\boldsymbol{C}^*(\mathfrak{M}_M; \mathfrak{M}_M)$ decomposes into $\partial = \partial_0 + \partial_1$ with $\partial_0(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes T^p(s\overline{\wedge V})$, and $\partial_1(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes T^{p-1}(s\overline{\wedge V})$.

We consider the quasi-isomorphism $f : \boldsymbol{C}^*(\mathfrak{M}_M; \mathfrak{M}_M) \to (\mathfrak{M}_M \otimes \bigwedge \overline{V}, \bar{d})$ defined in Lemma 5. If we apply Lemma 5, when $d = 0$ in $\wedge V$, we deduce that $\operatorname{Ker} f$ is ∂_1-acyclic.

Lemma 6. — *Let us define* $K^{(p)} := \operatorname{Ker} f \cap \big(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})\big)$.

1) *If* $\omega \in K^{(p)} \cap \operatorname{Ker}\partial$ *then there exists* $\omega' \in \bigoplus_{r \geq p+1} K^{(r)}$ *such that* $\partial \omega' = \omega$.

2) f *induces a surjective map*

$$\big(\mathfrak{M}_M \otimes T^{\geq p}(s\overline{\wedge V})\big) \cap \operatorname{Ker}\partial \longrightarrow\!\!\!\!\!\rightarrow \big(\mathfrak{M}_M \otimes \wedge^p sV\big) \cap \operatorname{Ker}\bar{d}.$$

Proof. — If $\omega \in K^{(p)} \cap \operatorname{Ker}\partial$ then $\omega = \partial(u+v)$ with $u \in K^{(p)}$ and $v \in K^{(\geq p+1)}$. Since $\partial_1 u = 0$ we have $u = \partial\beta_1$ some $\beta \in K^{(p+1)}$ and thus $\omega - d\beta_1 \in K^{(\geq p+1)}$. An induction on $n \geq 1$ we prove that there exists $\beta_n \in K^{(p+n)}$ such that $\omega - d\beta_n \in K^{(p+n)}$. Since $\bigwedge V$ is 1-connected $\big(\mathfrak{M}_M \otimes T^{p+n}(s\overline{\wedge V})\big)^{|\omega|} = 0$ for some integer n_0. We put $\omega' = \beta_{n_0}$.

In order to prove the second statement, we consider a $\bar{d}$-cocycle $\alpha \in \mathfrak{M}_M \otimes \bigwedge^p sV$ and we write $\alpha = f(\omega)$ for some $\omega \in \mathfrak{M}_M \otimes T^p(s\overline{\wedge V})$. It follows from the definition of f that $\partial\omega \in K^{(p-1)}$. Thus, by the first statement, $\partial\omega = \partial\omega'$ some $\omega' \in K^{(\geq p)}$. Then $\varpi = \omega - \omega'$ is ∂-cocycle of $K^{\geq p}$ such that $f(\varpi) = \alpha$. □

To end the proof of Theorem 2, let us consider $\alpha \in H^*_{[n]}(LM)$. By Lemma 6, α is the class of $f(\beta)$ where $\beta \in \mathfrak{M}_M \otimes T^{\geq n}(s\overline{\bigwedge V})$. Therefore $\Phi(\beta)$ belongs to $\bigoplus_{i+j \geq n} \big(\mathfrak{M}_M \otimes T^i(s\overline{\wedge V})\big) \otimes \big(\mathfrak{M}_M \otimes T^j(s\overline{\wedge V})\big)$ (see Lemma 2). Now since $f(\mathfrak{M}_M \otimes T^p(s\overline{\wedge V})) \subset \mathfrak{M}_M \otimes \bigwedge^p sV$,

$$\big[\Phi(\alpha)\big] \in \bigoplus_{i+j \geq n} H^*_{[i]}(LM) \otimes H^*_{[j]}(LM). \qquad \square$$

Now, as announced in the introduction (Proposition 3) there is an other interpretation of $H^n_{[p]}(LM)$ in terms of the cohomology of a differential graded Lie algebra.

Let L be a differential graded algebra L such that the cochain algebra $\mathcal{C}^*(L)$ is a Sullivan model of M, [8, p. 322]. In particular, the homology of the enveloping universal algebra of L, denoted UL, is a Hopf algebra isomorphic to $H_*(\Omega M)$. We consider the cochain complex $\mathcal{C}^*(L; UL^\vee_a)$ of L with coefficients in $UL^\vee$ considered as an L-module for the adjoint representation. We have shown (see [12, Lemma 4]) that the natural inclusion $\mathcal{C}^*(L) \hookrightarrow \mathcal{C}^*(L; UL^\vee_a)$ is a relative Sullivan model of the fibration $p_0 : LM \to M$. Write $\mathcal{C}^*(L) = (\bigwedge V, d)$, then $V = (sL)^\vee$ and $\overline{V} = L^\vee$. There is also (Poincaré-Birkoff-Witt Theorem) an isomorphism of graded coalgebras, [8, Prop. 21.2]:

$$\gamma : \bigwedge L \longrightarrow UL, \quad x_1 \wedge \cdots \wedge x_k \longmapsto \sum_{\sigma \in \mathfrak{S}_k} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

If we put $\Gamma^p = \gamma(\bigwedge^p V)$ we obtain the following isomorphisms of cochain complexes

$$\big(\bigwedge V \otimes \bigwedge \overline{V}, \bar{d}\big) \cong \mathcal{C}^*(L; UL^\vee_a), \quad G^p \cong \mathcal{C}^*\big(L; (\Gamma^p)^\vee\big)$$

which in turn induce the isomorphisms

$$\mathbb{H}^*(LM) \cong \mathrm{Ext}_{UL}(\boldsymbol{k}, UL^\vee_a), \quad \mathbb{H}^*_{[p]}(LM) \cong \mathrm{Ext}_{UL}(\boldsymbol{k}, \Gamma^p(L)^\vee)$$

and by duality,

$$\mathbb{H}_*(LM) \cong \mathrm{Tor}^{UL}(\boldsymbol{k}, UL_a), \quad \mathbb{H}^{[p]}_*(LM) \cong \mathrm{Tor}^{UL}(\boldsymbol{k}, \Gamma^p).$$

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