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VOLUME OF SPHERES IN DOUBLING METRIC MEASURED SPACES AND IN GROUPS OF POLYNOMIAL GROWTH

by Romain Tessera

Abstract. — Let G be a compactly generated locally compact group and let U be a compact generating set. We prove that if G has polynomial growth, then $(U^n)_{n\in\mathbb{N}}$ is a Følner sequence and we give a polynomial estimate of the rate of decay of $\frac{\mu(U^{n+1}\smallsetminus U^n)}{\mu(U^n)}$. Our proof uses only two ingredients: the doubling property and a weak geodesic property that we call Property (M). As a matter of fact, the result remains true in a wide class of doubling metric measured spaces including manifolds and graphs. As an application, we obtain a L^p-pointwise ergodic theorem $(1\leq p<\infty)$ for the balls averages, which holds for any compactly generated locally compact group G of polynomial growth.

Résumé (*Volume de sphères dans les espaces métriques mesurés doublants et dans les groupes à croissance polynomiale*)

Soit G un groupe localement compact, compactement engendré et U une partie compacte génératrice. On prouve que si G est à croissance polynomiale, alors la suite des puissances de U forme une suite de Følner et on montre que le rapport $\frac{\mu(U^{n+1}\smallsetminus U^n)}{\mu(U^n)}$ tend polynomialement vers 0. La démonstration n'utilise que deux ingrédients : le fait qu'un groupe à croissance polynomiale est doublant, et une propriété de faible géodésicité : la propriété (M). Par conséquent ce résultat s'étend à une large classe d'espaces métriques mesurés doublants, comme les graphes et les variétés riemanniennes. Comme application, nous obtenons un théorème ergodique presque sûr et dans L^p $(1\leq p<\infty)$ pour les moyennes sur les boules d'un groupe à croissance polynomiale.

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1. Introduction

Let G be a compactly generated, locally compact (cglc) group endowed with a left Haar measure μ. Recall that a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of a locally compact group G is said to be Følner if for any compact set K,

$$\mu(K\cdot A_n \bigtriangleup A_n) = o\big(\mu(A_n)\big).$$

Let U be a compact generating set of G (we mean by this that $\bigcup_{n\in\mathbb{N}} U^n = G$), not necessarily symmetric. If $\mu(U^n)$ grows exponentially, it is easy to see that the sequence $(U^n)_{n\in\mathbb{N}}$ *cannot* be Følner. On the other hand, if $\mu(U^n)$ grows subexponentially, then there exists trivially a sequence $(n_i)_{i\in\mathbb{N}}$ of integers such that $(U^{n_i})_{i\in\mathbb{N}}$ is Følner. But it is not clear whether the whole sequence $(U^n)_{n\in\mathbb{N}}$ is Følner. This was first conjectured for amenable groups by Greenleaf in 1969 (see [8, p. 69]), who also proved it with Emerson [7] in the abelian case, correcting a former proof of Kawada [12] (see also Proposition 21). The conjecture is actually not true for all finitely generated amenable groups since there exist amenable groups with exponential growth (for instance, all solvable groups which are not virtually nilpotent). Nevertheless, the conjecture is still open for groups with subexponential growth. In 1983, Pansu [17] proved it for nilpotent finitely generated groups. In [2], Breuillard recently generalized the theorem of Pansu, which now holds for general cglc groups of polynomial growth. In fact, they prove that $\mu(U^n)\sim Cn^d$, for a constant $C = C(U) > 0$, which clearly implies that $(U^n)_{n\in\mathbb{N}}$ is Følner. In this article, we prove the conjecture for all compactly generated groups with polynomial growth. More precisely, we prove the following theorem: there exist $\delta > 0$ and a constant $C < \infty$, such that

$$\mu\left(U^{n+1}\smallsetminus U^n\right) \le Cn^{-\delta}\mu(U^n).$$

Interestingly, our proof works in a much more general setting. Recall that a metric measure space (X, d, μ) satisfies the doubling condition (or "is doubling") if there exists a constant $C \ge 1$ such that

$$\forall r > 0,\ \forall x \in X,\quad \mu\big(B(x, 2r)\big) \le C\mu\big(B(x, r)\big)$$

where $B(x, r) = \{y \in X,\ d(x, y) \le r\}$. Let $S(x, r)$ denote the "1-sphere" of center x and radius r, *i.e.*, $S(x, r) = B(x, r + 1) \smallsetminus B(x, r)$. Actually, we prove a similar result for doubling metric measured spaces satisfying a weak geodesic property we will call Property (M) (see 5.2). In this setting, the result becomes: there exist $\delta > 0$ and a constant $C < \infty$, such that

$$\forall x \in X,\ \forall r > 0,\quad \mu\big(S(x, r)\big) \le Cr^{-\delta}\mu\big(B(x, r)\big).$$

In particular, the conclusion of this theorem holds for metric graphs and Riemannian manifolds satisfying the doubling condition.

In the case of metric measured spaces, our result is somewhat optimal, since in [21, Thm. 4.9], we build a graph X, quasi-isometric to $\mathbb{Z}^2$, such that there exist $0 < a < 1$, an increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ and $x \in X$ such (1) that

$$\forall i \in \mathbb{N}, \quad \big|S(x, n_i)\big| \geq c n_i^{-a} \big|B(x, n_i)\big|.$$

Note that easier counter examples can be obtained with trees with linear growth (see Remark 5). Moreover, we will see that our assumptions on X, that is, Doubling Property and Property (M) (see Definition 1 below) are also optimal in some sense.

An interesting and historical motivation (see for instance [8]) for finding Følner sequences in groups comes from ergodic theory. As a consequence of our result, we obtain a L^p-pointwise ergodic theorem ($1 \leq p < \infty$) for the balls averages, which holds for any cglc group G of polynomial growth (see Theorem 13). We refer to a recent survey of A. Nevo [16] for more details and complete proofs.

2. Main results

2.1. Property (M)

Definition 1. — We say that a metric space (X, d) has *Property (M)* if there exists $C < \infty$ such that the Hausdorff distance between any pair of balls with same center and any radii between r and $r + 1$ is less than C. In other words, for all $x \in X$, for all $r > 0$ and for all $y \in B(x, r+1)$, we have $d(y, B(x, r)) \leq C$.

Proposition 2. — *Let (X, d) be a metric space. The following properties are equivalent:*

1) *X has Property (M).*
2) *X has "monotone (2) geodesics",* i.e., *there exists $C < \infty$ such that, for all $x, y \in X$, $d(x, y) \geq 1$, there exists a finite chain $x_0 = y$, $x_1, \dots, x_m = x$ such that for $0 \leq i < m$*

$$d(x_i, x_{i+1}) \leq C \quad \text{and} \quad d(x_{i+1}, x) \leq d(x_i, x) - 1.$$

3) *There exists a constants $C < \infty$ such that for all $r > 0$, $s \geq 1$ and $y \in B(x, r + s)$*

$$d\big(y, B(x, r)\big) \leq Cs.$$

(1) In our example, $a = \log 2/\log 3$.

(2) This is why we call this property (M).

Proof

1) $\Rightarrow$ 2). Let $x, y \in X$ be such that $d(x,y) \geq 1$. Let us construct the sequence $y = x_0, x_1, \ldots, x_m = x$ inductively. First, by Property (M) and since $d(x,y) \geq 1$, there exists $x_1 \in B(x, d(x,y)-1)$ such that $d(y,x_1) \leq C$. Now, assume that we have constructed a sequence $y = x_0, x_1, \ldots, x_k$ such that $d(x_i, x_{i+1}) \leq C$ for $0 \leq i < k$, and $d(x_{i+1}, x) \leq d(x_i, x) - 1$. If $d(x_k, x) < 1$, then up to replace C by $C+1$, and x_k by x, the sequence $x_0, \ldots, x_k$ is a monotone geodesic between x and y. Otherwise, there exists $x_{k+1} \in B(x, d(x,x_k)-1)$ such that $d(x_k, x_{k+1}) \leq C$. Clearly this process has to stop after at most $[d(x,y)]$ steps, so we are done.

2) $\Rightarrow$ 3). Let $x_0 = y, x_1, \ldots, x_m = x$ be a monotone geodesic from y to x. There exists an integer $k \leq s+1$ such that $x_{m-k} \in B(x,r)$. Hence

$$d\big(y, B(x,r)\big) \leq d(y, x_k) \leq Ck \leq C(s+1) \leq 2Cs$$

which proves the implication.

3) $\Rightarrow$ 1). Just take $s = 1$. □

Invariance under Hausdorff equivalence. — Recall (see [10, p. 2]) that two metric spaces X and Y are said Hausdorff equivalent

$$X \sim_{\mathrm{Hau}} Y$$

if there exists a (larger) metric space Z such that X and Y are contained in Z and such that

$$\sup_{x \in X} d(x, Y) < \infty \quad \text{and} \quad \sup_{y \in Y} d(y, X) < \infty.$$

It is easy to see that Property (M) is invariant under Hausdorff equivalence. But on the other hand, Property (M) is unstable under quasi-isometry. To construct a counterexample, one can quasi-isometrically embed $\mathbb{R}_+$ into $\mathbb{R}^2$ such that the image, equipped with the induced metric does not have Property (M): consider a stairway-like curve starting from 0 and containing for every $k \in \mathbb{N}$ a half-circle of radius 2^k centered on 0. So (M) is strictly stronger than the quasi-geodesic property (see [10, p. 7]), which is invariant under quasi-isometry: X is quasi-geodesic if there exist two constants $d > 0$ and $\lambda > 0$ such that for all $(x,y) \in X^2$ there is a finite chain of points $x = x_0, \ldots, x_m = y$, of X such that

$$d(x_{i-1}, x_i) \leq d, \quad i = 1, \ldots, m,$$

and

$$\sum_{i=1}^{m} d(x_{i-1}, x_i) \leq \lambda d(x,y).$$

Note that a monotone geodesic is a quasi-geodesic.

EXAMPLES 3. — Recall that a metric space (X,d) is called b-geodesic if for any $x,y \in X$, there exists a finite chain $x = x_0, \ldots, x_m = y$ such that

$$\forall i = 0,\ldots,m-1,\ d(x_i, x_i+1) \le b \quad \text{and} \quad d(x,y) = \sum_{i=1}^{m} d(x_{i-1}, x_i).$$

Note that a b-geodesic space is b'-geodesic for any $b' \ge b$. For example, Riemannian manifolds and more generally geodesic spaces are b-geodesic for any $b > 0$. Clearly, 1-geodesic spaces satisfy Property (M). Other examples of 1-geodesic spaces are graphs. Namely, to any connected simplicial graph, we associate a metric space, whose elements are the vertices of the graph, the metric being the usual shortest path distance between two points. We simply call such a metric space a graph. By definition, graphs are 1-geodesic, so in particular they satisfy Property (M). Finally, a discretisation (*i.e.*, a discrete net) of a Riemannian manifold M has Property (M) for the induced distance (this is a consequence of the stability under Hausdorff equivalence).

2.2. The main theorem. — Let $X = (X,d,\mu)$ be a metric measured space. By metric measure space, we mean that μ is a Borel measure, supported on the metric space (X,d) satisfying $\mu(B(x,r) < \infty$ for all $x \in X$ and $r > 0$. Recall that X is said to be doubling if there exists $C \ge 1$ such that

$$\forall x \in X,\ \forall r > 0, \quad \mu\big(B(x,2r)\big) \le C\mu\big(B(x,r)\big).$$

Our main result says that in a doubling space with Property (M), balls are Følner sets.

THEOREM 4. — *Let $X = (X,\mu,d)$ be a doubling metric measured space with Property (M). Then, there exists $\delta > 0$ and a constant $C < \infty$ such that, for all $x \in X$ and for all $n \in \mathbb{N}$*

$$\mu\big(S(x,n)\big) \le Cn^{-\delta}\mu\big(B(x,n)\big).$$

In particular, the ratio $\mu(B(x,n+1) \smallsetminus B(x,n))/\mu(B(x,n))$ tends to 0 uniformly in x when n goes to infinity.

2.3. Optimality of the assumptions

The doubling property. — First, note that the doubling assumption cannot be replaced by polynomial growth. Indeed, for every integer n, consider the following finite rooted tree: first, take the standard ternary rooted tree of depth n. Then stretch it as follows: replace each edge connecting a $(k-1)$'th generation vertex to a k'th generation vertex by a (graph) interval of length 2^{n-k}. We obtain a rooted tree G_n of dept $2^n = \sum_{k=0}^{n-1} 2^k$. Then consider the graph G'_n obtained by taking two copies of G_n and identifying the vertices of last generation of the first copy with those of the second copy. Write r_n and r'_n for the two vertices of G_n' corresponding to the respective roots of the two copies of G_n. Finally, glue "linearly" the G_n' together identifying r_n' with r_{n+1}, for all n: it defines a infinite connected graph X.

Let us prove that this graph has polynomial growth. It is enough to look at radii of the form $r = 2^k$ where $k \in \mathbb{N}$. On the other hand, among the balls of radius 2^k, those which are centered in points of n'th generation of a G_n for n large enough are of maximal volume. Let us take such an x. Note that $B(x, 2^k)$ is isometric to G_{k-1}' glued with to segments of length 2^{k-1} at its extremities $r_{2^{k-1}}$ and $r_{2^{k-1}}'$. Hence,

$$\left|B(x,2^k)\right| = 2\Big(2^{k-1} + \sum_{j=0}^{k-2} 3^j 2^{k-j}\Big) \leq 2^k + 2^{k+2}(\tfrac{3}{2})^{k-1} \leq 83^k = 8r^{\log 3/\log 2}.$$

On the other hand, the sphere $S(r_{2^k}, 2^k) = B(r_{2^k}, 2^k) \smallsetminus B(r_{2^k}, 2^{k-1})$ has a volume larger than 3^k, so that

$$\frac{|S(r_{2^k}, 2^k)|}{|B(r_{2^k}, 2^k)|} \geq \frac{1}{8}.$$

REMARK 5. — We can generalize the above construction taking trees of valence $b \geq 2$ and replacing each edge connecting a $k-1$'th generation vertex to a k'th generation vertex by a (graph) interval of length a^{n-k}, for some $a \in \mathbb{N}$. If $a < b$, then we still obtain graph with polynomial growth, but not doubling which contradicts Theorem 4. But if $a > b$, then it is easy to see that we obtain a graph with linear growth, hence doubling. Moreover, there exists a sequence of vertices x_n and $c > 0$ such that

$$|S(x_n, a^n)| \geq cb^n.$$

Hence, in this case the δ of Theorem 4 is less than $1 - \log b/\log a$.

The property (M). — Another interesting point is the fact that Property (M) cannot be replaced by any quasi-isometry invariant property like quasi-geodesic property.

Indeed, one can very easily build a counterexample, embedding quasi-isometrically $\mathbb{R}_+ \times [0,1]$ into $\mathbb{R}^2$. First, consider a stairway-like curve starting from 0 and containing for every $k \in \mathbb{N}$ a half-circle of radius 2^k centered on 0 (cf. the counterexample to Property (M)), and then look at the closed 1-neighborhood of this curve. Denote by X the corresponding closed subset of $\mathbb{R}^2$, equipped with the Lesbegues measure and the induced distance. As X is quasi-isometric to $\mathbb{R}_+$, the volume of balls grows linearly. But observe that the volume of $B_X(0, 2^k + 1) \smallsetminus B_X(0, 2^k)$ is larger than $\pi 2^k$.

In particular, balls being Følner sets is not invariant under quasi-isometry.

2.4. An interesting particular case: locally compact groups with polynomial growth

Let (G,μ) be a cglc group endowed with a Haar measure μ. Let U be a compact generating set of G. Define a left invariant distance d on G by

$$\forall x,y\in G,\quad d(x,y)=\inf\left\{n\in\mathbb{N},\ x^{-1}y\in U^n\right\}.$$

Note that unless U is symmetric (*i.e.*, $U^{-1}=U$), d is not really a distance since we do not have $d(x,y)=d(y,x)$. Nevertheless, d is "weakly" symmetric, *i.e.*, there exists a constant $C<\infty$ such that

$$\forall x,y\in X,\quad d(x,y)\leq Cd(y,x)$$

In fact, we could prove Theorem 4 only supposing that d is weakly symmetric. But for simplicity, we only prove it in the true metric setting.

Let us start with some generalities. First, note that up to replacing U by U^m, for some fixed $m>0$, we can assume that $1\in U$, so that the sequence $(U^n)_{n\in\mathbb{N}}$ is nondecreasing. Moreover, we have the following simple fact.

PROPOSITION 6. — *Let G be a cglc group and let U and V be two compact sets such that U generates G and contains 1. Then there exists $m\in\mathbb{N}^*$ such that, for all $n\geq m$, $V\subset U^n$.*

Proof. — First, note that by a simple Baire argument, U^n contains a nonempty open set Ω for n big enough. On the other hand, for n big enough, U^n contains the inverse of a given element of Ω. Thus, U^{n+1} contains an open neighborhood of 1. Let $\Omega.x_i$ be a finite covering of V. For n big enough, we can suppose that $x_i\in U^n$, so actually, $V\subset U^{2n+2}$. □

DEFINITION 7. — Let G be a cglc group.

- We say that G has *polynomial growth* if there exist a compact generating set U, $D>0$ and a constant $C\geq 1$ such that
$$\mu(U^n)\leq Cn^D.$$
- We say that G has *strictly polynomial growth* if there exist a compact generating set U, a nonnegative number d, and a constant $C=C(U)\geq 1$ such that
$$C^{-1}n^d\leq\mu(U^n)\leq Cn^d. \tag{2.1}$$

Note that by Proposition 6, if G has strict polynomial growth, then the number d does not depend on U, provided that $\mu(U)\neq 0$. We call it the *growth exponent* of G.

THEOREM 8 (see [9, 11, 13, 15, 23]). — *Let G be a cglc group of polynomial growth, then it has strictly polynomial growth with integer exponent.*

Let us recall briefly how this result was proved. Using a structure theorem due to Wang [23] and Mostow [15], Guivarc'h [11, Cor. III.3] proved Theorem 8 for cglc solvable groups. Then a major step has been achieved by Gromov [9], who proved that a finitely generated group with polynomial growth is virtually a lattice in some nilpotent connected Lie group. Generalizing Gromov's approach, Losert [13, 14] proved a similar statement for general cglc groups with polynomial growth. According to Losert [13], G is quasi-isometric to a solvable cglc group S, and hence has strictly polynomial growth with integer exponent.

In the group setting, we obtain a slightly improved version of Theorem 4.

THEOREM 9. — *Let G be a cglc group with polynomial growth. Consider a sequence $(U_n)_{n\in\mathbb{N}}$ of measurable subsets such that there exists two generating compact sets K, K' such that, for all $n \in \mathbb{N}$,*

$$K \subset U_n \subset K'.$$

Write, for all $n \in \mathbb{N}$,

$$N_n = U_0 \cdots U_{n-1} U_n.$$

Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that for all $n \in \mathbb{N}^*$

$$\mu(N_{n+1} \smallsetminus N_n) \leq C n^{-\delta} \mu(N_n).$$

In particular, the sequence $(N_n)_{n\in\mathbb{N}}$ is Følner.

The following corollary is a also a corollary of Theorem 4.

COROLLARY 10. — *Let G be a cglc group with polynomial growth, and U be a compact generating set of G. Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that for all $n \in \mathbb{N}^*$*

$$\mu\left(U^{n+1} \smallsetminus U^n\right) \leq C n^{-\delta} \mu(U^n).$$

In particular, the sequence $(U^n)_{n\in N}$ is Følner.

In fact, we do not use the full contents of Theorem 8. We only need the fact that G satisfies a doubling property: there exists of a constant $C = C(U) \geq 1$ such that

$$\forall n \in \mathbb{N}, \quad \mu(U^{2n}) \leq C\mu(U^n).$$

It clearly results from Strictly Polynomial Growth. On the other hand, Doubling Property implies trivially Polynomial Growth. Unfortunately, there exist no elementary proofs of the converses, which require to prove Theorem 8.

3. Consequences in ergodic theory

Let G be a locally compact second countable (lcsc) group, X a standard Borel space on which G acts measurably by Borel automorphisms. Let m be a G-invariant probability measure on X ((X,m) is called a Borel probability G-space). The G-action on X gives rise to a strongly continuous representation π of G as a group of isometries of the Banach space $L^p(X)$ for $1 \leq p < \infty$, given by $\pi(g)f(x) = f(g^{-1}x)$. For any Borel probability measure β on G, and given some $p \geq 1$, we can consider the averaging operator given by

$$\pi(\beta)f(x) = \int_G f(g^{-1}x)\mathrm{d}\beta(g)$$

for all $f \in L^p(X)$. Let (β_n) be a sequence of probability measures on G. We say that (β_n) satisfies a pointwise ergodic theorem in $L^p(X)$ if

$$\lim_{n\to\infty} \pi(\beta_n)f(x) = \int_X f\mathrm{d}m$$

for almost every $x \in X$, and in the L^p-norm, for all $f \in L^p(X)$, where $1 \leq p < \infty$. Let μ be a Haar measure on G. We will be interested in the case when β is the normalized average on a set of finite measure N of G.

Definition 11 (Regular sequences). — A sequence of sets of finite measure N_k in G is called *regular* if

$$\mu(N_k^{-1} \cdot N_k) \leq C\mu(N_k).$$

Let us recall the following general result (also proved in the recent survey of Amos Nevo [16]).

Theorem 12 (see [1, 3, 6, 20]). — *Assume G is an amenable lcsc group, and $(N_n)_{n\in\mathbb{N}}$ is an increasing left Følner regular sequence, with $\bigcup_{n\in\mathbb{N}} N_n = G$. Then, the sequence $(\beta_n)_{n\in\mathbb{N}}$ (associated to (N_k)) satisfies the pointwise ergodic theorem in $L^p(X)$, for every Borel probability G-space (X,m) and every $1 \leq p < \infty$.*

Now, let us focus on the case when G is a locally compact, compactly generated group of polynomial growth. Consider a sequence $(U_n)_{n\in\mathbb{N}}$ satisfying the hypothesis of Theorem 9. According to Theorem 9 and Proposition 6, the sequence $N_n = U_0 \cdot U_1 \cdots U_n$ satisfies the hypothesis of Theorem 12. So we get the following corollary.

Theorem 13. — *Let G be a cglc group of polynomial growth. Consider a sequence $(U_n)_{n\in\mathbb{N}}$ of measurable subsets such that there exist two generating compact subsets K, K' such that, for all $n \in \mathbb{N}$*

$$K \subset U_n \subset K'.$$

Write $N_n = U_0U_1\cdots U_n$. *Then, the sequence* $(\beta_n)_{n\in\mathbb{N}}$ *(associated to* $(N_n)_{n\in\mathbb{N}}$*) satisfies the pointwise ergodic theorem in* $L^p(X)$*, for every Borel probability* G*-space* (X,m) *and every* $1\le p<\infty$.

4. Remarks and questions

In this section, we address a (non-extensive) list of remarks and problems related to the subject of this paper.

Question 14. — Is the Greenleaf localisation conjecture true for groups with subexponential growth?

Question 15 (Groups with exponential growth). — Let G be a finitely generated group with exponential growth and let U be a finite generating subset. Does there exist a constant $c>0$ such that [3]

$$\mu(U^{n+1}\smallsetminus U^n)\ge c\mu(U^n)?$$

Question 16 (Asymptotic isoperimetry). — Let G be a locally compact, compactly generated group and let U be a compact generating neighborhood of 1. If A is a subset of G, we call boundary of A and denote by ∂A the subset $UA\smallsetminus A$. Let μ be a Haar measure on G. Recall the definition of the monotone isoperimetric profile of G (see [19, 21])

$$I^{\uparrow}(t)=\inf_{\mu(A)\ge t}\mu(\partial A)/\mu(A)$$

where A runs over measurable subsets of finite measure of G. We can also define a (monotone) profile relatively to a family $\boldsymbol{A}$ of subsets of G

$$I^{\uparrow}_{\boldsymbol{A}}(t)=\inf_{\substack{\mu(A)\ge t\\ A\in\boldsymbol{A}}}\mu(\partial A)/\mu(A).$$

We say [21] that the family $\boldsymbol{A}$ is asymptotically isoperimetric if $I^{\uparrow}_{\boldsymbol{A}}\preceq I^{\uparrow}$.

By a theorem of Varopoulos (see [22, 5]), G has polynomial growth of exponent d if and only if $I^{\uparrow}(t)\approx t^{(d-1)/d}$.

An interesting question is for which groups do we have $I^{\uparrow}_{(U^n)_{n\in\mathbb{N}}}\preceq I^{\uparrow}$?

It is true for groups of polynomial growth as shown by the following proposition, valid for a general doubling metric measure space.

Proposition 17. — *Let* X *be a doubling metric measure space. There exists a sequence* $(r_i)_{i\in\mathbb{N}}$ *such that* $2^i\le r_i\le 2^{i+1}$ *and such that*

$$\forall i\in\mathbb{N},\ \forall x\in X,\quad \mu\big(S(x,r_i)\big)\le C\mu\big(B(x,r_i)/r_i\big).$$

(3) An erroneous proof of this statement is written in [18].

In particular, if G has polynomial growth of exponent d, and if U is a compact generating set of G, then there exists a subsequence n_i such that $2^i \leq n_{i+1} \leq 2^{i+1}$ and such that

$$\mu(U^{n_i+1} \smallsetminus U^{n_i}) \leq C n_i^{(d-1)/d}. \tag{4.1}$$

Proof of Proposition 17. — First, remark that

$$\forall n < m \in \mathbb{N}, \quad S(x,n) \cap S(x,m) = \varnothing$$

and that

$$\bigcup_{k=1}^{2^i} S(x, 2^i + k) \subset B(x, 2^{i+1}),$$

so that

$$2^i \inf_{1 \leq k \leq 2^i} \mu\big(S(x, 2^i + k)\big) \leq \mu\big(B(x, 2^{i+1})\big)$$

and finally, one can conclude thanks to doubling property. □

Remark 18. — In a general graph with strict polynomial growth of exponent d, the profile may sometimes be much smaller than $t^{(d-1)/d}$, so Proposition 17 does not necessarily imply that balls are asymptotically isoperimetric, *i.e.*, $I^{\uparrow}_{(B(x,k))_{x,r}} \preceq I^{\uparrow}$. This issue is studied quite extensively in [21].

Question 19. — One can make the last question more precise by asking if $I^{\uparrow}_{(U^k)} \preceq I^{\uparrow}$ implies that G has polynomial growth? Subexponential growth?

Question 20. — Let G be a cglc group with polynomial growth of exponent d. A very natural question is: does (4.1) hold for any integer n, or equivalently, is there a constant $C < \infty$ such that

$$\forall n \in \mathbb{N}, \quad \mu(U^{n+1} \smallsetminus U^n) \leq C \frac{\mu(U^n)}{n}? \tag{4.2}$$

This question is motivated by the following observation.

Proposition 21. — *Let G be a cglc abelian group and let U be a compact generating set of G. Then, (4.2) holds.*

Sketch of the proof. — First, note that it is an easy fact when $G = \mathbb{R}^d$ (the adaptation to $\mathbb{Z}^d$ is left to the reader): if K is convex, it is trivial (since $K + K = 2 \cdot K$); then show that $\widehat{K}^n \subset K^{n+k}$ where $\widehat{K}$ denotes the convex hull of K, and where k is a positive integer smaller than $d+1$ times the diameter of K. On the other hand, a cglc abelian group G is isomorphic to a direct product $K \times \mathbb{R}^a \times \mathbb{Z}^b$, with $a, b \in \mathbb{N}$, and K being a compact group. □

Remark 22. — Question 20 is also natural in the context of doubling graphs. In this setting, the question becomes: does there exist a constant C such that for all $x \in X$, and all $n \geq 1$,

$$\frac{|S(x,n)|}{|B(x,n)|} \leq \frac{C}{n}?$$

But as mentioned in the introduction, the answer is no in a very strong sense since in [21, Thm. 4.9], we construct a graph quasi-isometric to $\mathbb{Z}^2$ that does not satisfy this property.

5. Proofs

We will start proving Theorem 4 which is our "more general result". Nevertheless, Theorem 9 is not an immediate consequence of the group version of Theorem 4, that is, Corollary 10. So for the convenience of the reader, we will give a proof of Corollary 10 using notation adapted to the group setting, and then give the additional argument which is needed to obtain Theorem 9.

5.1. A preliminary observation. — The following observation is one of the main ingredients of the proofs.

Lemma 23. — *Let $X = (X, \mu)$ be a measured space. Let us consider an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of X. Define*

$$C_{n,n+k} = A_{n+k} \smallsetminus A_n.$$

We suppose that $\mu(A_n)$ is finite and unbounded with respect to $n \in \mathbb{N}$. Let us suppose that there exists a constant $\alpha > 0$ such that, for all integers $k \leq n$,

$$\mu(C_{n-k,n}) \geq \alpha \cdot \mu(C_{n,n+k}). \tag{5.1}$$

Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that for all $n \geq 1$

$$\frac{\mu(C_{n-1,n})}{\mu(A_n)} \leq Cn^{-\delta}.$$

Proof. — Write $i_n = [\log_2 n]$. For $i \leq i_n$, define $b_i = \mu(C_{n-2^i,n})$. Note that

$$C_{n-2^i,n} = C_{n-2^i,n-2^{i-1}} \cup \cdots \cup C_{n-1,n}$$

for all $i \leq i_n$ and that the union is disjoint. So we have

$$b_i = \mu(C_{n-2^i,n-2^{i-1}}) + \cdots + \mu(C_{n-1,n}).$$

On the other hand, by (5.1)

$$\begin{aligned}\mu(C_{n-2^i,n-2^{i-1}}) = \mu(C_{n-2^{i-1}-2^{i-1},n-2^{i-1}}) &\geq \alpha \cdot \mu(C_{n-2^{i-1},n-2^{i-1}+2^{i-1}}) \\ &= \alpha \cdot b_{i-1}.\end{aligned}$$

But note that

$$b_i = b_{i-1} + \mu(C_{n-2^i,n-2^{i-1}})$$

So $b_i \geq (1+\alpha)b_{i-1}$. Therefore

$$b_i \geq (1+\alpha)^i \mu(C_{n-1,n}).$$

Thus, it comes

$$\begin{aligned}\mu(A_n) \geq b_{i_n} \geq (1+\alpha)^{i_n}\mu(C_{n-1,n}) &\geq (1+\alpha)^{\log_2 n - 1}\mu(C_{n-1,n}) \\ &\geq \frac{1}{1+\alpha} n^{\log_2(1+\alpha)}\mu(C_{n-1,n}).\end{aligned}$$

So we are done. □

5.2. The case of metric measured spaces: proof of Theorem 4. — For all $x \in X$ and $r' > r > 0$, write

$$C_{r,r'}(x) = B(x,r') \smallsetminus B(x,r) \quad \text{and} \quad c_{r,r'}(x) = \mu\big(C_{r,r'}(x)\big).$$

Thanks to Lemma 23, we only need to prove that shells are doubling, *i.e.*, that there exists a constant $\alpha > 0$ such that, for all $x \in X$, and for any integers $n > k > 10C$ (where C is the constant that appears in the definition of Property (M))

$$c_{n-k,n}(x) \geq \alpha \cdot c_{n,n+k}(x).$$

So it is enough to prove the following lemma:

LEMMA 24. — *Let (X,d,μ) a doubling (M) space. Then, there exists $\alpha > 0$ such that for all $x \in X$ and for all couples of integers $4C < k \leq n$,*

$$c_{n-k,n}(x) \geq \alpha \cdot c_{n,n+k}(x).$$

Proof. — Let y be in $C_{n,n+k}(x)$. Consider a finite chain $x_0 = y, x_1, \ldots, x_m = x$ such that for $0 \leq i < m$,

$$d(x_i, x_{i+1}) \leq C \quad \text{and} \quad d(x_{i+1}, x) \leq d(x_i, x) - 1.$$

Let k_0 be the smallest integer such that $x_{k_0} \in B(x, n - \frac{1}{2}k)$. Since y belongs to $C_{n,n+k}(x)$, k_0 exists and is less than $2k$. Moreover, minimality of k_0 implies that $x_{k_0} \in C_{n-k/2-C,n-k/2}(x)$. So we have

$$d\big(y, C_{n-k/2-C,n-k/2}(x)\big) \leq 2Ck. \tag{5.2}$$

Let $(z_i)_i$ be a maximal family of k-separated points [(4)] in $C_{n-k/2-C,n-k/2}(x)$. Provided for instance that $k \geq 2C$, $C_{n-k/2-C,n-k/2}(x)$ is covered by the balls $B(z_i, 2k)$. Consequently, (5.2) implies that the balls $B(z_i, (2+2C)k)$ cover $C_{n,n+k}(x)$. On the other hand, if $k \geq 4C$, then for every i, z_i belongs

(4) As the space is doubling, such a family exists and is finite.

to $C_{n-3k/4,n-k/2}$ and hence the ball $B(z_i, \frac{1}{4}k)$ is included in $C_{n-k,n}(x)$. Moreover, these balls are disjoint. So we conclude by doubling property. $\square$

REMARK 25. — Note that a lower bound on k depending on C is necessary because otherwise, $C_{n-k,n}(x)$ could be empty for $k = 1$. For instance consider $\mathbb{Z}$ equipped with usual distance multiplied by 2: it satisfies Property (M) with $C = 2$. In this case, $C_{n-1,n}(0)$ is empty for odd n although $C_{n,n+1}$ is not empty.

5.3. The case of groups: proof of Theorem 9. — Note that to prove Theorem 9, we can assume that U_n contains 1, at least for n large enough, which ensures that (N_n) is nondecreasing. Indeed, choose an integer m such that $K' \cup \{1\} \subset K^m$. Then, write $n = qm + r$ with $r < m$ and for all $j \geq 1$, define

$$\widetilde{U}_j = U_{(j-1)m+r+1} \cdots U_{jm+r}.$$

Define also $\widetilde{U}_0 = U_0 \cdots U_r$. We therefore have

$$\widetilde{N}_q = N_n = \widetilde{U}_0 \cdots \widetilde{U}_q.$$

Finally, as $U_{n+1} \subset K' \subset \widetilde{U}_{n+1}$, it suffices to prove Theorem 9 for the sequence $(\widetilde{N}_q)$.

Actually, instead of directly proving Theorem 9, we will prove Corollary 10 and then explain how the proof can be generalized.

Proof of Corollary 10. — Let G be a cglc group of polynomial growth endowed with a Haar [(5)] measure μ. Let U be a compact generating subset containing 1. Recall that this implies that the sequence U^n is nondecreasing. Let us write

$$C_{n,n+k} = U^{n+k} \smallsetminus U^n \quad \text{and} \quad c_{n,n+k} = \mu(C_{n,n+k})$$

for all $n, k \in \mathbb{N}$. Recall that we want to find a constant α such that $c_{n-k,n} \geq \alpha \cdot c_{n,n+k}$ for k large enough. To simplify notation, let us assume that k is a positive multiple [(6)] of 4.

First, we have:

CLAIM 26. — $C_{n,n+k} \subset C_{n-k/2,n-k/2+1} U^{2k}$.

(5) Note that since G has subexponential growth, it is unimodular, so that the Haar measure is left and right invariant.
(6) If k is not a multiple of k, one has to assume at least that $k \geq 4$ and to replace everywhere in the proof $\frac{1}{4}k$ and $\frac{1}{2}k$ by their integer parts.

Proof. — Indeed, let y be in $C_{n,n+k}$, and let $(y_1,\ldots,y_{n+j})$ be a minimal sequence of elements of U such that $y = y_1\cdots y_{n+j}$. By definition of $C_{n,n+k}$ and by minimality, we have $1 \leq j \leq k$. Moreover, it is easy to see that minimality also implies

$$y_1\cdots y_{n-k/2+1} \in C_{n-k/2,n-k/2+1}.$$

So $y \in C_{n-k/2,n-k/2+1}y_{n-k/2+2}\cdots y_{n+j} \subset C_{n-k/2,n-k/2+1}U^{2k}$ and we are done. ☐

On the other hand, we have:

Claim 27. — $C_{n-k/2,n-k/2+1}U^{k/4} \subset C_{n-k,n}$.

Proof. — Since U contains 1, we have

$$C_{n-k/2,n-k/2+1}U^{k/4} \subset U^{n-k/2+k/4+1} \subset U^n.$$

Besides, let $x \in C_{n-k/2,n-k/2+1}U^{k/4}$, so that

$$x = yu_1\cdots u_{k/4}.$$

If we had $x \in U^{n-k}$, then, it would imply that $y \in U^{n-k+k/4} \subset U^{n-k/2}$: absurd. So $x \in C_{n-k,n}$. ☐

Now, let (x_i) be a maximal family of points of $C_{n-k/2,n-k/2+1}$ such that $x_iU^{k/4} \cap x_jU^{k/4} = \varnothing$ for $i \neq j$. By maximality of (x_i), we have

$$C_{n-k/2,n-k/2+1} \subset \cup_i x_iU^{k/4}U^{-k/4}.$$

So by Claim 26, we get

$$C_{n,n+k} \subset \bigcup_i x_iU^{k/4}U^{-k/4}U^{2k} \tag{5.3}$$

Let S be a symmetric compact neighborhood of 1 containing U^3. Then, since $U^{k/4}U^{-k/4}U^{2k}$ is included in S^k, Theorem 8 implies that there exists a constant $C < \infty$ such that for k large enough,

$$\mu(x_iU^{k/4}U^{-k/4}U^{2k}) \leq C\mu(x_iU^{k/4}). \tag{5.4}$$

Thus, since the $x_iU^{k/4}$ are disjoint and included in $C_{n-k,n}$, we get

$$c_{n-k,n} \geq \sum_i \mu(x_iU^{k/4}). \tag{5.5}$$

Finally, using (5.3), (5.4) and (5.5), we deduce

$$c_{n-k,n} \geq C^{-1}c_{n,n+k}.$$ ☐

Proof of Theorem 9. — The only significant modification we have to do in order to prove Theorem 10 concerns Claim 26. Actually, we have to show a kind of Property (M) adapted to this context.

Claim 28. — *There exists $j_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ and every $x \in U_0 \cdots U_{n+k} \smallsetminus U_0 \cdots U_n$, we have*

$$x \in (U_0 \cdots U_n \smallsetminus U_0 \cdots U_{n-kj_0} K'^k).$$

Proof. — Since K' contains U_i for every $i \in \mathbb{N}$, we have

$$x \in U_0 \cdots U_n K'^k.$$

On the other hand, let q be an integer such that

$$x \in U_0 \cdots U_{n-q} K'^k.$$

Then, let j_0 be such that $K' \subset K^{j_0}$ (see Proposition 6). Since $K \subset U_i$ for every i, it comes

$$x \in U_0 \cdots U_{n-q+kj_0}.$$

But this implies $q < kj_0$, so we are done. □

Let us finish the proof of Theorem 9. Write

$$C_{n,n+k} = U_0 \cdots U_{n+k} \smallsetminus U_0 \cdots U_n.$$

According to Claim 28, we have

$$C_{n,n+k} \subset C_{n-kj_0,n} K'^k$$

for every $k < n/j_0$.

Using the same arguments as in the proof of Theorem 10, we get

$$c_{n,n-j_0 k} \geq \alpha \cdot c_{n,n+k},$$

and we conclude thanks to Lemma 23. □

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