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ON A CERTAIN GENERALIZATION OF SPHERICAL TWISTS

BY YUKINOBU TODA

ABSTRACT. — This note gives a generalization of spherical twists, and describe the autoequivalences associated to certain non-spherical objects. Typically these are obtained by deforming the structure sheaves of $(0,-2)$-curves on threefolds, or deforming $\mathbb{P}$-objects introduced by D. Huybrechts and R. Thomas.

RÉSUMÉ (*Sur une généralisation des twists sphériques*). — Cette note donne une généralisation des twists sphériques et décrit des auto-équivalences associées à certains objets qui ne sont pas sphériques. Typiquement ces objets sont obtenus par déformation du faisceau structural d'une $(0,2)$-courbe dans une variété de dimension trois ou d'un $\mathbb{P}$-objet introduit par D. Huybrechts et R. Thomas.

1. Introduction

We introduce a new class of autoequivalences of derived categories of coherent sheaves on smooth projective varieties, which generalizes the notion of spherical twists given in [12]. Such autoequivalences are associated to a certain class of objects, which are not necessary spherical but are interpreted as "fat"

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version of them. We introduce the notion of R-spherical objects for a noetherian and artinian local $\mathbb{C}$-algebra R, and imitate the construction of spherical twists to give the associated autoequivalences.

Let X be a smooth complex projective variety, and $D(X)$ be a bounded derived category of coherent sheaves on X. When X is a Calabi-Yau 3-fold, $D(X)$ is considered to represent the category of D-branes of type B, and should be equivalent to the derived Fukaya category on a mirror manifold under Homological mirror symmetry [8]. On the mirror side, there are typical symplectic automorphisms by taking Dehn twists along Lagrangian spheres [11]. The notions of spherical objects and associated twists were introduced in [12] in order to realize Dehn twists under mirror symmetry. Recall that $E \in D(X)$ is called *spherical* if the following holds [12]:

- $\operatorname{Ext}^i_X(E,E) = \begin{cases} \mathbb{C} & \text{if } i=0 \text{ or } i=\dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

Then one can construct the autoequivalence $T_E\colon D(X) \to D(X)$ which fits into the distinguished triangle [12]:

$$\mathbb{R}\operatorname{Hom}(E,F) \otimes_{\mathbb{C}} E \longrightarrow F \longrightarrow T_E(F),$$

for $F \in D(X)$. The autoequivalence T_E is called a *spherical twist*. This is a particularly important class of autoequivalences, especially when we consider A_n-configulations on surfaces as indicated in [7]. On the other hand, it has been observed that there are some autoequivalences which are not described in terms of spherical twists. This occurs even in the similar situation discussed in [7] as follows. Let $X \to Y$ be a three dimensional flopping contraction which contracts a rational curve $C \subset X$, and $X^{\dagger} \to Y$ be its flop. Then one can construct the autoequivalence [1, 3, 4],

$$\Phi := \Phi^{\mathcal{O}_{X\times_Y X^{\dagger}}}_{X^{\dagger}\to X} \circ \Phi^{\mathcal{O}_{X\times_Y X^{\dagger}}}_{X\to X^{\dagger}} : D(X) \longrightarrow D(X^{\dagger}) \longrightarrow D(X).$$

If $C \subset X$ is not a $(-1,-1)$-curve, Φ is not written as a spherical twist, and our motivation comes from describing such autoequivalences. Let R be a noetherian and artinian local $\mathbb{C}$-algebra. We introduce the notion of R-spherical objects defined on $D(X \times \operatorname{Spec} R)$. In the above example, $\operatorname{Spec} R$ is taken to be the moduli space of $\mathcal{O}_C(-1)$, and the universal family gives the R-spherical object. Our main theorem is the following:

THEOREM 1.1. — *To any R-spherical object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}}\colon D(X) \to D(X)$, which fits into the distinguished triangle*

$$\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E},F) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. *Here* $\pi\colon X \times \operatorname{Spec} R \to X$ *is the projection.*

Using the notion of R-spherical objects and associated twists, we can also give the deformations of $\mathbb{P}$-twists in the case which is not treated in [5].

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Notations and conventions

- For a variety X, we denote by $D(X)$ its bounded derived category of coherent sheaves.
- Δ means the diagonal $\Delta \subset X \times X$ or the diagonal embedding $\Delta\colon X \to X \times X$.
- For another variety Y and an object $\mathcal{P} \in D(X \times Y)$, denote by $\Phi^{\mathcal{P}}_{X\to Y}$ the integral transform with kernel $\mathcal{P}$, *i.e.*,
$$\Phi^{\mathcal{P}}_{X\to Y}(*) := \mathbb{R}p_{Y*}(p_X^*(*) \overset{\mathbb{L}}{\otimes} \mathcal{P})\colon D(X) \longrightarrow D(Y).$$
Here p_X, p_Y are projections from $X \times Y$ onto corresponding factors.

2. Generalized spherical twists

Let X be a smooth projective variety over $\mathbb{C}$ and R be a noetherian and artinian local $\mathbb{C}$-algebra. We introduce the notion of R-spherical objects defined on $D(X \times \operatorname{Spec} R)$. Let $\pi\colon X \times \operatorname{Spec} R \to X$ and $\pi'\colon X \times \operatorname{Spec} R \to \operatorname{Spec} R$ be projections and $0 \in \operatorname{Spec} R$ be the closed point.

Definition 2.1. — An object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$ is called *R-spherical* if the following conditions hold:

- $\mathcal{E}$ is represented by a bounded complex $\mathcal{E}^\bullet$ with each $\mathcal{E}^i$ a coherent $\mathcal{O}_{X\times \operatorname{Spec} R}$-module flat over R. In particular we have the bounded derived restriction $E := \mathcal{E}^\bullet|_{X\times\{0\}} \in D(X)$.
- $\operatorname{Ext}^i_X(E,E) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } i = \dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

Remark 2.2. — If $R = \mathbb{C}$, then R-spherical objects coincide with usual spherical objects.

We imitate the construction of the spherical twists in the following theorem.

THEOREM 2.3. — *To any R-spherical object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}} \colon D(X) \to D(X)$, which fits into the distinguished triangle:*

$$\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, F) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. Here R-module structures on $\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, F)$ and $\pi_*\mathcal{E}$ are inherited from R-module structure on $\mathcal{E}$.

Proof. — First we construct the kernel of $T_{\mathcal{E}}$. Let p_{ij} and p_i be projections as in the following diagram

$$\begin{array}{ccccc}
 & & X \times \operatorname{Spec} R \times X & & \\
 & \swarrow^{p_{12}} & \downarrow{\scriptstyle p_{13}} & \searrow^{p_{23}} & \\
X \times \operatorname{Spec} R & & X \times X & & X \times \operatorname{Spec} R, \\
\downarrow{\scriptstyle \pi} & \swarrow^{p_1} & & \searrow^{p_2} & \downarrow{\scriptstyle \pi} \\
X & & & & X
\end{array}$$

and consider the object

$$\mathcal{Q} := \mathbb{R}p_{13*}\big(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}\big) \in D(X \times X).$$

Here $\check{\mathcal{E}}$ means its derived dual. Then for $F \in D(X)$, we can calculate $\Phi_{X\to X}^{\mathcal{Q}}(F)$ as follows:

$$\begin{aligned}
\Phi_{X\to X}^{\mathcal{Q}}(F) &\cong \mathbb{R}p_{2*}(\mathbb{R}p_{13*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}) \overset{\mathbb{L}}{\otimes} p_1^*F) \\
&\cong \mathbb{R}p_{2*}\mathbb{R}p_{13*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E} \overset{\mathbb{L}}{\otimes} p_{13}^*p_1^*F) \\
&\cong \pi_*\mathbb{R}p_{23*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E} \overset{\mathbb{L}}{\otimes} p_{12}^*\pi^*F) \\
&\cong \pi_*\big\{\mathcal{E} \overset{\mathbb{L}}{\otimes} \mathbb{R}p_{23*}p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} \pi^*F)\big\} \\
&\cong \pi_*\big\{\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi'^*\mathbb{R}\pi'_*\mathbb{R}\mathcal{H}om(\mathcal{E}, \pi^!F)\big\} \\
&\cong \pi_*\mathcal{E} \overset{\mathbb{L}}{\otimes}_R \mathbb{R}\operatorname{Hom}(\pi_*\mathcal{E}, F).
\end{aligned}$$

The fifth equality comes from the base change formula for the diagram below:

$$\begin{array}{ccc}
X \times \operatorname{Spec} R \times X & \xrightarrow{\;p_{12}\;} & X \times \operatorname{Spec} R \\
{\scriptstyle p_{23}}\downarrow & & \downarrow{\scriptstyle \pi'} \\
X \times \operatorname{Spec} R & \xrightarrow{\;\pi'\;} & \operatorname{Spec} R.
\end{array}$$

On the other hand, we have

$$\begin{aligned}
\operatorname{Hom}_{X\times X}(\mathcal{Q},\mathcal{O}_\Delta) &= \operatorname{Hom}_{X\times X}\big(\mathbb{R}p_{13*}\big(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}\big),\mathcal{O}_\Delta\big)\\
&= \operatorname{Hom}_X\big(\mathbb{L}\Delta^*\mathbb{R}p_{13*}\big(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}\big),\mathcal{O}_X\big)\\
&= \operatorname{Hom}_X\big(\pi_*\mathbb{L}(\Delta,\mathrm{id})^*(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}),\mathcal{O}_X\big)\\
&= \operatorname{Hom}_X\big(\mathbb{L}(\Delta,\mathrm{id})^*(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}),\pi^!\mathcal{O}_X\big)\\
&= \operatorname{Hom}_X(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} \mathcal{E},\pi^!\mathcal{O}_X).
\end{aligned}$$

The third equality comes from the base change formula for the diagram below:

$$\begin{array}{ccc}
X\times \operatorname{Spec} R & \xrightarrow{(\Delta,\mathrm{id})} & X\times \operatorname{Spec} R\times X\\
\pi\big\downarrow & & \big\downarrow p_{13}\\
X & \xrightarrow{\quad\Delta\quad} & X\times X.
\end{array}$$

Let $\mu\colon \mathcal{Q}\to\mathcal{O}_\Delta$ be the morphism which corresponds to the morphism

$$\mathrm{id}_{\pi^!\mathcal{O}_X}\otimes \mathrm{ev}\colon \pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} \mathcal{E} \longrightarrow \pi^!\mathcal{O}_X,$$

under the above isomorphisms. Let us take its cone $\mathcal{R} := \operatorname{Cone}(\mu)\in D(X\times X)$. Then the above calculation for $\Phi^{\mathcal{Q}}_{X\to X}$ implies the functor $T_{\mathcal{E}}\colon D(X)\to D(X)$ with kernel $\mathcal{R}$ fits into the triangle

$$\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E},F) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F\in D(X)$. We check $T_{\mathcal{E}}$ gives an equivalence. We follow the arguments of [10, 5]. Define $E^\perp$ to be the subcategory $\{F\in D(X)\mid \mathbb{R}\operatorname{Hom}(E,F)=0\}$. Then $\Omega := E\cup E^\perp$ is a spanning class in the sense of [2, Def. 2.1]. Let $\langle E\rangle$ be the minimum extension closed subcategory of $D(X)$ which contains E. Then since R is finite dimensional, we have $\pi_*\mathcal{E}\in\langle E\rangle$. Therefore if $F\in E^\perp$, then $\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E},F)=0$. Hence $T_{\mathcal{E}}(F)\cong F$ for $F\in E^\perp$. Next since $\mathcal{E}$ is R-spherical, we have the distinguished triangle

$$E \longrightarrow \mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E},E) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow E[-\dim X].$$

Then the following diagram

$$
\begin{array}{ccccc}
E & & & & \\
\downarrow & \searrow^{\mathrm{id}} & & & \\
\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, E) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} & \longrightarrow & E & \longrightarrow & T_{\mathcal{E}}(E) \\
\downarrow & & \downarrow & \nearrow & \\
E[-\dim X] & \longrightarrow & 0 & &
\end{array}
$$

shows $T_{\mathcal{E}}(E) \cong E[1-\dim X]$. Therefore $T_{\mathcal{E}}$ is fully faithful on Ω, hence fully faithful on $D(X)$. (cf. [2, Theorem 2.3]). Finally the assumption $E \otimes \omega_X \cong E$ implies $F \otimes \omega_X \in E^{\perp}$ for $F \in E^{\perp}$. Therefore $T_{\mathcal{E}}|_{\Omega}$ commutes with $\otimes \omega_X$. Hence $T_{\mathcal{E}}$ gives an equivalence by the argument of [2, Thm 5.4]. □

3. Flops at $(0,-2)$-curves

We give some examples of autoequivalences associated to R-spherical objects. Let $f\colon X \to Y$ be a three dimensional flopping contraction which contracts a rational curve $C \subset X$. Let $f^{\dagger}\colon X^{\dagger} \to Y$ be its flop, and $C^{\dagger} \subset X^{\dagger}$ be the flopped curve. Then in [1, 3, 4], the functor $\Phi_1\colon D(X^{\dagger}) \to D(X)$ with kernel $\mathcal{O}_{X\times_Y X^{\dagger}}$ gives an equivalence. Φ_1 satisfies the following (cf. [14, Lemma 5.1]):

- Φ_1 takes $\mathcal{O}_{C^{\dagger}}(-1)[1]$ to $\mathcal{O}_C(-1)$;
- Φ_1 commutes with derived push-forwards, *i.e.*, $\mathbb{R}f_* \circ \Phi_1 \cong \mathbb{R}f^{\dagger}_*$.

Similarly we can construct the equivalence $\Phi_2\colon D(X) \to D(X^{\dagger})$ with kernel $\mathcal{O}_{X\times_Y X^{\dagger}}$. Composing these, we obtain the autoequivalence

$$\Phi := \Phi_1 \circ \Phi_2\colon D(X) \longrightarrow D(X^{\dagger}) \longrightarrow D(X).$$

Note that $\Phi(\mathcal{O}_C(-1)) = \mathcal{O}_C(-1)[-2]$ and Φ commutes with $\mathbb{R}f_*$. If $C \subset X$ is a $(-1,-1)$-curve, then $\mathcal{O}_C(-1)$ is a spherical object and Φ coincides with the associated twist $T_{\mathcal{O}_C(-1)}$. But if C is not a $(-1,-1)$-curve, then $\mathcal{O}_C(-1)$ is no longer spherical, so we have to find some new descriptions of Φ. The idea is to consider the moduli problem of $\mathcal{O}_C(-1)$ and use the universal family.

Here we assume $C \subset X$ is a $(0,-2)$-curve, *i.e.*, normal bundle is $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$, and give the description of Φ. Let $\mathcal{M}$ be the connected component of the moduli space of simple sheaves on X, which contains $\mathcal{O}_C(-1)$. We define

$$R_m := \mathbb{C}[t]/(t^{m+1}), \qquad S_m := \operatorname{Spec} R_m.$$

Since $\operatorname{Ext}^1_X(\mathcal{O}_C(-1), \mathcal{O}_C(-1)) = \mathbb{C}$ and $C \subset X$ is rigid, we can write $\mathcal{M}$ as $\mathcal{M} = S_m$ for some $m \in \mathbb{N}$. Let $\mathcal{E} \in \operatorname{Coh}(X \times S_m)$ be the universal family.

THEOREM 3.1. — *$\mathcal{E}$ is a R_m-spherical object and the associated functor*

$$T_{\mathcal{E}}\colon D(X) \longrightarrow D(X)$$

coincides with Φ.

Proof. — For $n \le m$, define $\mathcal{E}_n$ to be

$$\mathcal{E}_n := \pi_{n*}(\mathcal{E}|_{X\times S_n}) \in \mathrm{Coh}(X),$$

where $\pi_n\colon X \times S_n \to X$ is a projection. Since we have the exact sequences of R_m-modules

$$0 \longrightarrow R_{n-1} \longrightarrow R_n \longrightarrow \mathbb{C} \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{C} \longrightarrow R_n \longrightarrow R_{n-1} \longrightarrow 0,$$

we have the exact sequences in $\mathrm{Coh}(X)$:

$$0 \longrightarrow \mathcal{E}_{n-1} \longrightarrow \mathcal{E}_n \longrightarrow E \longrightarrow 0, \tag{1}$$
$$0 \longrightarrow E \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow 0. \tag{2}$$

Here $E := \mathcal{O}_C(-1)$. Applying $\mathrm{Hom}(*, E)$ to the sequence (1), we obtain the long exact sequence

$$\mathrm{Hom}(\mathcal{E}_n, E) \longrightarrow \mathrm{Hom}(\mathcal{E}_{n-1}, E) \xrightarrow{\xi_n} \mathrm{Ext}^1(E,E) = \mathbb{C}$$
$$\longrightarrow \mathrm{Ext}^1(\mathcal{E}_n, E) \longrightarrow \mathrm{Ext}^1(\mathcal{E}_{n-1}, E) \xrightarrow{\eta_n} \mathrm{Ext}^2(E,E) = \mathbb{C}.$$

On the other hand, the sequence (2) determines the non-zero element

$$e_n \in \mathrm{Ext}^1(\mathcal{E}_{n-1}, E),$$

and $\eta_n(e_n) \in \mathrm{Ext}^2(E,E)$ gives the obstruction to deforming $\mathcal{E}|_{X\times S_n}$ to a coherent sheaf on $X \times S_{n+1}$ flat over S_{n+1} (cf. [13, Prop. 3.13]). Therefore $\eta_n(e_n) = 0$ for $n < m$ and $\eta_m(e_m) \neq 0$. On the other hand, we have the following morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_{n-1} & \longrightarrow & \mathcal{E}_n & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s_n} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \mathcal{E}_1 & \longrightarrow & E & \longrightarrow & 0,
\end{array}$$

where s_n is a natural surjection. Hence $\xi_n(s_n) \in \mathrm{Ext}^1(E,E)$ corresponds to the extension $\mathcal{E}_1$, which is a non-trivial first order deformation of E. Therefore $\xi_n(s_n) \neq 0$ and ξ_n is surjective. Combining these, we have

$$\mathrm{Ext}^1(\mathcal{E}_{n-1}, E) \cong \mathrm{Ext}^1(\mathcal{E}_n, E) \cong \mathbb{C} \;\; (\text{for } n < m), \quad \mathrm{Ext}^1(\mathcal{E}_m, E) \cong 0,$$
$$\mathrm{Hom}(\mathcal{E}_{n-1}, E) \cong \mathrm{Hom}(\mathcal{E}_n, E) \cong \mathbb{C}.$$

Similarly applying $\mathrm{Hom}(E,*)$ to the sequence (2), we obtain $\mathrm{Ext}^1(E,\mathcal{E}_m)=0$ and $\mathrm{Hom}(E,\mathcal{E}_m)=\mathbb{C}$. By Serre duality, we can conclude $\mathcal{E}$ is R_m-spherical.

Next let us consider the equivalence

$$\widetilde{\Phi} := T_{\mathcal{E}} \circ \Phi^{-1} \colon D(X) \longrightarrow D(X).$$

Then $\widetilde{\Phi}$ takes $\mathcal{O}_C(-1)$ to $\mathcal{O}_C(-1)$, and commutes with $\mathbb{R}f_*$. Therefore $\widetilde{\Phi}$ preserves perverse t-structure ${}^0\mathrm{Per}(X/Y)$ in the sense of [3]. Then the argument of [14, Thm 6.1] shows $\widetilde{\Phi}$ is isomorphic to the identity functor. □

4. Deformations of $\mathbb{P}$-twists

Review of $\mathbb{P}$-objects and associated twists. — R-spherical twists can also be used to construct deformations of $\mathbb{P}$-twists. Let us recall the definition of $\mathbb{P}$-objects and the associated autoequivalences introduced in [5]. Again we assume X is a smooth projective variety over $\mathbb{C}$.

Definition 4.1 (see [5]). — An object $E\in D(X)$ is called $\mathbb{P}^n$-*object* if it satisfies the following:

- $\mathrm{Ext}^*_X(E,E)$ is isomorphic to $H^*(\mathbb{P}^n,\mathbb{C})$ as a graded ring;
- $E\otimes\omega_X \cong E$.

Note that if $\mathbb{P}^n$-object exists, then $\dim X = 2n$ by Serre duality. D. Huybrechts and R. Thomas [5] constructed an equivalence $P_E\colon D(X)\to D(X)$ associated to E, which is described as follows. Let $h\in\mathrm{Ext}^2_X(E,E)$ be the degree two generator. First consider the morphism in $D(X\times X)$:

$$H := \check{h}\boxtimes\mathrm{id} - \mathrm{id}\boxtimes h\colon \check{E}\boxtimes E[-2]\longrightarrow \check{E}\boxtimes E.$$

Let us take its cone $\mathcal{H}\in D(X\times X)$. We can see the composition H with the trace map $\mathrm{tr}\colon \check{E}\boxtimes E\to\mathcal{O}_\Delta$ becomes zero. Therefore there exists a (in fact unique) morphism $t\colon\mathcal{H}\to\mathcal{O}_\Delta$ such that the following diagram commutes [5, Lemma 2.1]:

$$\begin{array}{ccccc} \check{E}\boxtimes E[-2] & \xrightarrow{H} & \check{E}\boxtimes E & \longrightarrow & \mathcal{H} \\ & & \mathrm{tr}\downarrow & \swarrow t & \\ & & \mathcal{O}_\Delta. & & \end{array}$$

Then define $\mathcal{Q}_{\mathcal{E}}$ to be the cone

$$\mathcal{Q}_{\mathcal{E}} := \mathrm{Cone}(t\colon\mathcal{H}\to\mathcal{O}_\Delta)\in D(X\times X).$$

Then in [5], it is shown that the functor $P_{\mathcal{E}}\colon D(X)\to D(X)$ with kernel $\mathcal{Q}_{\mathcal{E}}$ gives the equivalence.

Next let us consider a one parameter deformation of X. Let $f\colon \mathcal{X} \to C$ be a smooth family over a smooth curve C with a distinguished fibre $j\colon X = f^{-1}(0) \hookrightarrow \mathcal{X}$, $0 \in C$. Suppose $E \in D(X)$ is a $\mathbb{P}^n$-object and let be its Atiyah-class $A(E) \in \operatorname{Ext}^1_X(E, E \otimes \Omega_X)$. Then the obstruction to deforming E sideways to first order is given by the product

$$A(E) \cdot \kappa(X) \in \operatorname{Ext}^2_X(E, E),$$

where $\kappa(X) \in H^1(X, T_X)$ is the Kodaira-Spencer class of the family $f\colon \mathcal{X} \to C$. In [5], the case of $A(E) \cdot \kappa(X) \neq 0$ is studied. In that case, j_*E is a spherical object and the associated equivalence $T_{j_*E}\colon D(\mathcal{X}) \to D(\mathcal{X})$ fits into the commutative diagram [5, Prop. 2.7]

$$\begin{array}{ccc} D(X) & \xrightarrow{j_*} & D(\mathcal{X}) \\ {\scriptstyle P_E}\downarrow & & \downarrow{\scriptstyle T_{j_*E}} \\ D(X) & \xrightarrow[j_*]{} & D(\mathcal{X}). \end{array}$$

Our purpose is to treat the case of $A(E) \cdot \kappa(X) = 0$.

R-spherical objects via deformations of $\mathbb{P}$-objects. — Let $f\colon \mathcal{X} \to C$ and $E \in D(X)$ be as before, and assume $A(E) \cdot \kappa(X) = 0$. Note that j_*E is not spherical. In fact we have the distinguished triangle

$$E[1] \longrightarrow \mathbb{L}j^* j_* E \longrightarrow E \stackrel{A(E)\cdot\kappa(X)}{\longrightarrow} E[2],$$

by [5, Prop. 3.1]. Hence we have the decomposition $\mathbb{L}j^* j_* E \cong E \oplus E[1]$, and for $0 \le k \le 2n+1$ we calculate

$$\begin{aligned} \operatorname{Ext}^k_{\mathcal{X}}(j_*E, j_*E) &\cong \operatorname{Ext}^k_X(\mathbb{L}j^* j_* E, E) \\ &\cong \operatorname{Ext}^k_X(E, E) \oplus \operatorname{Ext}^{k-1}_X(E, E) \cong \mathbb{C}. \end{aligned}$$

As in the previous section, we are going to consider deformations of j_*E in $\mathcal{X}$. The moduli theories of complexes were carried out by [6, 9]. Following the notation used in [6], we consider the functor $\operatorname{Splcpx}_{\mathcal{X}/C}$ from the category of locally noetherian schemes over C to the category of sets,

$$\operatorname{Splcpx}_{\mathcal{X}/C}(T) := \left\{ \mathcal{F}^\bullet \;\middle|\; \begin{array}{l} \mathcal{F}^\bullet \text{ is a bounded complex of coherent sheaves on } \mathcal{X}_T \\ \text{such that each } \mathcal{F}^i \text{ is flat over } T \text{ and for any } t \in T, \\ \operatorname{Ext}^0_{X_t}(\mathcal{F}^\bullet(t), \mathcal{F}^\bullet(t)) \cong k(t),\ \operatorname{Ext}^{-1}_{X_t}(\mathcal{F}^\bullet(t), \mathcal{F}^\bullet(t)) = 0 \end{array} \right\} \Big/ \sim .$$

Here

$$\mathcal{X}_T := \mathcal{X} \times_C T, \quad \mathcal{F}^\bullet(t) := \mathcal{F}^\bullet \otimes_T k(t),$$

and $\mathcal{F}^\bullet \sim \mathcal{F}'^\bullet$ if and only if there exist $\mathcal{L} \in \operatorname{Pic}(T)$, a bounded complex of quasi-coherent sheaves $\mathcal{G}^\bullet$ and quasi-isomorphisms $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$, $\mathcal{G}^\bullet \to \mathcal{F}'^\bullet \otimes \mathcal{L}$.

Let $\operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/C}$ be the associated sheaf of $\operatorname{Splcpx}_{\mathcal{X}/C}$ in the étale topology. M. Inaba [6] showed the following:

Theorem 4.2 (see [6]). — *The functor* $\operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/C}$ *is represented by a locally separated algebraic space* $\mathcal{M}$ *over* C.

Let $S_m = \operatorname{Spec} \mathbb{C}[t]/(t^{m+1})$ be as before and $\gamma\colon S_1 \hookrightarrow C$ be an extension of $0 \hookrightarrow C$. Let r be the restriction,

$$r\colon \operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/C}(\gamma) \longrightarrow \operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/C}(0).$$

By the assumption $A(E) \cdot \kappa(X) = 0$, we have $r^{-1}(E) \neq \varnothing$. Moreover by [6, Prop. 2.3], there is a bijection between $r^{-1}(E)$ and $\operatorname{Ext}^1_X(E,E)$, which is zero. Therefore the map $T_{\mathcal{M},E} \to T_{C,0}$ is an isomorphism, hence $\dim \mathcal{M} \leq 1$ at $[E] \in \mathcal{M}$. Note that by taking push-forward along the inclusion $\mathcal{X} \times_C T \to \mathcal{X} \times T$, we get the morphism of functors:

$$\delta\colon \operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/C} \longrightarrow \operatorname{Splcpx}^{\text{ét}}_{\mathcal{X}/S_0}.$$

We put the following technical assumption:

(⋆) The morphism δ gives an isomorphism between connected components of both sides, which contain E and j_*E respectively. Let $[E] \in \mathcal{M}' \subset \mathcal{M}$ be the connected component. We assume $\mathcal{M}'$ is a zero-dimensional scheme.

Note that we can write $\mathcal{M}' = S_m$ for some m. Let

$$\mathcal{X}_m := \mathcal{X} \times_C \mathcal{M}' = \mathcal{X} \times_C S_m$$

and $\mathcal{E} \in D(\mathcal{X}_m)$ be the universal family. We use the following notations for morphism:

$$\begin{array}{ccc} \mathcal{X}_m & \xrightarrow{k} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S_m & \xrightarrow[k']{} & C, \end{array} \qquad \begin{array}{ccccc} \mathcal{X}_m & \xrightarrow{\ell} & \mathcal{X} \times S_m & \xrightarrow{\pi'} & S_m \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} & \searrow{\scriptstyle \pi} & \\ X & \xrightarrow[j]{} & \mathcal{X} & & \mathcal{X}. \end{array}$$

If there is no confusion, we will use the same notations for $n \leq m$. We show the following proposition:

Proposition 4.3. — *The object* $\ell_*\mathcal{E} \in D(\mathcal{X} \times S_m)$ *is* R_m*-spherical.*

Proof. — Since $\pi_*\ell_*\mathcal{E} \cong k_*\mathcal{E}$ and $\mathbb{L}i'^*\ell_*\mathcal{E} \cong j_*E$, we have to calculate $\operatorname{Ext}^i_{\mathcal{X}}(k_*\mathcal{E}, j_*E)$. By the assumption $(\star)$, we cannot deform $\ell_*\mathcal{E}$ to $(m+1)$-th order. For $n \leq m$, let $\mathcal{E}_n := \mathcal{E}|_{\mathcal{X}_n} \in D(\mathcal{X}_n)$ and $\widetilde{\mathcal{E}}_n := k_*\mathcal{E}_n \in D(\mathcal{X})$. We consider distinguished triangles:

$$\widetilde{\mathcal{E}}_{n-1} \longrightarrow \widetilde{\mathcal{E}}_n \longrightarrow j_*E \xrightarrow{e'_n} \widetilde{\mathcal{E}}_{n-1}[1], \tag{3}$$

$$j_*E \longrightarrow \widetilde{\mathcal{E}}_n \longrightarrow \widetilde{\mathcal{E}}_{n-1} \xrightarrow{e_n} j_*E[1]. \tag{4}$$

Then by the argument of [13, Prop. 3.13], we can see that the composition

$$e_n \circ e'_n \colon j_*E \longrightarrow \widetilde{\mathcal{E}}_{n-1}[1] \longrightarrow j_*E[2]$$

gives the obstruction to deforming $\ell_*\mathcal{E}_n$ to $(n+1)$-th order. If E is a sheaf, this is just [13, Prop. 3.13] and we can generalize this by replacing the exact sequences in [13, Prop. 3.13] by the exact sequences of representing complexes. We leave the detail to the reader. Hence $e_m \circ e'_m \neq 0$ and $e_n \circ e'_n = 0$ for $n < m$. Applying $\operatorname{Hom}(*, j_*E)$ to the triangle (3), we obtain the long exact sequence,

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{X}}(j_*E, j_*E) &\longrightarrow \operatorname{Ext}^1_{\mathcal{X}}(\widetilde{\mathcal{E}}_n, j_*E)\\ &\longrightarrow \operatorname{Ext}^1_{\mathcal{X}}(\widetilde{\mathcal{E}}_{n-1}, j_*E) \longrightarrow \operatorname{Ext}^2_{\mathcal{X}}(j_*E, j_*E) = \mathbb{C}.\end{aligned}$$

Then using the above sequence and the same argument as in Theorem 3.1, we can conclude $\operatorname{Ext}^1_{\mathcal{X}}(k_*\mathcal{E}, j_*E) = 0$.

Next we use the existence of the distinguished triangle [1, Lemma 3.3]:

$$\mathcal{E}[1] \longrightarrow \mathbb{L}k^*k_*\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}[2].$$

Pulling back to X, we have the triangle

$$E[1] \longrightarrow \mathbb{L}j^*k_*\mathcal{E} \longrightarrow E \xrightarrow{\theta} E[2]. \tag{5}$$

Since $\operatorname{Ext}^2_X(E,E)$ is one dimensional, θ is zero or non-zero multiple of h. Assume $\theta = 0$. Then $\mathbb{L}j^*k_*\mathcal{E} \cong E \oplus E[1]$, and

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{X}}(k_*\mathcal{E}, j_*E) &\cong \operatorname{Ext}^1_{\mathcal{X}}(\mathbb{L}j^*k_*\mathcal{E}, E)\\ &\cong \operatorname{Ext}^1_X(E,E) \oplus \operatorname{Hom}(E,E) \cong \mathbb{C},\end{aligned}$$

which is a contradiction. Hence we may assume $\theta = h$. Applying $\operatorname{Hom}(*, E)$ to the triangle (5), we obtain the long exact sequence

$$\to \operatorname{Ext}^i_X(E,E) \longrightarrow \operatorname{Ext}^i_{\mathcal{X}}(\mathbb{L}j^*k_*\mathcal{E}, E) \longrightarrow \operatorname{Ext}^{i-1}_X(E,E) \xrightarrow{h} \operatorname{Ext}^{i+1}_X(E,E) \to .$$

By the definition of $\mathbb{P}^n$-object, we obtain

$$\operatorname{Ext}^i_{\mathcal{X}}(k_*\mathcal{E}, j_*E) \cong \operatorname{Ext}^i_X(\mathbb{L}j^*k_*\mathcal{E}, E) = \begin{cases}\mathbb{C} & \text{if } i = 0 \text{ or } i = 2n+1,\\ 0 & \text{otherwise.}\end{cases}$$

□

REMARK 4.4. — Assumption $(\star)$ is satisfied if E is a sheaf and $\dim \mathcal{M}' = 0$. In fact suppose $\ell_*\mathcal{E}$ extends to a S_{m+1}-valued point of $\mathrm{Splcpx}^{\text{ét}}_{\mathcal{X}/S_0}$. Then as in [13, Prop. 3.13], there exists $\widetilde{\mathcal{E}}_{m+1} \in \mathrm{Coh}(\mathcal{X})$ such that there exists a morphism of exact sequences of $\mathcal{O}_{\mathcal{X}}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\mathcal{E}}_{m-1} & \longrightarrow & \widetilde{\mathcal{E}}_{m} & \longrightarrow & j_*E & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \mathrm{id} & & \\
0 & \longrightarrow & \widetilde{\mathcal{E}}_{m} & \xrightarrow{\nu} & \widetilde{\mathcal{E}}_{m+1} & \longrightarrow & j_*E & \longrightarrow & 0.
\end{array}$$

An easy diagram chase shows $\widetilde{\mathcal{E}}_{m+1}$ is a $\mathcal{O}_{\mathcal{X}}/(t^{m+2})$-module for the uniformizing parameter $t \in \mathcal{O}_{C,0}$. Moreover we have $t \cdot \widetilde{\mathcal{E}}_{m+1} = \mathrm{Im}\,\nu$. Therefore the map

$$\widetilde{\mathcal{E}}_{m+1} \otimes_{\mathcal{O}_C/(t^{m+2})} (t) \longrightarrow \widetilde{\mathcal{E}}_{m+1}$$

is a morphism from $\widetilde{\mathcal{E}}_m$ onto $\mathrm{Im}\,\nu \cong \widetilde{\mathcal{E}}_m$, hence injective. Then [13, Lemma 3.7] shows $\widetilde{\mathcal{E}}_{m+1}$ is flat over $\mathcal{O}_{C,0}/(t^{m+2})$ and gives a S_{m+1}-valued point of $\mathrm{Splcpx}^{\text{ét}}_{\mathcal{X}/C}$.

$\mathbb{P}$-twists and R-spherical twists. — By Proposition 4.3, we have the associated functor $T_{\ell_*\mathcal{E}} \colon D(\mathcal{X}) \to D(\mathcal{X})$ under assumption $(\star)$. The next purpose is to show the existence of the diagram as in [5, Prop. 2.7]. We use the following notations for morphisms:

$$\begin{array}{ccccc}
X \times X & & \xrightarrow{\widetilde{j}} & & \mathcal{X} \times_C \mathcal{X} \\
 & \searrow^{\widetilde{i}} & & \nearrow_{\widetilde{k}} & \\
 & & \mathcal{X}_m \times_{S_m} \mathcal{X}_m, & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{X} \times \mathcal{X}_m & \xrightarrow{\mathrm{id} \times \ell} & \mathcal{X} \times S_m \times \mathcal{X} \\
\uparrow \ell' & & \uparrow \ell \times \mathrm{id} \\
\mathcal{X}_m \times_{S_m} \mathcal{X}_m & \xrightarrow[\ell'']{} & \mathcal{X}_m \times \mathcal{X},
\end{array}$$

$$\begin{array}{ccccccc}
X \times X & \xrightarrow{\widetilde{i}} & \mathcal{X}_m \times_{S_m} \mathcal{X}_m & \xrightarrow{\widetilde{k}} & \mathcal{X} \times_C \mathcal{X} & \xrightarrow{\iota} & \mathcal{X} \times \mathcal{X} \\
\uparrow \Delta_0 & & \uparrow \Delta_m & & \uparrow \Delta' & \nearrow_{\Delta} & \\
X & \xrightarrow[i]{} & \mathcal{X}_m & \xrightarrow[k]{} & \mathcal{X}, & &
\end{array}$$

$$\begin{array}{ccccc}
 & & \mathcal{X} \times S_m \times \mathcal{X} & & \\
 & \swarrow^{p_{12}} & \downarrow p_{13} & \searrow^{p_{23}} & \\
\mathcal{X} \times S_m & & \mathcal{X} \times \mathcal{X} & & \mathcal{X} \times S_m,
\end{array}
\qquad
\begin{array}{ccccc}
 & & \mathcal{X}_m \times_{S_m} \mathcal{X}_m & & \\
 & \swarrow^{q_1} & & \searrow^{q_2} & \\
\mathcal{X}_m & & & & \mathcal{X}_m.
\end{array}$$

THEOREM 4.5. — *The functor $T_{\ell_*\mathcal{E}}$ fits into the following commutative diagram:*

$$\begin{array}{ccc} D(X) & \xrightarrow{j_*} & D(\mathcal{X}) \\ {\scriptstyle P_E}\big\downarrow & & \big\downarrow{\scriptstyle T_{\ell_*\mathcal{E}}} \\ D(X) & \xrightarrow[j_*]{} & D(\mathcal{X}). \end{array}$$

Proof. — We try to imitate the argument of [5, Prop. 2.7]. First we construct the morphism

$$\alpha\colon \widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1] \longrightarrow \Delta'_*\mathcal{O}_{\mathcal{X}}$$

in $D(\mathcal{X} \times_C \mathcal{X})$. This is constructed by the composition of $\widetilde{k}_*\,\mathrm{tr}$,

$$\widetilde{k}_*\,\mathrm{tr}\colon \widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1] \longrightarrow \widetilde{k}_*\Delta_{m*}\mathcal{O}_{\mathcal{X}_m}[-1] = \Delta'_*k_*\mathcal{O}_{\mathcal{X}_m}[-1],$$

with the morphism $\Delta'_*k_*\mathcal{O}_{\mathcal{X}_m}[-1] \to \Delta'_*\mathcal{O}_{\mathcal{X}}$ obtained by applying Δ'_* to the exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{X}_m) \longrightarrow k_*\mathcal{O}_{\mathcal{X}_m} \longrightarrow 0.$$

Let $\mathcal{L} := \mathrm{Cone}(\alpha) \in D(\mathcal{X} \times_C \mathcal{X})$. Applying Chen's lemma [4], it suffices to show

$$\iota_*\mathcal{L} \cong \mathrm{Cone}\big(\mathbb{R}p_{13*}(p_{12}^*(\ell_*\check{}\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi^!\mathcal{O}_{\mathcal{X}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\ell_*\mathcal{E}) \xrightarrow{\mu} \Delta_*\mathcal{O}_{\mathcal{X}}\big), \quad \mathbb{L}\widetilde{\jmath}^*\mathcal{L} \cong \mathcal{H}.$$

Here μ is the morphism constructed in the proof of Theorem 2.3 and $\mathcal{H}$ is the kernel of P_E. First we check $\iota_*\mathcal{L} \cong \mathrm{Cone}(\mu)$. Note that $\pi^!\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}\times S_m}$ and $\ell_*\check{}\mathcal{E} \cong \ell_*\check{\mathcal{E}}[-1]$ by the duality isomorphism. Hence we have

$$\begin{aligned}
\mathbb{R}p_{13*}\big(p_{12}^*(\ell_*\check{}\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi^!\mathcal{O}_{\mathcal{X}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\ell_*\mathcal{E}\big) &\cong \mathbb{R}p_{13*}(p_{12}^*\ell_*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} p_{23}^*\ell_*\mathcal{E})[-1] \\
&\cong \mathbb{R}p_{13*}\big\{(\ell \times \mathrm{id})_*r_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} (\mathrm{id} \times \ell)_*r_2^*\mathcal{E}\big\}[-1] \\
&\cong \mathbb{R}p_{13*}(\mathrm{id} \times \ell)_*\big\{\mathbb{L}(\mathrm{id} \times \ell)^*(\ell \times \mathrm{id})_*r_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} r_2^*\mathcal{E}\big\}[-1] \\
&\cong \mathbb{R}p_{13*}(\mathrm{id} \times \ell)_*(\ell'_*\mathbb{L}\ell''^*r_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} r_2^*\mathcal{E})[-1] \\
&\cong \mathbb{R}p_{13*}(\mathrm{id} \times \ell)_*\ell'_*(\mathbb{L}\ell''^*r_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} \mathbb{L}\ell'^*r_2^*\mathcal{E})[-1] \\
&\cong \iota_*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1].
\end{aligned}$$

Here r_1, r_2 are defined by the fiber squares:

$$\begin{array}{ccc} \mathcal{X}_m \times \mathcal{X} & \xrightarrow{\ell \times \mathrm{id}} & \mathcal{X} \times S_m \times \mathcal{X} \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle p_{12}} \\ \mathcal{X}_m & \xrightarrow[\ell]{} & \mathcal{X} \times S_m, \end{array} \qquad \begin{array}{ccc} \mathcal{X} \times \mathcal{X}_m & \xrightarrow{\mathrm{id} \times \ell} & \mathcal{X} \times S_m \times \mathcal{X} \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle p_{23}} \\ \mathcal{X}_m & \xrightarrow[\ell]{} & \mathcal{X} \times S_m. \end{array}$$

Under the above isomorphism, we can check $\iota_*\alpha = \mu$. Hence $\widetilde{\ell}_*\mathcal{L} \cong \mathrm{Cone}(\mu)$.

Next we check $\mathbb{L}\widetilde{\jmath}^*\mathcal{L} \cong \mathcal{H}$. Note that we have

$$\mathbb{L}\widetilde{\jmath}^*\mathcal{L} = \mathrm{Cone}\big(\mathbb{L}\widetilde{\jmath}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1] \xrightarrow{\mathbb{L}\widetilde{\jmath}^*\alpha} \mathbb{L}\widetilde{\jmath}^*\Delta'_*\mathcal{O}_{\mathcal{X}} = \Delta_{0*}\mathcal{O}_X\big),$$

and there exists the distinguished triangle

$$q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E}[-2] \longrightarrow q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E} \longrightarrow \mathbb{L}\widetilde{k}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1],$$

as in [1, Lemma 3.3]. Then applying $\mathbb{L}\widetilde{\imath}^*$, we have the triangle

$$\check{E} \boxtimes E[-2] \xrightarrow{u} \check{E} \boxtimes E \longrightarrow \mathbb{L}\widetilde{\jmath}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1].$$

We can easily check the following:

$$\begin{aligned} &\mathrm{Ext}^2_{X\times X}(\check{E} \boxtimes E, \check{E} \boxtimes E) \\ &\quad\cong \big(\mathrm{Ext}^2_X(E,E) \otimes \mathrm{Ext}^0_X(E,E)\big) \oplus \big(\mathrm{Ext}^0_X(E,E) \otimes \mathrm{Ext}^2_X(E,E)\big). \end{aligned}$$

Hence we can write $u = a(\check{h} \boxtimes \mathrm{id}) + b(\mathrm{id} \boxtimes h)$ for some $a, b \in \mathbb{C}$. On the other hand, we can check that the following diagram commutes:

$$\begin{array}{ccccc} \check{E} \boxtimes E[-2] & \xrightarrow{u} & \check{E} \boxtimes E & \longrightarrow & \mathbb{L}\widetilde{\jmath}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E}) \\ & & & \searrow^{\mathrm{tr}} & \downarrow{\scriptstyle \mathbb{L}\widetilde{\jmath}^*\alpha} \\ & & & & \Delta_{0*}\mathcal{O}_X. \end{array}$$

This is easily checked using the same argument of [5, Prop. 2.7], and leave the detail to the reader. Therefore $\mathrm{tr} \circ u = 0$, which implies $b = -a$. Hence if we show $u \neq 0$, then we can conclude $\mathbb{L}\widetilde{\jmath}^*\mathcal{L} \cong \mathcal{H}$. Assume $u = 0$. Then we have the decomposition

$$\mathbb{L}\widetilde{\jmath}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1] \cong (\check{E} \boxtimes E) \oplus (\check{E} \boxtimes E)[-1]. \tag{6}$$

Since $\mathrm{Hom}_{X\times X}(\check{E} \boxtimes E[-1], \Delta_*\mathcal{O}_X) = 0$, the morphism

$$\mathbb{L}\widetilde{\jmath}^*\alpha \colon \mathbb{L}\widetilde{\jmath}^*\widetilde{k}_*(q_1^*\check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} q_2^*\mathcal{E})[-1] \longrightarrow \Delta_{0*}\mathcal{O}_X$$

is a non-zero multiple of $(\mathrm{tr}, 0)$ under the decomposition (6). Let $\mathcal{S} \in D(X \times X)$ be the cone of the trace map:

$$\check{E} \boxtimes E \xrightarrow{\mathrm{tr}} \Delta_{0*}\mathcal{O}_X \longrightarrow \mathcal{S}.$$

Then we have the decomposition $\mathbb{L}\widetilde{\jmath}^*\mathcal{L} \cong \mathcal{S} \oplus (\check{E} \boxtimes E)$, and the following diagram commutes by Chen's lemma [4]:

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi^{\mathbb{L}\widetilde{\jmath}^*\mathcal{L}}_{X\to X}} & D(X) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ D(\mathcal{X}) & \xrightarrow[T_{\ell_*\mathcal{E}}]{} & D(\mathcal{X}). \end{array}$$

In particular we have

$$j_*\Phi^{\mathbb{L}\widetilde{\jmath}^*\mathcal{L}}_{X\to X}(E) \cong T_{\ell_*\mathcal{E}}(j_*E) \cong j_*E[1 - \dim \mathcal{X}],$$

which is indecomposable. It follows that

$$\Phi^{\mathcal{S}}_{X\to X}(E) \cong 0 \quad \text{or} \quad \Phi^{\check{E}\boxtimes E}_{X\to X}(E) \cong 0.$$

Since $\Phi^{\check{E}\boxtimes E}_{X\to X}(E) \cong \mathbb{R}\operatorname{Hom}(E,E)\otimes_{\mathbb{C}} E$, the latter is impossible by the definition of $\mathbb{P}^n$-object. Hence $\Phi^{\mathcal{S}}_{X\to X}(E)$ must be zero. Since we have the distinguished triangle:

$$\mathbb{R}\operatorname{Hom}(E,E) \otimes_{\mathbb{C}} E \longrightarrow E \longrightarrow \Phi^{\mathcal{S}}_{X\to X}(E) \cong 0,$$

we have $\mathbb{R}\operatorname{Hom}(E,E)\otimes_{\mathbb{C}} E \cong E$. But again this is impossible by the definition of $\mathbb{P}^n$-object. $\square$

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