Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

SEMISTABILITY OF FROBENIUS DIRECT IMAGES

Vikram B. Mehta & Christian Pauly

Tome 135
Fascicule 1

2007

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 1-

Bull. Soc. math. France
135 (1), 2007, p. 105–

SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

by Vikram B. Mehta & Christian Pauly

Abstract. — Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Given a semistable vector bundle E over X, we show that its direct image F_*E under the Frobenius map F of X is again semistable. We deduce a numerical characterization of the stable rank-p vector bundles F_*L, where L is a line bundle over X.

Résumé (*Semi-stabilité des images directes par Frobenius sur les courbes*)
Soit X une courbe projective lisse de genre ≥ 2 définie sur un corps k algébriquement clos de caractéristique $p > 0$. Étant donné un fibré vectoriel semi-stable E sur X, nous montrons que l'image directe F_*E par le morphisme de Frobenius F de X est aussi semi-stable. Nous déduisons une caractérisation numérique du fibré vectoriel stable F_*L de rang p, où L est un fibré en droites sur X.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$ and let $F : X \to X_1$ be the relative k-linear Frobenius map. It is by now a well-established fact that on

Texte reçu le 24 janvier 2006, révisé le 21 juillet 2006

Vikram B. Mehta, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005 (India) • *E-mail :* `vikram@math.tifr.res.in`
Christian Pauly, Université de Montpellier II, Département de Mathématiques, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5 (France)
E-mail : `pauly@math.univ-montp2.fr`

2000 Mathematics Subject Classification. — 14H40, 14D20, 14H40.
Key words and phrases. — Vector bundle, semistability, Frobenius.

any curve X there exist semistable vector bundles E such that their pull-back under the Frobenius map F^*E is not semistable [4, 5]. In order to control the degree of instability of the bundle F^*E, one is naturally lead (through adjunction $\mathrm{Hom}_{\mathcal{O}_X}(F^*E, E') = \mathrm{Hom}_{\mathcal{O}_{X_1}}(E, F_*E'))$ to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

THEOREM 1.1. — *Assume that $g \geq 2$. If E is a semistable vector bundle over X (of any degree), then F_*E is also semistable.*

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that F_*L is stable for a line bundle L (see [4, Proposition 1.2]) and that in small characteristics the bundle F_*E is stable for any stable bundle E of small rank (see [3]). The main ingredient of the proof is Faltings' cohomological criterion of semistability. We also need the fact that the generalized Verschiebung V, defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank-r vector bundles over X_1 with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map F,

$$V_r : \mathcal{M}_{X_1}(r) \dashrightarrow \mathcal{M}_X(r), \qquad E \longmapsto F^*E$$

is dominant for large r. We actually show a stronger statement for large r.

PROPOSITION 1.2. — *If $\ell \geq g(p-1)+1$ and ℓ prime, then the generalized Verschiebung V_ℓ is generically étale for any curve X. In particular V_ℓ is separable and dominant.*

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant ν of a vector bundle E, defined as

$$\nu(E) := \mu_{\max}(F^*E) - \mu_{\min}(F^*E),$$

where $\mu_{\max}$ (resp. $\mu_{\min}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of F^*E.

PROPOSITION 1.3. — *For any semistable rank-r vector bundle E*

$$\nu(E) \leq \min\big((r-1)(2g-2), (p-1)(2g-2)\big).$$

We note that the inequality $\nu(E) \leq (r-1)(2g-2)$ was proved in [10, Corollary 2], and in [11, Theorem 3.1]. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (= number of pieces) of the Harder-Narasimhan filtration of F^*E is at most p for semistable E.

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant ν.

Proposition 1.4. — *Let E be a stable rank-p vector bundle over X. Then the following statements are equivalent.*

1) *There exists a line bundle L such that $E = F_*L$.*
2) $\nu(E) = (p-1)(2g-2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case $\mu(E) = g - 1$

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles E with slope $\mu(E) = g - 1$.

Let E be a semistable vector bundle over X of rank r and let s be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}\cdot$$

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \to X_1$, we obtain

$$\mu(F_*E) = g - 1 + \frac{s}{pr}\cdot$$

Let $\pi : \widetilde{X} \to X$ be a connected étale covering of degree n and let $\pi_1 : \widetilde{X}_1 \to X_1$ denote its twist by the Frobenius of k (see [9, Section 4]). The diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{F} & \widetilde{X}_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ X & \xrightarrow{F} & X_1 \end{array} \tag{2.1}$$

is Cartesian and we have an isomorphism

$$\pi_1^*(F_*E) \cong F_*(\pi^*E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that π^*E is semistable. Moreover if $F_*(\pi^*E)$ is semistable, then F_*E is also semistable.

Let L be a degree d line bundle over $\widetilde{X}_1$. The projection formula

$$F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L$$

shows that semistability of $F_*(\pi^*E)$ is equivalent to semistability of

$$F_*(\pi^*E \otimes F^*L).$$

Let $\tilde{g}$ denote the genus of $\widetilde{X}$. By the Riemann-Hurwitz formula, one has $\tilde{g}-1=n(g-1)$. We compute

$$\mu(\pi^*E\otimes F^*L)=n(g-1)+n\frac{s}{r}+pd=\tilde{g}-1+n\frac{s}{r}+pd,$$

which gives

$$\mu\big(F_*(\pi^*E\otimes F^*L)\big)=\tilde{g}-1+n\frac{s}{pr}+d.$$

LEMMA 2.1. — *For any integer m there exists a connected étale covering $\pi:\widetilde{X}\to X$ of degree $n=p^km$ for some k.*

Proof. — If the p-rank of X is nonzero, the statement is clear. If the p-rank is zero, we know by [9, Corollaire 4.3.4], that there exist connected étale coverings $Y\to X$ of degree p^t for infinitely many integers t (more precisely for all t of the form $(\ell-1)(g-1)$ where ℓ is a large prime). Now we decompose $m=p^su$ with p not dividing u. We then take a covering $Y\to X$ of degree p^t with $t\geq s$ and a covering $\widetilde{X}\to Y$ of degree u. □

Now the lemma applied to the integer $m=pr$ shows existence of a connected étale covering $\pi:\widetilde{X}\to X$ of degree $n=p^km$. Hence $n\frac{s}{pr}$ is an integer and we can take d such that $n\frac{s}{pr}+d=0$.

To summarize, we have shown that for any semistable E over X there exists a covering $\pi:\widetilde{X}\to X$ and a line bundle L over $\widetilde{X}_1$ such that the vector bundle $\widetilde{E}:=\pi^*E\otimes F^*L$ is semistable with $\mu(\widetilde{E})=\tilde{g}-1$ and such that semistability of $F_*\widetilde{E}$ implies semistability of F_*E.

3. Proof of Theorem 1.1

In order to prove semistability of F_*E we shall use the cohomological criterion of semistability due to Faltings [2].

PROPOSITION 3.1 (see [6, Théorème 2.4]). — *Let E be a rank-r vector bundle over X with $\mu(E)=g-1$ and ℓ an integer $>\frac{1}{4}r^2(g-1)$. Then E is semistable if and only if there exists a rank-ℓ vector bundle G with trivial determinant such that*

$$h^0(X,E\otimes G)=h^1(X,E\otimes G)=0.$$

Moreover if the previous condition holds for one bundle G, it holds for a general bundle by upper semicontinuity of the function $G\mapsto h^0(X,E\otimes G)$.

REMARK. — The proof of this proposition (see [6, Section 2.4]) works over any algebraically closed field k.

By Proposition 1.2 (proved in Section 4) we know that V_ℓ is dominant when ℓ is a large prime number. Hence a general vector bundle $G \in \mathcal{M}_X(\ell)$ is of the form F^*G' for some $G' \in \mathcal{M}_{X_1}(\ell)$. Consider a semistable E with $\mu(E) = g-1$. Then by Proposition 3.1 $h^0(X, E \otimes G) = 0$ for general $G \in \mathcal{M}_X(\ell)$. Assuming G general, we can write $G = F^*G'$ and we obtain by adjunction

$$h^0(X, E \otimes F^*G') = h^0(X_1, F_*E \otimes G') = 0.$$

This shows that F_*E is semistable by Proposition 3.1.

4. Proof of Proposition 1.2

According to [7, Section 2], it will be enough to prove the existence of a stable vector bundle $E \in \mathcal{M}_{X_1}(\ell)$ satisfying F^*E stable and

$$h^0\big(X_1, B \otimes \mathrm{End}_0(E)\big) = 0,$$

because the vector space $H^0(X_1, B \otimes \mathrm{End}_0(E))$ can be identified with the kernel of the differential of V_ℓ at the point $E \in \mathcal{M}_{X_1}(\ell)$. Here B denotes the sheaf of locally exact differentials over X_1 (see [9, Section 4]).

Let $\ell \neq p$ be a prime number and let $\alpha \in JX_1[\ell] \cong JX[\ell]$ be a nonzero ℓ-torsion point. We denote by

$$\pi : \widetilde{X} \longrightarrow X \quad \text{and} \quad \pi_1 : \widetilde{X}_1 \longrightarrow X_1$$

the associated cyclic étale cover of X and X_1 and by σ a generator of the Galois group $\mathrm{Gal}(\widetilde{X}/X) = \mathbb{Z}/\ell\mathbb{Z}$. We recall that the kernel of the Norm map

$$\mathrm{Nm} : J\widetilde{X} \longrightarrow JX$$

has ℓ connected components and we denote by

$$i : P := \mathrm{Prym}(\widetilde{X}/X) \subset J\widetilde{X}$$

the associated Prym variety, *i.e.*, the connected component containing the origin. Then we have an isogeny

$$\pi^* \times i : JX \times P \longrightarrow J\widetilde{X}$$

and taking direct image under π induces a morphism

$$P \longrightarrow \mathcal{M}_X(\ell), \quad L \longmapsto \pi_*L.$$

Similarly we define the Prym variety $P_1 \subset JX_1$ and the morphism $P_1 \to \mathcal{M}_{X_1}(\ell)$ (obtained by twisting with the Frobenius of k). Note that $\pi_{1*}L$ is semistable for any $L \in P_1$ and stable for general $L \in P_1$ (see *e.g.*, [1]). Since $F^*(\pi_{1*}L) \cong \pi_*(F^*L)$ — see diagram (2.1) — and since F^* induces the Verschiebung $V_P : P_1 \to P$, which is surjective, we obtain that $\pi_{1*}L$ and $F^*(\pi_{1*}L)$ are stable for general $L \in P_1$.

Therefore Proposition 1.2 will immediately follow from the next proposition.

PROPOSITION 4.1. — *If $\ell \geq g(p-1)+1$ then there exists a cyclic degree ℓ étale cover $\pi_1 : \widetilde{X}_1 \to X_1$ with the property that*

$$h^0\big(X_1, B \otimes \mathrm{End}_0(\pi_{1*}L)\big) = 0$$

for general $L \in P_1$.

Proof. — By relative duality for the étale map π_1 we have $(\pi_{1*}L)^* \cong \pi_{1*}L^{-1}$. Therefore

$$\mathrm{End}(\pi_{1*}L) \cong \pi_{1*}L \otimes \pi_{1*}L^{-1} \cong \pi_{1*}(L^{-1} \otimes \pi_1^*\pi_{1*}L)$$

by the projection formula. Moreover since π_1 is Galois étale we have a direct sum decomposition

$$\pi_1^*\pi_{1*}L \cong \bigoplus_{i=0}^{\ell-1} (\sigma^i)^* L.$$

Putting these isomorphisms together we find that

$$\begin{aligned} H^0\big(X_1, B \otimes \mathrm{End}(\pi_{1*}L)\big) &= H^0(X_1, B \otimes \pi_{1*}\Big(\bigoplus_{i=0}^{\ell-1} L^{-1} \otimes (\sigma^i)^* L\Big) \\ &= \bigoplus_{i=0}^{\ell-1} H^0(X_1, B \otimes \pi_{1*}\big(L^{-1} \otimes (\sigma^i)^* L)\big) \\ &= H^0(X_1, B \otimes \pi_{1*}\mathcal{O}_{\tilde{X}_1}) \\ &\qquad \oplus \bigoplus_{i=1}^{\ell-1} H^0(X_1, B \otimes \pi_{1*}\big(L^{-1} \otimes (\sigma^i)^* L)\big). \end{aligned}$$

Moreover $\pi_*\mathcal{O}_{\tilde{X}_1} = \bigoplus_{i=0}^{\ell-1} \alpha^i$, which implies that

$$\begin{aligned} H^0(X_1, B \otimes \mathrm{End}_0(\pi_{1*}L)) &= \bigoplus_{i=1}^{\ell-1} H^0(X_1, B \otimes \alpha^i) \\ &\oplus \bigoplus_{i=1}^{\ell-1} H^0\big(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)\big). \end{aligned} \tag{4.1}$$

Let us denote for $i = 1, \dots, \ell-1$ by ϕ_i the isogeny

$$\phi_i : P_1 \longrightarrow P_1, \quad L \longmapsto L^{-1} \otimes (\sigma^i)^* L.$$

Since the function $L \mapsto h^0(X_1, B \otimes \mathrm{End}_0(\pi_{1*}L))$ is upper semicontinuous, it will be enough to show the existence of a cover $\pi_1 : \widetilde{X}_1 \to X_1$ satisfying

1) for $i = 1, \dots, \ell-1$, $h^0(X_1, B \otimes \alpha^i) = 0$ (or equivalently, P is an ordinary abelian variety);
2) for M general in P, $h^0(X_1, B \otimes \pi_{1*}M) = 0$.

Note that these two conditions implie that the vector space (4.1) equals $\{0\}$ for general $L \in P_1$, because the ϕ_i's are surjective.

We recall that $\ker(\pi_1^* : JX_1 \to J\widetilde{X}_1) = \langle \alpha \rangle \cong \mathbb{Z}/\ell\mathbb{Z}$ and that

$$P_1[\ell] = P_1 \cap \pi_1^*(JX_1) \cong \alpha^\perp / \langle \alpha \rangle$$

where $\alpha^\perp = \{\beta \in JX_1[\ell] \text{ with } \omega(\alpha, \beta) = 1\}$ and $\omega : JX_1[\ell] \times JX_1[\ell] \to \mu_\ell$ denotes the symplectic Weil form. Consider a $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. Then $\pi_1^* \beta \in P_1[\ell]$ and

$$\pi_{1*}\pi_1^*\beta = \bigoplus_{i=0}^{\ell=1} \beta \otimes \alpha^i.$$

Again by upper semicontinuity of the function $M \mapsto h^0(X_1, B \otimes \pi_{1*}M)$ one observes that conditions 1) and 2) are satisfied because of the following lemma (take $M = \pi_1^*\beta$).

Lemma 4.2. — *If $\ell \geq g(p-1)+1$ then there exists $(\alpha, \beta) \in JX_1[\ell] \times JX_1[\ell]$ satisfying*

1) *$\alpha \neq 0$ and $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$;*
2) *for $i = 1, \ldots, \ell - 1$, $h^0(X_1, B \otimes \alpha^i) = 0$;*
3) *for $i = 0, \ldots, \ell - 1$, $h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0$.*

Proof. — We adapt the proof of [9, Lemme 4.3.5]. We denote by $\mathbb{F}_\ell$ the finite field $\mathbb{Z}/\ell\mathbb{Z}$. Then there exists a symplectic isomorphism $JX_1[\ell] \cong \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$, where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in $JX_1[\ell]$ and additively in $\mathbb{F}_\ell^{2g}$. A quick computation shows that the number of isotropic 2-planes in $\mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ equals

$$N(\ell) = \frac{(\ell^{2g}-1)(\ell^{2g-2}-1)}{(\ell^2-1)(\ell-1)}\cdot$$

Let $\Theta_B \subset JX_1$ denote the theta divisor associated to B. Then by [9, Lemma 4.3.5], the cardinality $A(\ell)$ of the finite set

$$\Sigma(\ell) := JX_1[\ell] \cap \Theta_B$$

satisfies

$$A(\ell) \leq \ell^{2g-2} g(p-1).$$

Suppose that there exists an isotropic 2-plane $\Pi \subset \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ which contains $\leq \ell - 2$ points of $\Sigma(\ell)$. Then we can find a pair (α, β) satisfying the three properties of the lemma as follows: any nonzero point $x \in \Pi$ determines a line ($=\mathbb{F}_\ell$-vector space of dimension 1). Since a line contains $\ell - 1$ nonzero points, we obtain at most $(\ell-1)(\ell-2)$ nonzero points lying on lines generated by $\Sigma(\ell) \cap \Pi$. Since $(\ell-1)(\ell-2) < \ell^2 - 1$ there exists a nonzero α in the complement of these lines. Now we note that there are $\ell - 1$ affine lines parallel to the line generated by α and the ℓ points on any of these affine lines are of the form $\beta\alpha^i$ for $i = 0, \dots, \ell - 1$ for some $\beta \in \alpha^\perp \setminus \langle\alpha\rangle$. The points $\Sigma(\ell) \cap \Pi$ lie on at most $\ell - 2$ such affine lines, hence there exists at least one affine line parallel to $\langle\alpha\rangle$ avoiding $\Sigma(\ell)$. This gives β.

Finally let us suppose that any isotropic 2-plane contains $\geq \ell - 1$ points of $\Sigma(\ell)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \big\{(x, \Pi) \mid x \in \Pi \cap \Sigma(\ell) \text{ and } \Pi \text{ isotropic 2-plane}\big\}.$$

with cardinality $|S|$. Then by our assumption we have

$$|S| \geq (\ell - 1)N(\ell). \tag{4.2}$$

On the other hand, since any nonzero $x \in \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ is contained in $(\ell^{2g-2} - 1)/(\ell - 1)$ isotropic 2-planes, we obtain

$$|S| \leq \frac{\ell^{2g-2} - 1}{\ell - 1} A(\ell). \tag{4.3}$$

Putting (4.2) and (4.3) together, we obtain

$$A(\ell) \geq \frac{\ell^{2g} - 1}{\ell + 1}.$$

But this contradicts the inequality $A(\ell) \leq \ell^{2g-2} g(p-1)$ if $\ell \geq g(p-1) + 1$. □

This completes the proof of Proposition 4.1. □

Remark. — It has been shown [8, Theorem A.6], that V_r is dominant for any rank r and any curve X, by using a versal deformation of a direct sum a r line bundles.

Remark. — We note that V_r is not separable when p divides the rank r and X is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in \mathcal{M}_{X_1}(r)$ identifies with the quotient $H^1(X_1, \mathrm{End}_0(E))/\langle e\rangle$ where e denotes the nonzero extension class of $\mathrm{End}_0(E)$ by $\mathcal{O}_{X_1}$ given by $\mathrm{End}(E)$. Then the inclusion of homotheties $\mathcal{O}_{X_1} \hookrightarrow \mathrm{End}_0(E)$ induces an inclusion $H^1(X_1, \mathcal{O}_{X_1}) \subset H^1(X_1, \mathrm{End}_0(E))/\langle e\rangle$ and the restriction of the differential of V_r at the point E to $H^1(X_1, \mathcal{O}_{X_1})$ coincides with the non-injective Hasse-Witt map.

5. Proof of Proposition 1.3

Since we already know that $\nu(E) \leq (r-1)(2g-2)$ (see [10, 11]) it suffices to show that $\nu(E) \leq (p-1)(2g-2)$.

We consider the quotient $F^*E \to Q$ with minimal slope, *i.e.*, $\mu(Q) = \mu_{\min}(F^*E)$ and Q semistable. By adjunction we obtain a nonzero morphism $E \to F_*(Q)$, from which we deduce (using Theorem 1.1) that

$$\mu(E) \leq \mu(F_*Q) = \frac{1}{p}\big(\mu_{\min}(F^*E) + (p-1)(g-1)\big)$$

hence

$$\mu(F^*E) \leq \mu_{\min}(F^*E) + (p-1)(g-1).$$

Similarly we consider the subbundle $S \hookrightarrow F^*E$ with maximal slope, *i.e.*, $\mu(S) = \mu_{\max}(F^*E)$ and S semistable. Taking the dual and proceeding as above, we obtain that

$$\mu(F^*E) \geq \mu_{\max}(F^*E) - (p-1)(g-1).$$

Now we combine both inequalities and we are done.

Remark. — We note that the inequality of Proposition 1.3 is sharp. The maximum $(p-1)(2g-2)$ is obtained for the bundles $E = F_*E'$ (see [3, Theorem 5.3]).

6. Characterization of direct images

Consider a line bundle L over X. Then the direct image F_*L is stable (see [4], Proposition 1.2) and the Harder-Narasimhan filtration of F^*F_*L is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F^*F_*L, \quad \text{with} \quad V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}.$$

In particular $\nu(F_*L) = (p-1)(2g-2)$. In this section we will show a converse statement.

More generally let E be a stable rank-rp vector bundle with $\mu(E) = g-1+\frac{d}{rp}$ for some integer d and satisfying

1) the Harder-Narasimhan filtration of F^*E has ℓ terms.
2) $\nu(E) = (p-1)(2g-2)$.

Questions. — Do we have $\ell \leq p$? Is E of the form $E = F_*G$ for some rank-r vector bundle G? We will give a positive answer in the case $r = 1$ (Proposition 6.1).

Let us denote the Harder-Narasimhan filtration by

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = F^*E, \quad V_i/V_{i-1} = M_i$$

satisfying the inequalities

$$\mu_{\max}(F^*E) = \mu(M_1) > \mu(M_2) > \cdots > \mu(M_\ell) = \mu_{\min}(F^*E).$$

The quotient $F^*E \to M_\ell$ gives via adjunction a nonzero map $E \to F_*M_\ell$. Since F_*M_ℓ is semistable, we obtain that $\mu(E) \leq \mu(F_*M_\ell)$. This implies that $\mu(M_\ell) \geq g-1+d/r$. Similarly taking the dual of the inclusion $M_1 \subset F^*E$ gives a map $F^*(E^*) \to M_1^*$ and by adjunction $E^* \to F_*(M_1^*)$. Let us denote

$$\mu(M_1^*) = g-1+\delta,$$

so that $\mu(F_*(M_1^*)) = g-1+\delta/p$. Because of semistability of $F_*(M_1^*)$, we obtain

$$-\Big(g-1+\frac{d}{rp}\Big) = \mu(E^*) \leq \mu\big(F^*(M_1^*)\big),$$

hence $\delta \geq -2p(g-1) - d/r$. This implies $\mu(M_1) \leq (2p-1)(g-1)+d/r$. Combining this inequality with $\mu(M_\ell) \geq g-1+d/r$ and the assumption $\mu(M_1) - \mu(M_\ell) = (p-1)(2g-2)$, we obtain that

$$\mu(M_1) = (2p-1)(g-1) + \frac{d}{r}, \quad \mu(M_\ell) = g-1+\frac{d}{r}\cdot$$

Let us denote by r_i the rank of the semistable bundle M_i. We have the equality

$$\sum_{i=1}^{\ell} r_i = rp. \tag{6.1}$$

Since E is stable and $F_*(M_\ell)$ is semistable and since these bundles have the same slope, we deduce that $r_\ell \geq r$. Similarly we obtain that $r_1 \geq r$.

Note that it is enough to show that $r_\ell = r$. Since E is stable and F_*M_ℓ semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce for $i = 1, \ldots, \ell-1$ the integers

$$\delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g-1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i).$$

Then we have the equality

$$\sum_{i=1}^{\ell-1} \delta_i = \mu(M_\ell) - \mu(M_1) + 2(\ell-1)(g-1) = 2(\ell-p)(g-1). \tag{6.2}$$

We note that if $\delta_i < 0$, then $\mathrm{Hom}(M_i, M_{i+1} \otimes \omega) = 0$.

PROPOSITION 6.1. — *Let E be stable rank-p vector bundle with*

$$\mu(E) = g - 1 + d/p \quad \text{and} \quad \nu(E) = (p-1)(2g-2).$$

Then $E = F_* L$ for some line bundle L of degree $g - 1 + d$.

Proof. — Let us first show that $\ell = p$. We suppose that $\ell < p$. Then

$$\sum_{i=1}^{\ell-1} \delta_i = 2(\ell - p)(g-1) < 0$$

so that there exists a $k \leq \ell - 1$ such that $\delta_k < 0$. We may choose k minimal, *i.e.*, $\delta_i \geq 0$ for $i < k$. Then we have

$$\mu(M_k) > \mu(M_i) + 2(g-1) \quad \text{for } i > k. \tag{6.3}$$

We recall that $\mu(M_i) \leq \mu(M_{k+1})$ for $i > k$. The Harder-Narasimhan filtration of V_k is given by the first k terms of the Harder-Narasimhan filtration of F^*E. Hence $\mu_{\min}(V_k) = \mu(M_k)$.

Consider now the canonical connection ∇ on F^*E and its first fundamental form

$$\phi_k : V_k \hookrightarrow F^*E \xrightarrow{\nabla} F^*E \otimes \omega_X \longrightarrow (F^*E/V_k) \otimes \omega_X.$$

Since $\mu_{\min}(V_k) > \mu(M_i \otimes \omega)$ for $i > k$ we obtain $\phi_k = 0$. Hence ∇ preserves V_k and since ∇ has zero p-curvature, there exists a subbundle $E_k \subset E$ such that $F^*E_k = V_k$.

We now evaluate $\mu(E_k)$. By assumption $\delta_i \geq 0$ for $i < k$. Hence

$$\mu(M_i) \geq \mu(M_1) - 2(i-1)(g-1) \quad \text{for } i \leq k,$$

which implies that

$$\deg(V_k) = \sum_{i=1}^{k} r_i \mu(M_i) \geq \operatorname{rk}(V_k)\mu(M_1) - 2(g-1)\sum_{i=1}^{k} r_i(i-1).$$

Hence we obtain

$$p\mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g-1)C,$$

where C denotes the fraction $(\sum_{i=1}^{k} r_i(i-1))/\operatorname{rk}(V_k)$. We will prove in a moment that $C \leq \frac{1}{2}(p-1)$, so that we obtain by substitution

$$p\mu(E_k) \geq (2p-1)(g-1) + d - (g-1)(p-1) = p(g-1) + d = p\mu(E),$$

contradicting stability of E. Now let us show that $C \leq \frac{1}{2}(p-1)$ or equivalently

$$\sum_{i=1}^{k} i r_i \leq \tfrac{1}{2}(p+1)\sum_{i=1}^{k} r_i.$$

But that is obvious if $k \leq \frac{1}{2}(p-1)$. Now if $k > \frac{1}{2}(p-1)$ we note that passing from E to E^* reverses the order of the δ_i's, so that the index k^* for E^* satisfies $k^* \leq \frac{1}{2}(p-1)$. This proves that $\ell = p$.

Because of (6.1) we obtain $r_i = 1$ for all i and therefore $E = F_* M_p$. $\square$

7. Stability of F_*E?

Is stability also preserved by F_*?

We show the following result in that direction.

PROPOSITION 7.1. — *Let E be a stable vector bundle over X. Then F_*E is simple.*

Proof. — Using relative duality $(F_*E)^* \cong F_*(E^* \otimes \omega_X^{1-p})$ we obtain

$$H^0\big(X_1, \mathrm{End}(F_*E)\big) = H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}).$$

Moreover the Harder-Narasimhan filtration of F^*F_*E is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F^*F_*E, \quad \text{with} \quad V_i/V_{i-1} \cong E \otimes \omega_X^{p-i}.$$

We deduce that

$$H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, V_1 \otimes E^* \otimes \omega_X^{1-p}) = H^0\big(X, \mathrm{End}(E)\big),$$

and we are done. $\square$

BIBLIOGRAPHY

[1] A. BEAUVILLE – "On the stability of the direct image of a generic vector bundle", preprint available at http://math.unice.fr/~beauvill/pubs/imdir.pdf.

[2] G. FALTINGS – "Projective connections and G-bundles", *J. Algebraic Geom.* **2** (1993), p. 507–568.

[3] K. JOSHI, S. RAMANAN, E. Z. XIA & J. K. YU – "On vector bundles destabilized by Frobenius pull-back", *Compos. Math.* **142** (2006), p. 616–630.

[4] H. LANGE & C. PAULY – "On Frobenius-destabilized rank-2 vector bundles over curves", *Comm. Math. Helvetici* **83** (2008), p. 179–209.

[5] Y. LASZLO & C. PAULY – "The Frobenius map, rank 2 vector bundles and Kummer's quartic surface in characteristic 2 and 3", *Adv. Math.* **185** (2004), p. 246–269.

[6] J. Le Potier – "Module des fibrés semi-stables et fonctions thêta", in *Moduli of vector bundles (Sanda 1994, Kyoto 1994)*, Lect. Notes Pure Appl. Math., vol. 179, Dekker, New York, 1996, p. 83–101.

[7] V. B. Mehta & S. Subramanian – "Nef line bundles which are not ample", *Math. Z.* **219** (1995), p. 235–244.

[8] B. Osserman – "The generalized Verschiebung map for curves of genus 2", *Math. Ann.* **336** (2006), p. 963–986.

[9] M. Raynaud – "Sections des fibrés vectoriels sur une courbe", *Bull. Soc. Math. France* **110** (1982), p. 103–125.

[10] N. I. Shepherd-Barron – "Semistability and reduction mod p", *Topology* **37** (1998), p. 659–664.

[11] X. Sun – "Remarks on semistability of G-bundles in positive characteristic", *Compos. Math.* **119** (1999), p. 41–52.