SEMISTABILITY OF FROBENIUS DIRECT IMAGES

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SEMISTABILITY OF FROBENIUS DIRECT IMAGES  
OVER CURVES  

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1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Given a semistable vector bundle $E$ over $X$, we show that its direct image $F_*E$ under the Frobenius map $F$ of $X$ is again semistable. We deduce a numerical characterization of the stable rank-$p$ vector bundles $F_*L$, where $L$ is a line bundle over $X$. 

Résumé (Semi-stabilité des images directes par Frobenius sur les courbes)

Soit $X$ une courbe projective lisse de genre $\geq 2$ définie sur un corps $k$ algébriquement clos de caractéristique $p > 0$. Étant donné un fibré vectoriel semi-stable $E$ sur $X$, nous montrons que l’image directe $F_*E$ par le morphisme de Frobenius $F$ de $X$ est aussi semi-stable. Nous déduisons une caractérisation numérique du fibré vectoriel stable $F_*L$ de rang $p$, où $L$ est un fibré en droites sur $X$. 

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$ and let $F : X \to X_1$ be the relative $k$-linear Frobenius map. It is by now a well-established fact that on
any curve $X$ there exist semistable vector bundles $E$ such that their pull-back under the Frobenius map $F^* E$ is not semistable [4, 5]. In order to control the degree of instability of the bundle $F^* E$, one is naturally lead (through adjunction $\text{Hom}_{\mathcal{O}_X}(F^* E, E') = \text{Hom}_{\mathcal{O}_X}(E, F^* E)$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

**Theorem 1.1.** — Assume that $g \geq 2$. If $E$ is a semistable vector bundle over $X$ (of any degree), then $F_* E$ is also semistable.

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that $F^* L$ is stable for a line bundle $L$ (see [4, Proposition 1.2]) and that the bundle $F^* E$ is stable for any stable bundle $E$ of small rank (see [3]). The main ingredient of the proof is Faltings’ cohomological criterion of semistability. We also need the fact that the generalized Verschiebung $V_r$ defined as the rational map from the moduli space $M_{X_1}(r)$ of semistable rank-$r$ vector bundles over $X_1$ with fixed trivial determinant to the moduli space $M_X(r)$ induced by pull-back under the relative Frobenius map $F$,

$$V_r : M_{X_1}(r) \rightarrow M_X(r), \quad E \mapsto F^* E$$

is dominant for large $r$. We actually show a stronger statement for large $r$.

**Proposition 1.2.** — If $\ell \geq g(p-1) + 1$ and $\ell$ prime, then the generalized Verschiebung $V_\ell$ is generically étale for any curve $X$. In particular $V_\ell$ is separable and dominant.

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant $\nu$ of a vector bundle $E$, defined as

$$\nu(E) := \mu_{\text{max}}(F^* E) - \mu_{\text{min}}(F^* E),$$

where $\mu_{\text{max}}$ (resp. $\mu_{\text{min}}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of $F^* E$.

**Proposition 1.3.** — For any semistable rank-$r$ vector bundle $E$

$$\nu(E) \leq \min \left( (r - 1)(2g - 2), (p - 1)(2g - 2) \right).$$

We note that the inequality $\nu(E) \leq (r - 1)(2g - 2)$ was proved in [10, Corollary 2], and in [11, Theorem 3.1]. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (= number of pieces) of the Harder-Narasimhan filtration of $F^* E$ is at most $p$ for semistable $E$.

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant $\nu$.
Proposition 1.4.— Let $E$ be a stable rank-$p$ vector bundle over $X$. Then the following statements are equivalent.

1) There exists a line bundle $L$ such that $E = F_\ast L$.
2) $\nu(E) = (p - 1)(2g - 2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case $\mu(E) = g - 1$

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles $E$ with slope $\mu(E) = g - 1$.

Let $E$ be a semistable vector bundle over $X$ of rank $r$ and let $s$ be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}.$$ 

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \to X_1$, we obtain

$$\mu(F_\ast E) = g - 1 + \frac{s}{pr}.$$ 

Let $\pi : \tilde{X} \to X$ be a connected étale covering of degree $n$ and let $\pi_1 : \tilde{X}_1 \to X_1$ denote its twist by the Frobenius of $k$ (see [9, Section 4]). The diagram

\begin{equation}
\begin{array}{ccc}
\tilde{X} & \xrightarrow{F} & \tilde{X}_1 \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
X & \xrightarrow{F} & X_1
\end{array}
\end{equation}

(2.1)

is Cartesian and we have an isomorphism

$$\pi_1^\ast (F_\ast E) \cong F_\ast (\pi^\ast E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that $\pi^\ast E$ is semistable. Moreover if $F_\ast (\pi^\ast E)$ is semistable, then $F_\ast E$ is also semistable.

Let $L$ be a degree $d$ line bundle over $\tilde{X}_1$. The projection formula

$$F_\ast (\pi^\ast E \otimes F^\ast L) = F_\ast (\pi^\ast E) \otimes L$$

shows that semistability of $F_\ast (\pi^\ast E)$ is equivalent to semistability of

$$F_\ast (\pi^\ast E \otimes F^\ast L).$$
Let \( \tilde{g} \) denote the genus of \( \tilde{X} \). By the Riemann-Hurwitz formula, one has \( \tilde{g} - 1 = n(g - 1) \). We compute
\[
\mu(\pi^* E \otimes F^* L) = n(g - 1) + n \frac{s}{r} + pd = \tilde{g} - 1 + n \frac{s}{r} + pd,
\]
which gives
\[
\mu(F_*(\pi^* E \otimes F^* L)) = \tilde{g} - 1 + n \frac{s}{pr} + d.
\]

**Lemma 2.1.** — *For any integer \( m \) there exists a connected étale covering \( \pi : \tilde{X} \to X \) of degree \( n = p^k m \) for some \( k \).*

**Proof.** — If the \( p \)-rank of \( X \) is nonzero, the statement is clear. If the \( p \)-rank is zero, we know by \([9, \text{Corollaire 4.3.4}]\), that there exist connected étale coverings \( Y \to X \) of degree \( p^t \) for infinitely many integers \( t \) (more precisely for all \( t \) of the form \((\ell - 1)(g - 1)\) where \( \ell \) is a large prime). Now we decompose \( m = p^s u \) with \( p \) not dividing \( u \). We then take a covering \( Y \to X \) of degree \( p^t \) with \( t \geq s \) and a covering \( \tilde{X} \to Y \) of degree \( u \).

Now the lemma applied to the integer \( m = pr \) shows existence of a connected étale covering \( \pi : \tilde{X} \to X \) of degree \( n = p^k m \). Hence \( n \frac{s}{pr} \) is an integer and we can take \( d \) such that \( n \frac{s}{pr} + d = 0 \).

To summarize, we have shown that for any semistable \( E \) over \( X \) there exists a covering \( \pi : \tilde{X} \to X \) and a line bundle \( L \) over \( \tilde{X} \) such that the vector bundle \( \tilde{E} := \pi^* E \otimes F^* L \) is semistable with \( \mu(\tilde{E}) = \tilde{g} - 1 \) and such that semistability of \( F_* \tilde{E} \) implies semistability of \( F_* E \).

### 3. Proof of Theorem 1.1

In order to prove semistability of \( F_* E \) we shall use the cohomological criterion of semistability due to Faltings \([2]\).

**Proposition 3.1** (see \([6, \text{Théorème 2.4}]\)). — *Let \( E \) be a rank-\( r \) vector bundle over \( X \) with \( \mu(E) = g - 1 \) and \( \ell \) an integer \( > \frac{1}{4} r^2 (g - 1) \). Then \( E \) is semistable if and only if there exists a rank-\( \ell \) vector bundle \( G \) with trivial determinant such that*
\[
h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.
\]

Moreover if the previous condition holds for one bundle \( G \), it holds for a general bundle by upper semicontinuity of the function \( G \mapsto h^0(X, E \otimes G) \).

**Remark.** — The proof of this proposition (see \([6, \text{Section 2.4}]\)) works over any algebraically closed field \( k \).
By Proposition 1.2 (proved in Section 4) we know that $V_\ell$ is dominant when $\ell$ is a large prime number. Hence a general vector bundle $G \in \mathcal{M}_X(\ell)$ is of the form $F^*G'$ for some $G' \in \mathcal{M}_{X_1}(\ell)$. Consider a semistable $E$ with $\mu(E) = g-1$. Then by Proposition 3.1 $H^0(X, E \otimes G) = 0$ for general $G \in \mathcal{M}_X(\ell)$. Assuming $G$ general, we can write $G = F^*G'$ and we obtain by adjunction

$$H^0(X, E \otimes F^*G') = H^0(X_1, F_*E \otimes G') = 0.$$ 

This shows that $F_*E$ is semistable by Proposition 3.1.

4. Proof of Proposition 1.2

According to [7, Section 2], it will be enough to prove the existence of a stable vector bundle $E \in \mathcal{M}_{X_1}(\ell)$ satisfying $F^*E$ stable and $H^0(X_1, B \otimes \text{End}_0(E)) = 0$, because the vector space $H^0(X_1, B \otimes \text{End}_0(E))$ can be identified with the kernel of the differential of $V_\ell$ at the point $E \in \mathcal{M}_{X_1}(\ell)$. Here $B$ denotes the sheaf of locally exact differentials over $X_1$ (see [9, Section 4]).

Let $\ell \neq p$ be a prime number and let $\alpha \in JX_1[\ell] \cong JX[\ell]$ be a nonzero $\ell$-torsion point. We denote by

$$\pi : \widetilde{X} \longrightarrow X \quad \text{and} \quad \pi_1 : \widetilde{X}_1 \longrightarrow X_1$$

the associated cyclic étale cover of $X$ and $X_1$ and by $\sigma$ a generator of the Galois group $\text{Gal}(\widetilde{X}/X) = \mathbb{Z}/\ell\mathbb{Z}$. We recall that the kernel of the Norm map

$$\text{Nm} : J\widetilde{X} \longrightarrow JX$$

has $\ell$ connected components and we denote by

$$i : P := \text{Prym}(\widetilde{X}/X) \subset J\widetilde{X}$$

the associated Prym variety, i.e., the connected component containing the origin. Then we have an isogeny

$$\pi^* \times i : JX \times P \longrightarrow J\widetilde{X}$$

and taking direct image under $\pi$ induces a morphism

$$P \longrightarrow \mathcal{M}_X(\ell), \quad L \longmapsto \pi_*L.$$ 

Similarly we define the Prym variety $P_1 \subset JX_1$ and the morphism $P_1 \rightarrow \mathcal{M}_{X_1}(\ell)$ (obtained by twisting with the Frobenius of $k$). Note that $\pi_1 L$ is semistable for any $L \in P_1$ and stable for general $L \in P_1$ (see e.g., [1]). Since $F^*(\pi_1 L) \cong \pi_*(F^*L)$ — see diagram (2.1) — and since $F^*$ induces the Verschiebung $V_P : P_1 \rightarrow P$, which is surjective, we obtain that $\pi_1 L$ and $F^*(\pi_1 L)$ are stable for general $L \in P_1$. 

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Therefore Proposition 1.2 will immediately follow from the next proposition.

**Proposition 4.1.** — If \( \ell \geq g(p - 1) + 1 \) then there exists a cyclic degree \( \ell \) étale cover \( \pi_1 : \tilde{X}_1 \to X_1 \) with the property that

\[
h^0(\tilde{X}_1, B \otimes \text{End}_0(\pi_1_*L)) = 0
\]

for general \( L \in P_1 \).

**Proof.** — By relative duality for the étale map \( \pi_1 \) we have

\[
(\pi_1^*L)^* \cong \pi_1^*L - \pi_1^*L^{-1} - \pi_1^*(L^{-1} \otimes \pi_1^*\pi_1_*L)
\]

by the projection formula. Moreover since \( \pi_1 \) is Galois étale we have a direct sum decomposition

\[
\pi_1^*\pi_1_*L \cong \bigoplus_{i=0}^{\ell-1}(\sigma^i)^*L.
\]

Putting these isomorphisms together we find that

\[
H^0(\tilde{X}_1, B \otimes \text{End}_0(\pi_1_*L)) = H^0(\tilde{X}_1, B \otimes \pi_1_* \left( \bigoplus_{i=0}^{\ell-1} L^{-1} \otimes (\sigma^i)^*L \right))
\]

\[
= \bigoplus_{\ell-1} H^0(\tilde{X}_1, B \otimes \pi_1_* (L^{-1} \otimes (\sigma^i)^*L))
\]

\[
= H^0(\tilde{X}_1, B \otimes \pi_1_* \mathcal{O}_{\tilde{X}_1}) \oplus \bigoplus_{i=1}^{\ell-1} H^0(\tilde{X}_1, B \otimes \pi_1_* (L^{-1} \otimes (\sigma^i)^*L)).
\]

Moreover \( \pi_* \mathcal{O}_{\tilde{X}_1} = \bigoplus_{i=0}^{\ell-1} \alpha^i \), which implies that

\[
H^0(\tilde{X}_1, B \otimes \text{End}_0(\pi_1_*L)) = \bigoplus_{\ell-1} H^0(\tilde{X}_1, B \otimes \alpha^i)
\]

\[
\oplus \bigoplus_{i=1}^{\ell-1} H^0(\tilde{X}_1, B \otimes \pi_1_* (L^{-1} \otimes (\sigma^i)^*L)).
\]

Let us denote for \( i = 1, \ldots, \ell - 1 \) by \( \phi_i \) the isogeny

\[
\phi_i : P_1 \to P_1, \quad L \mapsto L^{-1} \otimes (\sigma^i)^*L.
\]

Since the function \( L \mapsto h^0(\tilde{X}_1, B \otimes \text{End}_0(\pi_1_*L)) \) is upper semicontinuous, it will be enough to show the existence of a cover \( \pi_1 : \tilde{X}_1 \to X_1 \) satisfying

1) for \( i = 1, \ldots, \ell - 1 \), \( h^0(\tilde{X}_1, B \otimes \alpha^i) = 0 \) (or equivalently, \( P \) is an ordinary abelian variety);

2) for \( M \) general in \( P \), \( h^0(\tilde{X}_1, B \otimes \pi_1_*M) = 0 \).
Note that these two conditions imply that the vector space (4.1) equals \{0\} for general \(L \in P_1\), because the \(\phi_i\)'s are surjective.

We recall that \(\ker (\pi_1^* : JX_1 \to J\tilde{X}_1) = \langle \alpha \rangle \cong \mathbb{Z}/\ell \mathbb{Z}\) and that

\[
P_1[\ell] = P_1 \cap \pi_1^*(JX_1) \cong \alpha^\perp/(\alpha)
\]

where \(\alpha^\perp = \{ \beta \in JX_1[\ell] \text{ with } \omega(\alpha, \beta) = 1 \}\) and \(\omega : JX_1[\ell] \times JX_1[\ell] \to \mu_\ell\) denotes the symplectic Weil form. Consider a \(\beta \in \alpha^\perp \setminus \langle \alpha \rangle\). Then \(\pi_1^*\beta \in P_1[\ell]\) and

\[
\pi_1^*\beta = \bigoplus_{i=0}^\ell \beta \otimes \alpha^i.
\]

Again by upper semicontinuity of the function \(M \mapsto h^0(X_1, B \otimes \pi_1^* M)\) one observes that conditions 1) and 2) are satisfied because of the following lemma (take \(M = \pi_1^*\beta\)).

**Lemma 4.2.** — If \(\ell \geq g(p - 1) + 1\) then there exists \((\alpha, \beta) \in JX_1[\ell] \times JX_1[\ell]\) satisfying

1. \(\alpha \neq 0\) and \(\beta \in \alpha^\perp \setminus \langle \alpha \rangle\);
2. for \(i = 1, \ldots, \ell - 1\), \(h^0(X_1, B \otimes \alpha^i) = 0\);
3. for \(i = 0, \ldots, \ell - 1\), \(h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0\).

**Proof.** — We adapt the proof of [9, Lemme 4.3.5]. We denote by \(\mathbb{F}_\ell\) the finite field \(\mathbb{Z}/\ell \mathbb{Z}\). Then there exists a symplectic isomorphism \(JX_1[\ell] \cong \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g\), where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in \(JX_1[\ell]\) and additively in \(\mathbb{F}_\ell^g\).

A quick computation shows that the number of isotropic 2-planes in \(\mathbb{F}_\ell^g \times \mathbb{F}_\ell^g\) equals

\[
N(\ell) = \frac{(\ell^g - 1)(\ell^{2g-2} - 1)}{(\ell^2 - 1)(\ell - 1)}.
\]

Let \(\Theta_B \subset JX_1\) denote the theta divisor associated to \(B\). Then by [9, Lemma 4.3.5], the cardinality \(A(\ell)\) of the finite set

\[
\Sigma(\ell) := JX_1[\ell] \cap \Theta_B
\]

satisfies

\[
A(\ell) \leq \ell^{2g-2}g(p - 1).
\]

Suppose that there exists an isotropic 2-plane \(\Pi \subset \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g\) which contains \(\leq \ell - 2\) points of \(\Sigma(\ell)\). Then we can find a pair \((\alpha, \beta)\) satisfying the three properties of the lemma as follows: any nonzero point \(x \in \Pi\) determines a line \((-\mathbb{F}_\ell\text{-vector space of dimension 1})\). Since a line contains \(\ell - 1\) nonzero points, we obtain at most \((\ell - 1)(\ell - 2)\) nonzero points lying on lines generated by \(\Sigma(\ell) \cap \Pi\). Since \((\ell - 1)(\ell - 2) < \ell^2 - 1\) there exists a nonzero \(\alpha\) in the complement of these lines. Now we note that there are \(\ell - 1\) affine lines parallel
to the line generated by $\alpha$ and the $\ell$ points on any of these affine lines are of the form $\beta \alpha_i$ for $i = 0, \ldots, \ell - 1$ for some $\beta \in \alpha^+ \setminus \langle \alpha \rangle$. The points $\Sigma(\ell) \cap \Pi$ lie on at most $\ell - 2$ such affine lines, hence there exists at least one affine line parallel to $\langle \alpha \rangle$ avoiding $\Sigma(\ell)$. This gives $\beta$.

Finally let us suppose that any isotropic 2-plane contains $\geq \ell - 1$ points of $\Sigma(\ell)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \{ (x, \Pi) \mid x \in \Pi \cap \Sigma(\ell) \text{ and } \Pi \text{ isotropic 2-plane} \}.$$ 

with cardinality $|S|$. Then by our assumption we have

$$|S| \geq (\ell - 1)N(\ell). \quad (4.2)$$

On the other hand, since any nonzero $x \in F_2^{\ell} \times F_2^{\ell}$ is contained in $(\ell^{2g-2} - 1)/(\ell - 1)$ isotropic 2-planes, we obtain

$$|S| \leq \frac{\ell^{2g-2} - 1}{\ell - 1} A(\ell). \quad (4.3)$$

Putting (4.2) and (4.3) together, we obtain

$$A(\ell) \geq \frac{\ell^{2g} - 1}{\ell + 1}.$$ 

But this contradicts the inequality $A(\ell) \leq \ell^{2g-2}g(p - 1)$ if $\ell \geq g(p - 1) + 1$. \hfill $\square$

This completes the proof of Proposition 4.1. \hfill $\square$

**Remark.** — It has been shown [8, Theorem A.6], that $V_r$ is dominant for any rank $r$ and any curve $X$, by using a versal deformation of a direct sum $r$ line bundles.

**Remark.** — We note that $V_r$ is not separable when $p$ divides the rank $r$ and $X$ is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in M_{X_1}(r)$ identifies with the quotient $H^1(X_1, \text{End}_0(E))/\langle e \rangle$ where $e$ denotes the nonzero extension class of $\text{End}_0(E)$ by $\mathcal{O}_{X_1}$ given by $\text{End}(E)$. Then the inclusion of homotheties $\mathcal{O}_{X_1} \hookrightarrow \text{End}_0(E)$ induces an inclusion $H^1(X_1, \mathcal{O}_{X_1}) \subset H^1(X_1, \text{End}_0(E))/\langle e \rangle$ and the restriction of the differential of $V_r$ at the point $E$ to $H^1(X_1, \mathcal{O}_{X_1})$ coincides with the non-injective Hasse-Witt map.
5. Proof of Proposition 1.3

Since we already know that $\nu(E) \leq (r - 1)(2g - 2)$ (see [10, 11]) it suffices to show that $\nu(E) \leq (p - 1)(2g - 2)$.

We consider the quotient $F^*E \to Q$ with minimal slope, i.e., $\mu(Q) = \mu_{\text{min}}(F^*E)$ and $Q$ semistable. By adjunction we obtain a nonzero morphism $E \to F_*(Q)$, from which we deduce (using Theorem 1.1) that

$$\mu(E) \leq \frac{1}{p} (\mu_{\text{min}}(F^*E) + (p - 1)(g - 1))$$

hence

$$\mu(F^*E) \leq \mu_{\text{min}}(F^*E) + (p - 1)(g - 1).$$

Similarly we consider the subbundle $S \to F^*E$ with maximal slope, i.e., $\mu(S) = \mu_{\text{max}}(F^*E)$ and $S$ semistable. Taking the dual and proceeding as above, we obtain that

$$\mu(F^*E) \geq \mu_{\text{max}}(F^*E) - (p - 1)(g - 1).$$

Now we combine both inequalities and we are done.

Remark. — We note that the inequality of Proposition 1.3 is sharp. The maximum $(p - 1)(2g - 2)$ is obtained for the bundles $E = F_*(E)$ (see [3, Theorem 5.3]).

6. Characterization of direct images

Consider a line bundle $L$ over $X$. Then the direct image $F_*L$ is stable (see [4], Proposition 1.2) and the Harder-Narasimhan filtration of $F^*F_*L$ is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F^*F_*L, \quad \text{with} \quad V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}.$$ 

In particular $\nu(F_*L) = (p - 1)(2g - 2)$. In this section we will show a converse statement.

More generally let $E$ be a stable rank-$rp$ vector bundle with $\mu(E) = g - 1 + \frac{d}{rp}$ for some integer $d$ and satisfying

1) the Harder-Narasimhan filtration of $F^*E$ has $\ell$ terms.

2) $\nu(E) = (p - 1)(2g - 2)$.

Questions. — Do we have $\ell \leq p$? Is $E$ of the form $E = F_*G$ for some rank-$r$ vector bundle $G$? We will give a positive answer in the case $r = 1$ (Proposition 6.1).
Let us denote the Harder-Narasimhan filtration by

\[ 0 = V_0 \subset V_1 \subset \cdots \subset V_{t-1} \subset V_t = F^* E, \quad V_i/V_{i-1} = M_i \]

satisfying the inequalities

\[ \mu_{\text{max}}(F^* E) = \mu(M_1) > \mu(M_2) > \cdots > \mu(M_t) = \mu_{\text{min}}(F^* E). \]

The quotient \( F^* E \to M_t \) gives via adjunction a nonzero map \( E \to F^* M_t \). Since \( F_t M_t \) is semistable, we obtain that \( \mu(E) \leq \mu(F^* M_t) \). This implies that \( \mu(M_t) \geq g - 1 + d/r \). Similarly taking the dual of the inclusion \( M_1 \subset F^* E \) gives a map \( F^*(E^*) \to M_1^* \) and by adjunction \( E^* \to F^*(M_1^*) \). Let us denote

\[ \mu(M_1^*) = g - 1 + \delta, \]

so that \( \mu(F^*(M_1^*)) = g - 1 + \delta/p \). Because of semistability of \( F^*(M_1^*) \), we obtain

\[ -\left( g - 1 + \frac{d}{rp} \right) = \mu(E^*) \leq \mu(F^*(M_1^*)), \]

hence \( \delta \geq -2p(g - 1) - d/r \). This implies \( \mu(M_1) \leq (2p - 1)(g - 1) + d/r \). Combining this inequality with \( \mu(M_t) \geq g - 1 + d/r \) and the assumption \( \mu(M_1) - \mu(M_t) = (p - 1)(2g - 2) \), we obtain that

\[ \mu(M_1) = (2p - 1)(g - 1) + \frac{d}{r}, \quad \mu(M_t) = g - 1 + \frac{d}{r}. \]

Let us denote by \( r_i \) the rank of the semistable bundle \( M_i \). We have the equality

\[ \sum_{i=1}^{t} r_i = rp. \]

Since \( E \) is stable and \( F^*(M_t) \) is semistable and since these bundles have the same slope, we deduce that \( r_t \geq r \). Similarly we obtain that \( r_1 \geq r \).

Note that it is enough to show that \( r_t = r \). Since \( E \) is stable and \( F_t M_t \) semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce for \( i = 1, \ldots, t - 1 \) the integers

\[ \delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g - 1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i). \]

Then we have the equality

\[ \sum_{i=1}^{t-1} \delta_i = \mu(M_t) - \mu(M_1) + 2(\ell - 1)(g - 1) = 2(\ell - p)(g - 1). \]

We note that if \( \delta_i < 0 \), then \( \text{Hom}(M_i, M_{i+1} \otimes \omega) = 0 \).
Proposition 6.1. — Let $E$ be a stable rank-$p$ vector bundle with 
\[ \mu(E) = g - 1 + d/p \quad \text{and} \quad \nu(E) = (p - 1)(2g - 2). \]
Then $E = F_*L$ for some line bundle $L$ of degree $g - 1 + d$.

Proof. — Let us first show that $\ell = p$. We suppose that $\ell < p$. Then 
\[ \sum_{i=1}^{\ell-1} \delta_i = 2(\ell - p)(g - 1) < 0 \]
so that there exists a $k \leq \ell - 1$ such that $\delta_k < 0$. We may choose $k$ minimal, i.e., $\delta_i \geq 0$ for $i < k$. Then we have
\[ (6.3) \quad \mu(M_k) > \mu(M_i) + 2(g - 1) \quad \text{for} \quad i > k. \]
We recall that $\mu(M_i) \leq \mu(M_{i+1})$ for $i > k$. The Harder-Narasimhan filtration of $V_k$ is given by the first $k$ terms of the Harder-Narasimhan filtration of $F^*E$.

Hence $\mu_{\text{min}}(V_k) = \mu(M_k)$.

Consider now the canonical connection $\nabla$ on $F^*E$ and its first fundamental form
\[ \phi_k : V_k \hookrightarrow F^*E \xrightarrow{\nabla} F^*E \otimes \omega_X \longrightarrow (F^*E/V_k) \otimes \omega_X. \]
Since $\mu_{\text{min}}(V_k) > \mu(M_i \otimes \omega)$ for $i > k$ we obtain $\phi_k = 0$. Hence $\nabla$ preserves $V_k$ and since $\nabla$ has zero $p$-curvature, there exists a subbundle $E_k \subset E$ such that $F^*E_k = V_k$.

We now evaluate $\mu(E_k)$. By assumption $\delta_i \geq 0$ for $i < k$. Hence
\[ \mu(M_i) \geq \mu(M_1) - 2(i - 1)(g - 1) \quad \text{for} \quad i \leq k, \]
which implies that 
\[ \deg(V_k) = \sum_{i=1}^{r_k} \mu(M_i) \geq \text{rk}(V_k) \mu(M_1) - 2(g - 1) \sum_{i=1}^{r_k}(i - 1). \]
Hence we obtain 
\[ p\mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g - 1)C, \]
where $C$ denotes the fraction $(\sum_{i=1}^{k} r_i(i - 1))/\text{rk}(V_k)$. We will prove in a moment that $C \leq \frac{1}{2}(p - 1)$, so that we obtain by substitution 
\[ p\mu(E_k) \geq (2p - 1)(g - 1) + d - (g - 1)(p - 1) = p(g - 1) + d = p\mu(E), \]
contradicting stability of $E$. Now let us show that $C \leq \frac{1}{2}(p - 1)$ or equivalently
\[ \sum_{i=1}^{k} i r_i \leq \frac{1}{2}(p + 1) \sum_{i=1}^{k} r_i. \]
But that is obvious if $k \leq \frac{1}{2}(p - 1)$. Now if $k > \frac{1}{2}(p - 1)$ we note that passing from $E$ to $E^*$ reverses the order of the $\delta_i$'s, so that the index $k^*$ for $E^*$ satisfies $k^* \leq \frac{1}{2}(p - 1)$. This proves that $\ell = p$.

Because of (6.1) we obtain $r_i = 1$ for all $i$ and therefore $E = F_iM_p$.

7. Stability of $F_*E$?

Is stability also preserved by $F_*$?

We show the following result in that direction.

**Proposition 7.1.** — Let $E$ be a stable vector bundle over $X$. Then $F_*E$ is simple.

**Proof.** — Using relative duality $(F_*E)^* \cong F_*(E^* \otimes \omega_X^{1-p})$ we obtain

$$H^0(X, \text{End}(F_*E)) = H^0(X, F_*F_*E \otimes E^* \otimes \omega_X^{1-p}).$$

Moreover the Harder-Narasimhan filtration of $F_*F_*E$ is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F_*F_*E, \quad \text{with} \quad V_i/V_{i-1} \cong E \otimes \omega_X^{p-i}.$$

We deduce that

$$H^0(X, F_*F_*E \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, V_1 \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, \text{End}(E)),$$

and we are done. □

**BIBLIOGRAPHY**


