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POINTED *k*-SURFACES

BY GRAHAM SMITH

ABSTRACT. — Let S be a Riemann surface. Let $\mathbb{H}^3$ be the 3-dimensional hyperbolic space and let $\partial_\infty\mathbb{H}^3$ be its ideal boundary. In our context, a Plateau problem is a locally holomorphic mapping $\varphi : S \to \partial_\infty\mathbb{H}^3 = \widehat{\mathbb{C}}$. If $i : S \to \mathbb{H}^3$ is a convex immersion, and if N is its exterior normal vector field, we define the Gauss lifting, $\hat{\imath}$, of i by $\hat{\imath} = N$. Let $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ be the Gauss-Minkowski mapping. A solution to the Plateau problem (S,φ) is a convex immersion i of constant Gaussian curvature equal to $k \in (0,1)$ such that the Gauss lifting $(S,\hat{\imath})$ is complete and $\overrightarrow{n} \circ \hat{\imath} = \varphi$. In this paper, we show that, if S is a compact Riemann surface, if $\mathcal{P}$ is a discrete subset of S and if $\varphi : S \to \widehat{\mathbb{C}}$ is a ramified covering, then, for all $p_0 \in \mathcal{P}$, the solution $(S\setminus\mathcal{P}, i)$ to the Plateau problem $(S \setminus \mathcal{P}, \varphi)$ converges asymptotically as one tends to p_0 to a cylinder wrapping a finite number, k, of times about a geodesic terminating at $\varphi(p_0)$. Moreover, k is equal to the order of ramification of φ at p_0. We also obtain a converse of this result, thus completely describing complete, constant Gaussian curvature, immersed hypersurfaces in $\mathbb{H}^3$ with cylindrical ends.

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GRAHAM SMITH, Équipe de Topologie et Dynamique, Laboratoire de Mathématiques, UMR 8628 du CNRS, Bâtiment 425, UFR des Sciences d'Orsay, 91405 Orsay Cedex (France).
E-mail : `smith@mis.mpg.de`

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Résumé (*k-surfaces à points*). — Soit S une surface de Riemann. Soit $\mathbb{H}^3$ l'espace hyperbolique de dimension 3 et soit $\partial_\infty\mathbb{H}^3$ son bord à l'infini. Dans le cadre de cet article, un problème de Plateau est une application localement holomorphe $\varphi : S \to \partial_\infty\mathbb{H}^3 = \hat{\mathbb{C}}$. Si $i : S \to \mathbb{H}^3$ est une immersion convexe, et si N est son champ de vecteurs normal, on définit $\hat{\imath}$, la relevée de Gauss de i, par $\hat{\imath} = N$. Soit $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ l'application de Gauss-Minkowski. Une solution au problème de Plateau (S,φ) est une immersion convexe i à courbure gaussienne constante égale à $k \in]0,1[$ telle que sa relevée de Gauss $(S,\hat{\imath})$ soit complète en tant que sous-variété immergée et que $\overrightarrow{n}\circ\hat{\imath} = \varphi$. Dans cet article, on montre que, si S est une surface de Riemannn compacte, si $\mathcal{P}$ est un sous-ensemble discret de S et si $\varphi : S \to \hat{\mathbb{C}}$ est un revêtement ramifié, alors, pour tout $p_0 \in \mathcal{P}$, la solution $(S \setminus \mathcal{P}, i)$ au problème de Plateau $(S \setminus \mathcal{P}, \varphi)$ converge asymptotiquement vers un cylindre qui s'enroule un nombre fini k de fois autour d'une géodésique ayant $\varphi(p_0)$ pour une de ses extrémités lorsqu'on s'approche de p_0. De plus, k est égale à l'ordre de ramification de φ en p_0. On obtient également une réciproque de ce résultat nous permettant de décrire entièrement les surfaces complètes immergées dans $\mathbb{H}^3$ à courbure gaussienne constante et aux bouts cylindriques.

1. Introduction

In this paper, by establishing a result permitting us to describe the behaviour "at infinity" of surfaces of constant Gaussian curvature immersed in 3-dimensional hyperbolic space, we obtain a complete geometric description of solutions to the Plateau problem for compact Riemann surfaces with marked points.

Let $\mathbb{H}^3$ be 3-dimensional hyperbolic space, and let $\partial_\infty\mathbb{H}^3$ be its ideal boundary (see, for example [1]). The ideal boundary of $\mathbb{H}^3$ may be identified canonically with the Riemann sphere $\widehat{\mathbb{C}}$. In this context, following [4] and [9], we define a *Plateau problem* to be a pair (S,φ) where S is a Riemann surface and $\varphi : S \to \partial_\infty\mathbb{H}^3$ is a locally conformal mapping (i.e., a locally homeomorphic holomorphic mapping). The Plateau problem (S,φ) is said to be of hyperbolic, parabolic or elliptic type depending on whether S is hyperbolic, parabolic or elliptic respectively.

Let $U\mathbb{H}^3$ be the unitary bundle over $\mathbb{H}^3$. For $i : S \to \mathbb{H}^3$ an immersion, using the canonical orientation of S, we may define the unit normal exterior vector field N over S. This field is a section of $U\mathbb{H}^3$ over i. We define the *Gauss lifting* $\hat{\imath}$ of i by $\hat{\imath} = \mathsf{N}$. We define a *$k$-surface* to be an immersed surface $\Sigma = (S,i)$ in $\mathbb{H}^3$ of constant Gaussian curvature k whose Gauss lifting $\widehat{\Sigma} = (S,\hat{\imath})$ is a complete immersed surface in $U\mathbb{H}^3$. For $k \in (0,1)$, a solution to the Plateau problem (S,φ) is a k-surface $\Sigma = (S,i)$ such that, if we denote by $\overrightarrow{n}$ the Gauss-Minkowski mapping of $\mathbb{H}^3$, then the Gauss lifting $\hat{\imath}$ of i satisfies

$$\varphi = \overrightarrow{n} \circ \hat{\imath}.$$

In [9] we show that, if (S,φ) is a hyperbolic Plateau problem, then, for all $k\in(0,1)$ there exists a unique solution i to the Plateau problem (S,φ) with constant Gaussian curvature k. Moreover, we show that i depends continuously on φ. In this paper, following on from these ideas, we study the structure of solutions to the Plateau problem (S,φ) when S is a compact Riemann surface with isolated marked points.

The following result, which provides the key to the rest of the paper, describes the behaviour "at infinity" of solutions to the Plateau problem.

THEOREM 1.1 (Boundary Behaviour Theorem). — *Let S be a hyperbolic Riemann surface and let $\varphi : S\to\widehat{\mathbb{C}}$ be a locally conformal mapping. For $k\in(0,1)$, let $i : S\to U\mathbb{H}^3$ be an immersion such that (S,i) is the unique solution to the Plateau problem (S,φ) with constant Gaussian curvature k. Let K be a compact subset of S and let Ω be a connected component of $S\setminus K$. Let q be an arbitrary point in the boundary of $\varphi(\Omega)$ that is not in $\varphi(\overline{\Omega}\cap K)$.*

If $(p_n)_{n\in\mathbb{N}}\in\Omega$ is a sequence of points such that $(\varphi(p_n))_{n\in\mathbb{N}}$ tends towards q, then the sequence $(i(p_n))_{n\in\mathbb{N}}$ also tends towards q.

REMARK. — This theorem confirms our intuition concerning solutions to the Plateau problem. In particular, if S is a Jordan domain in $\partial_\infty\mathbb{H}^3$, if φ is the canonical embedding and if $i : S\to\mathbb{H}^3$ is a solution to the Plateau problem (S,φ), then the ideal boundary of the immersed surface (S,i) coincides with ∂S.

We use this theorem to study the behaviour of solutions to the Plateau problem near to isolated singularities. We begin by a series of definitions concerning tubes about geodesics. For Γ a geodesic in $\mathbb{H}^3$, we define N_Γ to be the normal bundle over Γ in $U\mathbb{H}^3$:

$$N_\Gamma=\big\{n_p\in U\mathbb{H}^3 \text{ s.t. } p\in\Gamma,\ n_p\perp T_p\Gamma\big\}.$$

A *tube* about Γ is a pair $T=(S,\hat{\imath})$ where S is a complete surface and $\hat{\imath} : S\to N_\Gamma$ is a covering map. Since N_Γ is conformally equivalent to $S^1\times\mathbb{R}$, where S^1 is the circle of radius 1 in $\mathbb{C}$, we may assume either that $S=S^1\times\mathbb{R}$ or that $S=\mathbb{R}\times\mathbb{R}$. In the former case, $\hat{\imath}$ is a covering map of finite order, and, if k is the order of $\hat{\imath}$, then we say that the tube T is a *tube of order k*. The application $\hat{\imath}$ is then unique up to vertical translations and horizontal rotations of $S^1\times\mathbb{R}$. In the latter case, we say that T is a *tube of infinite order*. The application $\hat{\imath}$ is then unique up to translations of $\mathbb{R}\times\mathbb{R}$. In the sequel, we will only be interested in tubes of finite order.

Let S be a compact surface and let $\mathcal{P}$ be a finite set of points in S. Let $\hat{\imath} : S\setminus\mathcal{P}\to U\mathbb{H}^3$ be an immersion. Let p be an arbitrary point in $\mathcal{P}$. We say that $(S\setminus\mathcal{P},\hat{\imath})$ is *asymtotically tubular* of order k about p if and only if it is a bounded graph over a half tube of order k in $U\mathbb{H}^3$, which tends towards the tube itself as one tends towards infinity. More precisely, let $\mathrm{Exp} : TU\mathbb{H}^3\to U\mathbb{H}^3$

be the exponential mapping and let NN_Γ be the normal bundle of N_Γ. Then $(S \setminus \mathcal{P}, \hat{\imath})$ is asymptotically tubular of order k about p if there exists

(i) a geodesic Γ and a tube $T = (S^1 \times \mathbb{R}, \hat{\jmath})$ of order k about Γ,
(ii) a section λ of $\hat{\jmath}^* NN_\Gamma$ over $S^1 \times (0, \infty)$,
(iii) a neighbourhood Ω of p in S such that $\mathcal{P} \cap \Omega = \{p\}$, and
(iv) a diffeomorphism $\alpha : S^1 \times (0, \infty) \to \Omega \setminus \{p\}$,

such that

(i) $\hat{\imath} \circ \alpha = \mathrm{Exp} \circ \lambda$,
(ii) $\alpha(\mathrm{e}^{i\theta}, t) \to p$ as $t \to \infty$, and
(iii) for all $p \in \mathbb{N}$, the derivative $D^p \lambda(\mathrm{e}^{i\theta}, t)$ tends to zero as t tends to $+\infty$.

We now obtain the following result.

Theorem 1.2. — *Let S be a Riemann surface. Let $\mathcal{P}$ be a discrete subset of S such that $S \setminus \mathcal{P}$ is hyperbolic. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a ramified covering having critical points in $\mathcal{P}$. Let κ be a real number in $(0, 1)$. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be the unique solution to the Plateau problem $(S \setminus \mathcal{P}, \varphi)$ with constant Gaussian curvature κ. Let $\widehat{\Sigma} = (S \setminus \mathcal{P}, \hat{\imath})$ be the Gauss lifting of Σ. Let p_0 be an arbitrary point in $\mathcal{P}$.*

If φ has a critical point of order k at p_0, then $\widehat{\Sigma}$ is asymptotically tubular of order k at p_0.

Remark. — This means that if the mapping φ has a critical point of order k at p_0, and is thus equivalent to $z \mapsto z^k$, then the immersed surface $(S \setminus \mathcal{P}, i)$ wraps k times about a geodesic which terminates at $\varphi(p_0)$. We observe that critical points of order 1 are admitted, even though they are not, strictly speaking, critical points.

We also obtain a converse to this result:

Theorem 1.3. — *Let S be a surface and let $\mathcal{P} \subseteq S$ be a discrete subset of S. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be an immersion such that $\Sigma = (S \setminus \mathcal{P}, i)$ is a k-surface (and is thus the solution to a Plateau problem). Let $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$ be the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3$. Let $\hat{\imath}$ be the Gauss lifting of i so that $\varphi = \overrightarrow{n} \circ \hat{\imath}$ defines the Plateau problem to which i is the solution. Let $\mathcal{H}$ be the holomorphic structure generated over $S \setminus \mathcal{P}$ by the local homeomorphism φ. Let p_0 be an arbitrary point in $\mathcal{P}$, and suppose that Σ is asymptotically tubular of order k about p_0.*

Then there exists a unique holomorphic structure $\widetilde{\mathcal{H}}$ over $(S \setminus \mathcal{P}) \cup \{p_0\}$ and a unique holomorphic mapping $\tilde{\varphi} : (S \setminus \mathcal{P}) \cup \{p_0\} \to \widehat{\mathbb{C}}$ such that $\widetilde{\mathcal{H}}$ and $\tilde{\varphi}$ extend $\mathcal{H}$ and φ respectively. Moreover, $\tilde{\varphi}$ has a critical point of order k at p_0.

REMARK. — Together, these two theorems provide a complete geometric descrition of solutions to the Plateau problem (S, φ) when S is a compact Riemann surface with a finite number of marked points.

Throughout this paper, we will use the convention that $0 \notin \mathbb{N}$.

In the first section, we provide an overview of the definitions and notations that will be used in the sequel. In the second section, we study the differential geometry of the unitary bundle of a Riemannian manifold, focusing, in particular, on the canonical contact and complex structures of this bundle. In the third section, we define the Plateau problem, providing various auxiliary definitions and recalling existing results of [4] and [9] which will be required in the sequel. In the fifth section, we prove Theorem 1.1. In the sixth section, we study the geometry of the Plateau problem $(\mathbb{D}^*, z \mapsto z)$, which provides a model for the study of all other cases. In the seventh section, we prove Theorem 1.2, and in the final section we prove Theorem 1.3.

These results provoke the following reflections concerning potential future avenues of research: first, we obtain a homeomorphism between the space of meromorphic mappings over compact Riemann surfaces with a finite number of marked points on the one hand and complete positive pseudo-holomorphic curves immersed in $U\mathbb{H}^3$ with cylindrical ends on the other. These pseudo-holomorphic curves project down to surfaces of constant Gaussian curvature immersed into $\mathbb{H}^3$. Such an equivalence may well permit us to better understand the structure of either one or both of these two spaces. Second, by integrating primitives of the canonical volume form of $\mathbb{H}^3$ over these immersed surfaces, one obtains a "volume" bounded by these surfaces. If this volume can be shown to be finite, then we would obtain a new function over the Teichmüller space of compact Riemann surfaces with marked points. We would then be interested in the properties of such a function. Finally, since the reasoning employed is essentially geometric in nature, and does not appear to rely on the precise analytic structure of $\mathbb{H}^3$, it seems reasonable to expect an analogous result in the case where $\mathbb{H}^3$ is replaced by a Hadamard manifold whose curvature lies in the range $[-K, -k]$, where $K \geqslant k > 0$ are two positive real numbers.

I would like to thank François Labourie for having initially brought my attention to this problem.

2. Immersed surfaces – Definitions and notations

2.1. Definitions. — In this section we will review basic definitions from the theory of immersed submanifolds and establish the notations that will be used throughout this article.

Let M be a smooth manifold. An *immersed submanifold* is a pair $\Sigma = (S, i)$ where S is a smooth manifold and $i : S \to M$ is a smooth immersion. An *immersed hypersurface* is an immersed submanifold of codimension 1.

Let g be a Riemannian metric on M. We give S the unique Riemannian metric i^*g which makes i into an isometry. We say that Σ is *complete* if and only if the Riemannian manifold (S, i^*g) is.

2.2. Normal vector fields, second fundamental form, convexity

Let Σ be a hypersurface immersed in the Riemannian manifold M. There exists a canonical embedding i_* of the tangent bundle TS of S into the pull-back i^*TM of the tangent bundle of M. This embedding may be considered as a section of $\mathrm{End}(TS, i^*TM)$. We denote by $T\Sigma$ the image of TS under the action of this embedding.

Let us suppose that both M and S are oriented. We define $N\Sigma \subseteq i^*TM$, the *normal bundle* of Σ, by

$$N\Sigma = T\Sigma^{\perp}.$$

$N\Sigma$ is a 1-dimensional subbundle of i^*TM from which it inherits a canonical Riemannian metric. Using the orientations of S and M, we define the *exterior unit normal vector field*, $\mathsf{N}_\Sigma \in \Gamma(S, N\Sigma)$, over Σ in M. This is a global section of $N\Sigma$ which consequently trivialises this bundle. We define the *Weingarten operator*, A_Σ, which is a section of $\mathrm{End}(TS, T\Sigma)$, by

$$A_\Sigma(X) = (i^*\nabla)_X \mathsf{N}_\Sigma.$$

Since there exists a canonical isomorphism (being i_*) between $T\Sigma$ and TS, we may equally well view A_Σ as a section of $\mathrm{End}(TS)$. This section is self-adjoint with respect to the canonical Riemannian metric over S. We thus define the *second fundamental form*, II_Σ, which is a symmetric bilinear form over TS by

$$\mathrm{II}_\Sigma(X, Y) = \langle A_\Sigma X, Y\rangle.$$

Σ is said to be *convex* at $p \in S$ if and only if the bilinear form II_Σ is either positive or negative definite at p. Σ is then said to be *locally convex* if and only if it is convex at every point. Through a slight abuse of language, we will say that Σ is *convex* in this case. Bearing in mind that the sign of II_Σ depends on the sign of N_Σ, which in turn depends on the choice of orientation of S, if Σ is convex, then we may choose the orientation of S such that II_Σ is positive definite. Consequently, in the sequel, if Σ is convex, then we will assume that II_Σ is positive definite.

2.3. Curvature. — Let $\Sigma = (S, i)$ be an oriented hypersurface immersed in an oriented Riemannian manifold M. We define the *Gaussian curvature* k_Σ of Σ by

$$k_\Sigma = \mathrm{Det}(A_\Sigma).$$

In this paper, we study oriented surfaces of constant Gaussian curvature immersed into 3-dimensional hyperbolic space.

Let p be an arbitrary point in S. Let $\Sigma' = (S', i')$ be another oriented immersed hypersurface in M. We say that Σ' is *tangent* to Σ at p if there exists $p' \in S'$ such that $i(p) = i'(p')$ and

$$T_p\Sigma = T_{p'}\Sigma'.$$

We call p' a *point of tangency* of Σ' on Σ. For such a pair of points (p, p'), we may show that there exists:

(i) a neighbourhood U of p in S and a neighbourhood U' of p' in S',

(ii) a diffeomorphism $\varphi : (U, p) \to (U', p')$, and

(iii) a function $\lambda : U \to \mathbb{R}$,

such that, if $\mathsf{N} : S \to TM$ is the exterior unit normal vector field over Σ in M, and if $\mathrm{Exp} : TM \to M$ is the exponential mapping of M, then, for all $x \in U$

$$(i' \circ \varphi)(x) = \mathrm{Exp}\big(\lambda(x)\mathsf{N}(x)\big).$$

In otherwords Σ' is locally a graph over Σ near p. Moreover, since Σ' is tangent to Σ at p, we obtain

$$\mathrm{d}\lambda(p) = 0.$$

If φ' is another diffeomorphism defined in a neighbourhood of p such that $\varphi'(p) = p'$ and if λ' is another function defined in a neighbourhood of p such that $(i' \circ \varphi')(x) = \mathrm{Exp}(\lambda'(x)\mathsf{N}(x))$ for all x in a neighbourhood of p, then the pairs of functions (φ, λ) and (φ', λ') coincide in a neighbourhood of p.

If Σ' is tangent to Σ at p, then we say that Σ' is an *exterior tangent* (resp. *interior tangent*) to Σ at p if and only if $\lambda \geqslant 0$ (resp. $\lambda \leqslant 0$) in a neighbourhood of p. We now obtain the following *weak geometric maximum principle*:

LEMMA 2.1 (Weak Geometric Maximum Principle). — *Let M be an oriented manifold. Let $\Sigma = (S, i)$ and $\Sigma' = (S', i')$ be two convex oriented immersed hypersurfaces in M. Let p be a point in S and suppose that Σ' is an exterior tangent to Σ at p. Let $p' \in S'$ be a point of exterior tangency of Σ' on Σ. If $k_\Sigma(p)$ and $k_{\Sigma'}(p')$ denote the Gaussian curvatures of Σ at p and Σ' at p' respectively, we obtain*

$$k_\Sigma(p) \leqslant k_{\Sigma'}(p').$$

A proof of this result may be found in [4]. An analogous result exists when Σ' is an interior tangent to Σ at p.

2.4. Hausdorff convergence. — In the sequel, we will make use of the notion of Hausdorff convergence of sequences of compact sets contained within a given metrisable space. The following lemmata will permit us to better understand the nature of the Hausdorff topology. First, we recall a classical result which tells us that the Hausdorff topology of a compact metrisable space does not depend on the metric chosen over that space.

Lemma 2.2. — *Let X be a compact metrisable space. Let g_1 and g_2 be two metrics over X compatible with the topology of X. Let $(A_n)_{n\in\mathbb{N}}, A_0$ be compact subsets of X. The sequence $(A_n)_{n\in\mathbb{N}}$ converges to A_0 in the g_1-Hausdorff topology if and only if it converges to A_0 in the g_2-Hausdorff topology.*

In particular, the Hausdorff topology of $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ is well defined. Next, we have a result concerning the relationship between the Hausdorff topology and the topology of uniform convergence for homeomorphisms of a given compact metric space.

Lemma 2.3. — *Let (X,d) be a compact metric space. Let $(Y_n)_{n\in\mathbb{N}}$, $Y_0 \subset X$ be subsets of X such that $(Y_n)_{n\in\mathbb{N}}$ converges to Y_0 in the Hausdorff topology. Let $(\alpha_n)_{n\in\mathbb{N}}, \alpha_0$ be homeomorphisms of X such that $(\alpha_n)_{n\in\mathbb{N}}$ converges to α_0 in the compact-open topology* (*i.e., the topology of uniform convergence*)*. The sequence $(\alpha_n(Y_n))_{n\in\mathbb{N}}$ converges to $\alpha_0(Y_0)$ in the Hausdorff topology.*

Finally, we have a result concerning the intersections of two sequences of compact sets that converge.

Lemma 2.4. — *Let (X,d) be a compact metric space. Let $(A_n)_{n\in\mathbb{N}}$, $A_0 \subseteq X$ and $(B_n)_{n\in\mathbb{N}}$, $B_0 \subseteq X$ be compact sets such that $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ converge to A_0 and B_0 respectively in the Hausdorff topology. If, for all n $A_n \cap B_n \neq \varnothing$. then $A_0 \cap B_0 \neq \varnothing$.*

Proofs of Lemmata 2.3 and 2.4 may be found in appendix A of [8].

2.5. Pointed manifolds, convergence. — In the sequel, we will use the concept of Cheeger-Gromov convergence for complete pointed immersed submanifolds.

A *pointed Riemannian manifold* is a pair (M,p) where M is a Riemannnian manifold and p is a point in M. If (M,p) and (M',p') are pointed manifolds then a *mapping* from (M,p) to (M',p') is a (not necessarily even continuous) function from M to M' which sends p to $p'$and is of type C^∞ in a neighbourhood of p. In this section, we will discuss a notion of convergence for this family. It should be borne in mind that, even though this family is not a set, we may consider it as such. Indeed, since every manifold may be plunged into an infinite dimensional real vector space, we may discuss, instead, the equivalent family of pointed finite dimensional submanifolds of this vector space, and this is a set.

Let $(M_n,p_n)_{n\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds. For all n, we denote by g_n the Riemannian metric over M_n. We say that the sequence $(M_n,p_n)_{n\in\mathbb{N}}$ *converges* to the complete pointed manifold (M_0,p_0) in the *Cheeger-Gromov topology* if and only if, for all n, there exists a mapping

$\varphi_n : (M_0, p_0) \to (M_n, p_n)$ such that, for every compact subset K of M_0, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$:

(i) the restriction of φ_n to K is a C^∞-diffeomorphism onto its image, and

(ii) if we denote by g_0 the Riemannian metric over M_0, then the sequence of metrics $(\varphi_n^* g_n)_{n \geqslant N}$ converges to g_0 in the C^∞ topology over K.

We refer to the sequence $(\varphi_n)_{n\in\mathbb{N}}$ as a sequence of *convergence mappings* of the sequence $(M_n)_{n\in\mathbb{N}}$ *with respect to* the limit (M_0, p_0). The convergence mappings are trivially not unique. However, two sequences of convergence mappings $(\varphi_n)_{n\in\mathbb{N}}$ and $(\varphi'_n)_{n\in\mathbb{N}}$ are equivalent in the sense that there exists an isometry ϕ of (M_0, p_0) such that, for every compact subset K of M_0, there exists $N \in \mathbb{N}$ such that:

(i) for $n \geqslant N$, the mapping $(\varphi_n^{-1} \circ \varphi'_n)$ is well defined over K, and

(ii) the sequence $(\varphi_n^{-1} \circ \varphi'_n)_{n \geqslant N}$ converges to ϕ in the C^∞ topology over K.

One may verify that this mode of convergence does indeed arise from a topological structure over the space of complete pointed manifolds. Moreover, this topology is Hausdorff (up to isometries).

Most topological properties are unstable under this limiting process. For example, the limit of a sequence of simply connected manifolds is not necessarily simply connected. On the other hand, the limit of a sequence of surfaces of genus k is a surface of genus at most k (but quite possibly with many holes).

Let M be a complete Riemannian manifold. A *pointed immersed submanifold* in M is a pair (Σ, p) where $\Sigma = (S, i)$ is an immersed submanifold in M and p is a point in S.

Let $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, p_n, i_n)_{n\in\mathbb{N}}$ be a sequence of complete pointed immersed submanifolds in M. We say that $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ *converges* to $(\Sigma_0, p_0) = (S_0, p_0, i_0)$ in the *Cheeger-Gromov topology* if $(S_n, p_n)_{n\in\mathbb{N}}$ converges to (S_0, p_0) in the Cheeger-Gromov topology, and, for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ of convergence mappings with respect to this limit, and for every compact subset K of S_0, the sequence of functions $(i_n \circ \varphi_n)_{n \geqslant N}$ converges to the function $(i_0 \circ \varphi_0)$ in the C^∞ topology over K.

As before, this mode of convergence arises from a topological structure over the space of complete immersed submanifolds. Moreover, this topology is Hausdorff (up to isometries).

2.6. "Common sense" lemmata. — In order to make good use of the concept of Cheeger-Gromov convergence, it is helpful to recall some basic lemmata concerning the topological properties of functions acting on open subsets of $\mathbb{R}^n$. The results that follow are essentially formal expressions of "common sense". To begin with, we recall a result concerning the inverses of a sequence of functions that converges.

LEMMA 2.5. — *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}}, f_0 : \Omega \to \mathbb{R}^n$ be such that, for every n, the function f_i is a homeomorphism onto its image. Let Ω' be the image of Ω under the action of f_0. If $(f_n)_{n\in\mathbb{N}}$ converges towards f_0 locally uniformly in Ω, then the sequence $(f_n^{-1})_{n\in\mathbb{N}}$ converges towards f_0^{-1} locally uniformly in Ω'.*

To be precise, for every compact subset K in Ω', there exists $N \in \mathbb{N}$ such that, for every $n \geqslant N$, the set K is contained within $f_n(\Omega)$ and $(f_n^{-1})_{n\geqslant N}$ converges towards f_0^{-1} uniformly over K.

Moreover, if every f_n is of type C^m and if $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the C^m_{loc} topology, then $(f_n^{-1})_{n\in\mathbb{N}}$ also converges towards f_0^{-1} in the C^m_{loc} topology.

We recall a result concerning the injectivity of the limit.

LEMMA 2.6. — *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}}, f_0 : \Omega \to \mathbb{R}^n$ be such that for every $n > 0$, the function f_n is a homeomorphism onto its image. If $(f_n)_{n\in\mathbb{N}}$ tends towards f_0 locally uniformly, and if, moreover, f_0 is a local homeomorphism, then f_0 is injective.*

We recall a converse of this result for C^2 functions.

LEMMA 2.7. — *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}}$, $f_0 : \Omega \to \mathbb{R}^n$ be C^2 functions such that $(f_n)_{n\in\mathbb{N}}$ converges towards f_0 in the C^2_{loc} topology. If f_0 is a diffeomorphism onto its image, then for every compact subset K in Ω, there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$, the restriction of f_n to K is injective.*

Finally, we have a result concerning the images of a sequence of functions.

LEMMA 2.8. — *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}} : \Omega \to \mathbb{R}^n$ be such that for every n, the function f_n is a homeomorphism onto its image. If there exists a local homeomorphism $f_0 : \Omega \to \mathbb{R}^n$ such that $(f_n)_{n\in\mathbb{N}}$ converges to f_0 locally uniformly, then, for every compact subset $K \subseteq f_0(\Omega)$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, $K \subseteq f_n(\Omega)$.*

The interested reader may find a discussion and proofs of these results in the appendix A of [8].

3. The unitary bundle of a Riemannian manifold

3.1. Geometric structures over TM. — Let M be a Riemannian manifold. We define $\pi : TM \to M$ to be the canonical projection of the tangent space of M onto M. We denote by $HTM \subseteq TTM$ the *horizontal bundle* of the Levi-Civita covariant derivative of M. We denote by $VTM \subseteq TTM$ the *vertical bundle* over TM. To be precise, VTM is defined to be the kernel of the projection π within TTM. The tangent bundle of TM is the direct sum of these two subbundles:

$$TTM = HTM \oplus VTM.$$

Each of HTM and VTM is canonically isomorphic to π^*TM. We denote by i_H (resp. i_V), which is a section of $\mathrm{End}(HTM, \pi^*TM)$ (resp. $\mathrm{End}(VTM, \pi^*TM)$), the canonical isomorphism sending HTM (resp. VTM) to π^*TM. We obtain the isomorphism

$$i_H \oplus i_V : TTM \longrightarrow \pi^*TM \oplus \pi^*TM.$$

For every pair of vector fields $X, Y \in \Gamma(M, TM)$ over M we define the vector field $\{X, Y\}$ over TM such that

$$(i_H \oplus i_V)\big(\{X, Y\}\big) = (\pi^*X, \pi^*Y).$$

Trivially, every vector field over TM may be expressed (at least locally) in terms of a linear combination of such vector fields. In the same way, for a given point $p \in M$ and for a given triplet of vectors $X, Y, q \in T_pM$ over p, we may define $\{X, Y\}_q \in T_qTM$ such that

$$(i_H \oplus i_V)_q\{X, Y\}_q = (\pi_q^*X, \pi_q^*Y).$$

Finally, for a given vector field X over M, we may define

$$X^H = \{X, 0\}, \quad X^V = \{0, X\}.$$

3.2. Geometric structures over *UM*. — For M a Riemannian manifold, we define UM, the *unitary bundle* over M by

$$UM = \big\{X \in TM \text{ s.t. } \|X\| = 1\big\}.$$

We define the *tautological vector fields* T^H and T^V over the tangent space TM to M such that, for all $q \in TM$

$$T^H(q) = \{q, 0\}_q, \quad T^V(q) = \{0, q\}_q.$$

Let $i : UM \to TM$ be the canonical embedding. Let HUM (resp. VUM) be the restriction of HTM (resp. VTM) to UM:

$$HUM = i^*HTM, \quad VUM = i^*VTM.$$

The section i^*T^H (resp. i^*T^V) is nowhere vanishing. It consequently defines a 1-dimensional subbundle of HUM (resp. VUM). In order to simplify the notation we will also denote this section by T^H (resp. T^V). We denote the 1-dimensional subbundle that it generates by $\langle q\rangle_H$ (resp. $\langle q\rangle_V$). We define the subbundles $\langle q\rangle_H^{\perp}$ and $\langle q\rangle_V^{\perp}$ to be the orthogonal complements of $\langle q\rangle_H$ and $\langle q\rangle_V$ in HUM and VUM respectively.

Since parallel transport preserves the length of vectors and thus sends UM onto itself, the immersion i induces the following isomorphism of vector bundles:

$$i_* : TUM \longrightarrow HUM \oplus \langle q\rangle_V^{\perp}.$$

In order to simplify our notation, we consider HUM, $\langle q\rangle_H^\perp$ and $\langle q\rangle_V^\perp$ as subbundles of TUM. In particular, we define WUM by

$$WUM = \langle q\rangle_H^\perp \oplus \langle q\rangle_V^\perp.$$

The subbundle WUM defines, in fact, a contact structure over UM, and we will consequently refer to it as the *contact bundle* over UM. In the sequel we will denote the bundles HUM, VUM, WUM, etc. by H, V, W, etc. For $k>0$, we write $\nu=\sqrt{k}$ and we define the metric g^ν over TTM such that, for every pair of vector fields $X,Y\in\Gamma(M,TM)$ over M we have

$$g^\nu\big(\{X,Y\},\{X,Y\}\big) = \langle X,X\rangle + \nu^{-2}\langle Y,Y\rangle.$$

We denote also by g^ν the metric induced over UM by g^ν and the canonical embedding of UM into TM.

From now on, we will suppose that M is oriented and 3-dimensional. This allows us to canonically identify TM and $TM\wedge TM$ and consequently to define a vector product $\times$ over TM. We then define the canonical *complex structures* J^H (resp. J^V) over $\langle q\rangle_H^\perp$ (resp. $\langle q\rangle_V^\perp$) such that for every $q\in UM$ and for every vector X orthogonal to q:

$$J_q^H\{X,0\}_q = \{q\times X,0\}_q\,,\quad J_q^V\{0,X\}_q = \{0,q\times X\}_q\,.$$

In order to simplify notation we refer to both J^H and J^V by J. We define the isomorphism $j: HTM\to VTM$ by

$$j = i_V^{-1}\circ i_H.$$

This isometry sends $\langle q\rangle_H$ onto $\langle q\rangle_V$ and consequently $\langle q\rangle_H^\perp$ onto $\langle q\rangle_V^\perp$. Moreover, we trivially obtain the commutative diagram

$$\begin{tikzcd}
\langle q\rangle_H^\perp \arrow[r, "j"] \arrow[d, "J"] & \langle q\rangle_V^\perp \arrow[d, "J"] \\
\langle q\rangle_H^\perp \arrow[r, "j"] & \langle q\rangle_V^\perp.
\end{tikzcd}$$

We identify $\langle q\rangle_H^\perp$ and $\langle q\rangle_V^\perp$ through the isomorphism j, and we define the *complex structure* J^ν over $W=\langle q\rangle_H^\perp\oplus\langle q\rangle_V^\perp$ by

$$J^\nu = \begin{pmatrix} 0 & \nu^{-1}J \\ \nu J & 0\end{pmatrix}.$$

By composing this form with the orthogonal projection of TUM onto W, we may extend it to a form defined on TUM.

Let q be a point in UM. Let $\Sigma\subseteq W_q$ be a plane in W_q. We say that Σ is the *graph of the matrix A over $\langle q\rangle_H^\perp$* if

$$\Sigma = \big\{\{V,AV\} \text{ s.t. } V\in\langle q\rangle^\perp\big\}.$$

We say that the plane Σ is *k-complex* if and only if it is stable under the action of J^ν. In this case, if it is the graph of a matrix A, we may trivially show that

A is symmetric and of determinant equal to k. The plane Σ is then said to be *positive* if and only if it is the graph of a positive definite matrix.

3.3. Holomorphic curves, k-surfaces. — Let M be a compact oriented 3-dimensional Riemannian manifold and let $\Sigma = (S, i)$ be a convex hypersurface immersed in M. Let N_Σ be the normal exterior vector field to Σ. We define the *Gauss lifting* $\widehat{\Sigma} = (S, \hat{\imath})$ of Σ by

$$(S, \hat{\imath}) = (S, \mathsf{N}_\Sigma).$$

For $k > 0$, we say that Σ is a *k-surface* if and only if $\widehat{\Sigma}$ is complete and the Gaussian curvature of Σ is always equal to k.

We say that $\widehat{\Sigma}$ is a *k-holomorphic curve* if and only if all its tangent planes are k-complex planes, and we say that it is *positive* if and only if all its tangent planes are positive.

These concepts are related by the following elementary result:

Lemma 3.1. — *Let M be an oriented 3-dimensional Riemannian manifold. Let $\Sigma = (S, i)$ be a convex hypersurface immersed in M. Σ is a k-surface if and only if $\widehat{\Sigma}$ is a complete positive k-holomorphic curve.*

Proof. — See, for example [3]. □

We now consider the case where $M = \mathbb{H}^3$. Let $\overrightarrow{n}$ be the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3 \cong \widehat{\mathbb{C}}$. For $\widehat{\Sigma} = (S, \hat{\imath})$ a k-holomorphic curve in $U\mathbb{H}^3$, we define $\varphi : S \to \widehat{\mathbb{C}}$ by

$$\varphi = \overrightarrow{n} \circ \hat{\imath}.$$

Let $\mathcal{H}$ be the canonical holomorphic structure over $\widehat{\mathbb{C}}$. We obtain the following result:

Lemma 3.2. — *Let $\widehat{\Sigma} = (S, \hat{\imath})$ be a positive k-holomorphic curve in $U\mathbb{H}^3$. Let $\overrightarrow{n}$ be the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3 \cong \widehat{\mathbb{C}}$. Let $\mathcal{H}$ be the canonical conformal structure over $\widehat{\mathbb{C}}$. Let $\mathcal{H}'$ be the conformal structure generated over S by $\hat{\imath}^* g^\nu$ and the canonical orientation of S. The two structures $\mathcal{H}'$ and $\varphi^* \mathcal{H}$ are quasiconformally equivalent.*

Proof. — See [9]. □

4. The Plateau problem

4.1. Definitions. — A *Hadamard manifold* is a complete, connected and simply connected manifold of negative sectional curvature. The manifold $\mathbb{H}^3$ is an example of a 3-dimensional Hadamard manifold. In [4], Labourie studies the *Plateau problem* for constant Gaussian curvature hypersurfaces immersed in a 3-dimensional Hadamard manifold M. In the language of this paper, a Plateau problem is a pair (S,φ) where S is a Riemann surface and $\varphi : S \to \partial_\infty\mathbb{H}^3$ is a locally conformal mapping. A solution to this Plateau problem is an immersion $i : S \to \mathbb{H}^3$ such that the immersed hypersurface (S,i) is a k-surface and, if we denote by $\hat{\imath}$ the Gauss lifting of i and by $\overrightarrow{n}$ the Gauss-Minkowski mapping, then

$$\overrightarrow{n} \circ \hat{\imath} = \varphi.$$

4.2. Tubes, tubular surfaces, asymptotically tubular surfaces. — In this section we will define tubes about geodesics which, as will be shown in the following sections, play a special role in the study of k-surfaces.

For Γ a geodesic in $\mathbb{H}^3$, we define N_Γ to be the normal bundle over Γ in $U\mathbb{H}^3$:

$$N_\Gamma = \big\{ n_p \in U\mathbb{H}^3 \text{ s.t. } p \in \Gamma, n_p \perp T_p\Gamma \big\}.$$

A *tube* about Γ is a pair $T = (S,\hat{\imath})$ where S is a complete surface and $\hat{\imath} : S \to N_\Gamma$ is a locally conformal covering map. Since N_Γ is conformally equivalent to $S^1 \times \mathbb{R}$, where S^1 is the circle of radius 1 in $\mathbb{C}$, we may assume either that $S = S^1 \times \mathbb{R}$ or that $S = \mathbb{R} \times \mathbb{R}$. In the former case $\hat{\imath}$ is a covering map of finite order, and, if k is the order of $\hat{\imath}$, then we say that the tube T is a *tube of order k*. The application $\hat{\imath}$ is then unique up to vertical translations and horizontal rotations of $S^1 \times \mathbb{R}$. In the latter case, we say that the tube T is a *tube of infinite order*. The application $\hat{\imath}$ is then unique up to translations of $\mathbb{R} \times \mathbb{R}$. In either case, we call the point $(0,0)$ the *origin* of the tube T.

In the sequel, we will only be interested in tubes of finite order.

Let $T = (S^1 \times \mathbb{R}, \hat{\imath})$ be a tube of order k. We define the fields ∂_θ and ∂_t over T by

$$\partial_\theta(\mathrm{e}^{i\theta}, t) = \partial_\phi(\mathrm{e}^{i\theta + i\phi}, t)_{|\phi=0}, \quad \partial_t(\mathrm{e}^{i\theta}, t) = \partial_s(\mathrm{e}^{i\theta}, t+s)_{|s=0}.$$

Using the definition of g^ν, we find that every fibre of N_Γ is a circle of length $2\pi\nu^{-1}$. Consequently, since $\hat{\imath}$ is a covering map of order k, and since S^1 is of length 2π, it follows by homogeneity that

$$\|T\hat{\imath} \cdot \partial_\theta\| = k\nu^{-1}.$$

Since $\hat{\imath}$ is locally conformal, we obtain

$$\|T\hat{\imath} \cdot \partial_t\| = k\nu^{-1}.$$

Let $\mathrm{Exp} : TU\mathbb{H}^3 \to U\mathbb{H}^3$ be the exponential mapping over $U\mathbb{H}^3$. Let NN_Γ be the normal bundle over N_Γ in $TU\mathbb{H}^3$. Let $T = (S^1 \times \mathbb{R}, \hat{\imath})$ be a tube of order k about Γ. We define the normal bundle NT over T by

$$NT = \hat{\imath}^* NN_\Gamma.$$

For $r \in \mathbb{R}$ we define T_r by

$$T_r = \big(S^1 \times (-r,r), \hat{\imath}\big).$$

We define NT_r to be the restriction of NT to the set $S^1 \times (-r,r)$.

Let $(\widehat{\Sigma}, p) = (S, \hat{\imath}, p)$ be a pointed immersed surface in $U\mathbb{H}^3$. We say that $(\widehat{\Sigma}, p)$ is a *graph* over T *of half length* r if and only if there exist:

(i) a neighbourhood Ω of S about p,

(ii) a diffeomorphism $\varphi : S^1 \times (-r, r) \to \Omega$, and

(iii) a section $\lambda \in \Gamma(S^1 \times (-r, r), NT_r)$, such that $\varphi(0,0) = p$ and

$$\mathrm{Exp} \circ \lambda = \hat{\imath} \circ \varphi.$$

We call φ a *graph diffeomorphism* of $(\widehat{\Sigma}, p)$ over T_r and we call λ a *graph function* of $(\widehat{\Sigma}, p)$ over T_r.

For $\epsilon \in \mathbb{R}^+$, we define

$$N_\epsilon N_\Gamma = \big\{ v_p \in NN_\Gamma \text{ s.t. } \|v_p\| \leqslant \epsilon \big\}.$$

Since $U\mathbb{H}^3$ is homogeneous, there exists $\epsilon \in \mathbb{R}^+$ independent of Γ such that the restriction of Exp to $N_\epsilon N_\Gamma$ is a diffeomorphism onto its image. It follows that if S is a graph over a tube of finite order and of half length r with graph diffeomorphism φ and graph function λ, and if $\|\lambda\| < \epsilon$, then λ and φ are unique.

We define the *upper half tube* T_+ of T by

$$T_+ = \big(S^1 \times (0, \infty), \hat{\imath}\big).$$

We define NT_+ to be the restriction of NT to the set $S^1 \times (0, \infty)$.

Let S be surface and let p be a point in S. Let $\hat{\imath} : S \setminus \{p\} \to U\mathbb{H}^3$ be an immersion. We define the immersed surface

$$\widehat{\Sigma} = \big(S \setminus \{p\}, \hat{\imath}\big).$$

We say that $\widehat{\Sigma}$ is a *graph* over T_+ near p if and only if there exists:

(i) a neighbourhood Ω of S about p,

(ii) a diffeomorphism $\varphi : S^1 \times (0, \infty) \to \Omega \setminus \{p\}$, and

(iii) a section $\lambda \in \Gamma(S^1 \times (0, \infty), NT_+)$,

such that $\varphi(\mathrm{e}^{i\theta}, t)$ tends to p as t tends to ∞ and

$$\mathrm{Exp} \circ \lambda = \hat{\imath} \circ \varphi.$$

As before, we call φ a *graph diffeomorphism* of $\widehat{\Sigma}$ over T_+ and we call λ a *graph function* of $\widehat{\Sigma}$ over T_+. Similarly, if $\|\lambda\| < \epsilon$, then λ and φ are unique up to composition with an affine transformation of $S^1 \times \mathbb{R}$.

There exists a canonical trivialisation $\tau : NT \to (S_1 \times \mathbb{R}) \times \mathbb{R}^3$ which is unique up to composition by an endomorphism in $\mathrm{SO}(3)$. Consequently, we may interpret a graph funtion λ as a function on a subset of $S^1 \times \mathbb{R}$ taking values in $\mathbb{R}^3$. We say that $\widehat{\Sigma}$ is *asymptotically tubular of order* k about p if and only if there exists a tube T of order k such that:

(i) $\widehat{\Sigma}$ is a graph over T_+, and

(ii) if λ is a graph function of $\widehat{\Sigma}$ over T_+, then, for all $p \in \mathbb{N}\cup\{0\}$:
$$\big\|D^p\lambda(\mathrm{e}^{i\theta}, t)\big\| \longrightarrow 0 \quad \text{as } t \to +\infty.$$

4.3. The space of solutions. — We define $\mathcal{L}$ to be the set of Gauss liftings of pointed k-surfaces in $\mathbb{H}^3$:
$$\mathcal{L} = \big\{(\widehat{\Sigma}, p) \text{ s.t. } \Sigma \text{ is a } k\text{-surface in } \mathbb{H}^3,\ p \in \widehat{\Sigma}\big\}.$$
We define $\mathcal{L}_\infty$ to be the set of pointed tubes in $U\mathbb{H}^3$:
$$\mathcal{L}_\infty = \big\{(T, p) \text{ s.t. } T \text{ is a tube about a geodesic } \gamma \text{ in } \mathbb{H}^3,\ p \in T\big\}.$$
We define $\bar{\mathcal{L}}$ to be the union of these two sets:
$$\bar{\mathcal{L}} = \mathcal{L} \cup \mathcal{L}_\infty.$$
The justification for this notation will become clear presently. In [9], we proved the existence of solutions to Plateau problems of hyperbolic type. We quote Theorem 1.1 of this paper:

Theorem 4.1 (Hyperbolic Existence Theorem, [8]). — *Let $\varphi : \mathbb{D} \to \widehat{\mathbb{C}}$ be locally conformal. Then, for every $k \in (0,1)$, there exists a unique solution $i_k : \mathbb{D} \to \mathbb{H}^3$ to the Plateau problem $(\mathbb{D}, \varphi)$.*

We now introduce a definition. If $\Sigma = (S, i) \subseteq \mathbb{H}^3$ is a complete, convex, immersed surface and if N_Σ is the exterior, normal vector field over Σ, we define $\Phi : S \times (0, \infty) \to \mathbb{H}^3$ by
$$\Phi(p, t) = \mathrm{Exp}_{i(p)}\big(tN_\Sigma(p)\big).$$
Φ is everywhere a local diffeomorphism. Thus, if g is the canonical Riemannian metric over $\mathbb{H}^3$, then Φ^*g defines a complete Riemannian metric over $S\times[0,\infty)$. Moreover, this metric is trivially of constant curvature equal to -1. We refer to the Riemannian manifold with boundary $(S \times [0, \infty), \Phi^*g)$ as the *extension* of Σ.

We may now quote the following result of Labourie concerning the relationship between two solutions to the Plateau problem (Theorem 7.2.1 of [4]):

Theorem 4.2 (Structure of Solutions, [4]). — *Let (S, φ) be a Plateau problem. If there exists a solution to this problem, then it is unique. Moreover, let $\Sigma = (S, i)$ be a complete, immersed, convex surface in $\mathbb{H}^3$ of constant Gaussian curvature equal to k, and let $\widehat{\Sigma} = (S, \hat{\imath})$ be its Gauss lifting. Define φ by $\varphi = \overrightarrow{n} \circ \hat{\imath}$ such that i is the solution to the Plateau problem (S, φ). Let Ω by an open subset of S. There exists a solution to the Plateau problem (Ω, φ) which is a graph over Ω in the extension of Σ. In other words, there exists $f : \Omega \to \mathbb{R}^+$ such that the solution coincides (up to reparametrisation) with $(\Omega, \mathrm{Exp}(fN_\Sigma))$.*

Finally, as a compactness result, we use the principal result of [3], which translates into our framework as follows:

THEOREM 4.3 (Compactness, [4]). — *Let* $(\Sigma_n)_{n\in\mathbb{N}} = (S_n, i_n)_{n\in\mathbb{N}}$ *be a sequence of* k*-surfaces in* $\mathbb{H}^3$ *and, for every* n*, let* $\widehat{\Sigma}_n = (S_n, \hat{\imath}_n)$ *be the Gauss lifting of* Σ_n*. For every* n*, let* $p_n \in S_n$ *be an arbitrary point of* S_n*. If there exists a compact subset* $K \subseteq U\mathbb{H}^3$ *such that* $\hat{\imath}_n(p_n) \in K$ *for every* n*, then there exists* $(\widehat{\Sigma}_0, p_0) \in \overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{L}_\infty$ *such that, after extraction of a subsequence,* $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ *converges to* $(\widehat{\Sigma}_0, p_0)$ *in the Cheeger-Gromov topology.*

REMARK. — One may also obtain this theorem as a special case of [10].

5. The behaviour at infinity

5.1. The key result. — The following theorem provides the key to the rest of this paper:

THEOREM 1.1 (Boundary Behaviour Theorem). — *Let* S *be a hyperbolic Riemann surface and let* $\varphi : S \to \widehat{\mathbb{C}}$ *be a locally conformal mapping. For* $k \in (0, 1)$*, let* $i : S \to U\mathbb{H}^3$ *be an immersion such that* (S, i) *is the unique solution to the Plateau problem* (S, φ) *with constant Gaussian curvature* k*. Let* K *be a compact subset of* S *and let* Ω *be a connected component of* $S \setminus K$*. Let* q *be an arbitrary point in the boundary of* $\varphi(\Omega)$ *that is not in* $\varphi(\overline{\Omega} \cap K)$*.*

If $(p_n)_{n\in\mathbb{N}} \in \Omega$ *is a sequence of points such that* $(\varphi(p_n))_{n\in\mathbb{N}}$ *tends to* q*, then the sequence* $(i(p_n))_{n\in\mathbb{N}}$ *also tends towards* q*.*

Proof. — We will assume the contrary in order to obtain a contradiction. Let us denote by $\hat{\imath}$ the Gauss lifting of i. We identify $\mathbb{H}^3$ with $\mathbb{C} \times (0, \infty)$ and $\partial_\infty \mathbb{H}^3$ with $\widehat{\mathbb{C}}$. After applying an isometry of $\mathbb{H}^3$ if necessary, we may identify q with ∞. For all n, we define q_n by $q_n = \varphi(p_n)$. Let $(z_n, \lambda_n)_{n\in\mathbb{N}}$ be a sequence such that, for all n,

$$i(p_n) = (z_n, \lambda_n).$$

Since $(i(p_n))_{n\in\mathbb{N}}$ does not tend towards infinity, after extraction of a subsequence, we may assume that there exists $R > 0$ such that, for all n,

$$\|z_n, \lambda_n\| < R.$$

Let us define the sequence of isometries $(A_n)_{n\in\mathbb{N}}$ of $\mathbb{H}^3$ such that, for all n,

$$A_n(z, \lambda) = \frac{1}{\lambda_n}(z - z_n, \lambda).$$

In particular, for all n, we have

$$A_n(z_n, \lambda_n) = (1, 0).$$

For all n, we note also by A_n the automorphism of $\partial_\infty \mathbb{H}^3 = \widehat{\mathbb{C}}$ induced by the action of A_n. If we denote by $\|.\|$ the Euclidean norm over $\mathbb{C}$, we obtain

$$\big\|A_n(q_n)\big\| \geqslant \frac{1}{R}\big(\|q_n\| - R\big).$$

In particular, since $(q_n)_{n\in\mathbb{N}}$ tends to infinity, the sequence of points $(A_n(q_n))_{n\in\mathbb{N}}$ also tends to infinity. For all n, we define $i_n : S \to \mathbb{H}^3$ by

$$i_n = A_n \circ i.$$

For every n, we denote the Gauss lifting of i_n by $\hat{\imath}_n$. Since $i_n(p_n) = (0,1)$, by Labourie's compactness theorem (Theorem 4.3), there exists an immersed surface $(S_0, \hat{\imath}_0, p_0)$ in $U\mathbb{H}^3$ (which may be a tube) such that $(S, \hat{\imath}_n, p_n)_{n\in\mathbb{N}}$ tends towards this surface. Let $\overrightarrow{n}$ be the Gauss-Minkowksi mapping which sends $U\mathbb{H}^3$ to $\partial_\infty\mathbb{H}^3 = \widehat{\mathbb{C}}$. We obtain

$$\begin{aligned}(\overrightarrow{n} \circ \hat{\imath}_0)(p_0) &= \operatorname*{Lim}_{n\to\infty}(\overrightarrow{n} \circ \hat{\imath}_n)(p_n) &&= \operatorname*{Lim}_{n\to\infty}(\overrightarrow{n} \circ A_n \circ \hat{\imath})(p_n)\\ &= \operatorname*{Lim}_{n\to\infty}(A_n \circ \overrightarrow{n} \circ \hat{\imath})(p_n) &&= \operatorname*{Lim}_{n\to\infty}(A_n \circ \varphi)(p_n)\\ &= \operatorname*{Lim}_{n\to\infty} A_n(q_n) &&= \infty.\end{aligned}$$

For $p \in S$ an arbitary point, $\epsilon \in (0,\infty)$ a positive real number, and g a metric over S, we define $B_\epsilon(p;g)$ to be the ball of radius ϵ about p in S with respect to the metric g. Let us furnish S with the metric $\hat{\imath}^*g^\nu$. For all $\epsilon \in (0,\infty)$, since the surface $(S, \hat{\imath}^*g^\nu)$ is complete, there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$,

$$B_\epsilon(p_n; \hat{\imath}^*g^\nu) \subseteq \Omega.$$

Indeed, otherwise, since these balls are connected, we may assume that, for all n,

$$B_\epsilon(p_n; \hat{\imath}^*g^\nu) \cap K \neq \varnothing.$$

It thus follows that the sequence $(p_n)_{n\in\mathbb{N}}$ is contained in the ball of radius ϵ about the compact set K. Since this ball is also compact, we may assume that there exists $p_0' \in \overline{\Omega}$ such that the sequence $(p_n)_{n\in\mathbb{N}}$ converges to p_0'. By continuity $\varphi(p_0') = q$ and consequently q is either in the image of Ω or in the image of $\overline{\Omega} \cap K$. In either case, this contradicts the hypotheses on q.

For all n, let us define the metric g_n over S by

$$g_n = \hat{\imath}_n^* g^\nu = \hat{\imath}^* A_n^* g^\nu.$$

For all n, since A_n is an isometry, the metric g_n coincides with $\hat{\imath}^*g^\nu$. For all n, let us define

$$B_n = B_\epsilon(p_n; g_n).$$

We may thus assume that B_n is contained in Ω for all n, and consequently that ∞ is not in $\varphi(B_n)$. Since A_n preserves ∞, we obtain

$$\infty \notin (A_n \circ \varphi)(B_n) = (\overrightarrow{n} \circ \hat{\imath}_n)(B_n).$$

By choosing ϵ to be sufficiently small, we may assume that the restriction of $\overrightarrow{n} \circ \hat{\imath}_0$ to $B_0 = B_\epsilon(p_0; g_0)$ is a homeomorphism onto its image. Consequently, by common sense Lemma 2.7, we may assume that, for all n, the restriction of the mapping $(\overrightarrow{n} \circ \hat{\imath}_n)$ to B_n is a homeomorphism onto its image. Thus, by common sense Lemma 2.8, ∞ is not in $(\overrightarrow{n}\circ\hat{\imath}_0)(B_0)$. We thus obtain the desired contradiction and the result follows. □

6. The Geometry of the problem $(\mathbb{D}^*, z\mapsto z)$

6.1. Overview of geometric properties of the solution. — The solution to the problem $(\mathbb{D}, z\mapsto z)$ will serve as a model for the study of the general case. In this section we establish some of its geometric properties.

We identify $\mathbb{H}^3$ with $\mathbb{C}\times(0,\infty)$ and $\partial_\infty\mathbb{H}^3$ with $\widehat{\mathbb{C}}$.

We define $\mathbb{D}^*$ and $\varphi:\mathbb{D}^*\to\widehat{\mathbb{C}}$ by

$$\mathbb{D}^* = \left\{z\in\mathbb{C} \text{ s.t. } 0<|z|<1\right\} \quad\text{and}\quad \varphi(z)=z.$$

Since $\mathbb{D}^*$ is hyperbolic, by the hyperbolic existence theorem (Theorem 4.1), there exists a unique solution to the Plateau problem $(\mathbb{D}^*,\varphi)$. Let us denote this solution by $i:\mathbb{D}^*\to\mathbb{H}^3$, and let $\hat{\imath}$ be its Gauss lifting. We define the immersed surface Σ by $\Sigma=(\mathbb{D}^*, i)$.

The following result gives us a better idea of the shape of Σ:

Lemma 6.1 (First Structure Lemma). — *There exists $f:\mathbb{D}^*\to(0,\infty)$ which only depends on $r=|z|$ such that Σ coincides with the graph of f over $\mathbb{D}^*$. Moreover $f(r)$ tends towards 0 as r tends towards 0 and 1.*

This result is proven in Section 6.2. We denote by h the metric on $\mathbb{H}^3$ and we define the metric g over Σ by $g=i^*h$. By the uniqueness of solutions to the Plateau problem, g is invariant under rotations and reflections of $\mathbb{D}^*$. In Section 6.3, we prove the following result concerning g.

Lemma 6.2 (Second Structure Lemma). — *The Riemannian manifold $(\mathbb{D}^*, g)$ is complete. Moreover, if we define the vector field ∂_θ over $\mathbb{D}^*$ by*

$$\partial_\theta(r\mathrm{e}^{i\theta}) = \partial_t\, r\mathrm{e}^{i(\theta+t)}|_{t=0},$$

then $g(\partial_\theta,\partial_\theta)$ tends towards 0 as r tends to 0.

Remark. — For $r\in\,]0,1[$ we define the curve $c_r:\,]0,2\pi[\to\mathbb{D}^*$ by $c_r(\theta)=r\mathrm{e}^{i\theta}$. We define $\mathrm{Len}(c_r,g)$ to be the length of c_r with respect to the metric g. Since g is rotationally invariant, we obtain

$$\mathrm{Len}(c_r,g)=\int_0^{2\pi}\sqrt{g(\partial_\theta,\partial_\theta)}\mathrm{d}\theta = 2\pi\sqrt{g(\partial_\theta,\partial_\theta)}.$$

It follows that $g(\partial_\theta,\partial_\theta)$ tends to 0 as r tends to 0 if and only if $\mathrm{Len}(c_r,g)$ tends to 0 as r tends to 0.

We define $T=i(S)$ to be the image in $\mathbb{H}^3$ of the mapping i. Let $\overline{T}$ be the closure of T in $\mathbb{H}^3\cup\partial_\infty\mathbb{H}^3$. We obtain the following result.

LEMMA 6.3 (Third Structure Lemma). — *Let K be a compact subset of $\mathbb{H}^3$. Let $(p_n)_{n\in\mathbb{N}} \in \mathbb{H}^3$ be a sequence which converges towards $0 \in \partial_\infty\mathbb{H}^3 \cong \widehat{\mathbb{C}}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of isometries of $\mathbb{H}^3$ such that, for all n,*

$$A_n p_n \in K.$$

If, for every n, we define $\overline{T}_n$ by $\overline{T}_n = A_n\overline{T}$, then there exists $\overline{T}_0 \subseteq \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ which is either the closure in $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ of a geodesic in $\mathbb{H}^3$, or a point in $\partial\mathbb{H}^3$, such that, after extraction of a subsequence, $(\overline{T}_n)_{n\in\mathbb{N}}$ converges to $\overline{T}_0$ in the Hausdorff topology.

We define $\overline{\Gamma}_{0,\infty}$ to be the closure in $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ of the geodesic joining 0 and ∞. If, for every n, we define $\overline{\Gamma}_n$ by $\overline{\Gamma}_n = A_n\overline{\Gamma}_{0,\infty}$, then the sequence $(\overline{\Gamma}_n)_{n\in\mathbb{N}}$ also converges to $\overline{T}_0$ in the Hausdorff topology.

This result is proven in Section 6.4.

6.2. A graph over $\mathbb{D}^*$. — By symmetry of $\mathbb{D}^*$ with respect to reflections and rotations, and by the uniqueness of solutions to the Plateau problem, there exist functions $i_1 : (0,1) \to \mathbb{R}$ and $i_2 : (0,1) \to (0,\infty)$ such that

$$i(r\mathrm{e}^{i\theta}) = \big(i_1(r)\mathrm{e}^{i\theta}, i_2(r)\big).$$

Using the boundary behaviour theorem (Theorem 1.1), we obtain the following result.

PROPOSITION 6.4. — *Let S be a surface and let $\varphi : S \to \widehat{\mathbb{C}}$ be a local diffeomorphism. Let $i : S \to \mathbb{H}^3$ be an immersion such that (S,i) is the unique solution to the Plateau problem (S,φ). Let q be a point in the boundary of $\varphi(S)$. If $(p_n)_{n\in\mathbb{N}}$ is a sequence of points in S such that $(\varphi(p_n))_{n\in\mathbb{N}}$ converges to q, then $(i(p_n))_{n\in\mathbb{N}}$ also converges to q.*

Proof. — We take $K = \varnothing$ in Theorem 1.1 and the result follows. □

This permits us to establish certain properties of i_1 and i_2:

PROPOSITION 6.5. — *One has*

$$i_1(r) \to \begin{cases} 0 & \textit{as } r \to 0,\\ 1 & \textit{as } r \to 1,\end{cases} \qquad i_2(r) \to 0 \ \textit{ as } \ r \to 0,1.$$

Proof. — By the preceding proposition, $i(z) \to (0,0)$ as z tends to 0 and, for all $\theta \in [0,2\pi]$, $i(z)$ converges to $(\mathrm{e}^{i\theta},0)$ as z tends to $\mathrm{e}^{i\theta}$. The result now follows. □

Next we have the following result:

PROPOSITION 6.6. — *The function i_1 is strictly increasing.*

Proof. — Since i is smooth, the function i_1 is also. We recall that the intersection of a strictly convex surface with a geodesic consists of isolated points. Moreover, in the canonical identification of $\mathbb{H}^3$ with $\mathbb{C} \times (0, \infty)$, the vertical lines are geodesics. Thus, since Σ is a strictly convex surface of revolution, the critical points of i_1 are either strict local maxima or strict local minima (see Figure 1), and, in particular, they are isolated. We denote the Gauss-Minkowksi application that sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3$ by $\overrightarrow{n}$. We identify $\partial_\infty \mathbb{H}^3$ with $\widehat{\mathbb{C}}$ and, for $t \in (0,1)$, we find that $\overrightarrow{n} \circ \hat{\imath}(t) \in \widehat{\mathbb{R}} \subseteq \widehat{\mathbb{C}}$. Since Σ is strictly convex, Figure 1 illustrates how, if $t \in (0,1)$ is a local minimum of i_1, then

$$\overrightarrow{n} \circ \hat{\imath}(t) < i_1(t).$$

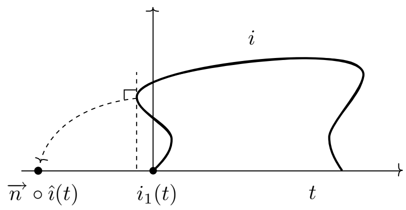

FIGURE 1

Likewise, if $t \in (0,1)$ is a local maximum of i_1, then

$$\overrightarrow{n} \circ \hat{\imath}(t) > i_1(t).$$

Since $\overrightarrow{n} \circ \hat{\imath}(t) = t \in \,]0,1[$ and since $i_1(t)$ tends to 0 and 1 as t tends to 0 and 1 respectively, the function i_1 takes values in the interval $[0,1]$. Indeed, otherwise, by compactness, there exists $t_0 \in (0,1)$ such that, either $i_1(t_0) < 0$ and t_0 is a minimum of i_1 or $i_1(t_0) > 1$ and t_0 is a maximum of i_1. In the first instance, we obtain

$$t_0 = (\overrightarrow{n} \circ \hat{\imath})(t_0) < i_1(t_0) < 0.$$

This is absurd. Likewise, the second possibility is absurd, and we thus obtain the desired contradiction.

Suppose that $t_0 \in (0,1)$ is a strict local maximum of i_1. Since $i_1(t_0) \leqslant 1$, and since $i_1(t)$ tends to 1 as t tends to 1, there exists a strict local minimum t_1 of i_1 in the open interval $(t_0, 1)$ such that $i_1(t_1) < i_1(t_0)$. However:

$$t_1 = \overrightarrow{n} \circ \hat{\imath}(t_1) < i_1(t_1) < i_1(t_0) < \overrightarrow{n} \circ \hat{\imath}(t_0) = t_0.$$

This is absurd. Consequently, there are no strict local maxima of i_1 in $(0,1)$. For the same reasons, there are no strict local minima of i_1 in $(0,1)$. Consequently i_1 does not have any critical points in $(0,1)$ and the result follows. $\square$

We may now prove the first structure lemma.

LEMMA 6.1 (First Structure Lemma). — *There exists $f : \mathbb{D}^* \to (0,\infty)$ which only depends on $r = |z|$ such that Σ coincides with the graph of f over $\mathbb{D}^*$. Moreover $f(r)$ tends towards 0 as r tends towards 0 and 1.*

Proof. — By the preceding proposition, the application i_1 does not have critical points and is thus strictly increasing and invertible. We define $\alpha : \mathbb{D}^* \to \mathbb{D}^*$ by

$$\alpha(r\mathrm{e}^{i\theta}) = i_1^{-1}(r)\mathrm{e}^{i\theta}.$$

We define $f : \mathbb{D}^* \to (0,\infty)$ by

$$f(r\mathrm{e}^{i\theta}) = i_2\big(i_1^{-1}(r)\big).$$

The mapping α is a diffeomorphism of $\mathbb{D}^*$ and:

$$(i\circ\alpha)(r\mathrm{e}^{i\theta}) = i\big(i_1^{-1}(r)\mathrm{e}^{i\theta}\big) = \big(i_1(i_1^{-1}(r))\mathrm{e}^{i\theta}, i_2(i_1^{-1}(r))\big) = \big(r\mathrm{e}^{i\theta}, f(r\mathrm{e}^{i\theta})\big).$$

The surface Σ thus coincides with the graph of f above $\mathbb{D}^*$. By definition, the function f is independent of θ, and by Proposition 6.5, $f(r)$ tends to 0 as r tends to 0 and 1. The result now follows. $\square$

6.3. The properties of i^*g. — For all $\theta \in [0,2\pi]$, we define D_θ by

$$D_\theta = \big\{z\in\mathbb{C}\,;\ |z - \tfrac{1}{2}\mathrm{e}^{i\theta}| < \tfrac{1}{2}\big\}.$$

For all θ, let i_θ be the unique solution to the Plateau problem $(D_\theta, z\mapsto z)$ with constant Gaussian curvature equal to k. For all θ, we define Σ_θ by $\Sigma_\theta = (D_\theta, i_\theta)$. The surface Σ_θ is a surface equidistant from the (Euclidian) hemisphere of radius $\frac{1}{2}$ centred on $\frac{1}{2}\mathrm{e}^{i\theta}$ in the complex plane. In fact, Σ_θ is the intersection with the upper half space of a Euclidean sphere whose centre lies in the lower half space. Let Ω_θ be the region exterior to this surface (i.e., Ω_θ is the intersection of the interior of this sphere with the upper half space). We define

$$\Omega = \bigcap_{\theta\in[0,2\pi]} \Omega_\theta.$$

We obtain the following result.

PROPOSITION 6.7. — *The surface Σ is contained in the complement of Ω.*

Proof. — Let $\theta\in[0,2\pi]$ be arbitrary. For $t\in(0,1)$ we define $D_{t,\theta}$ by

$$D_{t,\theta} = \big\{z\in\mathbb{C}\,;\ |w - \tfrac{1}{2}\mathrm{e}^{i\theta}| < \tfrac{1}{2}\big\}.$$

For all t, let $i_{t,\theta}$ be the unique solution to the Plateau problem $(D_{t,\theta}, z\mapsto z)$ with constant Gaussian curvature equal to $k_t = (1-t) + tk > k$. For all t, we define $\Sigma_{t,\theta}$ by $\Sigma_{t,\theta} = (D_{t,\theta}, i_{t,\theta})$. By considering the foliation of $\mathbb{H}^3$ defined by solutions of constant Gaussian curvature k_t to the Plateau problems given by all the discs centred on $\frac{1}{2}\mathrm{e}^{i\theta}$, using the weak geometric maximum principle (Lemma 2.1), we may show that, for all $s<t$, $\Sigma_{s,\theta}$ lies in the exterior of $\Sigma_{t,\theta}$ and so the family $(\Sigma_{t,\theta})_{t\in(0,1)}$ is a foliation of Ω_θ.

Let us denote by $\overline{\Sigma}_{t,\theta}$ the closure of the image of $\Sigma_{t,\theta}$ in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$. Since $\partial_\infty \Sigma = \partial \mathbb{D}^*$, there exists $\epsilon \in (0,1)$ such that, for all $t < \epsilon$, $\overline{\Sigma}_{t,\theta} \cap \overline{\Sigma} = \varnothing$. Let us define t_0 by

$$t_0 = \operatorname{Sup}\big\{t \text{ s.t. } \overline{\Sigma}_{s,\theta} \cap \overline{\Sigma} = \varnothing,\ \forall s \in \left]0,t\right[\big\}.$$

We aim to show that $t_0 = 1$. We will assume the contrary in order to obtain a contradiction. Since $\overline{\Sigma}_\theta$ and $\overline{\Sigma}_{t_0,\theta}$ are compact, and since the foliation is continuous, we obtain

$$\overline{\Sigma}_{t_0,\theta} \cap \overline{\Sigma} \neq \varnothing.$$

Now $\partial_\infty \Sigma_{t_0,\theta} = \partial D_{t_0,\theta} \subseteq \widehat{\mathbb{C}}$ and $\partial_\infty \Sigma = \partial \mathbb{D}^* \subseteq \widehat{\mathbb{C}}$. Thus

$$\partial_\infty \Sigma_{t_0,\theta} \cap \partial_\infty \Sigma = \varnothing \implies \Sigma_{t_0,\theta} \cap \Sigma \neq \varnothing.$$

Let p_0 be in the intersection of $\Sigma_{t_0,\theta}$ and Σ, and let us denote by $\operatorname{Ext}(\Sigma_{t_0,\theta})$ the exterior of $\Sigma_{t_0,\theta}$. We obtain

$$\operatorname{Ext}(\Sigma_{t_0,\theta}) = \bigcap_{0<t<t_0} \Sigma_{t,\theta} \implies \Sigma \cap \operatorname{Ext}(\Sigma_{t_0,\theta}) = \varnothing.$$

It follows that Σ is tangent to $\Sigma_{t_0,\theta}$ within the interior of this surface at p_0. However, since the Gaussian curvature of $\Sigma_{t_0,\theta}$ is strictly greater than that of Σ, we obtain a contradiction by the weak geometric maximum principle (Lemma 2.1). It thus follows that $t_0 = 1$ and we obtain $\Sigma \subseteq \Omega_\theta^c$. Since $\theta \in [0, 2\pi]$ is arbitrary, the result follows. □

We now obtain the following result concerning the behaviour of i and f.

Corollary 6.8. — *There exists $B \in \left]0,\infty\right[$ such that*

$$\operatorname*{LimSup}_{r\to 0} \frac{i_1(r)}{i_2(r)} \leqslant B.$$

In other words

$$\operatorname*{LimSup}_{r\to 0} \frac{r}{f(r)} \leqslant B.$$

Proof. — Since $f(r) = (i_2 \circ i_1^{-1})(r)$ and $i_1(0) = 0$, these two results are equivalent. Let $\Sigma_{1,0}$ and D_0 be as in the proof of the preceding propostion. Let $\tilde{f} : D_0 \to \left]0,\infty\right[$ be such that $\Sigma_{1,0}$ is the graph of $\tilde{f}$ over D_0. By the preceding proposition, for all $p \in D_0$,

$$f(p) \geqslant \tilde{f}(p).$$

However, by considering the restriction of $\tilde{f}$ to $(0,1)$, there exists $B \in (0,\infty)$ such that

$$\operatorname*{LimSup}_{r\to 0} \frac{r}{\tilde{f}(r)} \leqslant B.$$

The result now follows. □

Let $\Gamma \subseteq \mathbb{H}^3$ be the geodesic going from 0 to ∞. Let $\delta : \mathbb{H}^3 \to (0,\infty)$ be such that, for all r,θ and λ,

$$\delta(r\mathrm{e}^{i\theta}, \lambda) = d\big((r\mathrm{e}^{i\theta}, \lambda), \Gamma\big).$$

Since δ is invariant under the action of isometries of $\mathbb{H}^3$ which preserve Γ, we find that it only depends on r/λ. By the preceding result, there exists $B \in (0,\infty)$ such that

$$\operatorname*{LimSup}_{z\to 0} \delta\big(i(z)\big) \leqslant B.$$

We may now refine this estimate as follows:

Proposition 6.9. — $\operatorname*{Lim}_{z\to 0} \delta\big(i(z)\big) = 0.$

Proof. — Let $(p_n)_{n\in\mathbb{N}} \in \mathbb{D}^*$ be a sequence of points that converges to 0 such that

$$\delta\big(i(p_n)\big) \longrightarrow \operatorname*{LimSup}_{z\to 0} \delta\big(i(z)\big).$$

For all n, let us define $(z_n, \lambda_n)_{n\in\mathbb{N}} \in \mathbb{H}^3$ by

$$(z_n, \lambda_n) = i(p_n).$$

For all n, we define the isometry $A_n : \mathbb{H}^3 \to \mathbb{H}^3$ by

$$A_n(z,\lambda) = \frac{1}{\lambda_n}(z,\lambda).$$

In particular,

$$(A_n \circ i)(p_n) = A_n(z_n, \lambda_n) = \Big(\frac{z_n}{\lambda_n}, 1\Big).$$

Since $\operatorname{LimSup}_{n\to\infty} |z_n/\lambda_n| \leqslant B$, there exists a compact subset K of $\mathbb{H}^3$ such that, for all n,

$$(A_n \circ i)(p_n) \in K.$$

For all n, we define i_n by $i_n = A_n \circ i$ and we denote the Gauss lifting of i_n by $\hat{\imath}_n$. For all n, we define the immersed surface Σ_n by $\Sigma_n = (\mathbb{D}^*, i_n)$ and we denote the Gauss lifting of Σ_n by $\widehat{\Sigma}_n$. By Labourie's compactness theorem (Theorem 4.3), there exists a (possibly tubular) pointed immersed surface $(\widehat{\Sigma}_0, p_0) = (S_0, \hat{\imath}_0, p_0)$ in $U\mathbb{H}^3$ such that $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to $(\widehat{\Sigma}_0, p_0)$ in the Cheeger-Gromov topology.

We define $\urcorner$ by

$$\urcorner = \operatorname*{LimSup}_{z\to 0} \delta\big(i(z)\big).$$

Let $\eta \in (0,\infty)$ be an arbitrary positive real number. By definition, there exists a positive real number $\epsilon \in (0,\infty)$ such that, for $0 < |z| < \epsilon$,

$$\delta(i(z)) \leqslant \urcorner + \eta.$$

For g an arbitrary metric over $\mathbb{D}^*$, for $p \in \mathbb{D}^*$ an arbitrary point and for $R \in (0,\infty)$ an arbitrary positive real number, we define d_g to be the metric (i.e., the distance function) generated over $\mathbb{D}^*$ by g and we define $B_R(p;g)$ by

$$B_R(p;g) = \big\{q \in \mathbb{D}^* \text{ s.t. } d_g(p,q) < R\big\}.$$

For all n, since A_n is an isometry, we obtain

$$\hat{\imath}_n^* g^\nu = \hat{\imath}^* g^\nu \implies B_R(p_n;\hat{\imath}_n^* g^\nu) = B_R(p_n;\hat{\imath}^* g^\nu).$$

We fix R. Since $(\mathbb{D}^*, \hat{\imath}^* g^\nu)$ is complete, if we define D_ϵ^* by

$$D_\epsilon^* = \big\{z \in \mathbb{D}^* \text{ s.t. } |z| < \epsilon\big\},$$

then there exists a positive integer $N \in \mathbb{N}$ such that for all $n \geqslant N$:

$$B_R(p_n;\hat{\imath}^* g^\nu) \subseteq D_\epsilon^*.$$

Consequently, for $n \geqslant N$,

$$B_R(p_n;\hat{\imath}_n^* g^\nu) \subseteq D_\epsilon^* \implies \operatorname{Sup}\big\{\delta(i_n(q)) \text{ s.t. } q \in B_R(p_n;\hat{\imath}_n^* g^\nu)\big\} \leqslant \daleth + \eta.$$

Thus, after taking limits, we obtain

$$\operatorname{Sup}\{\delta(i_0(q)) \text{ s.t. } q \in B_R(p_0;\hat{\imath}_0^* g^\nu)\} \leqslant \daleth + \eta.$$

Since $\eta, R \in (0,\infty)$ are both arbitrary, we have

$$\operatorname{Sup}\{\delta(i_0(q)) \text{ s.t. } q \in S_0\} \leqslant \daleth.$$

However, by definition $\delta(i_n(p_n)) \to \daleth$ as $n \to \infty$. Consequently

$$\delta\big(i_0(p_0)\big) = \daleth.$$

We will show that $\widehat{\Sigma}_0$ is a tube. Indeed, suppose the contrary, in which case it is the Gauss lifting of a k-surface $\Sigma_0 = (S, i_0)$. The surface Σ_0 is an interior tangent at the point p_0 to the surface $\delta^{-1}(B)$. However, the Gaussian curvature of $\delta^{-1}(B)$ is equal to 1 and is thus strictly greater than k. The desired contradiction now follows by the weak geometric maximum principle (Lemma 2.1). Consequently $\widehat{\Sigma}_0$ must be a tube about a geodesic Δ.

For all $p \in \Delta$, $\delta(p) \leqslant \daleth$. Consequently, Δ remains within a fixed distance of Γ. It thus follows that Δ and Γ coincide and that, for all $p \in \Delta$, $\delta(p) = 0$. In particular, we obtain $\daleth = \delta(i_0(p_0)) = 0$. The desired result now follows. □

In particular, we obtain:

Corollary 6.10. — $\displaystyle\operatorname*{Lim}_{r\to 0} \frac{i_1(r)}{i_2(r)} = \operatorname*{Lim}_{r\to 0} \frac{r}{f(r)} = 0.$

We may now prove the second structure lemma.

LEMMA 6.2 (Second Structure Lemma). — *The Riemannian manifold* $(\mathbb{D}^*, g)$ *is complete. Moreover, if we define the vector field* ∂_θ *over* $\mathbb{D}^*$ *by*

$$\partial_\theta(r\mathrm{e}^{i\theta}) = \partial_t r\mathrm{e}^{i(\theta+t)}|_{t=0},$$

then $g(\partial_\theta, \partial_\theta)$ *tends towards* 0 *as* r *tends to* 0.

Proof. — Let h be the Riemannian metric over $\mathbb{H}^3$. By the boundary behaviour theorem (Theorem 1.1), the immersion i is proper, and thus the metric $g = i^*h$ is complete. For $r \in (0,1)$, we define the curve $c_r : (0, 2\pi) \to \mathbb{D}^*$ such that, for all θ, $c_r(\theta) = r\mathrm{e}^{i\theta}$. We have

$$\begin{aligned}(i \circ c_r)(\theta) = (i_1(r)\mathrm{e}^{i\theta}, i_2(r)) &\implies \mathrm{Len}_h(i \circ c_r) = 2\pi i_1(r)/i_2(r) \\ &\implies \mathop{\mathrm{Lim}}_{r\to 0} \mathrm{Len}_h(i \circ c_r) = 0.\end{aligned}$$

However, for all r, $\mathrm{Len}_h(i \circ c_r) = \mathrm{Len}_g(c_r)$. The result now follows. ☐

6.4. Convergence in the Hausdorff topology. — We now prove the

LEMMA 6.3 (Third Structure Lemma). — *Let* K *be a compact subset of* $\mathbb{H}^3$. *Let* $(p_n)_{n\in\mathbb{N}} \in \mathbb{H}^3$ *be a sequence which converges towards* $(0,0) \in \partial_\infty\mathbb{H}^3$. *Let* $(A_n)_{n\in\mathbb{N}}$ *be a sequence of isometries of* $\mathbb{H}^3$ *such that, for all* n,

$$A_n p_n \in K.$$

If, for every n, *we define* $\overline{T}_n$ *by* $\overline{T}_n = A_n\overline{T}$, *then there exists* $\overline{T}_0 \subseteq \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ *which is either the closure in* $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ *of a geodesic in* $\mathbb{H}^3$, *or a point in* $\partial\mathbb{H}^3$, *such that* $(\overline{T}_n)_{n\in\mathbb{N}}$ *converges to* $\overline{T}_0$ *in the Hausdorff topology. We define* $\overline{\Gamma}_{0,\infty}$ *to be the closure in* $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ *of the geodesic joining* 0 *and* ∞.

If, for every n, *we define* $\overline{\Gamma}_n$ *by* $\overline{\Gamma}_n = A_n\overline{\Gamma}_{0,\infty}$, *then the sequence* $(\overline{\Gamma}_n)_{n\in\mathbb{N}}$ *converges to* $\overline{T}_0$ *in the Hausdorff topology.*

Proof. — Let $(p_n)_{n\in\mathbb{N}} \in \mathbb{H}^3$ be a sequence in $\mathbb{H}^3$ that converges to 0. For all n, let us define (w_n, λ_n) by

$$p_n = (w_n, \lambda_n).$$

For all n, we define the isometry M_n of $\mathbb{H}^3$ by

$$M_n(z,\lambda) = \frac{1}{\lambda_n}(z,\lambda).$$

For all $R \in (0,\infty)$, we define $B_R \subseteq \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ by

$$B_R = \left\{(z,\lambda) \in \mathbb{C}\times[0,\infty) \text{ s.t. } |z|^2 + \lambda^2 \geqslant R^2\right\} \cup \{\infty\}.$$

For all $r \in (0,\infty)$ we define $C_r \subseteq \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ by

$$C_r = \left\{(z,\lambda) \in \mathbb{C}\times[0,\infty) \text{ s.t. } |z|^2 \leqslant r^2\lambda^2\right\} \cup \{\infty\}.$$

We now define the mushroom $\mathrm{Mush}_{R,r} \subseteq \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ by

$$\mathrm{Mush}_{R,r} = B_R \cup C_r.$$

The mushrooms $\mathrm{Mush}_{R,r}$ converge to $\overline{\Gamma}_{0,\infty}$ in the Hausdorff topology as R tends to infinity and r tends to 0.

Suppose that there exists $B \in (0,\infty)$ such that, for all n, $|w_n/\lambda_n| < B$. There exists a compact subset L of $\mathbb{H}^3$ such that, for all n, $M_n p_n \in L$. By the first and second structure lemmata (Lemmata 6.1 and 6.2), for all $r \in \mathbb{R}^+$, there exists $R \in \mathbb{R}^+$ such that

$$T \subseteq \mathrm{Mush}_{R,r}\,.$$

After taking a subsequence if necessary, we may thus construct sequences of positive real numbers, $(R_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}} \in (0,\infty)$ such that $(R_n/\lambda_n)_{n\in\mathbb{N}}$ tends to infinity, $(r_n)_{n\in\mathbb{N}}$ tends to 0, and, for all n,

$$\overline{T} \subseteq \mathrm{Mush}_{R_n,r_n}\,.$$

Consequently,

$$M_n\overline{T} \subseteq \mathrm{Mush}_{R_n/\lambda_n,r_n}\,.$$

It follows that $M_n\overline{T}$ converges towards $\overline{\Gamma}_{0,\infty}$ in the Hausdorff topology. For all n, we define the application B_n by $B_n = A_n M_n^{-1}$ and we obtain

$$B_n(M_n p_n) \in K.$$

However, since L and K are both compact, the family of isometries of $\mathbb{H}^3$ which send a point of L onto a point of K is also compact. It follows that, after taking a further subsequence if necessary, we may assume that there exists B_0 such that $(B_n)_{n\in\mathbb{N}}$ converges to B_0. By Lemma 2.3, it follows that $(A_n\overline{T})_{n\in\mathbb{N}} = (B_n M_n(\overline{T}))_{n\in\mathbb{N}}$ and $(A_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}} = (B_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}}$ both converge to $A_0\Gamma_{0,\infty}$ in the Hausdorff topology.

We now suppose that no such B exists. For all n, we define $\rho_n = |w_n/\lambda_n|$. After taking a subsequence if necessary, we may assume that $(\rho_n)_{n\in\mathbb{N}}$ tends to infinity. As before, after taking a further subsequence if necessary, we may construct sequences $(R_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}} \in \,]0,\infty[$ such that:

(i) the sequence $(K_n)_{n\in\mathbb{N}}$ tends to infinity,

(ii) the sequence $(r_n)_{n\in\mathbb{N}}$ tends to zero,

(iii) for all n, $R_n \geqslant K_n\,|w_n| = K_n\rho_n\lambda_n$, and

(iv) for all n, $\overline{T} \subseteq \mathrm{Mush}_{R_n,r_n} \Rightarrow M_n\overline{T} \subseteq \mathrm{Mush}_{R_n/\lambda_n,r_n}$. For all n, we define the application N_n by

$$N_n(z,\lambda) = \Big(z - \frac{w_n}{\lambda_n}, \lambda\Big).$$

For all sufficiently large n, we obtain

$$N_n\,\mathrm{Mush}_{R_n/\lambda_n,r_n} \subseteq B_{\rho_n/2}.$$

Consequently, for sufficiently large n,

$$N_nM_n\overline{T}_n,\ N_nM_n\overline{\Gamma}_{0,\infty} \subseteq B_{\rho_n/2}.$$

Since $(B_{\rho_n/2})_{n\in\mathbb{N}}$ converges towards $\{\infty\}$ in the Hausdorff topology, it follows that $(N_nM_n\overline{T}_n)_{n\in\mathbb{N}}$ and $(N_nM_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}}$ also converge to $\{\infty\}$ in the Hausdorff topology. For all n, we define $B_n = A_n(N_nM_n)^{-1}$. By following the same reasoning as before, after taking a subsequence if necessary, we may assume that there exists $p_0 \in \partial_\infty\mathbb{H}^3$ such that $(A_n\overline{T}_n)_{n\in\mathbb{N}}$ and $(A_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}}$ both converge towards $\{p_0\}$ in the Hausdorff topology, and the result follows. $\square$

7. Ramified Coverings

7.1. Introduction. — In this section we prove Theorem 1.2:

THEOREM 1.2. — *Let S be a Riemann surface. Let $\mathcal{P}$ be a discrete subset of S such that $S\setminus\mathcal{P}$ is hyperbolic. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a ramified covering having critical points in $\mathcal{P}$. Let κ be a real number in $(0,1)$. Let $i : S\setminus\mathcal{P} \to \mathbb{H}^3$ be the unique solution to the Plateau problem $(S\setminus\mathcal{P},\varphi)$ with constant Gaussian curvature κ. Let $\widehat{\Sigma} = (S\setminus\mathcal{P},\hat{\imath})$ be the Gauss lifting of Σ.*

Let p_0 be an arbitrary point in $\mathcal{P}$. If φ has a critical point of order k at p_0, then $\widehat{\Sigma}$ is asymptotically tubular of order k at p_0.

We proceed in many steps. We first prove that if $(p_n)_{n\in\mathbb{N}}$ is a sequence of points in $S\setminus\mathcal{P}$ which converges to p_0, then $(\widehat{\Sigma},p_n)_{n\in\mathbb{N}}$ converges to a tube in the Cheeger-Gromov topology. We show that this tube is necessarily of order k. We then show how convergence in the Cheeger-Gromov topology allows us to deduce that (Σ,p_n) is a graph over a tube of given finite length for all sufficiently large n. Finally, by gluing together such graphs, we obtain the desired result.

7.2. The position of $\hat{\imath}(p)$ near to ramification points. — Let S be a Riemann surface and let $\mathcal{P}\subseteq S$ be a discrete subset. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a ramified covering of S over $\widehat{\mathbb{C}}$ with critical points in $\mathcal{P}$. Let $\kappa\in(0,1)$ be a real number. Let $i : S\setminus\mathcal{P}\to\mathbb{H}^3$ be the unique solution to the Plateau problem $(S\setminus\mathcal{P},\varphi)$ with constant Gaussian curvature κ, and let us define the immersed surface Σ by $\Sigma = (S\setminus\mathcal{P}, i)$. Let $\hat{\imath}$ be the Gauss lifting of i, and let $\widehat{\Sigma} = (S\setminus\mathcal{P},\hat{\imath})$ be the Gauss lifting of Σ. Let $p_0\in\mathcal{P}$ be a ramification point of φ. Let $\Gamma_{0,\infty}$ be the unique geodesic in $\mathbb{H}^3$ going from 0 to ∞.

We begin by recalling the following result which gives a local description of ramified coverings near to ramification points:

Lemma 7.1. — *Suppose that $\varphi(p_0) = 0$. There exist a chart $(f, \Omega, \mathbb{D})$ of S about p_0, a real number $\lambda \in (0, \infty)$ and a positive integer $k \in \mathbb{N}$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[d, "f"'] \arrow[dr, "\varphi"] & \\ \mathbb{D} \arrow[r, "z \longmapsto \lambda z^k"'] & \widehat{\mathbb{C}} \end{tikzcd}$$

Remark. — This lemma is usually quoted without the factor λ. We include λ so that the image of f may be chosen to be $\mathbb{D}$, which is more convenient for our purposes.

Using the boundary behaviour theorem (Theorem 1.1), we obtain the following result.

Proposition 7.2. — *Let $p_0 \in \mathcal{P}$ be a ramification point of φ. If $(p_n)_{n\in\mathbb{N}}$ is a sequence of $S \setminus \mathcal{P}$ which converges to p_0, then $(i_n(p_n))_{n\in\mathbb{N}}$ tends towards $\varphi(p_0)$ in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$.*

Proof. — Let us denote by $2\mathbb{D}$ the disc of radius 2 about the origin in $\mathbb{C}$. By the preceding lemma, after composing φ with an isometry of $\mathbb{H}^3$ if necessary, we may assume that there exists a chart $(f, U, 2\mathbb{D})$ of $S \setminus \mathcal{P}$ about p_0 and $k \in \mathbb{N}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, "f"'] \arrow[dr, "\varphi"] & \\ 2\mathbb{D} \arrow[r, "z \longmapsto z^k"'] & \widehat{\mathbb{C}} \end{tikzcd}$$

We define the compact subset K of $S \setminus \mathcal{P}$ by $K = f^{-1}(\{|z| = 1\})$ and we define the connected component Ω of $(S \setminus \mathcal{P}) \setminus K$ by $\Omega = f^{-1}(\{0 < |z| < 1\})$. The result now follows by the boundary behaviour theorem (Theorem 1.1). $\square$

7.3. Tubes about geodesics. — We begin by controlling the geometry of Σ near to p_0. For simplicity, we will assume that $\varphi(p_0) = 0$. Let $(p_n)_{n\in\mathbb{N}} \in S \setminus \mathcal{P}$ be a sequence which tends to p_0. By identifying $\mathbb{H}^3$ with $\mathbb{C} \times (0, \infty)$, for all n, we define $(w_n, \lambda_n)_{n\in\mathbb{N}} \in \mathbb{H}^3$ by

$$(w_n, \lambda_n) = i(p_n).$$

For all n, we define the isometry A_n of $\mathbb{H}^3$ by

$$A_n(w, \lambda) = \frac{1}{\lambda_n}(w, \lambda).$$

For all n, we define the immersion i_n by

$$i_n = A_n i.$$

For all n, let $\hat{\imath}_n$ be the Gauss lifting of i_n. For all n, we define the immersed surface Σ_n by $\Sigma_n = (S \setminus \mathcal{P}, i_n)$ and we define $\widehat{\Sigma}_n = (S \setminus \mathcal{P}, \hat{\imath}_n)$ to be the Gauss lifting of Σ.

We obtain the following result:

Proposition 7.3. — *After extraction of a subsequence, $(i_n(p_n))_{n\in\mathbb{N}}$ converges towards $(0,1)$ and $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to a tube about $\Gamma_{0,\infty}$.*

Proof. — By Lemma 7.1, after applying an isometry of $\mathbb{H}^3$ if necessary, we may find a chart $(z, U, \mathbb{D})$ of S about p_0 and $k \in \mathbb{N}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, "z"'] \arrow[dr, "\varphi"] & \\ \mathbb{D} \arrow[r, "z\longmapsto z^k"'] & \widehat{\mathbb{C}} \end{tikzcd}$$

For all ℓ, we define the function $\varphi_\ell : \mathbb{D}^* \to \widehat{\mathbb{C}}$ by

$$\varphi_\ell(z) = z^\ell.$$

For all ℓ, let us define the immersion $f_\ell : \mathbb{D}^* \to \widehat{\mathbb{C}}$ to be the unique solution of the Plateau problem $(\mathbb{D}^*, \varphi_\ell)$. For all ℓ, we define $\Sigma_\ell = (\mathbb{D}^*, f_\ell)$ and we define $\mathcal{T}_\ell = f_\ell(\mathbb{D}^*)$ to be the image of f_ℓ in $\mathbb{H}^3$. Let us define T by $T = \mathcal{T}_1$. Then $\mathcal{T}_\ell$ coincides with T for all ℓ. Let $\overline{T}$ be the closure of T in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$.

By Labourie's uniqueness theorem (Theorem 4.2), the immersed surface Σ_k is a graph above $U \setminus \{p_0\}$ in the extension of Σ. By controlling the behaviour of Σ near p_0 in terms of Σ_k, we will be able to conclude.

For all n, let us define the isometry M_n of $\mathbb{H}^3$ by

$$M_n(w, \lambda) = \Big(w - \frac{w_n}{\lambda_n}, \lambda\Big).$$

In particular, for all n, $M_n(i_n(p_n)) = (0,1)$. For all n we define the immersion j_n by $j_n = M_n \circ i_n$ and we denote the Gauss lifting of j_n by $\hat{\jmath}_n$. For all n, we define the immersed surface Σ'_n by $\Sigma'_n = (S \setminus \mathcal{P}, j_n)$ and we define $\widehat{\Sigma}'_n = (S \setminus \mathcal{P}, \hat{\jmath}_n)$ to be the Gauss lifting of Σ'_n.

By Labourie's compactness theorem 4.3, after extracting a subsequence if necessary, we may assume that there exists a (possibly tubular) pointed immersed surface $(\widehat{\Sigma}'_0, \hat{p}_0) = (S_0, \hat{\jmath}_0, \hat{p}_0)$ in $U\mathbb{H}^3$ such that $(\widehat{\Sigma}'_n, p_n)_{n\in\mathbb{N}}$ converges towards $(\widehat{\Sigma}'_0, \hat{p}_0)$ in the Cheeger-Gromov topology.

For all n, we define $\overline{T}_n$ by $\overline{T}_n = M_n A_n \overline{T}$. By the Third Structure Lemma (Lemma 6.3), after extracting a subsequence if necessary, we may assume that there exists $\overline{T}_0 \subseteq \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ which is either the closure in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ of a geodesic in $\mathbb{H}^3$ or a point in $\partial_\infty \mathbb{H}^3$ such that $(\overline{T}_n)_{n\in\mathbb{N}}$ converges towards $\overline{T}_0$ in the Hausdorff topology. At the same time $(M_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}}$ converges to $\overline{T}_0$ in the Hausdorff topology.

We now show that every normal geodesic leaving $\widehat{\Sigma}_0$ intersects $\overline{T}_0$ non-trivially. This permits us to show that $\widehat{\Sigma}_0$ is a tube about a geodesic.

For M a manifold, for $p \in M$ an arbitrary point, g a Riemannian metric over M and R a positive real number, we define $B_R(p;g)$ to be the ball of radius R in M with respect to the metric g. Let q_0 be an arbitrary point in S_0. For all n, since A_n and M_n are isometries, we observe that $\hat{\jmath}_n^* g^\nu = \hat{\imath}^* g^\nu$. Let $R \in (0,\infty)$ be a positive real number such that $q_0 \in B_R(p_0;\hat{\jmath}_0^* g^\nu)$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of points in $S \setminus \mathcal{P}$ which tends to q_0 such that, for all n,

$$q_n \in B_R(p_n;\hat{\jmath}_n^* g^\nu) = B_R(p_n;\hat{\imath}^* g^\nu).$$

Since the surface $(S\setminus\mathcal{P},\hat{\imath}^* g^\nu)$ is complete, for all sufficently large n, we obtain

$$q_n \in B_R(p_n;\hat{\jmath}_n^* g^\nu) = B_R(p_n;\hat{\imath}^* g^\nu) \subseteq U \setminus \{p_0\}.$$

Consequently, for all sufficiently large n, the point q_n is in $U \setminus \{p_0\}$. Let $\mathrm{Exp} : T\mathbb{H}^3 \to \mathbb{H}^3$ be the exponential mapping over $\mathbb{H}^3$. For v_p a vector in $U\mathbb{H}^3$, we define the set $X(v_p) \subseteq \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ by

$$X(v_p) = \big\{\mathrm{Exp}(tv_p) \text{ s.t. } t \in [0,\infty]\big\}.$$

The sequence $(X(\hat{\jmath}_n(q_n)))_{n\in\mathbb{N}}$ converges towards $X(\hat{\jmath}_0(q_0))$ in the Hausdorff topology. Since Σ_k is a graph above $U \setminus \{p_0\}$ in the extension of Σ, it follows that for every n,

$$X\big(\hat{\jmath}_n(q_n)\big) \cap \overline{T}_n \neq \varnothing.$$

Consequently, by Lemma 2.4:

$$X\big(\hat{\jmath}_0(q_0)\big) \cap \overline{T}_0 \neq \varnothing$$

We will begin by showing that $\widehat{\Sigma}_0'$ is a tube. Let us assume the contrary in order to obtain a contradiction. There exists an immersion $j_0 : S_0 \to \mathbb{H}^3$ such that $\hat{\jmath}_0$ is the Gauss lifting of j_0. The immersed surface $\Sigma_0' = (S_0, j_0)$ is everywhere locally convex. Let V be a sufficently small open subset of S_0 such that the immersed surface (V, j_0) coincides with a portion of the boundary of a strictly convex subset of $\mathbb{H}^3$. Let us define the applications $\mathcal{E} : V \times [0,\infty) \to \mathbb{H}^3$ and $\mathcal{E}_\infty : V \to \partial_\infty \mathbb{H}^3$ by

$$\mathcal{E}(p,t) = \mathrm{Exp}\big(t\hat{\jmath}_0(p)\big), \quad \mathcal{E}_\infty(p) = \operatorname*{Lim}_{t\to\infty} \mathrm{Exp}\big(t\hat{\jmath}_0(p)\big).$$

Let us denote by W the set $\mathcal{E}(U \times [0,\infty))$. The application $\mathcal{E}$ defines a diffeomorphism of $V \times [0,\infty)$ onto W. Let us define $\pi_1 : V \times [0,\infty) \to V$ to be the projection onto the first coordinate, and let us define $\pi : W \to V$ by

$$\pi = \pi_1 \circ \mathcal{E}^{-1}.$$

In particular π is smooth. Let us denote by W_∞ the set $\mathcal{E}_\infty(U)$. The application $\mathcal{E}_\infty$ defines a homeomorphism of V onto W_∞.

We now have two possibilities. Either $\overline{T}_0$ is a point in $\partial_\infty\mathbb{H}^3$, or it is the closure in $\mathbb{H}^3\cup\partial_\infty\mathbb{H}^3$ of a geodesic in $\mathbb{H}^3$. If $\overline{T}_0$ is a point $\{t_0\}$ in $\partial_\infty\mathbb{H}^3$, then, since, for all $q\in V$, the intersection of $X(\hat{\jmath}_0(q))$ with $\overline{T}_0$ is non-empty, we obtain

$$\mathcal{E}_\infty(V)=\{t_0\}\,.$$

This is absurd, since $\mathcal{E}_\infty$ is a homeomorphism. Let us now assume that $\overline{T}_0$ is the closure in $\mathbb{H}^3\cup\partial_\infty\mathbb{H}^3$ of a geodesic in $\mathbb{H}^3$. Let $\gamma:\mathbb{R}\to\mathbb{H}^3$ be a parametrisation of this geodesic. Let $(I_n)_{n\in\mathbb{N}}$ be a collection of disjoint subintervals of $\mathbb{R}$ such that

$$\gamma(\mathbb{R})\cap W=\bigcap_{n\in\mathbb{N}}\gamma(I_n).$$

For every n, the application $\pi\circ\gamma$ is smooth over I_n. Consequently, if we denote by μ the 2-dimensional measure generated over V by the metric $\hat{\jmath}_0^*g^\nu$, we obtain

$$\mu\Big(\bigcap_{n\in\mathbb{N}}(\pi\circ\gamma)(I_n)\Big)=0.$$

Let us denote by $\gamma(\pm\infty)$ the boundary of the image of γ in $\mathbb{H}^3\cup\partial_\infty\mathbb{H}^3$. Since $\mathcal{E}_\infty$ is a homeomorphism, the set $\mathcal{E}_\infty^{-1}(\gamma(\pm\infty))$ consists of at most two points. Consequently

$$\mu\big(\mathcal{E}_\infty^{-1}\big(\gamma(\pm\infty)\big)\big)=0.$$

Let $q\in V$ be an arbitrary point in V. Since the intersection of $X(\hat{\jmath}_0(q))$ with T_0 is non-empty, q must be in the union of $\bigcap_{n\in\mathbb{N}}(\pi\circ\gamma)(I_n)$ with $\mu(\mathcal{E}_\infty^{-1}(\gamma(\pm\infty))$. Consequently, V is the union of these two sets and is thus of measure zero. This is absurd, and it follows that $\widehat{\Sigma}'_0$ is not the Gauss lifting of a k-surface, and is consequently a tube about a geodesic in $\mathbb{H}^3$. Let us denote this geodesic by Γ, and let us denote by $\overline{\Gamma}$ the closure of Γ in $\mathbb{H}^3\cup\partial_\infty\mathbb{H}^3$. Since every normal geodesic leaving Γ intersects $\overline{T}_0$ non-trivially, a similar reasoning permits us to conclude that T_0 coincides with Γ.

For all n, $M_n\Gamma_{0,\infty}$ is the unique vertical geodesic joining $-w_n/\lambda_n$ to ∞. Since $(M_n\overline{\Gamma}_{0,\infty})_{n\in\mathbb{N}}$ converges towards $\overline{T}_0=\overline{\Gamma}$ in the Hausdorff topology, and since $\overline{\Gamma}$ passes by $(0,1)$, we conclude that $\overline{\Gamma}$ is the unique vertical geodesic passing by $(0,1)$. Consequently

$$\overline{\Gamma}=\overline{\Gamma}_{0,\infty}.$$

It follows that $(w_n/\lambda_n)_{n\in\mathbb{N}}$ tends towards 0. Since $(w_n/\lambda_n)_{n\in\mathbb{N}}$ converges towards 0, it follows that the sequence $(M_n)_{n\in\mathbb{N}}$ converges to the identity. Consequently, the sequence of immersions $(\hat{\imath}_n)_{n\in\mathbb{N}}=(M_n^{-1}\circ\hat{\jmath}_n)_{n\in\mathbb{N}}$ converges towards $\hat{\jmath}_0$. In otherwords, the sequence of immersed surfaces $(\Sigma_n,p_n)_{n\in\mathbb{N}}$ converges in the Cheeger-Gromov topology towards (Σ'_0,p_0), which is itself a tube about $\Gamma_{0,\infty}$, and the result now follows. ☐

7.4. Tubes of finite order about geodesics. — By continuing to identify $\mathbb{H}^3$ with the upper half space $\mathbb{C} \times \mathbb{R}^+$ and $T\mathbb{H}^3$ with $(\mathbb{C} \times \mathbb{R})_{(\mathbb{C}\times\mathbb{R}^+)}$, we define $n_{0,1} \in U_{(0,1)}\mathbb{H}^3$ by

$$n_{(0,1)} = (1,0)_{(0,1)}.$$

Let $N_{0,\infty}$ be the normal circle bundle over $\Gamma_{0,\infty}$ in $\mathbb{H}^3$. Let $N_{0,\infty}(0,1)$ be the fibre above $(0,1)$. Every subsequence of $(\hat{\imath}_n(p_n))_{n\in\mathbb{N}}$ has a subsubsequence converging to a point in $N_{0,\infty}(0,1)$. It follows that there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of rotations about $\Gamma_{0,\infty}$ such that the sequence $(R_n \circ \hat{\imath}_n(p_n))_{n\in\mathbb{N}}$ converges to $n_{(0,1)}$. By replacing A_n with $R_n \circ A_n$ for all n, we may assume that the sequence $(\hat{\imath}_n(p_n))_{n\in\mathbb{N}}$ converges to $n_{(0,1)}$.

We now obtain the following stronger version of the previous result.

Proposition 7.4. — *After extracting a subsequence, $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to a tube of order k about $\Gamma_{0,\infty}$ with base point $n_{(0,1)}$.*

Proof. — Let $N_{0,\infty}$ be the normal unitary bundle over $\Gamma_{0,\infty}$ in $U\mathbb{H}^3$. The sequence $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to a tube about $\Gamma_{0,\infty}$ in the Cheeger-Gromov topology. Let $(\widehat{\Sigma}_0, \hat{p}_0) = (S_0, \hat{\imath}_0, \hat{p}_0)$ be the limit of this sequence. Since $\hat{\imath}_0 : S_0 \to N_{0,\infty}$ is a local isometry between two complete surfaces, there exists $m \in \mathbb{N} \cup \{\infty\}$ such that $\hat{\imath}_0$ is an m-fold covering of $N_{0,\infty}$. We thus aim to show that $m = k$.

As before, after applying an isometry of $\mathbb{H}^3$ if necessary, by Lemma 7.1, we may find a chart $(z, U, \mathbb{D})$ of S about p_0 and $k \in \mathbb{N}$ such that the following diagram commutes:

For all n, recalling that $i(p_n) = (w_n, \lambda_n)$, we define z_n and D_n by

$$z_n = \lambda_n^{-1/k} z, \quad D_n = \{z \in \mathbb{C} \text{ s.t. } 0 < |z| < \lambda_n^{-1/k}\}.$$

For all n, $(z_n, U \setminus \{p_0\}, D_n)$ defines a chart of $S \setminus \mathcal{P}$ such that, if $\overrightarrow{n}$ is the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3$, then, for all $p \in U \setminus \{p_0\}$:

$$\varphi_n(p) = \overrightarrow{n} \circ \hat{\imath}_n(p) = z_n(p)^k.$$

Using these charts, we will construct a sequence of pointed tubes of order k about $\Gamma_{0,\infty}$ which converges to $(\widehat{\Sigma}_0, \hat{p}_0)$ in the Cheeger-Gromov topology. The Hausdorff property of the Cheeger-Gromov topology will then permit us to conclude.

We begin by constructing a number of coordinate charts that are well adapted to our problem. To begin with, we may assume that $(S_0, \hat{\imath}_0^* g^\nu)$ is equal either to $S^1 \times \mathbb{R}$ or to $\mathbb{R}^2$, both of these spaces being furnished with the canonical Euclidean metric. Let $NN_{0,\infty}$ be the normal bundle over $N_{0,\infty}$ in $U\mathbb{H}^3$. $NN_{0,\infty}$ is trivial and there exists a canonical vector bundle isomorphism $\tau : (S^1 \times \mathbb{R}) \times \mathbb{R}^3 \to NN_{0,\infty}$ which is unique up to composition with an element of SO(3). For $\epsilon \in \mathbb{R}^+$ we define

$$N_\epsilon N_{0,\infty} = \{v \in NN_{0,\infty} \text{ s.t. } \|v\| < \epsilon\}.$$

Let $\mathrm{Exp} : NN_{0,\infty} \to U\mathbb{H}^3$ be the exponential mapping. Since $U\mathbb{H}^3$ is homogeneous, there exists $\epsilon \in \mathbb{R}^+$ such that the restriction of Exp to $N_\epsilon N_{0,\infty}$ is a diffeomorphism onto its image. We define the mapping ω by $\omega = \mathrm{Exp} \circ \tau$ and we define $\Omega \subseteq U\mathbb{H}^3$ by

$$\Omega = \omega\big((S^1 \times \mathbb{R}) \times B_\epsilon(0)\big).$$

We may assume that ω sends the origin to $\hat{\imath}_0(\hat{p}_0) = n_{(0,1)}$. The triple $(\omega^{-1}, \Omega, (S^1 \times \mathbb{R}) \times B_\epsilon(0))$ provides a coordinate chart of $U\mathbb{H}^3$ which is well adapted to our problem. Let $\pi_1 : (S^1 \times \mathbb{R}) \times B_\epsilon(0) \to S^1 \times \mathbb{R}$ be the projection onto the first factor. In the sequel, we will identify $S^1 \times \mathbb{R}$ with the zero section $(S^1 \times \mathbb{R}) \times \{0\}$ of the trivial bundle $(S^1 \times \mathbb{R}) \times \mathbb{R}^3$.

Let us denote by $\overrightarrow{n}$ the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ onto $\partial_\infty \mathbb{H}^3$. We identify $\partial_\infty \mathbb{H}^3$ with the Riemann sphere $\widehat{\mathbb{C}}$. Since the group of isometries of $\mathbb{H}^3$ which preserve $\Gamma_{0,\infty}$ acts transitively over $N_{0,\infty}$, by reducing ϵ if necessary, we may assume that

$$\overrightarrow{n}(\Omega) = \mathbb{C}^* = \widehat{\mathbb{C}} \setminus \{0, \infty\}.$$

We define $\tilde{n} : (S^1 \times \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{C}^*$ by

$$\tilde{n} = \overrightarrow{n} \circ w.$$

The application $\tilde{n}$ defines a diffeomorphism between $S^1 \times \mathbb{R}$ and $\mathbb{C}^*$. Let us denote the inverse of this mapping by $\hat{\pi}_{\mathrm{Cyl}}$.

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of convergence mappings of $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ with respect to $(\widehat{\Sigma}_0, \hat{p}_0)$. Let $R \in \mathbb{R}^+$ be a positive real number. Let $N \in \mathbb{N}$ be such that, for all $n \geqslant N$:

(i) the restriction of ψ_n to $B_{R+1}(\hat{p}_0)$ is a diffeomorphism onto its image,

(ii) $B_{R+1/2}(p_n)$ is contained in the image of $B_{R+1}(\hat{p}_0)$ under ψ_n,

(iii) the image of $B_{R+1/4}(\hat{p}_0)$ under ψ_n is contained in $B_{R+1/2}(p_n)$,

(iv) $B_{R+1/2}(p_n)$ is contained in $U \setminus \{p_0\}$, and

(v) $i_n(B_{R+1/2}(\hat{p}_0))$ is contained in Ω.

In particular, for all $n \geqslant N$, the fourth condition permits us to identify φ_n with $\alpha_k \circ z_n$ over $B_{R+1/2}(p_n)$, and the fifth condition allows us to define $\tilde{\imath}_n : B_{R+1/2}(p_n) \to (S^1 \times \mathbb{R}) \times \mathbb{R}^3$ by

$$\tilde{\imath}_n = \omega^{-1} \circ \hat{\imath}_n.$$

We define $\tilde{\imath}_0$ in a similar manner. For all $p \in \mathbb{N} \cup \{0\}$, let $\|.\|_{C^p,R}$ be a C^p norm over $B_R(\hat{p}_0)$. Since Σ_0 is a tube, $\tilde{\imath}_0$ takes values in $S^1 \times \mathbb{R}$. Consequently

$$\pi_1 \circ \tilde{\imath}_0 = \tilde{\imath}_0.$$

Thus $(\|\pi_1\circ(\tilde{\imath}_n\circ\psi_n)-(\tilde{\imath}_n\circ\psi_n)\|_{C^p,R})_{n\geqslant N}\to 0$. Applying $\tilde{n}$, we obtain

$$\big(\|\tilde{n}\circ\pi_1\circ(\tilde{\imath}_n\circ\psi_n)-\tilde{n}\circ(\tilde{\imath}_n\circ\psi_n)\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

However, for all $n\geqslant N$,

$$\varphi_n=\overrightarrow{n}\circ\hat{\imath}_n=(\overrightarrow{n}\circ\omega)\circ(\omega^{-1}\circ\hat{\imath}_n)=\tilde{n}\circ\tilde{\imath}_n.$$

Thus

$$\big(\|\tilde{n}\circ\pi_1\circ(\tilde{\imath}_n\circ\psi_n)-(\varphi_n\circ\psi_n)\|_{C^p,R}\big)_{n\geqslant N}\to 0.$$

We now apply $\hat{\pi}_{\mathrm{Cyl}}$ to obtain

$$(\|\hat{\pi}_{\mathrm{Cyl}}\circ\tilde{n}\circ\pi_1\circ(\tilde{\imath}_n\circ\psi_n)-\hat{\pi}_{\mathrm{Cyl}}\circ(\varphi_n\circ\psi_n)\|_{C^p,R})_{n\geqslant N}\longrightarrow 0.$$

Since the restriction of $\hat{\pi}_{\mathrm{Cyl}}\circ\tilde{n}$ to $S^1\times\mathbb{R}$ is equal to the identity, we obtain

$$\big(\|\pi_1\circ(\tilde{\imath}_n\circ\psi_n)-\hat{\pi}_{\mathrm{Cyl}}\circ(\varphi_n\circ\psi_n)\|_{C^p,R}\big)_{n\in\mathbb{N}}\longrightarrow 0.$$

By considering the metric (ω^*g^ν) over $(S^1\times\mathbb{R})\times B_\epsilon(0)$, the first and last of these limits permit us to obtain

$$\big(\|(\tilde{\imath}_n\circ\psi_n)^*\pi_1^*(\omega^*g^\nu)-(\tilde{\imath}_n\circ\psi_n)^*(\omega^*g^\nu)\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0,$$

$$\big(\|(\tilde{\imath}_n\circ\psi_n)^*\pi_1^*(\omega^*g^\nu)-(\varphi_n\circ\psi_n)^*\hat{\pi}_{\mathrm{Cyl}}^*(\omega^*g^\nu)\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

Combining these two limits, we obtain

$$\big(\|(\tilde{\imath}_n\circ\psi_n)^*(\omega^*g^\nu)-(\varphi_n\circ\psi_n)^*\hat{\pi}_{\mathrm{Cyl}}^*(\omega^*g^\nu)\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

Since $\hat{\imath}=\omega\circ\tilde{\imath}$, we have $\hat{\imath}^*=\tilde{\imath}^*\omega^*$. Let us define $\pi_{\mathrm{Cyl}}:\mathbb{C}^*\to N_{0,\infty}$ by

$$\pi_{\mathrm{Cyl}}=\omega\circ\hat{\pi}_{\mathrm{Cyl}}.$$

This gives us $\pi_{\mathrm{Cyl}}^*=\hat{\pi}_{\mathrm{Cyl}}^*\omega^*$. Thus

$$\big(\|(\hat{\imath}\circ\psi_n)^*g^\nu-(\varphi_n\circ\psi_n)^*\pi_{\mathrm{Cyl}}^*g^\nu\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

Consequently, since $((\hat{\imath}_n\circ\psi_n)^*g^\nu)_{n\in\mathbb{N}}$ converges to $\hat{\imath}_0^*g^\nu$, we obtain

$$\big(\|\hat{\imath}_0^*g^\nu-(\varphi_n\circ\psi_n)^*(\pi_{0,\infty})^*g^\nu\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

Let us define $\alpha_k:\mathbb{C}^*\to\mathbb{C}^*$ by

$$\alpha_k(z)=z^k.$$

We define g_k over $\mathbb{C}^*$ by

$$g_k=\alpha_k^*\pi_{\mathrm{Cyl}}^*g^\nu.$$

Since $\varphi_n=\alpha_k\circ z_n$, we have

$$\big(\|\hat{\imath}_0^*g^\nu-(z_n\circ\psi_n)^*g_k\|_{C^p,R}\big)_{n\geqslant N}\longrightarrow 0.$$

For all $n\geqslant N$, the function $(z_n\circ\psi_n)$ is defined and is smooth over $B_R(\hat{p}_0)$, and its restriction to $B_R(\hat{p}_0)$ is a diffeomorphism onto its image. Since $R\in\mathbb{R}^+$ is arbitrary, it follows that $(z_n\circ\psi_n)_{n\in\mathbb{N}}$ defines a sequence of convergence mappings for the sequence $(\mathbb{C}^*,\varphi_n(p_n),g_k)_{n\in\mathbb{N}}$ with respect to the limit $(S_0,p_0,\hat{\imath}_0^*g^\nu)$.

Since $(\varphi_n(p_n))_{n\in\mathbb{N}} = (\overrightarrow{n}\circ\hat{\imath}_n(p_n))_{n\in\mathbb{N}}$ converges to $\overrightarrow{n}(n_{0,1}) = 1$ and since the Cheeger-Gromov topology is Hausdorff, it follows that $(S_0, p_0, \hat{\imath}_0^* g^\nu)$ is isometric to $(\mathbb{C}^*, 1, g_k)$. By considering, for example, the length of the shortest homotopically non-trivial curve in $\mathbb{C}^*$, we find that, for $k \neq k'$, the manifolds $(\mathbb{C}^*, 1, g_k)$ and $(\mathbb{C}^*, 1, g_{k'})$ are not isometric. Consequently $m = k$, and the result now follows. □

Since the same result holds for every subsequence of $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$, we obtain the following stronger version of this result.

Corollary 7.5. — *$(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to a tube of order k about $\Gamma_{0,\infty}$ with base point $n_{(0,1)}$.*

Proof. — We assume the contrary. Let (T_0, p_0) be the tube of order k about $\Gamma_{0,\infty}$ with base point $n_{(0,1)}$. We may assume that there exists a neighbourhood Ω of (T_0, p_0) in the Cheeger-Gromov topology such that, after extraction of a subsequence, for all n,

$$(\widehat{\Sigma}_n, p_n) \notin \Omega.$$

However, by the preceding result, there exists a subsequence of $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ which converges towards (T_0, p_0). We thus have a contradiction and the result now follows. □

Expressing this result in terms of graphs over tubes, we obtain:

Proposition 7.6. — *Let $r \in \mathbb{R}^+$ be a positive real number. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, the pointed surface $(\widehat{\Sigma}_n, p_n)$ is locally a graph over a tube about $\Gamma_{0,\infty}$ of order k and of half length r.*

Moreover, if, for all $n \geqslant N$, we denote by λ_n the graph function of $(\widehat{\Sigma}_n, p_n)$ over $S^1\times]-r, r[$, then $(\lambda_n)_{n\geqslant N}$ converges to 0 in the C^∞_{loc} topology.

Proof. — Let $\mathrm{Exp} : TU\mathbb{H}^3 \to U\mathbb{H}^3$ be the exponential mapping over $U\mathbb{H}^3$. Let $N_{0,\infty}$ be the normal circle bundle over $\Gamma_{0,\infty}$ in $U\mathbb{H}^3$. Let $NN_{0,\infty}$ be the normal bundle over $N_{0,\infty}$ in $TU\mathbb{H}^3$. For $\epsilon \in \mathbb{R}^+$, we define $N_\epsilon N_{0,\infty}$ by

$$N_\epsilon N_{0,\infty} = \big\{v \in NN_{0,\infty} \text{ s.t. } \|v\| < \epsilon\big\}.$$

Since $U\mathbb{H}^3$ is homogeneous, there exists $\epsilon \in \mathbb{R}^+$ such that the restriction of Exp to $N_\epsilon N_{0,\infty}$ is a diffeomorphism onto its image. Let us define $U \subseteq U\mathbb{H}^3$ by $U = \mathrm{Exp}(N_\epsilon N_{0,\infty})$. Let $\pi : U \to N_{0,\infty}$ be the orthogonal projection onto $N_{0,\infty}$.

Let $T = (S^1 \times \mathbb{R}, \hat{\imath}_0)$ be a tube of order k about $\Gamma_{0,\infty}$. Let $\hat{p}_0$ be the origin of T. By the preceding result, we may assume that $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges towards $(T, \hat{p}_0)$ in the Cheeger-Gromov topology. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of convergence mappings for $(\widehat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ with respect to $(T, \hat{p}_0)$.

Let us define the application $\alpha : (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R}) \to (S^1 \times \mathbb{R}) \times N_{0,\infty}$ by

$$\alpha(x, y) = \big(x, \hat{\imath}_0(y)\big).$$

Let Δ be the diagonal in $(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$:

$$\Delta = \big\{(x,x) \text{ s.t. } x \in (S^1 \times \mathbb{R})\big\}.$$

For all $\rho \in \mathbb{R}^+$, let $B_\rho(\Delta)$ be the tubular neighbourhood of radius ρ about Δ in $(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$. Since α is a local diffeomorphism, it follows by homogeneity that there exists $\rho \in \mathbb{R}^+$ such that the restriction of α to $B_\rho(\Delta)$ is a diffeomorphism onto its image. We will use this mapping to unravel other mappings that wrap k times round $N_{0,\infty}$. We define $V \subseteq (S^1 \times \mathbb{R}) \times N_{0,\infty}$ by

$$V = \alpha\big(B_\rho(\Delta)\big).$$

Let $\pi_1, \pi_2 : (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R}) \to S^1 \times \mathbb{R}$ be the projections onto the first and second factors respectively.

Let $R > 0$ be a positive real number such that $T_{4r} \subseteq B_R(\hat{p}_0) \subseteq T$. Let $N_1 \in \mathbb{N}$ be such that for all $n \geqslant N_1$:

(i) the restriction of φ_n to $B_{R+1}(\hat{p}_0)$ is a diffeomorphism onto its image,

(ii) $(\hat{\imath}_n \circ \varphi_n)(B_{R+1}(\hat{p}_0))$ is contained within U, and

(iii) $(x, \pi \circ \hat{\imath}_n \circ \varphi_n(x)) \in V$ for all $x \in B_{R+1}(\hat{p}_0)$.

For all $n \geqslant N_1$, we define $\beta_n : B_{R+1}(\hat{p}_0) \to S^1 \times \mathbb{R}$ by

$$\beta_n(x) = \pi_2 \circ \alpha^{-1}\big(x, \pi \circ \hat{\imath}_n \circ \varphi_n(x)\big).$$

We define $\beta_0 : B_{R+1}(\hat{p}_0) \to S^1 \times \mathbb{R}$ by

$$\beta_0(x) = \pi_2 \circ \alpha^{-1}\big(x, \pi \circ \hat{\imath}_0(x)\big).$$

Trivially $\beta_0(x) = x$. Since $(\beta_n)_{n \geqslant N_1}$ converges to β_0 in the C^∞_{loc} topology, it follows by the common sense Lemmata 2.7 and 2.8 that there exists $N_2 \geqslant N_1$ such that for all $n \geqslant N_2$:

(i) the restriction of β_n to $B_{R+1/2}(\hat{p}_0)$ is a diffeomorphism onto its image,

(ii) $(S^1 \times (-3r, 3r))$ is contained in $\beta_n(B_{R+1/2}(\hat{p}_0))$.

For all $n \geqslant N_2$, we define $\psi_n : S^1 \times (-2r, 2r) \to B_{R+1/2}(\hat{p}_0) \subseteq S^1 \times \mathbb{R}$ and λ_n by

$$\psi_n = \beta_n^{-1}|_{S^1 \times]-2r,2r[} \quad \text{and} \quad \lambda_n = \mathrm{Exp}^{-1} \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n.$$

Trivially, for all $n \geqslant N_2$, $\varphi_n \circ \psi_n$ is a diffeomorphism onto its image, and

$$\mathrm{Exp} \circ \lambda_n = \hat{\imath}_n \circ (\varphi_n \circ \psi_n).$$

We now show that, for all $n \geqslant N_2$, λ_n is a section of $NN_{0,\infty}$ above $\hat{\imath}_0$. For all $x \in S^1 \times (-2r, 2r)$ and for all $n \geqslant N_2$, we have $\beta_n \circ \psi_n(x) = x$. Thus, by definition of β_n

$$\pi_2 \circ \alpha^{-1}\big(\psi_n(x), \pi \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n(x)\big) = x.$$

Since $\pi_1 \circ \alpha = \pi_1$, and thus $\pi_1 \circ \alpha^{-1} = \pi_1$, we obtain

$$\begin{aligned}\alpha^{-1}\big(\psi_n(x), \pi \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n(x)\big) &= \big(\psi_n(x), x\big) \\ \implies \big(\psi_n(x), \pi \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n(x)\big) &= \big(\psi_n(x), \hat{\imath}_0(x)\big) \\ \implies \pi \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n(x) &= \hat{\imath}_0(x).\end{aligned}$$

Thus, if $\hat{\pi}$ denotes the canonical projection from $N_\epsilon N_{0,\infty}$ onto $N_{0,\infty}$, then, since $\pi = \hat{\pi} \circ \mathrm{Exp}$,

$$\hat{\pi} \circ \mathrm{Exp}^{-1} \circ \hat{\imath}_n \circ \varphi_n \circ \psi_n = \hat{\imath}_0 \implies \hat{\pi} \circ \lambda_n = \hat{\imath}_0.$$

It follows that λ_n is a section of $\hat{\imath}_0^* N N_{0,\infty}$.

Since $(\beta_n)_{n\in\mathbb{N}}$ converges to the identity, it follows that $(\beta_n(\hat{p}_0))_{n\in\mathbb{N}}$ converges to $\hat{p}_0$. For all $n \geqslant N_2$, let $\phi_n : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be the unique conformal mapping which sends $\hat{p}_0$ to $\beta_n(\hat{p}_0)$. There exists $N_3 \geqslant N_2$ such that for all $n \geqslant N_3$,

$$\phi_n\big(S^1 \times (-r, r)\big) \subseteq S^1 \times (-2r, 2r).$$

For all $n \geqslant N_3$, we define

$$\psi'_n = \varphi_n \circ \psi_n \circ \phi_n, \quad \lambda'_n = \lambda_n \circ \phi_n, \quad \hat{\imath}'_{0,n} = \hat{\imath}_0 \circ \phi_n.$$

For all $n \geqslant N_3$, we obtain:

(i) $(S^1 \times (-r, r), \hat{\imath}'_{0,n})$ is a tube of order k about $\Gamma_{0,\infty}$,

(ii) $\psi'_n : (S^1 \times (-r, r), 0) \to (S_n, p_n)$ is a diffeomorphism onto its image,

(iii) λ'_n is a section of $(\hat{\imath}'_{0,n})^* N N_{0,\infty}$ over $S^1 \times \mathbb{R}$, and

(iv) $\hat{\imath}_n \circ \psi'_n = \mathrm{Exp} \circ \lambda'_n$.

Consequently, for all $n \geqslant N_3$, the immersed surface $(\widehat{\Sigma}_n, p_n)$ is locally a graph over a tube about $\Gamma_{0,\infty}$ of order k and of half length r, and the first result follows. Moreover, we find that $(\lambda'_n)_{n\in\mathbb{N}}$ converges to 0 over $S^1 \times (-r, r)$ in the C^∞_{loc} topology, and the second result follows. □

Since the property of being locally a graph over $\Gamma_{0,\infty}$ is invariant under isometries of $\mathbb{H}^3$ which preserve $\Gamma_{0,\infty}$, we immediately obtain the following result:

Corollary 7.7. — *Let $r, \epsilon \in \mathbb{R}^+$ be positive real numbers. There exists an integer $N \in \mathbb{N}$ such that for $n \geqslant N$ the pointed immersed surface $(\widehat{\Sigma}, p_n)$ is locally a graph over a tube about $\Gamma_{0,\infty}$ of order k and of half length r and if λ_n is the graph function of $(\widehat{\Sigma}, p_n)$ over $S^1 \times]-r, r[$ then $\|\lambda_n\| < \epsilon$. Moreover $(\lambda_n)_{n\in\mathbb{N}}$ tends to 0 in the C^∞_{loc} topology.*

If ϵ is sufficiently small, then the graph functions and the graph diffeomorphisms are unique. Since the same result holds for every sequence of points in $S \setminus \mathcal{P}$ which tends towards p_0, we obtain:

Corollary 7.8. — *Let $r, \epsilon \in \mathbb{R}^+$ be positive real numbers. There exists an open set Ω of p_0 in S such that, if $p \in \Omega \setminus \{p_0\}$, then $(\widehat{\Sigma}, p)$ is locally a graph over a tube about $\Gamma_{0,\infty}$ of order k and of half length r and if λ_p is the graph function of $(\widehat{\Sigma}, p)$ over $S^1 \times (-r, r)$ then $\|\lambda_p\| < \epsilon$. Moreover λ_p tends to 0 in the C^∞_{loc} topology as p tends to p_0.*

By gluing these graphs together, we now obtain Theorem 1.2:

Theorem 1.2. — *Let S be a Riemann surface. Let $\mathcal{P}$ be a discrete subset of S such that $S \setminus \mathcal{P}$ is hyperbolic. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a ramified covering having critical points in $\mathcal{P}$. Let κ be a real number in $(0,1)$. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be the unique solution to the Plateau problem $(S \setminus \mathcal{P}, \varphi)$ with constant Gaussian curvature κ. Let $\widehat{\Sigma} = (S \setminus \mathcal{P}, \hat{\imath})$ be the Gauss lifting of Σ. Let p_0 be an arbitrary point in $\mathcal{P}$.*

If φ has a critical point of order k at p_0, then $\widehat{\Sigma}$ is asymptotically tubular of order k at p_0.

Proof. — Let ϵ be such that the restriction of Exp to $N_\epsilon N_{0,\infty}$ is a diffeomorphism onto its image. Let us define $U \subseteq U\mathbb{H}^3$ by $U = \mathrm{Exp}(N_\epsilon N_{0,\infty})$. Let $\pi : U \to N_{0,\infty}$ be the orthogonal projection onto $N_{0,\infty}$.

Let r be a positive real number. By Corollary 7.8, there exists a connected neighbourhood Ω of p_0 in S such that if $p \in \Omega \setminus \{p_0\}$, then $(\widehat{\Sigma}, p)$ is locally a graph over a tube about $\Gamma_{0,\infty}$ of order k and of half length $2r$ and if λ is the graph function of $(\widehat{\Sigma}, p)$ over $S^1 \times (-2r, 2r)$, then $\|\lambda\| < \epsilon$.

By using foliations, we will construct a chart over an open set about p_0 which is well adapted to our problem. Let $\mathcal{F}_{0,\infty}$ be the canonical circle foliation of $N_{0,\infty}$ arising from its structure as a circle bundle over $\Gamma_{0,\infty}$. Let $\mathcal{F}$ be a the canonical circle foliation of $S^1 \times (-2r, 2r)$.

For p a point in $\Omega \setminus \{p_0\}$, let $T_p = (S^1 \times (-2r, 2r), \hat{\imath}_p)$ be the tube of order k and of half length $2r$ over which (Σ, p) is a locally a graph. Let $\varphi_p : S^1 \times (-2r, 2r) \to S \setminus \mathcal{P}$ be the graph diffeomorphism of (Σ, p) over T_p. We define

$$\Omega_p = \varphi_p\big(S^1 \times (-r, r)\big).$$

Let $\lambda_p : S^1 \times (-2r, 2r) \to \mathbb{R}$ be the graph function of (Σ, p) over T_p. Since $\|\lambda_p\| < \epsilon$, we have $\hat{\imath}(q) \in U$ for all $q \in \Omega_p$. Moreover, we may assume that $(\pi \circ \hat{\imath})^* g^\nu$ defines a metric over Ω_p.

We remark that, by the uniqueness of graph diffeomorphisms and graph functions, for all $p, q \in \Omega \setminus \{p_0\}$, $q \in \Omega_p$ if and only if $p \in \Omega_q$. We define

$$\widehat{\Omega} = \bigcap_{p \in \Omega \setminus \{p_0\}} \Omega_p.$$

Since Ω is connected, so is $\widehat{\Omega}$. For all p, $(\varphi_p)_* \mathcal{F}$ defines a smooth circle foliation of Ω_p. This circle foliation coincides with $(\pi \circ \hat{\imath})^* \mathcal{F}_{0,\infty}$. It thus follows that $\widehat{\Omega}$ is foliated by $(\pi\circ\hat{\imath})^*\mathcal{F}_{0,\infty}$. Using the definition of g^ν, and recalling that $\pi\circ\hat{\imath}\circ\varphi_p=\hat{\imath}_p$ is a k-fold covering map, we find that every leaf of this foliation is of length $2\pi k\nu^{-1}$ with respect to the metric $(\pi\circ\hat{\imath})^*g^\nu$.

Let us define L to be the quotient manifold given by

$$L=\widehat{\Omega}/(\pi\circ\hat{\imath})^*\mathcal{F}_{0,\infty}.$$

L is a smooth connected 1-dimensional manifold without boundary and is thus diffeomorphic to an open interval $I=(a,b)$ in $\mathbb{R}$. The set $\widehat{\Omega}$ is thus diffeomorphic to a smooth circle bundle over I. We thus obtain a diffeomorphism $\psi_1:S^1\times I\to\widehat{\Omega}$. We will show that by modifying this diffeomorphism we obtain the desired chart.

Let δ be an arbitrary metric over S compatible with its topology. For $p\in\Omega$, we define

$$\Delta(p)=\operatorname{Inf}\bigl\{\delta(\varphi_p(\mathrm{e}^{i\theta},t),p_0)\ \text{s.t.}\ (\mathrm{e}^{i\theta},t)\in S^1\times(-r,r)\bigr\}.$$

By uniqueness of graph diffeomorphisms and graph functions, for all $(\mathrm{e}^{i\theta},t)$ in $S^1\times(-r,r)$, we obtain

$$\Delta\bigl(\varphi_p(\mathrm{e}^{i\theta},t)\bigr)=\operatorname{Inf}\bigl\{\delta(\varphi_p(\mathrm{e}^{i\phi},s),p_0)\ \text{s.t.}\ (\mathrm{e}^{i\phi},s)\in S^1\times(t-r,t+r)\bigr\}.$$

Consequently, Δ is continuous. Since $S^1\times(-r,r)$ is compact, we find that $\Delta(p)>0$ for all $p\in\Omega\setminus\{p_0\}$.

Let Ω_1 be connected neighbourhood of p_0 contained in $\widehat{\Omega}$. Let us define

$$\Delta_1=\operatorname{Inf}\bigl\{\Delta(p)\ \text{s.t.}\ p\in\partial\Omega_1\bigr\}.$$

Let us define

$$\Omega_2=\bigl\{p\in S\ \text{s.t.}\ \delta(p,p_0)<\Delta_1\bigr\}.$$

We define $\widehat{\Omega}_2$ in the same way as $\widehat{\Omega}$. Since $q\in\Omega_p$ if and only if $p\in\Omega_q$, we obtain $\widehat{\Omega}_2\cap\partial\Omega_1=\varnothing$. Consequently

$$\widehat{\Omega}_2\subseteq\Omega_1.$$

Since $\widehat{\Omega}_2$ is connected and foliated by $(\pi\circ\hat{\imath})^*\mathcal{F}_{0,\infty}$, there exists an open subinterval $I'\subseteq I$ such that

$$\widehat{\Omega}_2=\varphi_1(S^1\times I').$$

Moreover, since p_0 is contained in the closure of $\widehat{\Omega}_2$, it follows that the closure of I' in I is not compact. Consequently, we may assume that there exists $a'\in(a,b)$ such that

$$I'=(a',b).$$

Since we may choose Ω_1 arbitrarily small about p_0, we find that $\psi_1(\mathrm{e}^{i\theta},t)$ tends to p_0 as t tends to b.

Let $p:N_{0,\infty}\to\Gamma_{0,\infty}$ be the canonical projection. Let us also denote by p the composition $p\circ\pi:NN_{0,\infty}\to\Gamma_{0,\infty}$. Let t_0 be an arbitrary point in (a,b).

Let $\gamma : \mathbb{R} \to \Gamma_{0,\infty}$ be a unit speed parametrisation of $\Gamma_{0,\infty}$ such that $\gamma(t) \to 0$ as t tends to $+\infty$ and

$$\gamma(0) = (p \circ \hat{\imath} \circ \psi_1)(\mathrm{e}^{i\theta}, t_0).$$

Since ψ_1 respects the foliation $(\pi \circ \hat{\imath})^*\mathcal{F}_{0,\infty}$, the mapping

$$t \longmapsto (\gamma^{-1} \circ p \circ \hat{\imath} \circ \psi_1)(\mathrm{e}^{i\theta}, t)$$

is independent of θ and is everywhere a local diffeomorphism. Consequently, it defines a diffeomorphism. By Proposition 7.2, $\hat{\imath}(p) \to 0$ as $p \to p_0$. Consequently $(\gamma^{-1} \circ p \circ \hat{\imath} \circ \varphi_1)(\mathrm{e}^{i\theta}, t) \to +\infty$ as $t \to b$. We may thus reparametrise ψ_1 to obtain a diffeomorphism $\psi_2 : S^1 \times (0, \infty) \to \widehat{\Omega}$ such that, for all θ and for all t,

$$(p \circ \hat{\imath} \circ \psi_2)(\mathrm{e}^{i\theta}, t) = \gamma(t).$$

We define the vector fields ∂_θ and ∂_t over $S^1 \times (-2r, 2r)$ by

$$\partial_\theta(\mathrm{e}^{i\theta}, t) = \partial_\phi(\mathrm{e}^{i(\theta+\phi)}, t)_{|\phi=0}, \quad \partial_t(\mathrm{e}^{i\theta}, t) = \partial_s(\mathrm{e}^{i\theta}, t+s)_{|s=0}.$$

For all p, we may orient φ_p in such a manner that there exists $T_p \in \mathbb{R}$ such that for all $(t, \mathrm{e}^{i\theta})$,

$$(p \circ \hat{\imath} \circ \varphi_p)(t, \mathrm{e}^{i\theta}) = \gamma(t + T_p).$$

We then define X_p and Y_p over Ω_p by

$$X_p = (\varphi_p)_*\partial_\theta, \quad Y_p = (\varphi_p)_*\partial_t.$$

By the uniqueness of graph diffeomorphisms, for all $p, q \in \Omega$, the mapping $\varphi_p^{-1} \circ \varphi_q$ defined over $\varphi_q^{-1}(\Omega_p \cap \Omega_q)$ is an affine mapping (i.e., a rotation followed by a translation). Since $\varphi_p^{-1} \circ \varphi_q$ preserves orientation, we obtain, for all $p, q \in \Omega$,

$$X_{p|\Omega_p \cap \Omega_q} = X_{q|\Omega_p \cap \Omega_q}, \quad Y_{p|\Omega_p \cap \Omega_q} = Y_{q|\Omega_p \cap \Omega_q}.$$

We may thus define X and Y over the whole of $\widehat{\Omega}$ such that, for all p,

$$X_{|\Omega_p} = X_p, \quad Y_{|\Omega_p} = Y_p.$$

In particular $[XY] = 0$. Let Φ and Ψ be the flows of X and Y. $(\Phi_t)_{t\in\mathbb{R}}$ is a flow that moves along the leaves of the foliation $(\pi \circ \hat{\imath})^*\mathcal{F}_{0,\infty}$ with speed $k\nu^{-2}$ with respect to the metric $(\pi \circ \hat{\imath})^*g^\nu$. In particular Φ_t is defined over $\widehat{\Omega}$ for all $t \in \mathbb{R}$. Moreover, since every leaf of $(\pi \circ \hat{\imath})^*\mathcal{F}_{0,\infty}$ is of length $2\pi k\nu^{-1}$, it follows that $(\Phi_t)_{t\in\mathbb{R}}$ is periodic with period 2π. Let $(G_t)_{t\in\mathbb{R}}$ be the geodesic flow along $\Gamma_{0,\infty}$ in the positive direction (i.e., towards 0) with constant speed $k\nu^{-2}$. For all $t \geqslant 0$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{\Omega} & \xrightarrow{\Psi_t} & \widehat{\Omega} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \Gamma_{0,\infty} & \xrightarrow{G_t} & \Gamma_{0,\infty}. \end{array}$$

It follows that Ψ_t is defined over $\widehat{\Omega}$ for all $t \geqslant 0$. We define $\psi_3 : \mathbb{R}\times(0,\infty) \to \widehat{\Omega}$ by

$$\psi_3(s,t) = \Phi_{2\pi\nu^{-1}ks}\Psi_t\big(\psi_2(0,0)\big).$$

Since $(\Phi_t)_{t\in\mathbb{R}}$ is periodic with period 2π, the mapping ψ_3 quotients to an application $\psi : S^1 \times (0,\infty) \to \widehat{\Omega}$. We define $\hat{\jmath}$ and λ by

$$\hat{\jmath} = \pi \circ \hat{\imath} \circ \psi, \quad \text{and} \quad \lambda = \mathrm{Exp}^{-1} \circ \hat{\imath} \circ \psi.$$

For all $p \in \Omega$, we find that $\varphi_p^{-1} \circ \psi_{|\psi^{-1}(\Omega_p)}$ is the restriction of an affine transformation ϕ_p of $S^1 \times \mathbb{R}$ to $\psi^{-1}(\Omega_p)$. Consequently

$$\begin{aligned}\hat{\jmath}_{|\psi^{-1}(\Omega_p)} &= \pi \circ \hat{\imath} \circ \psi_{|\psi^{-1}(\Omega_p)} \\ &= (\pi \circ \hat{\imath} \circ \varphi_p) \circ (\varphi_p^{-1} \circ \psi)_{|\psi^{-1}(\Omega_p)} = \hat{\imath}_p \circ \phi_p.\end{aligned}$$

We recall that $\hat{\imath}_p$ is a locally conformal k-fold covering map. It thus follows that $\hat{\jmath} : S^1 \times (0,\infty) \to N_{0,\infty}$ is a locally conformal k-fold covering map. We thus have:

(i) $(S^1 \times (0,\infty), \hat{\jmath})$ defines a half tube of order k about $\Gamma_{0,\infty}$,

(ii) λ is a section of $\hat{\jmath}^*NN_{0,\infty}$, and

(iii) $\hat{\imath} \circ \psi = \mathrm{Exp} \circ \lambda$.

We have thus shown that $\widehat{\Sigma}$ is a graph over a half tube of order k about $\Gamma_{0,\infty}$. Moreover, by Corollary 7.8, and the uniqueness of graph functions, we find that for all $p \in \mathbb{R}$, $\|D^p\lambda(\mathrm{e}^{i\theta},t)\|$ converges to 0 as t tends to $+\infty$, and the result follows. □

8. Asymptotically tubular surfaces of finite order

8.1. Introduction. — In this section, we will prove Theorem 1.3:

THEOREM 1.3. — *Let S be a surface and let $\mathcal{P} \subseteq S$ be a discrete subset of S. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be an immersion such that $\Sigma = (S \setminus \mathcal{P}, i)$ is a k-surface (and is thus the solution to a Plateau problem). Let $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ be the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty\mathbb{H}^3$. Let $\hat{\imath}$ be the Gauss lifting of i so that $\varphi = \overrightarrow{n} \circ \hat{\imath}$ defines the Plateau problem to which i is the solution. Let $\mathcal{H}$ be the holomorphic structure generated over $S \setminus \mathcal{P}$ by the local homeomorphism φ. Let p_0 be an arbitrary point in $\mathcal{P}$, and suppose that Σ is asymptotically tubular of order k about p_0.*

Then there exists a unique holomorphic structure $\widetilde{\mathcal{H}}$ over $(S\setminus\mathcal{P})\cup\{p_0\}$ and a unique holomorphic mapping $\tilde{\varphi} : (S \setminus \mathcal{P})\cup\{p_0\} \to \widehat{\mathbb{C}}$ such that $\widetilde{\mathcal{H}}$ and $\tilde{\varphi}$ extend $\mathcal{H}$ and φ respectively. Moreover, $\tilde{\varphi}$ has a critical point of order k at p_0.

This result will be proven in two stages. First, by using the properties of the modules of conformal rings, we obtain:

PROPOSITION 8.1. — *Let S be a surface. Let $\mathcal{P}$ be a discrete subset of S. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be an immersion such that the immersed surface $\Sigma = (S \setminus \mathcal{P}, i)$ is a k-surface. Let $\widehat{\Sigma} = (S, \hat{\imath})$ be the Gauss lifting of Σ. Let $\overrightarrow{n}$ be the Gauss-Minkowski mapping that sends $U\mathbb{H}^3$ into $\partial_\infty \mathbb{H}^3 = \widehat{\mathbb{C}}$. Let us define $\varphi = \overrightarrow{n} \circ \hat{\imath}$. Let $\mathcal{H}$ be the canonical conformal structure over $\widehat{\mathbb{C}}$. Let p_0 be a point in $\mathcal{P}$.*

If $\widehat{\Sigma}$ is asymptotically tubular of finite order near p_0, then $\varphi^*\mathcal{H}$ extends to a unique conformal structure on $(S \setminus \mathcal{P}) \cup \{p_0\}$.

Next, by showing that there exists q_0 such that $\varphi(p)$ tends to q_0 as p tends to p_0, applying Cauchy's removable singularity theorem, we obtain:

PROPOSITION 8.2. — *With the same hypotheses as in Proposition* 8.1*, let p_0 be a point in $\mathcal{P}$. If $\widehat{\Sigma}$ is asymptotically tubular of order k near p_0, then φ extends to a unique holomorphic function over $(S \setminus \mathcal{P}) \cup \{p_0\}$ having a critical point of order k at p_0.*

Theorem 1.3 now follows as a direct corollary to these two propositions.

8.2. Conformal rings. — In this section we will recall various properties of holomorphic rings. We define a *(conformal) ring* to be a Riemann surface A whose fundamental group is isomorphic to $\mathbb{Z}$. For R a real number greater than 1, let us define the ring

$$A_R = \{z \text{ s.t. } 1 < |z| < R\}.$$

The uniformisation principle permits us to show that an arbitrary ring is biholomorphic to one of $\mathbb{C}^*$, $\mathbb{D}^*$ or A_R for some $R \in (1, \infty)$. Let Γ be the family of curves in A which are freely homotopic to a generator of $\pi_1(A)$. For g a conformal Riemannian metric over A, and for $\gamma \in \Gamma$ an arbitrary curve in Γ, we define $\mathrm{Len}_g(\gamma)$ to be the length of γ with respect to g, and we define

$$\mathcal{L}_g(\Gamma) = \operatorname*{Inf}_{\gamma \in \Gamma} \mathrm{Len}_g(\gamma).$$

For g a conformal metric over A, we define $\mathrm{Area}_g(A)$ to be the area of A with respect to g. We define $\mathrm{Mod}(A)$, the *module* of A, by

$$\mathrm{Mod}(A) = \operatorname*{Sup}_{\substack{\mathrm{Area}_g(A)=1 \\ g \text{ conformal}}} \mathcal{L}_g(\Gamma).$$

By definition $\mathrm{Mod}(A)$ only depends on the conformal class of A. $\mathrm{Mod}(A)$ may be calculated in certain cases, and, in particular, we have the following result:

LEMMA 8.3. — *For all $R \in (0, \infty)$,*

$$\mathrm{Mod}(A_R) = \sqrt{\frac{2\pi}{\mathrm{Log}(R)}}, \quad \mathrm{Mod}\big(S^1 \times (0, R)\big) = \sqrt{\frac{2\pi}{R}}.$$

Proof. — Let g be a conformal metric of area 1 over A_R. Let g_{Euc} be the Euclidean metric over A_R and let $\lambda : A_R \to (0,\infty)$ be such that

$$g = \lambda g_{\text{Euc}}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\text{Area}_g(A_R) = \int_1^R \int_0^{2\pi} \lambda r \,\mathrm{d}r\,\mathrm{d}\theta &\geqslant \int_1^R \frac{1}{2\pi r}\Big(\int_0^{2\pi} r\lambda^{1/2}\,\mathrm{d}\theta\Big)^2 \mathrm{d}r \\ &\geqslant \int_1^R \frac{1}{2\pi r}\mathcal{L}_g(\Gamma)^2\,\mathrm{d}r = \frac{\text{Log}(R)}{2\pi}\mathcal{L}_g(\Gamma)^2.\end{aligned}$$

Since $\text{Area}_g(A_R) = 1$, we obtain

$$\mathcal{L}_g(\Gamma)^2 \leqslant \frac{2\pi}{\text{Log}(R)}.$$

We obtain equality if and only if $\lambda = Kr^{-2}$ for some normalising factor $K \in (0,\infty)$, and an explicit calculation of K permits us to obtain the first result. The second result follows by a similar reasoning. □

The following lemma permits us to compare the modules of two rings of which one is contained inside the other:

Lemma 8.4. — *Let A_1 and A_2 be two rings. Let $i : A_1 \to A_2$ be an embedding. If $i_*\pi_1(A_1) = \pi_1(A_2)$, then*

$$\text{Mod}(A_2) \leqslant \text{Mod}(A_1).$$

Proof. — Let Γ_2 be the family of curves in A_2 which are freely homotopic to a generator of $\pi_1(A_2)$. By the proof of the preceding lemma, there exists a conformal metric g over A_2 such that

$$\text{Area}_g(A_2) = 1, \quad \mathcal{L}_g(\Gamma_2) = \text{Mod}(A_2).$$

Let Γ_1 be the family of curves in A_1 which are freely homotopic to a generator of $\pi_1(A_1)$. The mapping i_* sends Γ_1 into Γ_2. We thus obtain

$$\mathcal{L}_g(i_*\Gamma_1) \geqslant \mathcal{L}_g(\Gamma_2).$$

Let us define h by

$$h = \frac{1}{\text{Area}_{i^*g}(A_1)} i^*g.$$

Since $\text{Area}_g(A_1) \leqslant \text{Area}_g(A_2) = 1$, it follows that $h \geqslant i^*g$ and consequently

$$\mathcal{L}_h(\Gamma_1) \geqslant \mathcal{L}_{i^*g}(\Gamma_1) \geqslant \mathcal{L}_g(\Gamma_2) = \text{Mod}(A_2).$$

Since $\text{Area}_h(A_1) = 1$, the result follows. □

In particular, we obtain:

Corollary 8.5. — *One has $\text{Mod}(A) = 0$ if and only if A is conformally equivalent to $\mathbb{D}^*$ or to $\mathbb{C}^*$.*

Proof. — Let $R \in (0,\infty)$ be a positive real number. We have

$$\left\{1/R < |z| < 1\right\} \subseteq \mathbb{D}^* \subseteq \mathbb{C}^*.$$

Using the previous result, we thus obtain

$$\mathrm{Mod}(\mathbb{C}^*) \leqslant \mathrm{Mod}(\mathbb{D}^*) \leqslant \sqrt{\frac{2\pi}{\mathrm{Log}(R)}}.$$

By letting R tend to infinity, we obtain

$$\mathrm{Mod}(\mathbb{C}^*) = \mathrm{Mod}(\mathbb{D}^*) = 0.$$

The converse follows by the uniformisation principle. □

8.3. Extending the complex structure. — Let S be a surface and let $\mathcal{P}$ be a discrete subset of S. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be an immersion such that the immersed surface $\Sigma = (S \setminus \mathcal{P}, i)$ is a k-surface. Let $\hat{\imath}$ be the Gauss lifting of i and let us denote by $\vec{n}$ the Gauss-Minkowski mapping that sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3 = \widehat{\mathbb{C}}$. Let $\mathcal{H}$ be the canonical conformal structure over $\widehat{\mathbb{C}}$. Let us define $\varphi = \vec{n} \circ \hat{\imath}$. Since φ is a local homeomorphism $\varphi^*\mathcal{H}$ defines a conformal structure over $S \setminus \mathcal{P}$. In this section, we will prove Proposition 8.1.

Let p_0 be an arbitrary point in $\mathcal{P}$. We suppose that $\widehat{\Sigma}$ is asymptotically tubular of finite order about p_0. We obtain the following result:

Proposition 8.6. — *For every sufficiently small neighbourhood U of p_0 in S which is homeomorphic to a disc, the Riemann surface $(U \setminus \{p_0\}, \varphi^*\mathcal{H})$ is conformally equivalent to $\mathbb{D}^*$.*

Proof. — Let $\mathcal{H}'$ be the conformal structure generated over $S \setminus \mathcal{P}$ by the metric $\hat{\imath}^* g^\nu$ and the canonical orientation of S. By Lemma 3.2, $\mathcal{H}'$ is k-quasi-conformally equivalent to $\varphi^*\mathcal{H}$. It thus suffices to show that $(U \setminus \{p_0\}, \mathcal{H}')$ is conformally equivalent to $\mathbb{D}^*$.

Let $\Gamma_{0,\infty}$ be the unique geodesic in $\mathbb{H}^3$ joining 0 to ∞. We may assume that $\widehat{\Sigma}$ is asymptotically tubular about $\Gamma_{0,\infty}$. Let $N_{0,\infty}$ be the normal circle bundle over $\Gamma_{0,\infty}$ in $U\mathbb{H}^3$. Let $\mathrm{Exp} : TU\mathbb{H}^3 \to U\mathbb{H}^3$ be the exponential mapping over $U\mathbb{H}^3$.

Let $T = (S^1 \times (0,\infty), \hat{\jmath})$ be a half tube of order k about $\Gamma_{0,\infty}$ such that there exist

(i) a neighbourhood Ω of p_0 in S,

(ii) a graph diffeomorphism $\phi : S^1 \times (0,\infty) \to \Omega \setminus \{p_0\}$, and

(iii) a graph function $\lambda \in \Gamma(\hat{\jmath}^* NN_{0,\infty})$,

such that

(i) $\phi(\mathrm{e}^{i\theta}, t) \to p_0$ as t tends to $+\infty$,

(ii) for all $p \in \mathbb{N}\cup\{0\}$, $\|D^p\lambda(\mathrm{e}^{i\theta},t)\|$ tends to 0 in the C^1 norm as t tends to $+\infty$, and

(iii) $\hat{\imath}\circ\phi = \operatorname{Exp}\circ\lambda$.

For $R, T > 0$, we define the set

$$A_{R,T} = S^1 \times (T, T+R).$$

Since $\|D^1\lambda(\mathrm{e}^{i\theta},t)\|$ tends to 0 as t tends to $+\infty$, we have

$$|\phi^*\hat{\imath}^*g^\nu - \hat{\jmath}^*g^\nu|\,(\mathrm{e}^{i\theta},t) \longrightarrow 0 \quad \text{as } t\to\infty.$$

Consequently, if we denote by $d_{R,T}$ the L^∞ norm of the complex dilatation of the metric $\phi^*\hat{\imath}^*g^\nu$ relative to $\hat{\jmath}^*g^\nu$ over $A_{R,T}$, we find that for all R,

$$d_{R,T} \longrightarrow 0 \quad \text{as } T\to+\infty.$$

Thus, by the translation invariance of $\hat{\jmath}^*g^\nu$, for all R we obtain

$$\operatorname{Mod}(A_{R,T};\phi^*\hat{\imath}^*g^\nu) \longrightarrow \operatorname{Mod}(A_{0,R};\hat{\jmath}^*g^\nu) \quad \text{as } T\to+\infty.$$

It follows by Lemma 8.4 that, for all R,

$$\operatorname{Mod}\big(S^1\times(0,\infty);\phi^*\hat{\imath}^*g^\nu\big) \leqslant \operatorname{Mod}(A_{0,R};\hat{\jmath}^*g^\nu).$$

Thus

$$\operatorname{Mod}\big(S^1\times(0,\infty);\phi^*\hat{\imath}^*g^\nu\big) \leqslant \operatorname{Mod}\big(S^1\times(0,\infty);\hat{\jmath}^*g^\nu\big) = 0.$$

Consequently, by Corollary 8.5, $(\Omega\setminus\{p_0\},\hat{\imath}^*g^\nu)$ is biholomorphic either to $\mathbb{C}^*$ or to $\mathbb{D}^*$. Thus, by reducing Ω if necessary, we obtain the desired result. □

We now obtain Proposition 8.1 as a corollary to this result:

Proposition 8.1. — *Let S be a surface. Let $\mathcal{P}$ be a discrete subset of S. Let $i : S\setminus\mathcal{P}\to\mathbb{H}^3$ be an immersion such that the immersed surface $\Sigma = (S\setminus\mathcal{P}, i)$ is a k-surface. Let $\widehat{\Sigma} = (S,\hat{\imath})$ be the Gauss lifting of Σ. Let $\overrightarrow{n}$ be the Gauss-Minkowski mapping that sends $U\mathbb{H}^3$ into $\partial_\infty\mathbb{H}^3 = \widehat{\mathbb{C}}$. Let us define $\varphi = \overrightarrow{n}\circ\hat{\imath}$. Let $\mathcal{H}$ be the canonical conformal structure over $\widehat{\mathbb{C}}$. Let p_0 be a point in $\mathcal{P}$.*

If $\widehat{\Sigma}$ is asymptotically tubular of finite order near p_0, then $\varphi^*\mathcal{H}$ extends to a unique conformal structure on $(S\setminus\mathcal{P})\cup\{p_0\}$.

Proof. — Let U be a neighbourhood of p_0 in S such that $U\setminus\{p_0\}$ is biholomorphic to $\mathbb{D}^*$ and let $\alpha : U\setminus\{p_0\}\to\mathbb{D}^*$ be this biholomorphism. Let Ω be a neighbourhood of zero in $\mathbb{D}$ and let γ be a simple closed curve in $\Omega\setminus\{0\}$ such that $0\in\operatorname{Int}(\gamma)$.

Let us define the curve $\tilde{\gamma}$ by $\tilde{\gamma} = \alpha^{-1}\circ\gamma$. It is a simple closed curve in $U\setminus\{p_0\}$. It follows that the complement of $\tilde{\gamma}$ in U consists of two connected components U_1 and U_2. We may assume that $p_0\in U_1$. Since α is a homeomorphism, it sends $U_1\setminus\{p_0\}$ either onto $\operatorname{Int}(\gamma)\setminus\{0\} = \Omega$ or onto $\operatorname{Ext}(\gamma)\cap\mathbb{D}$. However, by the preceding lemma

$$\operatorname{Mod}\big(U_1\setminus\{p_0\}\big) = 0.$$

Consequently, the set $U_1 \setminus \{p_0\}$ is not biholomorphic to $\mathrm{Ext}(\gamma)\cap\mathbb{D}$, and so

$$\alpha\big(U_1 \setminus \{p_0\}\big) = \mathrm{Int}(\gamma)\setminus\{0\}.$$

It follows that $\alpha(p)$ tends to zero as p tends to p_0 and we may thus extend α to a continuous mapping over U by defining

$$\alpha(p_0) = 0.$$

Since α is bijective, by the principle of invariance of domains, it is a homeomorphism. We thus obtain a holomorphic chart $(\alpha, U, \mathbb{D})$ of $(S\setminus\mathcal{P})\cup\{p_0\}$ about p_0 which extends the conformal structure of $S\setminus\mathcal{P}$, and we thus obtain existence. Uniqueness follows from the Cauchy removable singularity theorem. □

8.4. Extending the holomorphic function. — We continue to work with the construction of the previous section. We now obtain the following result:

Proposition 8.7. — *If $\widehat{\Sigma}$ is asymptotically tubular of finite order about p_0, then there exists a point $q_0 \in \widehat{\mathbb{C}}$ such that $\varphi(p)$ tends to q_0 as p tends to p_0.*

Proof. — Let $\Gamma_{0,\infty}$ be the geodesic joining 0 to infinity. We may assume that Σ is asymptotically tubular about $\Gamma_{0,\infty}$. Let $N_{0,\infty}$ be the normal circle bundle of $\Gamma_{0,\infty}$ in $U\mathbb{H}^3$. Let $\pi : N_{0,\infty} \to \Gamma_{0,\infty}$ be the canonical projection. Let $\gamma : \mathbb{R} \to \Gamma_{0,\infty}$ be a unit speed parametrisation of $\Gamma_{0,\infty}$ such that $\gamma(t)$ tends to 0 as t tends to $+\infty$. By identifying $\mathbb{H}^3$ with $\mathbb{C}\times(0,\infty)$, we define $h : \mathbb{R} \to (0,\infty)$ such that, for all t,

$$\gamma(t) = \big(0, h(t)\big).$$

For all $t \in \mathbb{R}$, we define $A_t \in \mathrm{Isom}(\mathbb{H}^3)$ by

$$A_t(z,s) = \Big(\frac{z}{h(t)}, \frac{s}{h(t)}\Big).$$

We also denote by A_t the actions of A_t on $U\mathbb{H}^3$ and $\partial_\infty\mathbb{H}^3$.

Let $T = (S^1\times(0,\infty), \hat{\jmath})$, Ω, ϕ and λ be as in the proof of Proposition 8.6. We may suppose that $\hat{\jmath}$ is normalised such that, for all $t \geqslant 0$,

$$\pi\circ A_t\circ\hat{\jmath}(\mathrm{e}^{i\theta}, t) = (0,1) \implies \big|\overrightarrow{n}\circ A_t\circ\hat{\jmath}(\mathrm{e}^{i\theta}, t)\big| = 1.$$

Since $\lambda(\mathrm{e}^{i\theta}, t)$ tends to 0 in the C^1 norm as t tends to $+\infty$, we have

$$\begin{aligned}
&\big|\overrightarrow{n}\circ A_t\circ\hat{\jmath}(\mathrm{e}^{i\theta}, t) - \overrightarrow{n}\circ A_t\circ\hat{\imath}\circ\phi(\mathrm{e}^{i\theta}, t)\big| \longrightarrow 0 \quad \text{as } t\to+\infty,\\
&\implies \big|\overrightarrow{n}\circ A_t\circ\hat{\imath}\circ\phi(\mathrm{e}^{i\theta}, t)\big| \longrightarrow 1 \quad \text{as } t\to+\infty,\\
&\implies \big|A_t\circ\overrightarrow{n}\circ\hat{\imath}\circ\phi(\mathrm{e}^{i\theta}, t)\big| \longrightarrow 1 \quad \text{as } t\to+\infty,\\
&\implies \big|\overrightarrow{n}\circ\hat{\imath}\circ\phi(\mathrm{e}^{i\theta}, t)\big| \longrightarrow 0 \quad \text{as } t\to+\infty,\\
&\implies \varphi(p) \longrightarrow 0 \quad \text{as } p\to p_0.
\end{aligned}$$

The result now follows. □

We now obtain Proposition 8.2 as a corollary to this result:

PROPOSITION 8.2. — *With the same hypotheses as in Proposition* 8.1*, let* p_0 *be a point in* $\mathcal{P}$*. If* $\widehat{\Sigma}$ *is asymptotically tubular of order* k *near* p_0*, then* φ *extends to a unique holomorphic function over* $(S \setminus \mathcal{P}) \cup \{p_0\}$ *having a critical point of order* k *at* p_0*.*

Proof. — It follows by Cauchy's removable singularity theorem and the preceding proposition that φ extends to a unique holomorphic function over $(S \setminus \mathcal{P}) \cup \{p_0\}$.

Using the same reasoning and notation as in the preceding proposition, we find that there exists $T > 0$ such that for $t \geqslant T$, and for all θ:

$$\begin{aligned} &\left|(A_t \circ \varphi \circ \phi)(\mathrm{e}^{i\theta}, t) - (A_t \circ \overrightarrow{n} \circ \hat{\jmath})(\mathrm{e}^{i\theta}, t)\right| < 1 \\ &\qquad \Longrightarrow \left|(A_t \circ \varphi \circ \phi)(\mathrm{e}^{i\theta}, t) - \mathrm{e}^{ik\theta}\right| < 1. \end{aligned}$$

It follows that, for $t \geqslant T$, the curve $\theta \mapsto (\varphi \circ \phi)(\mathrm{e}^{i\theta}, t)$ is homotopic in $\mathbb{C}^*$ to the curve $\theta \mapsto \mathrm{e}^{ik\theta}$, and thus turns k times round the origin.

By Proposition 8.1, we may assume that $(\phi(S^1 \times [T, \infty)), \phi^*\varphi^*\mathcal{H})$ is biholomorphic to $\mathbb{D}^*$. Let $\alpha : \phi(S^1 \times [T, \infty)) \to \mathbb{D}^*$ be a biholomorphic mapping. For $t \geqslant T$, since α is a homeomorphism, the curve $\theta \mapsto (\alpha \circ \phi)(\mathrm{e}^{i\theta}, t)$ turns once around the origin. The result now follows by considering $(\varphi \circ \alpha^{-1})$. $\square$

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