ON COVERINGS OF SIMPLE ABELIAN VARIETIES

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ABSTRACT. — To any finite covering \( f : Y \to X \) of degree \( d \) between smooth complex projective manifolds, one associates a vector bundle \( E_f \) of rank \( d-1 \) on \( X \) whose total space contains \( Y \). It is known that \( E_f \) is ample when \( X \) is a projective space ([9]), a Grassmannian ([11]), or a Lagrangian Grassmannian ([7]). We show an analogous result when \( X \) is a simple abelian variety and \( f \) does not factor through any nontrivial isogeny \( X' \to X \). This result is obtained by showing that \( E_f \) is \( M \)-regular in the sense of Pareschi-Popa, and that any \( M \)-regular sheaf is ample.

Résumé (Sur les revêtements des variétés abéliennes simples). — On associe à tout revêtement fini \( f : Y \to X \) de degré \( d \) entre variétés projectives lisses complexes un fibré vectoriel \( E_f \) de rang \( d-1 \) sur \( X \) dont l'espace total contient \( Y \). On sait que \( E_f \) est ample lorsque \( X \) est un espace projectif ([9]), une grassmannienne ([11]) ou une grassmannienne lagrangienne ([7]). Nous montrons un résultat analogue lorsque \( X \) est une variété abélienne simple et que \( f \) ne se factorise par aucune isogénie non triviale \( X' \to X \). Ce résultat est obtenu en montrant que \( E_f \) est \( M \)-régulier au sens de Pareschi-Popa, puis que tout faisceau \( M \)-régulier est ample.

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1. Introduction

We work over the complex numbers. Let $f : Y \to X$ be a finite surjective morphism of degree $d$ between smooth projective varieties of the same dimension $n$. The morphism $f$ is flat, hence the sheaf $f_* \mathcal{O}_Y$ is locally free. We may define a locally free sheaf $E_f$ of rank $d - 1$ on $X$ as the dual of the kernel of the trace map $\text{Tr}_{Y/X} : f_* \mathcal{O}_Y \to \mathcal{O}_X$, so that

$$f_* \mathcal{O}_Y = \mathcal{O}_X \oplus E_f^*$$

By duality for a finite flat morphism, we have

$$f_* \omega_{Y/X} = \mathcal{O}_X \oplus E_f$$

Our aim is to prove the following statement conjectured in [1].

**Theorem 1.1.** — Let $X$ be a simple abelian variety, let $Y$ be a smooth connected projective variety, and let $f : Y \to X$ be a finite cover. If $f$ does not factor through any nontrivial isogeny $X' \to X$, the vector bundle $E_f$ is ample.

For a more general statement, see Theorem 4.1. See also the remarks at the end of this article for more comments. Even if $X$ is not simple, the vector bundle $E_f$ is known to be nef (see [14, Theorem 1.17], [10, Example 6.3.59]) and its restriction to a general complete intersection curve in $X$ to be ample (see [6, Lemma 2.7]).

The ampleness of $E_f$ has a number of consequences, as explained in [10, Example 6.3.56]. In our case, one new statement beyond the Fulton-Hansen-type results already obtained in [1] is the following: under the hypotheses of the theorem, the induced morphism

$$H^i(f, \mathcal{C}) : H^i(X, \mathcal{C}) \longrightarrow H^i(Y, \mathcal{C})$$

is bijective for $i \leq n - d + 1$ (see [10, Theorem 7.1.16]).

When moreover $d \leq n$, the morphism $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ is bijective.\(^{(1)}\)

In particular, the group $H_1(Y, \mathbb{Z})$ is isomorphic to $H_1(X, \mathbb{Z})$, hence is torsion-free, and so is $H^2(Y, \mathbb{Z})$ by the universal coefficient theorem.

When $d \leq n - 1$, the morphism $H^2(f, \mathcal{Z}) : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is injective with finite cokernel, hence so is $	ext{Pic}(f) : \text{Pic}(X) \to \text{Pic}(Y)$. It seems likely that those two maps are bijective.

The proof is a simple application of the results of [13] about global generation of sheaves on an abelian variety. More precisely, it is based on the remark that any $M$-regular sheaf ($\S$ 3) on an abelian variety is ample (Corollary 3.2).

\(^{(1)}\)For algebraic fundamental groups, this is [1, Corollaire 6.2]; for topological fundamental groups, this is [2, Exercice VIII.5], where the hypothesis $d \leq n$ is unfortunately missing.
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2. Ample sheaves

To any coherent sheaf $F$ on a scheme $X$ of finite type over $\mathbb{C}$, one associates the $X$-scheme

$$P(F) = \text{Proj} \left( \bigoplus_{m \geq 0} \text{Sym}^m F \right)$$

and an invertible sheaf $\mathcal{O}_{P(F)}(1)$ on $P(F)$. The sheaf $F$ is said to be ample if $\mathcal{O}_{P(F)}(1)$ is.

Well-known properties of ampleness for locally free sheaves (see for example [10, Chapter 6]) still hold in this general setting:

- a) the sheaf $F$ is ample if and only if, for any coherent sheaf $G$ on $X$, the sheaf $G \otimes \text{Sym}^m F$ is globally generated for all $m \gg 0$ (see [8, Theorem 1]);
- b) any quotient of an ample sheaf is ample (see [8, Proposition 1]);
- c) if $\pi : Y \to X$ is a finite morphism, $F$ is ample if and only if $\pi^* F$ is (this is because $P(\pi^* F) = P(F) \times_X Y$ and $\mathcal{O}_{P(F)}(1)$ pulls back, by a finite morphism, to $\mathcal{O}_{P(\pi^* F)}(1)$);
- d) if $X$ is proper and $F$ is globally generated, $F$ is ample if and only if, for any curve $C$ in $X$, the restriction $F \otimes \mathcal{O}_C$ has no trivial quotient (Gieseker’s Lemma).

3. Continuously generated sheaves

Following [13, Definition 2.10], we say that a coherent sheaf $F$ on an irreducible projective variety $X$ is continuously globally generated if, for any nonempty subset $U$ of $\text{Pic}^0(X)$, the sum of the twisted evaluation maps

$$\bigoplus_{\xi \in U} H^0(X, F \otimes P_{\xi}) \otimes P_{\xi}^\vee \longrightarrow F$$

is surjective, where, for any element $\xi$ of $\text{Pic}^0(X)$, we denote by $P_{\xi}$ the corresponding numerically trivial line bundle on $X$. This property is equivalent to the existence of a positive integer $N$ such that for $(\xi_1, \ldots, \xi_N)$ general in $\text{Pic}^0(X)^N$, the analogous map

$$\bigoplus_{i=1}^N H^0(X, F \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \longrightarrow F$$

is surjective.
is surjective. Being a quotient of a direct sum of numerically trivial line bundles, a continuously globally generated sheaf is nef. Our aim is to show that under certain circumstances, it is ample.

**Proposition 3.1.** — A coherent sheaf $\mathcal{F}$ on an irreducible projective variety $X$ is continuously globally generated if and only if there exists a connected abelian Galois étale cover $\pi: Y \to X$ such that $\pi^*(\mathcal{F} \otimes P_\xi)$ is globally generated for all $\xi \in \text{Pic}^0(X)$.

**Proof.** — Assume $\mathcal{F}$ is continuously globally generated and let $\xi \in \text{Pic}^0(X)$. Since torsion points are dense in $\text{Pic}^0(X)$, the open subset of $\text{Pic}^0(X)^N$ of points for which the map (1) is surjective and all $h^0(X, \mathcal{F} \otimes P_\xi)$ are minimal contains a point of the type

$$(\xi + \eta_1(\xi), \ldots, \xi + \eta_N(\xi))$$

where $(\eta_1(\xi), \ldots, \eta_N(\xi))$ is torsion, hence contains also $U_\xi + (\eta_1(\xi), \ldots, \eta_N(\xi))$, where $U_\xi$ is a neighborhood of $\xi$ in $\text{Pic}^0(X)$. Since $\text{Pic}^0(X)$ is quasi-compact, it is covered by finitely many such neighborhoods, say $U_{\xi_1}, \ldots, U_{\xi_M}$.

Let $\pi: Y \to X$ be a connected abelian Galois étale cover such that the kernel of $\text{Pic}^0(\pi): \text{Pic}^0(X) \to \text{Pic}^0(Y)$ contains all $\eta_i(\xi_j)$, for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$; the map

$$\bigoplus_{i=1}^N H^0(X, \mathcal{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)}) \otimes \pi^* P_\xi \otimes \pi^* P_{\eta_i(\xi_j)} \to \pi^* \mathcal{F}$$

is surjective for all $\xi \in U_{\xi_j}$. But this map is

$$\bigoplus_{i=1}^N H^0(X, \mathcal{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)}) \otimes \pi^* P_\xi \to \pi^* \mathcal{F}$$

and since each $H^0(X, \mathcal{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)})$ is a vector subspace of $H^0(Y, \pi^*(\mathcal{F} \otimes P_\xi))$, the sheaf $\pi^*(\mathcal{F} \otimes P_\xi)$ is globally generated for all $\xi \in U_{\xi_j}$, hence for all $\xi$ in $\text{Pic}^0(X)$.

For the converse, assume that there exists a connected abelian Galois étale cover $\pi: Y \to X$ such that the evaluation map

$$H^0(Y, \pi^*(\mathcal{F} \otimes P_\xi)) \otimes \mathcal{O}_Y \to \pi^*(\mathcal{F} \otimes P_\xi)$$

is surjective for all $\xi \in \text{Pic}^0(X)$. Since $\pi$ is finite, the map

$$H^0(X, \mathcal{F} \otimes P_\xi \otimes \pi_* \mathcal{O}_Y) \otimes \pi_* \mathcal{O}_Y \to \mathcal{F} \otimes P_\xi \otimes \pi_* \mathcal{O}_Y$$

is also surjective. If we let $\text{Ker}(\text{Pic}^0(\pi)) = \{\eta_1, \ldots, \eta_N\}$, we have $\pi_* \mathcal{O}_Y = \bigoplus_{i=1}^N P_{\eta_i}$, the map

$$\left( \bigoplus_{i=1}^N H^0(X, \mathcal{F} \otimes P_\xi \otimes P_{\eta_i}) \right) \otimes \left( \bigoplus_{i=1}^N P_{\eta_i} \right) \to \mathcal{F} \otimes P_\xi \otimes \left( \bigoplus_{i=1}^N P_{\eta_i} \right)$$
is surjective, and so is
\[ \bigoplus_{i=1}^{N} H^0(X, \mathcal{F} \otimes P_{\xi} \otimes P_{\eta_i}) \otimes P_{\eta_i}^* \rightarrow \mathcal{F} \otimes P_{\xi}. \]
In other words, the map (1) is surjective for \((\xi_1, \ldots, \xi_N) = (\xi + \eta_1, \ldots, \xi + \eta_N)\), for all \(\xi \in \text{Pic}^0(X)\). Choosing \(\xi_0\) such that \(h^0(X, \mathcal{F} \otimes P_{\xi_0} + \eta_i)\) takes the general (minimal) value for each \(i\) in \(\{1, \ldots, N\}\), we obtain that the map (1) is still surjective for \((\xi_1, \ldots, \xi_N) = (\xi_0 + \eta_1, \ldots, \xi_0 + \eta_N)\). This proves that \(\mathcal{F}\) is continuously globally generated.

**Corollary 3.2.** — Let \(X\) an irreducible projective variety with a finite map to an abelian variety. Any continuously globally generated coherent sheaf on \(X\) is ample.

**Proof.** — Let \(\mathcal{F}\) be a continuously globally generated coherent sheaf on \(X\). By Proposition 3.1, there exists a connected abelian Galois étale cover \(\pi: Y \rightarrow X\) such that \(\pi^*(\mathcal{F} \otimes P_{\xi})\) is globally generated for all \(\xi \in \text{Pic}^0(X)\).

Let \(C\) be a curve in \(Y\). If there is a trivial quotient \(\pi^*[\mathcal{F}]_{|C} \rightarrow \mathcal{O}_{C}\), we have also surjections \(\pi^*(\mathcal{F} \otimes P_{\xi})_{|C} \rightarrow \pi^*P_{\xi}^*_{|C}\) for each \(\xi \in \text{Pic}^0(X)\). Since \(\pi^*(\mathcal{F} \otimes P_{\xi})\) is globally generated, so is \(\pi^*P_{\xi}^*_{|C}\). This implies that the composition \(\text{Pic}^0(X) \rightarrow \text{Pic}^0(Y) \rightarrow \text{Pic}^0(C)\) is zero, hence that \(\pi(C)\) is contracted by any map from \(X\) to an abelian variety. This contradicts our hypothesis, hence \(\pi^*[\mathcal{F}]_{|C}\) has no trivial quotient.

By Gieseker’s Lemma, \(\pi^*[\mathcal{F}]\) is ample, and so is \(\mathcal{F}\) (§ 2).

### 4. The main theorem

Following [13, Definition 2.1], we say that a coherent sheaf \(\mathcal{F}\) on an abelian variety \(A\) is \(M\)-regular if

\[ \text{codim}_{\text{Pic}^0(A)} \text{Supp} \left( R^i \hat{S}(\mathcal{F}) \right) > i \]

for all \(i > 0\) (\(R^i \hat{S}\) is the \(i\)th Fourier-Mukai functor). This is the case if

\[ \text{codim}_{\text{Pic}^0(A)} \{ \xi \in \text{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes P_{\xi}) \neq 0 \} > i \]

for all \(i > 0\). We refer to [12] and [13] for more details. For our purposes, the main result of [13] (Proposition 2.13) is that an \(M\)-regular coherent sheaf on an abelian variety is continuously globally generated.

**Theorem 4.1.** — Let \(X\) be a smooth connected projective variety with a finite map to a simple abelian variety, let \(Y\) be a smooth connected projective variety with a finite surjective map \(f: Y \rightarrow X\). If \(f\) factors through no nontrivial connected abelian Galois étale covering of \(X\), the vector bundle \(E_f \otimes \omega_X\) is ample.
Proof. — Let \( n \) be the common dimension of \( X \) and \( Y \), and let \( \alpha : X \to A \) be a finite map to a simple abelian variety such that \( \Pic^0(\alpha) : \Pic^0(A) \to \Pic^0(X) \) is injective. Set \( g = \alpha \circ f \). By [4, Theorem 1], [5, Theorem 0.1], and [3, Remark 1.6] (see also [3, Theorem 1.2]), every irreducible component of the set

\[
V_i = \{ \xi \in \Pic^0(A) \mid H^{n-i}(Y, g^* P_{\xi}^e) \neq 0 \}
\]

is a translated abelian subvariety of \( \Pic^0(A) \) of codimension at least \( i \). In particular, since \( A \) is simple, \( V_i \) is finite for \( i > 0 \).

Since \( Y \) is connected, we have

\[
V_n = \{ \xi \in \Pic^0(A) \mid H^0(Y, g^* P_{\xi}^e) \neq 0 \}
= \{ \xi \in \Pic^0(A) \mid g^* P_{\xi}^e \cong \mathcal{O}_Y \}
= \text{Ker}(\Pic^0(g) : \Pic^0(A) \to \Pic^0(Y)),
\]

hence \( V_n = \{0\} \) since both \( \Pic^0(\alpha) \) and \( \Pic^0(f) \) are injective (\( f \) factors through nontrivial abelian étale covering of \( X \)). Consider now

\[
W_i = \{ \xi \in \Pic^0(A) \mid H^i(X, E_f \otimes \omega_X \otimes \alpha^* P_{\xi}) \neq 0 \}
= \{ \xi \in \Pic^0(A) \mid H^i(A, \alpha_*(E_f \otimes \omega_X) \otimes P_{\xi}) \neq 0 \}.
\]

By Serre duality on \( Y \),

\[
V_i = \{ \xi \in \Pic^0(A) \mid H^i(Y, \omega_Y \otimes g^* P_{\xi}) \neq 0 \}
= \{ \xi \in \Pic^0(A) \mid H^i(X, f_* \omega_Y \otimes \alpha^* P_{\xi}) \neq 0 \}.
\]

Since \( f_* \omega_Y = f_* \omega_Y \otimes \omega_X = \omega_X \otimes (E_f \otimes \omega_X) \), we have \( W_i \subset V_i \) and \( W_n = \emptyset \).

It follows that \( W_i \) is finite, hence \( \text{codim}(W_i) > i \) for each \( i > 0 \), so that the sheaf \( \alpha_*(E_f \otimes \omega_X) \) on \( A \) is \( M \)-regular, hence continuously globally generated. It is therefore ample by Corollary 3.2, and, since \( \alpha \) is finite, so are \( \alpha^*(\alpha_*(E_f \otimes \omega_X)) \) and its quotient \( E_f \otimes \omega_X \) (§2).

In the following remarks, we keep the hypotheses and notation of the theorem and its proof.

Remark 4.2. — The proof of the theorem shows that the sheaf \( \alpha_*(E_f \otimes \omega_X) \) is continuously globally generated. In particular, if \( f \) is not an isomorphism, \( E_f \otimes \omega_X \) has nonzero sections, hence \( p_g(Y) > p_g(X) \).

Remark 4.3. — The simplicity of the abelian variety in the theorem is essential: if \( B \) is an abelian variety and \( g = (f, \text{Id}_B) : Y \times B \to X \times B \), we have \( E_g = p^* E_f \), where \( p : X \times B \to X \) is the first projection, hence \( E_g \otimes \omega_{X \times B} = p^*(E_f \otimes \omega_X) \) is not ample if \( B \) is nonzero. The locus \( W_i \) for \( g \) contains \( \Pic^0(A) \times \{0\} \) for \( i \leq \text{dim}(B) \); in particular, for \( i = \text{dim}(B) \), it is an abelian subvariety of codimension \( i \) of \( \Pic^0(A \times B) \).

Note however that if \( A \) is not simple but \( \alpha(X) \) is not ruled by nonzero abelian subvarieties of \( A \), the end of the proof of Theorem 3 of [3] implies \( \text{codim}(V_i) > i \).
for each $i > 0$. The proof above shows that the conclusion of Theorem 4.1 still holds.

Remark 4.4. — If $X$ is not an abelian variety, $\omega_X$ is already ample (see, e.g., [1, Théorème 6.9]) and one can show that the hypothesis that $f$ does not factor through a nontrivial connected abelian Galois étale covering of $X$ is unnecessary. If $X$ is a (simple) abelian variety, any finite cover $Y \to X$ factorizes as $Y \xrightarrow{f} X' \xrightarrow{\rho} X$ where $\rho$ is an isogeny and $f$ satisfies the hypotheses of the theorem.

Remark 4.5. — Assume $X = A$ and let $d$ be the degree of $f$. We want to prove that for all $i \geq d - 1$, the set $W_i$ is empty, i.e.,

$$H^i(A, E_f \otimes P_\xi) = 0 \quad \text{for all } \xi \in \text{Pic}^0(A).$$

By a theorem of Simpson [15], all points of $V_i$, hence a fortiori all points of $W_i$, are torsion points. As explained in the introduction, Theorem 4.1 implies that the morphism

$$H^{n-i}(f, C) : H^{n-i}(A, C) \to H^{n-i}(Y, C)$$

is bijective for $i \geq d - 1$. Using the Hodge decomposition, this implies $0 \notin W_i$. For any isogeny $\pi : A' \to A$, the smooth variety $Y' = Y \times_A A'$ is connected and if $f' : Y' \to A'$ is the second projection, we have $E_{f'} = \pi^* E_f$. It follows that for $i \geq \deg(f') - 1 = d - 1$, we have

$$0 = H^i(A', E_{f'}) = H^i(A, E_f \otimes \pi_* O_A') = \bigoplus_{\xi \in \text{Ker}(\text{Pic}^0(\pi))} H^i(A, E_f \otimes P_\xi).$$

In particular, $W_i$ contains no torsion points, hence is empty.

Equality (2) does not hold in general for $0 \leq i < d - 1$, as shown by the following example. Take an elliptic curve $C$, with origin $o_C$. Let $L$ be a very ample line bundle on $C$ and let $Y \subset C \times A$ be a general (smooth) element of $[O_C((n + 1)o_C) \otimes L]$. Following the proof of [10, Lemma 6.3.43], one sees that the second projection $f : Y \to A$ is finite (of degree $d = n + 1$). By the Lefschetz theorem, the induced morphism

$$H^{n-i}(C \times A, O_{C \times A}) \to H^{n-i}(Y, O_Y)$$

is bijective for $i > 0$ and injective for $i = 0$. In particular, $H^{n-i}(f, O)$ is not surjective for $0 \leq i < n$, hence $0 \notin W_i$, i.e.,

$$H^i(A, E_f) \neq 0 \quad \text{for } 0 \leq i < d - 1 = n.$$
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