POSITIVITY OF QUADRATIC BASE CHANGE
L-FUNCTIONS

BY Hervé Jacquet & Chen Nan

Abstract. — We show that certain quadratic base change L-functions for $GL(2)$ are non-negative at their center of symmetry.

Résumé (Positivité des fonctions $L$ du changement de base quadratique)
On montre que certaines des fonctions $L$ de $GL(2)$ obtenues par changement de base quadratique sont positives en leur centre de symétrie.

Contents

1. The main theorem .............................................. 34
2. Choice of measures and local matching .......................... 42
3. Global matching ................................................ 47
4. Factorization of $\Theta_{\Pi}$ over $E$ ............................... 49
5. Local comparisons .............................................. 54
6. Global comparison .............................................. 68
7. Concluding remarks ............................................ 69
8. Appendix : Continuous spectrum over $E$ ........................ 70
9. Appendix : Continuous spectrum over $F$ ........................ 87
Bibliography ...................................................... 88

Texte reçu le 23 septembre 1999, révisé le 1er mars 2000
Hervé Jacquet, Department of Mathematics, Columbia University, 2990 Broadway, MC 4408, New York, NY 10027 (USA) • E-mail : hj@math.columbia.edu
Chen Nan, Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027 (USA) • E-mail : nanchen@math.columbia.edu
2000 Mathematics Subject Classification. — 11F67, 11F70, 11F72, 22E55.
Key words and phrases. — $L$-functions, positivity, quadratic base change.
H.J.: Partially supported by NSF Grant DMS 9619766.
1. The main theorem

Let $E/F$ be a quadratic extension of number fields and $\eta_{E/F}$ or simply $\eta$ the quadratic character of $F$ attached to $E$, $\{1, \tau\}$ the Galois group of $E/F$. We will often write

$$\tau(z) = z^2.$$

We will denote by $U_1$ the unitary group in one variable, that is, the group of elements of norm 1 in $E^\times$. Suppose that $\pi$ is an automorphic cuspidal representation of $\text{Gl}(2,F)$ whose central character $\omega$ is trivial on the group of norms. In other words $\omega = 1$ or $\omega = \eta$. We assume that $\pi$ is not dihedral with respect to $E$ so that the base change representation $\Pi$ of $\pi$ to $\text{Gl}(2,E)$ is still automorphic and cuspidal. Let $\Omega$ be an idele class character of $E$ whose restriction to $F_\ell^\times$ is equal to $\omega$. Our main result is the following theorem:

**Theorem 1.** — *With the previous notations: $L\left(\frac{1}{2}, \Pi \otimes \Omega^{-1}\right) \geq 0$.*

If $\omega = 1$ and $\Omega = 1$ then $L(s, \Pi) = L(s, \pi)L(s, \pi \otimes \eta)$ and the result has been established by Guo (see [G1], under some restrictions on $E/F$). As a matter of fact, by using results on averages of $L$-functions (see [FH]), Guo is able to prove that $L\left(\frac{1}{2}, \pi\right) \geq 0$, which then implies our result for $\omega = 1$, $\Omega = 1$, without restriction on $E/F$. At any rate, Baruch and Mao [BM] have independently established that $L\left(\frac{1}{2}, \pi\right) \geq 0$ if $\omega = 1$. However, the present result—where $\Omega$ needs not be trivial—is more general, even in the case $\omega = 1$.

Results on the positivity of $\text{Gl}(2)$ $L$-functions have been considered by many mathematicians (see, for instance, [BFH], [Gr], [K], [Kk], [KS], [KZ], [S], [R], [S], [W3], [Ya]). Specially, the positivity of the twisted $L$-function at hand has been investigated (for holomorphic forms) in [GZ].

We note that $\Omega^\tau = \Omega^{-1}$ and $\Pi$ is self-contragredient: $\tilde{\Pi} = \Pi$. Thus

$$L(s, \Pi \otimes \Omega^{-1}) = L(s, \Pi^\tau \otimes (\Omega^{-1})^\tau) = L(s, \Pi \otimes \Omega) = L(s, \tilde{\Pi} \otimes \Omega).$$

Likewise,

$$\varepsilon(s, \Pi \otimes \Omega^{-1})\varepsilon(1 - s, \Pi \otimes \Omega^{-1}) = \varepsilon(s, \Pi \otimes \Omega^{-1})\varepsilon(1 - s, \tilde{\Pi} \otimes \Omega) = 1.$$

In particular $\varepsilon\left(\frac{1}{2}, \Pi \otimes \Omega^{-1}\right) = \pm 1$. Thus, despite the fact that $\Pi \otimes \Omega^{-1}$ is not necessarily self-contragredient, the $L$-function $L(s, \Pi \otimes \Omega^{-1})$ is symmetric:

$$L(s, \Pi \otimes \Omega^{-1}) = \varepsilon(s, \Pi \otimes \Omega^{-1})L(1 - s, \Pi \otimes \Omega^{-1}).$$

The following lemma is easily verified:

**Lemma 1.** — *Let $v_0$ be a place of $F$. If $v_0$ is inert and $v$ is the corresponding place of $E$ then:

$L\left(\frac{1}{2}, \Pi_v \otimes \Omega_v^{-1}\right) > 0$. If $v_0$ splits into $v_1$ and $v_2$ then:

$L\left(\frac{1}{2}, \Pi_{v_1} \otimes \Omega_{v_1}^{-1}\right)L\left(\frac{1}{2}, \Pi_{v_2} \otimes \Omega_{v_2}^{-1}\right) > 0.$*
Let $S_0$ be a finite set of places of $F$ and $S$ the corresponding set of places of $E$. Set

$$L^S(s, \Pi \otimes \Omega^{-1}) = \prod_{v \in S} L(s, \Pi_v \otimes \Omega_v^{-1}).$$

In view of the lemma, the statement of the theorem is equivalent to the positivity of $L^S(\frac{1}{2}, \Pi \otimes \Omega^{-1})$.

If $\Pi$ is dihedral with respect to $E$, then $\Pi$ is associated with an idele class character $\Xi$ of $E$ whose restriction to $\mathbb{F}^\times$ is $\omega \eta$. Thus $\Xi' = \Xi^{-1}$ and

$$L(s, \Pi \otimes \Omega^{-1}) = L(s, \Xi \Omega^{-1}) L(s, \Xi' \Omega^{-1}) = L(s, \Xi \Omega^{-1}) L(s, \Xi^{-1} \Omega^{-1}).$$

If $\Omega$ is trivial or even quadratic this is $\geq 0$. At any rate, in general, $\Xi \Omega^{-1}$ and $\Xi^{-1} \Omega^{-1}$ have $\eta$ for restriction to $\mathbb{F}^\times$. Thus there are cuspidal representations $\pi_1$ and $\pi_2$ of $\text{Gl}(2, \mathbb{F}^\times)$ with trivial central character such that:

$$L(s, \Xi \Omega^{-1}) = L(s, \pi_1), L(s, \Xi^{-1} \Omega^{-1}) = L(s, \pi_2)$$

and by the results already quoted each factor is $\geq 0$ at $s = \frac{1}{2}$. We will not discuss this case but remark that, by considering the discrete but non-cuspidal terms in our trace formula, we could probably handle this case as well.

The proof of the theorem is based on a careful analysis of the relative trace formula of [J2] (In the case $\Omega = 1$ we could, like Guo, use the simpler trace formula of [J1].) Namely, we consider an inner form $G$ of $\text{Gl}(2, F)$ which contains a torus $T$ isomorphic to $E^\times$. There is then an $\epsilon \in F^\times$, uniquely determined modulo $\text{Norm}(E^\times)$, such that the pair $(G, T)$ is isomorphic to the pair $(G_{\epsilon}, T)$ defined as follows. We denote by $\mathbb{H}_{\epsilon}$ the semi-simple algebra of matrices $g \in M(2, E)$ of the form

$$(1) \quad g = \begin{pmatrix} a & \epsilon b \\ 0 & \pi \end{pmatrix}$$

and by $G_{\epsilon}$ its multiplicative group. Then

$$T = \left\{ t = \begin{pmatrix} a & 0 \\ 0 & \pi \end{pmatrix} \right\}.$$ 

We let $Z$ be the center of $G_{\epsilon}$. We regard $\Omega$ as a character of $T(\mathbb{F}_{\kappa})$: $t \mapsto \Omega(t)$. Suppose that $f$ is a smooth function of compact support on $G_{\epsilon}(\mathbb{F})$. We form as usual a kernel

$$K_f(x, y) := \int_{Z(\mathbb{F}_{\kappa})/\mathbb{F}^\times} \sum_{\xi \in G_{\epsilon}(\mathbb{F})} f(x^{-1} z \xi y) \omega(z) \, dz$$

and a distribution

$$J_{\epsilon}(f) := \int_{(Z(\mathbb{F}_{\kappa})/T(\mathbb{F}_{\kappa}))^2} K_f(t_1, t_2) \Omega(t_1)^{-1} dt_1 \Omega(t_2) dt_2.$$
We have a spectral decomposition of the kernel:

\[ K_f = \sum_{\sigma} K_{f,\sigma} + K_{f,\text{cont}}; \]

the sum on the right is over all irreducible (cuspidal) automorphic representations \( \sigma \) of \( G_\epsilon \): if \( G_\epsilon \) is not split, by cuspidal we mean an irreducible automorphic representation which is not one-dimensional. The term \( K_{f,\text{cont}} \) represents the contribution of the one dimensional representations and the continuous spectrum which is present only if \( G_\epsilon \) is split, that is, \( \epsilon \) is a norm. For every \( \sigma \) the kernel \( K_{f,\sigma} \) is defined by

\[ K_{f,\sigma}(x, y) = \sum_{\phi} \rho(f)\phi(x)\overline{\phi(y)}, \]

the sum over an orthonormal basis of the space of \( \sigma \). We define then:

\[ J_\sigma(f) := \int_{(Z(F)\cap T(F))T(F)^2} K_{f,\sigma}(t_1, t_2) \Omega(t_1)^{-1} dt_1 \Omega(t_2) dt_2. \]

This is a distribution of positive type: if \( f = f_1 * f_1^* \) where \( f_1^*(g) := \overline{f_1}(g^{-1}) \) then

\[ J_\sigma(f) = \sum \nu(\rho(f_1)\phi) \overline{\nu(\rho(f_1)\phi)}, \]

where we have set

\[ \nu(\phi) := \int_{Z(F)\cap T(F)} \phi(t)\Omega(t)^{-1} dt, \]

thus \( J_\sigma(f) \geq 0 \). Moreover, if \( \nu \) is not identically zero on the space of \( \sigma \), or as we shall say, if \( \sigma \) is distinguished by \( (T, \Omega) \), then every local component \( \sigma_\nu \) of \( \sigma \) is distinguished by \( (T_\nu, \Omega_\nu) \), that is, admits a non-zero continuous linear form \( \nu_\nu \) such that \( \nu_\nu(\pi_\nu(t)u) = \Omega_\nu(t)\nu_\nu(u) \) for all \( t \in T_\nu \) and all smooth vectors \( u \). The dimension of the space of such linear forms is one. One can then define a local distribution

\[ J_{\sigma_\nu}(f_\nu) = \sum \nu_\nu(\rho(f_\nu)u) \overline{\nu_\nu(u)}, \]

the sum over an orthonormal basis. The distribution \( J_{\sigma_\nu} \) is defined within a positive factor. It is of positive type. Normalizing in an appropriate way we get

\[ J_\sigma(f) = C(\sigma) \prod_{\nu_0} J_{\sigma_\nu}(f_{\nu_0}), \]

where the constant \( C(\sigma) \) is positive. Assuming that \( L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \neq 0 \) we can find an \( \epsilon \) such that there is an automorphic representation \( \sigma \) of \( G_\epsilon \) corresponding to \( \pi \) and distinguished by \( (T, \Omega) \) (see [J2], [W4]). Another goal of the paper is to obtain an explicit decomposition of the above form, with a specific normalization (Theorem 2). The crux of the matter is then to show that \( C(\sigma) \) is essentially equal to \( L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \) which gives the positivity result. Possibly, this can be used to provide lower bounds for \( L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \).
We note that if $\epsilon$ exists then it is unique. Indeed, this follows at once from the following local fact: if $v_0$ is inert, $\epsilon$ not a norm at $v_0$, $\pi$ is a square integrable representation of $G_{\epsilon}(F_{v_0})$, and $\sigma$ the representation of $G_{\epsilon}(F_{v_0})$ corresponding to $\pi$ then $\pi$ and $\sigma$ cannot be both distinguished by $(T_{v_0}, \Omega_{v_0})$ (see [W4]).

We stress that there is no direct way to compute the constant $C(\sigma)$ because there is no direct relation between the global linear form $\nu$ and the local linear forms $\nu_{v_0}$. The situation at hand (a globally defined distribution of positive type decomposed as a product over all places of $F$ of local distributions of positive type, times the appropriate special values of $L$-functions) is, conjecturally, quite general. In this situation, the positivity of the special value of the $L$-function follows. One can view this question as a generalization of the problem of computing the Tamagawa number. This is our motivation for investigating in detail the present situation.

We proceed as follows. We introduce the matrices
\begin{equation}
(4) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}.
\end{equation}

It will be more convenient to consider instead the distributions $\theta_{\epsilon}(f) := \int_{(Z(F)\cap T(F))\backslash T(F)} K_f(t_1, t_2) \Omega(t_1 t_2)^{-1} dt_1 dt_2$, and, for $\sigma$ an automorphic representation of $G_{\epsilon}$, $\theta_{\sigma}(f) := \int_{(Z(F)\cap T(F))\backslash T(F)} K_{f, \sigma}(t_1, t_2) \Omega(t_1 t_2)^{-1} dt_1 dt_2.$

Thus $J_{\sigma}(f) = \theta_{\sigma}(\rho(\pi) f)$ and likewise for $J_\epsilon$. We will decompose explicitly $\theta_\sigma$ into a product over all places $v_0$ of $F$ of local distributions $\theta_{\sigma_{v_0}}$.

To that end, we compute the geometric expression for $\theta_\epsilon(f)$. A set of representatives for the double cosets of $T(F)$ in $G_{\epsilon}(F)$ is given by the matrices:
\begin{equation}
1_2, \quad \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \epsilon \beta \\ \beta & 1 \end{pmatrix}, \quad \beta \in E^\times / U_1(F).
\end{equation}

We define orbital integrals. For $\xi \neq 1$ in $\text{Norm}(E^\times) \epsilon$ we write $\xi = \beta \overline{\epsilon}$ and set:
\begin{equation}
(5) \quad H(\xi; f) = \int f \begin{pmatrix} 1 & \frac{1}{\beta-1} \\ \beta & 1 \end{pmatrix} \Omega(t_1 t_2^{-1}) dt_1 dt_2.
\end{equation}

Note that the right hand side of the integral depends only on $\beta \overline{\epsilon}$, which justifies the notations. In addition, we define
\begin{equation}
(6) \quad H(\infty; f) := \int f \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \Omega(t_1) dt_1,
\end{equation}

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE
and, if $\Omega^2$ is trivial,

\begin{equation}
H(0; f) := \int f(t_1) \Omega(t_1) dt_1.
\end{equation}

It will be useful to observe that the condition $\Omega^2 = 1$ is equivalent to $\Omega^2 = \Omega$ or $\Omega_{U_1}(F_k) = 1$. The integrals are for $t_1 \in T(F_k)$ and $t_2 \in T(F_k)/Z(F_k)$.

An idele class character $\chi$ of $E$ is normalized if it is trivial on the subgroup $E^+_{\infty}$ of ideles with all finite components trivial and all infinite components equal to some common positive number. For such a character we set $\delta(\chi) = 1$ if $\chi = 1$ and $\delta(\chi) = 0$ otherwise. All idele class characters will be assumed to be normalized. We have then

\begin{equation}
\theta_{\epsilon}(f) = \sum_{\xi \in \text{Norm}(E)} H(\xi; f) + \left(\delta(\Omega^2)H(0; f) + H(\infty; f)\right) \text{vol}(T(F_k)/T(F)Z(F_k)).
\end{equation}

In what follows we denote by $G$ the group $\text{Gl}(2)$, by $A$ the group of diagonal matrices, by $Z$ the group of scalar matrices, by $P$ the group of upper triangular matrices, by $P_1$ the subgroup of matrices with $(0, 1)$ for second row and by $N$ the subgroup of triangular matrices with unit diagonal. Depending on the context these groups are regarded as algebraic groups over $F$ or $E$. We now view the group $G$ as an algebraic group defined over $E$ and we let $H_0$ be the unitary group for the matrix $w$. We denote by $H$ the corresponding similitude group and by $\kappa$ the similitude ratio. It is a result of [HLR] that $\Pi$ is distinguished by $(H, \eta \omega)$, in the sense that there is a vector $\phi$ in the space of $\Pi$ such that

\begin{equation}
\mathcal{P}(\phi) := \int_{H(F_k)Z(E_k) \setminus H(F_k)} \phi(h) \eta \omega(\kappa(h)) dh \neq 0.
\end{equation}

This condition characterizes representations which are base change of representations of $\text{Gl}(2, F_k)$ with central character $\omega$ (loc. cit.).

We may regard $\Omega$ as a character of $A(E_k)$ trivial on $Z(E_k)$:

$\Omega([\text{diag}(a_1, a_2)] = \Omega(a_1a_2^{-1})$.

Then $L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \neq 0$ if and only if there is $\phi$ in the space of $\Pi$ such that

\begin{equation}
\lambda(\phi) := \int_{Z(E_k)A(E) \setminus A(E_k)} \phi(a) \Omega^{-1}(a) da \neq 0.
\end{equation}

These facts suggest the following construction. Let $f$ be a smooth function of compact support on $G(E_k)$. Define a kernel

\begin{equation}
K_f(x, y) = \int_{Z(E_k)/Z(E)} \sum_{\xi \in G(E)} f(x^{-1}z\xi y) dz.
\end{equation}
and a distribution:

$$
\Theta(f) := \int_{Z(E_\Lambda)A(E) \setminus A(E_\Lambda)} \int_{Z(E_\Lambda)H(E) \setminus H(E_\Lambda)} K_f(a,h)\Omega^{-1}(a)\eta\omega(\kappa(h))dhda. 
$$

The outer integral is not convergent and must be regularized. If \( \Pi \) is a cuspidal automorphic representation of \( G(E_\Lambda) \) with trivial central character then we define similarly the distribution \( \Theta_\Pi \). This distribution is non-zero if and only if \( \Pi \) is distinguished by \( (H,\omega) \) and \( L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \neq 0 \). It can be decomposed explicitly into a product of local distributions.

The distribution \( \Theta \) can be computed in terms of orbital integrals in the following way. We let \( S \) be the space of invertible Hermitian matrices and \( S_s \) the space of split Hermitian matrices. The group \( \text{Gl}(2,E) \) operates on \( S \) by \( s \mapsto gs \). We let \( \Phi(f) \) or simply \( \Phi \) be the function on the space \( S(F_\Lambda) \) such that

$$
\Phi(f(gw^t)) = \int_{H_0(F_\phi)} \Phi(f(gh_0))dh_0
$$

and \( \Phi \) vanishes outside \( S_s(F_\Lambda) \). We have

$$
\int K_f\left[\left[\begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array}\right], h\right]\eta\omega(\kappa(h))dh = \int \sum_{\xi \in S(F)} \Phi\left[\left[\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right] \xi \left[\begin{array}{cc} \xi & 0 \\ 0 & \xi \end{array}\right] \omega\eta(z)\right]dz.
$$

This leads us to introduce the action of \( E^\times \times F^\times \) defined by:

$$
(\alpha, \zeta) \rightarrow \left(\begin{array}{cc} \alpha & 0 \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} \xi & 0 \\ 0 & \xi \end{array}\right) \left(\begin{array}{cc} \epsilon & 0 \\ 0 & 1 \end{array}\right).
$$

A system of representatives for the orbits of \( E^\times \times F^\times \) on \( S(F) \) is constituted of the following matrices:

$$
(11) \left(\begin{array}{cc} \xi^{-1} & 0 \\ 1 & 1 \end{array}\right), \xi \in F^\times - 1, \quad (12) \left(\begin{array}{cc} \epsilon^{-1} & 0 \\ 1 & 1 \end{array}\right), \epsilon \in F^\times / \text{Norm}(E^\times),
$$

$$
(13) \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right), \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right).
$$

For \( \xi \in F^\times - \{1\} \), we set

$$
U(\xi; \Phi) := \int_{E_\Lambda^\times} \int_{F_\Lambda^\times} \Phi\left[\left[\begin{array}{cc} z & 0 \\ 0 & z \end{array}\right] \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) \left(\begin{array}{cc} \xi^{-1} & 0 \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} \overline{\gamma} & 0 \\ 0 & 1 \end{array}\right)\right] \times \eta\omega(z)d^\times \Omega(y)d^\times y.
$$
If $\Omega$ is trivial on $U_1$ we define

$$U \left[ \left( \begin{array}{cc} \epsilon^{-1} & 0 \\ 0 & 1 \end{array} \right); \Phi \right] := \int_{F_A^\times} \int_{E_A^\times / U_1(F_A)} \Phi \left[ \left( \begin{array}{cc} \epsilon^{-1}y & \bar{\eta}z \\ \eta & 0 \end{array} \right) \right] \omega \eta(z) \Omega(y) d^\times y d^\times z.$$  

The stabilizer of (12) in $E_A^\times \times F_A^\times$ is the set of pairs $(y, z)$ with $yz = 1$. However, because of our assumptions, the character $(y, z) \mapsto \Omega(y)\omega\eta(z)$ is non-trivial on that subgroup and so the element (12) does not contribute to the sum below.

We also introduce two \textit{unipotent} orbital integrals

$$U \left[ \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right); \Phi \right] := \int \int \Phi \left[ \left( \begin{array}{cc} 0 & y \\ \eta & z \end{array} \right) \right] \Omega(y) d^\times y \eta(z) d^\times z,$$

$$U \left[ \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right); \Phi \right] := \int \int \Phi \left[ \left( \begin{array}{cc} z & y \\ \bar{\eta} & 0 \end{array} \right) \right] \Omega(y) d^\times y \eta(z) d^\times z.$$  

These integrals are improper integrals. For the first one, for instance, we remark that the function $\phi$ defined by:

$$\phi(z) := \int \Phi \left[ \left( \begin{array}{cc} 0 & y \\ \eta & z \end{array} \right) \right] \Omega(y) d^\times y$$

is a Schwartz-Bruhat function on $F_A$. The unipotent integral is then the analytic continuation to the point $s = 0$ of the Tate integral:

$$\int_{F_A^\times} \phi(z) \eta(z) |z|^s d^\times z.$$  

We have then

\begin{equation}
\Theta(f) = \sum_{\xi \in F_A^\times \setminus \{1\}} U(\xi; \Phi) + \delta(\Omega^2) \operatorname{vol}(U_1(F_A)/U_1(F)) \times \sum_{\epsilon \in F^\times / \operatorname{Norm}(E^\times)} U \left[ \left( \begin{array}{cc} \epsilon^{-1} & 0 \\ 0 & 1 \end{array} \right); \Phi \right] + U \left[ \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right); \Phi \right] + U \left[ \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right); \Phi \right].
\end{equation}

We say that $f$ and a family of functions $(f_\epsilon)$, $\epsilon \in F^\times / \operatorname{Norm}(E^\times)$, have \textit{matching orbital integrals} if

$$U(\xi; \Phi) = H(\xi; f),$$

for $\xi \in \epsilon \operatorname{Norm}(E^\times) \setminus \{1\}$. Implicit in this definition is the fact that $f_\epsilon = 0$ for almost all $\epsilon$. We show that for any $f$ there is a family $(f_\epsilon)$ with matching orbital integrals. Note that the converse is not true: to have a converse one would need to consider all unitary groups. When $f$ and the family $(f_\epsilon)$ have matching orbital integrals, we have then

$$\Theta(f) = \sum_{\epsilon} \theta_\epsilon(f_\epsilon),$$

that is, the terms corresponding to all orbital integrals match (see [J2]).
Now we apply standard arguments: by identifying the contribution of the continuous (and residual) spectrum on both sides we arrive at the identity

$$\Theta_\Pi(f) = \sum \theta_{\sigma,}(f_\epsilon).$$

In a precise way, the representation $\Pi$ is the base change of $\pi$ and $\pi \otimes \eta$. The sum on the right is over all representations $\sigma_\epsilon$ which correspond to $\pi$ or $\pi \otimes \eta$. As noted above, the sum on the right reduces to two terms, that is, there is a unique $\epsilon$ and a representation $\sigma_\epsilon$ of $G_\epsilon(F_A)$ which corresponds to $\pi$ such that

$$\Theta_\Pi(f) = \theta_{\sigma,}(f_\epsilon) + \theta_{\sigma,\otimes \eta}(f_\epsilon).$$

The distribution $\Theta_\Pi$ can be decomposed explicitly into a product; the above identity allows us to decompose $\theta_{\sigma,}$ and, finally, prove the theorem.

The method is quite general and should apply to many situations. For instance, the work of [G2] suggests a possible direct generalization of the present set-up to $\text{Gl}(2n)$, the group $T$ being replaced by the group $\text{Gl}(n, E)$ embedded in $\text{Gl}(2n, F)$ and some inner forms of it. [G2] is concerned with the generalization of the simpler trace formula in [J1].

We remark that it would be interesting to compare our explicit result with the results of [W4].

The paper is arranged as follows. In sections 2 and 3 we review the results of [J2] on the matching of orbital integrals, reformulating the results in terms of symmetric spaces; we carefully normalize the various Haar measures. In section 4 we decompose the distribution $\Theta_\Pi$ explicitly into a product of local distributions. This is mainly a review of the material in [HLR]. The heart of the paper is section 5 where we compare the local distributions at hand.

The proof of the main theorem is given in section 6, with some additional comments in section 7. The paper concludes with sections 8 and 9, an appendix where we discuss the absolute convergence of the term coming from the continuous spectrum in the trace formulas at hand. Unfortunately, reference [J2] (also [J1]) is somewhat deficient on this point: the main point is that [J2] omits infinite sums on idele class characters unramified at all places. To make the argument rigorous we introduce, in the case at hand, a new form of truncation which may be more appropriate than the standard truncation operator for the investigation of period integrals (cf. [JLRo]) and a new device (special to $\text{Gl}(2)$ or closely related groups) to estimate some period integrals. Note that [G1] which is based on [J1] is also deficient and so is [BM]. Thus we take care at once of a gap in several papers.

Finally, we would like to thank the referee for a careful and thorough reading of the manuscript.
2. Choice of measures and local matching

We review the results of [J2] on local matching. It will be essential to keep track of the choice of the various Haar measures.

2.1. Inert case. — We consider a local quadratic extension $E/F$. We choose an additive character $\psi_F$ of $F$ and set

$$\psi_E(z) = \psi_F(z + \pi).$$

On the additive groups $F$ and $E$ we consider the self-dual Haar measures. In particular, if we write $E = F[\sqrt{\delta}]$ and $z = a + b\sqrt{\delta}$ then $dz = dadb |2|_F |\delta|^{1/2}_F$.

The multiplicative Tamagawa measures on $F^\times$ and $E^\times$ are respectively:

$$d^\times x = L(1,1_F) \frac{dx}{|x|_F}, \quad d^\times z = L(1,1_E) \frac{dz}{|z|_E}.$$

The following integration formula will be used below:

**Lemma 2.** — One has

$$\int_{E^\times} \phi(x) \frac{dz}{|z|_F} = \int_{\text{Norm}(E^\times)} \tilde{\phi}(x) \frac{dx}{|x|_F},$$

where $\tilde{\phi}(x)$ is the function on $\text{Norm}(E^\times)$ defined by

$$\tilde{\phi}(x) := \int_{E^\times/F^\times} \phi \left( \frac{z}{n} \right) d^0 u, \quad x = z \pi,$$

and the measure $d^0 u$ is the quotient of $dz/|z|_E$ by $dx/|x|_F$.

Since $T(F)$ is isomorphic to $E^\times$ via the map $\text{diag}(a, \pi) \mapsto a$, we obtain the Tamagawa measure on $T(F)$. Likewise the center $Z$ of $\text{Gl}(2)$ is isomorphic to $\text{Gl}(1)$ and so we have the Tamagawa measure on $Z(F)$ and $Z(E)$.

On the group $\text{Gl}(2, F)$ we have the Tamagawa measure

$$d^\times g = L(1,1_F) \frac{dpdqdrds}{|\det g|^2_F}, \quad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$  

Using the Iwasawa decomposition we can write

$$d^\times g = d^\times a d^\times b dx dk,$$

if

$$g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k,$$

where $dk$ is a Haar measure on the standard maximal compact subgroup $K_F$ of $\text{Gl}(2, F)$. Let $\Phi$ be a Schwartz-Bruhat function on $F^2$. The following lemma is easily verified:
Lemma 3. — One has
\[
\int \int \Phi(x, y) \, dx \, dy = \int \int \Phi(0, t) \, k \cdot |t|^2 \, dt \, dk.
\]

In particular, in the unramified situation, we can take for \( \Phi \) the characteristic function of the set \( \mathcal{O}_F^2 \) and obtain \( \text{vol}(K_F) = L(2, 1_F)^{-1} \). On the group \( \text{Gl}(2, E) \) we also have the Tamagawa measure
\[
d^*g = L(1, 1_E) \frac{dg}{|\text{det}g|_E} \quad dg = dpdqdrds
\]
if
\[
g = \left( \begin{array}{c} p \\ q \\ \end{array} \right).
\]

On the group \( G_\varepsilon \) we take for the Tamagawa measure the measure:
\[
d^*g = L(1, 1_F)|\varepsilon|_F \frac{da \, db}{|a^\varepsilon - \varepsilon bb|_F^2}
\]
if
\[
g = \left( \begin{array}{c} a \\ \varepsilon b \\ \end{array} \right).
\]

If \( \varepsilon \) is a norm the group \( G_\varepsilon(F) \) is isomorphic to \( \text{Gl}(2, F) \) and the isomorphism takes the above measure to the Tamagawa measure on \( \text{Gl}(2, F) \). From Lemma 2 we get the following integration formula:
\[
(15) \quad \int_{G_\varepsilon} F(g) \, d^*g = \frac{1}{L(1, \eta)^2} \int_{x = \varepsilon a \pi} \{ \int \int F\left[t_1 \left( \begin{array}{cc} a^{-1} \varepsilon & 1 \\ \pi & 1 \end{array} \right) t_2 \right] dt_1 \, dt_2 \} \frac{dx}{|1 - x|_F^2}
\]
with \( t_1 \in T(F), t_2 \in T(F)/Z(F) \).

In view of the isomorphism
\[
\text{Gl}(2, F)Z(E)/Z(E) \cong \text{Gl}(2, F)/Z(F)
\]
we give to \( \text{Gl}(2, F)Z(E) \) the measure defined by
\[
dg = L(1, 1_E)\frac{da \, db}{|a|_F \, |b|_E} \, dx \, dy
\]
if
\[
g = \left( \begin{array}{c} ab \\ 0 \\ \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ y & 1 \end{array} \right) \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right).
\]

On the group \( H \) isomorphic to \( \text{Gl}(2, F)Z(E) \) the corresponding measure is given by
\[
L(1, 1_E)\frac{da \, db}{|a|_F \, |b|_E} \, dx_1 \, dy_1
\]
for
\[ g = \begin{pmatrix} ab & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \]

with \( a \in F^\times, \ b \in E^\times, \ x = x_1 \sqrt{\delta}, \ y = y_1 \sqrt{\delta^{-1}}, \ x_1, y_1 \in F \). Of course, this measure does not depend on the choice of \( \delta \). Using the exact sequence
\[ 1 \rightarrow H_0 \rightarrow H \xrightarrow{\kappa} F^\times \rightarrow 1, \]

we get the Tamagawa measure on \( H_0 \):
\[ dh_0 = L(1, \eta_{|b|E}) \frac{db}{|b|} \ dx_1 \ dy_1 \]
if
\[ h_0 = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \]

We need the following integration formula:

**Lemma 4.** — The quotient of the Tamagawa measure on \( \text{Gl}(2, E) \) by the Tamagawa measure on \( H_0 \) is the following measure on the symmetric space \( S_{\text{s}} \):
\[ ds = L(1, 1_F) \frac{dx \ dy \ dz}{|xy - z|^F}. \]

For \( f \) a smooth function of compact support on \( G_{\epsilon}(F) \) we define
\[ \Phi(x; f) := \int_{E^\times} \int_{F^\times} \Phi \left( \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} \right) \times \eta \omega(z) \ dx \ dy \ omega(y). \]

Now let \( \{ \epsilon_1, \epsilon_2 \} \) be a system of representatives for the classes of \( \text{Norm}(E^\times) \) in \( F^\times \), where \( \epsilon_1 \) is a norm. Let \( \Phi \in C_0^\infty(S(F)) \) and \( f_{\epsilon_i}, \ i = 1, 2, \) be functions in \( C_0^\infty(G_{\epsilon_i}(F)) \). We say that \( \Phi \) and the pair \( (f_{\epsilon_1}, f_{\epsilon_2}) \) have matching orbital integrals if
\[ U(x; \Phi) = H(x; f_{\epsilon_i}) \quad \text{for} \quad x \in \epsilon_i \text{Norm}(E^\times). \]

**Proposition 1.** — Given \( \Phi \) there is a pair \( (f_{\epsilon_1}, f_{\epsilon_2}) \) with matching orbital integrals.
To prove the result one needs to consider the behavior at infinity of the orbital integrals defined above as well as the other orbital integrals. This is done in [J2]. We will use this result in the following way: $f$ will be a smooth function of compact support on $\text{Gl}(2,E)$ and we will set

$$\Phi_f(s) = \int_{H_0(F)} f(gh_0) dh_0, \quad s = gw^t \varphi,$$

$$\Phi_f(s) = 0, \quad s \notin S_0(F).$$

If the pair $(f_{\epsilon_1}, f_{\epsilon_2})$ has matching orbital integrals with $\Phi_f$ we shall say that it has matching orbital integrals with $f$.

Consider now the unramified situation: the residual characteristic is odd, the extension $E/F$ is unramified, the conductor of $\psi_F$ is $O_F$, $\epsilon_1$ is a unit, $\Omega$ is unramified. Then

$$K_0 := \text{Gl}(2, O_E) \cap H_0(F)$$

is a maximal compact subgroup of $H_0(F)$ with volume $L(2,1_F)^{-1}$ and

$$K_\epsilon := \text{Gl}(2, O_E) \cap G_\epsilon(F)$$

is a maximal compact subgroup of $G_\epsilon$ with volume $L(2,1_F)^{-1}$. The unramified Tamagawa measures are obtained by multiplying the Tamagawa measures by the inverse of those volumes. Using the Cartan decomposition and the methods of [J1, pp.199–204], one can prove the following proposition:

**Proposition 2.** — Suppose the situation is unramified. Let $q$ be the cardinality of the residue field of $F$. For $n \geq 0$ let $\Phi_{2n}$ be the characteristic function of the set of matrices $s \in S(F)$, with integral entries such that $|\det s|_F = q^{-2n}$. Let $f_{2n}$ be the characteristic function of the set of matrices $g \in G_{\epsilon_1}(F)$ with integral entries such that $|\det g|_F = q^{-2n}$. Then $\Phi_{2n}$ and the pair $(f_{2n}, 0)$ have matching orbital integrals.

Finally, we recall the fundamental lemma for the unramified situation. Since $\epsilon_1$ is a norm, there is an isomorphism of $G_{\epsilon_1}(F)$ onto $G(F)$ taking $K_{\epsilon_1}$ to $K_F$ and the Hecke algebra $\mathcal{H}(G_{\epsilon_1}(F), K_{\epsilon_1})$ to the Hecke algebra $\mathcal{H}(\text{Gl}(2,F), K_F)$. We have thus a base change homomorphism $b$ from $\mathcal{H}(\text{Gl}(2,F), K_F)$ to $\mathcal{H}(G_{\epsilon_1}(F), K_{\epsilon_1})$.

**Proposition 3.** — For $f \in \mathcal{H}(K)$ the function $\Phi_f$ and the pair $(b(f), 0)$ have matching orbital integrals.

**Proof.** — This is really a reformulation of the corresponding result of [J2]. It can derived more directly from the previous proposition by using the methods of [JLR, pp.318–322].
2.2. Split case. — *Mutatis mutandis*, the above discussion applies to the split case where the quadratic extension is replaced by the algebra $E = F \oplus F$ with $F$ embedded diagonally in $E$ and Galois action $\tau(x, y) = (y, x)$. If we write $z = (x, y)$ then $\psi_E(z) = \psi(x + y)$ and so the self dual Haar measure on $E$ is $dz = dx dy$. To obtain the Tamagawa measure on $\text{Gl}(2, E)$ we must divide the self-dual additive measure $dg = dg_1 dg_2$ by $|\det g_1 g_2|_F$ and then multiply by $L(1, 1_F)^2$. So the isomorphism $\text{Gl}(2, E) \simeq \text{Gl}(2, F) \times \text{Gl}(2, F)$ preserves the Tamagawa measures.

Likewise, the group $H_0$ becomes the group of pairs $(h_1, h_2)$ with $h_2 = w^{-1} h_1 w$. Choosing the first factor in the decomposition $E = F \oplus F$, we have an isomorphism $H_0(F) \simeq \text{Gl}(2, F)$ which is also compatible with the Tamagawa measures.

The group $G_\epsilon(F)$ is the group of invertible matrices of the form

$$\begin{pmatrix} a & b \\ \frac{\epsilon}{a} & \frac{1}{a} \end{pmatrix},$$

with $a, b$ in $E$. We consider the measure $|\epsilon|_F L(1, 1_F) \frac{dadb}{|a\xi - \epsilon b\eta|_F}$. Thus $G_\epsilon(F)$ is the group of pairs

$$(g_1, g_2)$$

with $g_2 = w \epsilon g_1 w^{-1}$. Again $(g_1, g_2) \mapsto g_1$ defines an isomorphism $G_\epsilon(F) \simeq \text{Gl}(2, F)$ which takes the above measure to the Tamagawa measure.

The manifold $S(F)$ is the submanifold of pairs

$$S(F) = \{(s_1, s_2) \mid s_2 = ^t s_1\}$$

and is again isomorphic to $G(F)$ via $(s_1, s_2) \mapsto s_1$.

The torus $T(F)$ is the subgroup of pairs:

$$t = \left( \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \right).$$

A character $\Omega$ of $T(F)$ trivial on $Z(F)$ has the form

$$\Omega(t) = \Omega_1(a_1) \Omega_2(a_2)$$

with $\Omega_1 \Omega_2 = 1$.

In particular, if $f_1 \otimes f_2$ is a product function on $G(E)$ then

$$\Phi_{f_1 \otimes f_2}(s_1, s_2) = \int f_1(g_1 h_1) f_2(g_2 h_2) dh_0$$

becomes

$$\Phi_{f_1 \otimes f_2}(g) = \int f_1(gwh) f_2(w^{-1} h^{-1} w) dh = f_1 \ast \tilde{f}_2(gw)$$
where we have set
\[ \tilde{f}_2(g) = f_2(w^t gw). \]
With this identification, the orbital integrals for \( \Phi \in \mathcal{C}_0^\infty(S(F)) \) become:
\[
U(x; \Phi) = \int \int \int \Phi \left[ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w^{-1} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \right] \times \Omega_1(a) \Omega_2(b) \, da \, db \, dz.
\]
Similarly, for \( f_\epsilon \in \mathcal{C}_0^\infty(G_\epsilon(F)) \), the orbital integrals take the form:
\[
H(x; f_\epsilon) = \int \int \int f_\epsilon \left[ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} w_1^{-1} \epsilon \\ 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right] \times \Omega_1(a) \Omega_2(b) \, da \, db \, \Omega_1(c)^{-1} \, dc,
\]
with \( x = u_1 w_2 \epsilon \). The condition of matching \( U(x; \Phi) = H(x; f_\epsilon) \) is trivially verified with
\[
f_\epsilon(g) = \Phi \left[ g \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right].
\]
If \( \Phi = \Phi_{f_1 \otimes f_2} \) this becomes the condition of matching for the split case:
\[ (17) \quad f_\epsilon(g) = f_1 \ast \tilde{f}_2(gw_\epsilon). \]
If \( F \) is non-Archimedean, \( \Omega_1 \) unramified, the functions \( f_i \) bi-invariant under \( K_F = \text{Gl}(2, \mathcal{O}_F) \), and the Haar measure of \( K_F \) is 1, then \( f_2 = f_2, w_\epsilon \in K_F \) and\[
f = \Phi = f_1 \ast f_2.
\]

### 3. Global matching

We now consider an extension \( E/F \) of number fields.

#### 3.1. Global Haar measures.

On the group \( F_\mathbb{A}^\times \) we consider the Tamagawa measure \( d^\times x \) which is the (convergent) product of the local Tamagawa measures. We let \( F^1 \) be the group of ideles of norm 1 and use the exact sequence
\[
1 \rightarrow F^1 \rightarrow F_\mathbb{A}^\times \rightarrow \mathbb{R}_+^\times
\]
to define a measure on \( F^1 \) for which
\[
\text{vol}(F^1 / F_\mathbb{A}^\times) = \text{res}_{s=1} L(s, 1_F).
\]
Likewise for \( E \).

We also consider a finite set places \( S_0 \) of \( F \) and the let \( S \) be the corresponding set of places of \( E \). We assume that \( S_0 \) contains all the places at infinity, the places of even residual characteristic, the finite places which ramify in \( E \), and all the places where the character \( \psi \) is ramified (that is, the conductor is not the ring of integers). We also assume that \( \Omega \) is unramified outside \( S \). We enlarge \( S_0 \) as the need arises.

*Bulletin de la Société Mathématique de France*
On the group $\text{Gl}(2, F_{\lambda})$ we consider the Tamagawa measure $d^2g$ which is the convergent product of the local Tamagawa measures and multiply it by the factor $L^{S_0}(2, 1_F) := \prod_{v_0 \in S_0} L(2, 1_{F_{v_0}})$. In other words, the measure we are considering is the product of the Tamagawa measures $d^2g_{v_0}$ for $v_0 \in S_0$ times the measures $dg_{v_0}$, $v_0 \not\in S_0$, for which the measure of $\text{Gl}(2, O_{v_0})$ is one. Using the Iwasawa decomposition
\[ g = \text{diag}(a, b) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k \]
we obtain a measure on the standard maximal compact $K$:
\[ dg = d^2ad^2bdxdk. \]

We have then the following identity for a Schwartz-Bruhat function $\Phi$ in two variables:
\[ \int \int \Phi(x, y) dx dy = \int \int \Phi([0, t]^k) \cdot |t|^2 d^2x \, dt \, dk \frac{1}{L^{S_0}(2, 1_F)}. \]

If $\Phi = \prod \Phi_{v_0}$ with $\Phi_{v_0}$ the characteristic function of $O_{v_0}^2$ outside $S_0$ we get then:
\[ \int \int \Phi(x, y) dx dy = \int_{K_{S_0}} \int_{F_{S_0}^x} \Phi_{v_0}([0, t]^k) \cdot |t|^2 d^2x \, dt \, dk. \]

We use the similar measure on $\text{Gl}(2, E_{\lambda})$. Likewise for $H_0(F_{\lambda})$ and $H(F_{\lambda})$ we multiply the Tamagawa measure by $L^{S_0}(2, 1_F)$.

3.2. Matching. — As explained in the introduction, we consider a smooth function of compact support $f$ on $G(E_{\lambda})$ and the corresponding function $\Phi = \Phi_f$ on $S(F_{\lambda})$. The function $f$ is a product of local functions $f_v$. For $v \not\in S$ the function $f_v$ is bi-invariant under $K_v := \text{Gl}(2, O_v)$. We choose a set of representative $\{ \epsilon \}$ for $F^\times / \text{Norm}(E^\times)$. For each $\epsilon$ we choose a smooth function of compact support $f_\epsilon$ on $G_{\epsilon}(F_{\lambda})$. It is a product of local functions that we choose as follows. For a place $v_0$ inert in $F$ and the corresponding place $v$ of $E$ we demand that $f_{\epsilon, v_0}$ and $\Phi_{v_0}$ have matching orbital integrals, that is, $H(x; f_{\epsilon, v_0}) = U(x; \Phi_{v_0})$ for $x \in \epsilon \text{Norm}(E_{\lambda})$. If $v_0 \not\in S_0$ and $\epsilon$ is not a norm at the place $v_0$, we have seen that $H(x; \Phi_{v_0}) = 0$ for $x \in \epsilon \text{Norm}(E_{\lambda})$ and so we take $f_{\epsilon, v_0} = 0$. Thus we have $f_\epsilon = 0$ unless $\epsilon$ is a norm at all places not in $S_0$. Thus there is only a finite set $\Xi$ of $\epsilon$ such that $f_\epsilon \neq 0$. There is a finite set $S_0' \supseteq S_0$ of places of $F$ such that the $\epsilon \in \Xi$ are unit at all places not in $S_0'$. We let $S'$ be the set of places of $E$ above a place in $S_0'$.

For $v_0$ inert not in $S_0'$ we may and do assume that $f_{\epsilon, v_0} = b(f_v)$ where $v$ is the corresponding place of $E$ and $b$ is the base change homomorphism. For $v_0$ split into $v_1$ and $v_2$ in $E$ we assume that
\[ f_{\epsilon, v_0}(g) = \Phi_{v_0} \left[ g \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right]. \]
In particular, for $\epsilon \in \Xi$ and $v_0 \notin S'_0$ this means that (with the usual identifications) $f_{v_0} = f_1 * f_2$. This being so, we change our notations and now take the set $S_0$ to be the set $S'_0$.

We compare the geometric terms in the two relative trace formulas. As usual we set

$$e^{H(g)} = |a_1a_2^{-1}|$$

if $g = n a k$, $a = \text{diag}(a_1, a_2)$.

As in [J2], we consider a compact truncation along the diagonal. Namely, if $\phi$ is a function on $A(E_k)$ invariant under $A(E)$ and $Z(E_k^+)$, we set:

$$\Lambda_T^\epsilon \phi(a) = \phi(a) \quad \text{if} \quad -T < H(a) < T, \quad (18)$$

$$\Lambda_T^\epsilon \phi(a) = 0 \quad \text{otherwise.} \quad (19)$$

We define then

$$\Theta_T(f) := \int_{Z(E_k)H(F) \setminus H(F_k)} \Lambda_T^\epsilon K_f(a, h)\Omega^{-1}(a) \omega h(\epsilon(h)) \, dh.$$ 

The outer integral is over $A(E_k) \setminus A(E)Z(E_k)$. This integral is absolutely convergent and equal to:

$$\int_{e^{-T} \leq |a| \leq e^T} \int_{F_k^+ / F^+} \sum_{\xi \in \xi \xi_0} \Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \xi \left( \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} z \right) \Omega^{-1}(a) \omega h(\xi(z)) \right) \, d^d z.$$ 

In [J2] it is shown that as $T \to +\infty$ this tends to the right hand side of (14). Thus we define

$$\Theta(f) = \lim_{T \to +\infty} \Theta_T(f),$$

and we arrive at (14). Comparing with (8) we get:

$$\Theta(f) = \sum_{\epsilon \in \Xi} \theta(f_\epsilon).$$

Indeed given $\xi \in F^\times \setminus \{1\}$ we have $\xi \in \epsilon \text{Norm}(E^\times)$ for a unique $\epsilon$ and

$$U(\xi; \Phi) = H(\xi; f_\epsilon).$$

The matching of the other terms is explained in [J2].

4. Factorization of $\Theta$ over $E$

Most of the material of this section is a review of [HLR] (see also [A], [F2], [FZ]).

We now consider the Epstein Eisenstein series associated to a Schwartz-Bruhat function $\Phi$ on $F_k^\times$: we assume that $\Phi$ is a product, the local component
\( \Phi_{v_0} \) being the characteristic function of \( \mathcal{O}^2_{v_0} \) for all \( v_0 \notin S_0 \):

\[
E(g, \Phi, s) = \int_{F^2 \backslash F} \sum_{\xi \in F^2 - \{(0,0)\}} \Phi[\xi t g] \cdot |\det t|^{2s} d^2 t |\det g|^s.
\]

It is meromorphic, with simple pole at \( s = 1 \) and residue:

\[
\frac{1}{2} \text{vol} \left( \frac{F^1}{F^\times} \right) \int_{F^1 \backslash F} \Phi(x, y) dx dy = \frac{1}{2} \left( \frac{F^1}{F^\times} \right) \int_{K_{S_0}} \int_{F^\times_{S_0}} \Phi_{S_0}[(0, t) k] \cdot |t|^{2s} d^2 t dk.
\]

Let \( \Pi \) be the base change of a representation \( \pi \) with central character \( \eta \) and \( \phi \) a cusp form in the space of \( \Pi \) which is invariant under translations by the compact subgroup \( K^S := \prod_{v \notin S} K_{E_v} \). Let us compute the integral:

\[
\Psi(s, \phi, \Phi) := \int_{G(F)Z(F) \backslash G(F)} E(g, \Phi, s) \phi(g) dg.
\]

It is a meromorphic function of \( s \), with simple pole at \( s = 1 \) and residue:

\[\text{(20)} \quad \frac{1}{2} \text{vol} \left( \frac{F^1}{F^\times} \right) \int_{K_{S_0}} \int_{F^\times_{S_0}} \Phi_{S_0}[(0, t) k] \cdot |t|^{2s} d^2 t dk \int \phi(g) dg.\]

On the other hand, for \( \text{Re } s \gg 0 \),

\[\Psi(s, \phi, \Phi) = \Psi(s, W, \Phi)\]

where we have set:

\[W(g) := \int_{E_v} \phi \left( \begin{pmatrix} 1 & \bar{t} \\ 0 & 1 \end{pmatrix} \right) \psi_E(-x) dx\]

and, writing \( E = F[\sqrt{\delta}] \),

\[\Psi(s, \phi, \Phi) := \int_{N(F) \backslash \text{GL}(2, F)} W \left( \begin{pmatrix} \sqrt{\delta}^{-1} & 0 \\ 0 & 1 \end{pmatrix} g \right) \Phi[(0, 1) g] \cdot |\det g|^s d^2 g.\]

We assume that \( W \) is a product

\[W(g) = \prod_v W_v(g_v)\]

over all places of \( E \), with \( W_v|_{K_{E_v}} = 1 \) outside \( S \). The local component is an element of the local Whittaker model \( W(\Pi_v, \psi_v) \). If \( v_0 \) is a place of \( F \) we write \( E_{v_0} = E \otimes F_{v_0} \). If \( v_0 \) is inert in \( E \) we also write \( W_{v_0} \) for \( W_v \). If \( v_0 \) splits into \( v_1, v_2 \) we write \( W_{v_0} \) for \( W_{v_1} \otimes W_{v_2} \), a function on \( \text{GL}(2, E_{v_0}) \). We can then set in all cases:

\[\Psi(s, W_{v_0}, \Phi_{v_0}) = \int_{N(F_{v_0}) \backslash \text{GL}(2, F_{v_0})} W_{v_0} \left( \begin{pmatrix} \sqrt{\delta}^{-1} & 0 \\ 0 & 1 \end{pmatrix} g \right) \Phi_{v_0}[(0, 1) g] \cdot |\det g|^s d^2 g.\]
Explicitly, if \( v_0 \) splits, that is, \( \delta \) is the square of two elements \( \epsilon_1 \) and \( \epsilon_2 \) of \( F_{v_0} \), the above integral is in fact equal to:

\[
\int W_{v_0} \left[ \begin{pmatrix} \epsilon_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} g \right] W_{v_0} \left[ \begin{pmatrix} \epsilon_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} g \right] \Phi_{v_0} [(0, 1)g] \cdot | \det g |^s \mathrm{d}^x g.
\]

We recall that \( S_0 \) is so large that all the data is unramified outside \( S_0 \) (or \( S \)).

We introduce the partial Asai \( L \)-function attached to \( \Pi \) noted \( L^{S_0}(s, \Pi, \text{Asai}) \) (see [HLR]). For every place \( v_0 \not\in S_0 \) the integral \( \Psi(s, W_{v_0}, \Phi_{v_0}) \) is equal to the corresponding local \( L \)-factor of the Asai \( L \)-function. Thus

\[
\Psi(s, \phi, \Phi) = L^{S_0}(s, \Pi, \text{Asai}) \prod_{v_0 \in S_0} \Psi(s, W_{v_0}, \Phi_{v_0}).
\]

Taking the residue at \( s = 1 \) we obtain that (20) is equal to:

\[
\text{res}_{s=1} L^{S_0}(s, \Pi, \text{Asai}) \prod_{v_0 \in S_0} \Psi(s, W_{v_0}, \Phi_{v_0}).
\]

This equality implies that, as a linear form in \( W_{v_0} \), \( \Psi(1, W_{v_0}, \Phi_{v_0}) \) is invariant under right shifts by \( \text{Gl}(2, F_{v_0}) \). As a linear form in \( \Phi_{v_0} \), it is proportional to

\[
\int_{K_{v_0}} \int_{F_{v_0}^x} \Phi_{v_0} [(0, t)k] \cdot |t|^2 \mathrm{d}^y t \mathrm{d}k.
\]

Now let us set:

\[
\mathcal{P}_{v_0}(W_{v_0}) := \int_{F_{v_0}^x} W_{v_0} \left( \begin{pmatrix} \sqrt{\epsilon_1^{-1}} & 0 \\ 0 & 1 \end{pmatrix} a \right) \mathrm{d}^y a.
\]

This is a linear form invariant under \( \text{Gl}(2, F_{v_0}) \). Moreover, in the Archimedean case, it is continuous for the topology of the smooth vectors. The invariance is well known for \( v_0 \) split (see also [B]) and is established in [H] for \( v_0 \) inert. At any rate, it follows from the above considerations. Then

\[
\Psi(1, W_{v_0}, \Phi_{v_0}) = \mathcal{P}_{v_0}(W_{v_0}) \int_{K_{v_0}} \int_{F_{v_0}^x} \Phi_{v_0} [(0, t)k] \cdot |t|^2 \mathrm{d}^y t \mathrm{d}k.
\]

The above residue becomes:

\[
\text{res}_{s=1} L^{S_0}(s, \Pi, \text{Asai}) \prod_{S_0} \mathcal{P}_{v_0}(W_{v_0}) \int_{K_{v_0}} \int_{F_{v_0}^x} \Phi_{S_0} [(0, t)k] \cdot |t|^2 \mathrm{d}^y t \mathrm{d}k.
\]

Comparing with the previous residue computation, we get:

\[
(21) \int \phi(g) \mathrm{d}g = \prod_{v_0 \in S_0} \mathcal{P}_{v_0}(W_{v_0}) \frac{2 \text{res}_{s=1} L^{S_0}(s, \Pi, \text{Asai})}{\text{vol}(F^1/F^x)}.
\]

The identity is to be interpreted as the statement that the period integral of \( \phi \) is non-zero if and only the Asai \( L \)-function has a pole at \( s = 1 \) and each of the local integral \( \mathcal{P}_{v_0} \) does not vanish on \( W_{v_0} \).
Remark. — The convergence of the integral defining \( \mathcal{P}_{v_0} \) follows from the local analog of the integral representation for Whittaker functions given below in (44). Formula (21) implies that the linear form \( \mathcal{P}_{v_0} \) is continuous for the topology of smooth vectors, since the left hand side is.

Recall that we write \( E = F[\sqrt{\delta}] \). Then

\[
\begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1 \end{pmatrix} H(F) \begin{pmatrix} \sqrt{\delta}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = Z(E) \text{ Gl}(2, F).
\]

Accordingly:

\[
\int_{H(F)Z(E_\pi)\backslash H(F_\pi)} \phi(h) \mathrm{d}h = \int_{Z(F_\pi) \text{ Gl}(2,F) \backslash \text{ Gl}(2,F_\pi)} \phi \left[ g \begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1 \end{pmatrix} \right] \mathrm{d}g
\]

or

\[
\int_{H(F)Z(E_\pi)\backslash H(F_\pi)} \phi(h) \mathrm{d}h = \prod_{S_0} \mathcal{P}_{v_0}(W_{v_0}) \frac{2 \operatorname{res}_{s=1} L_{S_0}^S(s, \Pi, \text{Asai})}{\operatorname{vol}(F^1/F^\times)}
\]

where here we have set:

\[
\mathcal{P}_{v_0}(W_{v_0}) = \int_{F_{v_0}^\times} W_{v_0} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \mathrm{d}^\times a.
\]

We also denote by \( L_{S_0}^S(s, \Pi, \text{Asai};1) \) the previous Asai \( L \)-function and by \( L(s, \Pi, \text{Asai}; \eta) \) the function \( L(s, \Pi \otimes \mu, \text{Asai};1) \) where \( \mu \) is an idele class character of \( E \) whose restriction to \( F \) is \( \eta \). Suppose that now \( \Pi \) is the base change of \( \pi \) with trivial central character. Then:

\[
\int_{H(F)Z(E_\pi)\backslash H(F_\pi)} \phi(h) \eta(\kappa(h)) \mathrm{d}h = \prod_{S_0} \mathcal{P}_{v_0}(W_{v_0}) \frac{2 \operatorname{res}_{s=1} L_{S_0}^S(s, \Pi, \text{Asai}; \eta)}{\operatorname{vol}(F^1/F^\times)}
\]

where here we have set:

\[
\mathcal{P}_{v_0}(W_{v_0}) = \int_{F_{v_0}^\times} W_{v_0} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \eta(a) \mathrm{d}^\times a.
\]

From now on we consider both cases at once, that is, \( \Pi \) is the base change of \( \pi \) with central character \( \omega = 1 \) or \( \omega = \eta \).

A similar discussion applies to the computation of the scalar product of two forms \( \phi_1, \phi_2 \). As usual we assume that the corresponding functions \( W_i \) are product and the data is unramified outside \( S \). We introduce a scalar product \( \mathcal{B}_v \) on \( \mathcal{W}(\Pi_v, \psi_v) \) for any place \( v \in S \):

\[
\mathcal{B}_v(W_1, W_2) := \int_{E_v^\times} W_1 \left[ \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \right] W_2 \left[ \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathrm{d}^\times a_1.
\]
Then:
\[ \int \phi_1(g) \bar{\phi}_2(g) \, dg = \prod_{v \in S} B_v((W_1)_v, (W_2)_v) \left( \frac{\text{res}_{s=1} L^S(s, \Pi \times \tilde{\Pi})}{\text{vol}(E^1/E^\vee)} \right). \]

We set:
\[ \lambda(\phi) := \int_{E_v^1/E_v^\vee} \phi \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \Omega^{-1}(a) \, d^v a. \]

At a place \( v \in S \), we define
\[ \lambda_v(W_v) := \int_{E_v^1/E_v^\vee} W_v \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \Omega^{-1}(a) \, d^v a. \]

Then
\[ \lambda(\phi) = \prod_{v \in S} \lambda_v(W_v) L^S \left( \frac{1}{2}, \Pi \otimes \Omega^{-1} \right). \]

We consider the global distribution
\[ \Theta_\Pi(f) = \sum_{\phi} \lambda(\Pi(f) \phi) \int \phi(h) \omega \eta(\kappa(h)) \, dh, \]
where the sum is over an orthonormal basis. Likewise, for a place \( v_0 \) inert in \( E \) with corresponding place \( v \) of \( E \) we define
\[ \Theta_{\Pi, v_0}(f_{v_0}) := \sum_{W_{v_1}, W_{v_2}} \lambda_{v_1}(\Pi_{v_1}(f_{v_1}W_{v_1})) \lambda_{v_2}(\Pi_{v_2}(f_{v_2}W_{v_2})) \overline{P_{v_0}(W_{v_1} \otimes W_{v_2})}, \]
the sum over an orthonormal basis for the scalar product \( B_v \). Whenever convenient, we write \( f_{v_0} \) for \( f_v \).

Similarly, at a place \( v_0 \) which splits into \( v_1, v_2 \) we define
\[ \Theta_{\Pi, v_0}(f_{v_1} \otimes f_{v_2}) := \sum_{W_{v_1}, W_{v_2}} \lambda_{v_1}(\Pi_{v_1}(f_{v_1}W_{v_1})) \lambda_{v_2}(\Pi_{v_2}(f_{v_2}W_{v_2})) \overline{P_{v_0}(W_{v_1} \otimes W_{v_2})}. \]

We again write \( f_{v_0} \) for \( f_{v_1} \otimes f_{v_2} \).

Now we assume that the function \( f \) is a product with \( f^S \) the characteristic function of \( R^S \). Then:
\[ \Theta_{\Pi, v_0}(f_{v_0}) = \prod_{v_0 \in S_0} \Theta_{\Pi, v_0}(f_{v_0}) \left( \frac{\text{vol}(E^1/E_v^\vee) \text{res}_{s=1} L^S_0(s, \Pi, \text{Asai}; \omega \eta) L^S \left( \frac{1}{2}, \Pi \otimes \Omega^{-1} \right)}{\text{vol}(F^1/F^\vee) \text{res}_{s=1} L^S(s, \Pi \times \tilde{\Pi})} \right). \]

Recall that here \( \tilde{\Pi} = \Pi = \Pi^\tau \) and \( \Pi \) is the base change of \( \pi \) with central character \( \omega \). We have the relation
\[ L^S(s, \Pi \times \Pi^\tau) = L^S_0(s, \pi \times \pi) L(s, \pi \times \pi \otimes \eta). \]

If \( \omega = 1 \) we have the relation
\[ L^S_0(s, \Pi, \text{Asai}; \eta) = L^S_0(s, \pi \times \pi \otimes \eta) L^S_0(s, 1_{F'}). \]
Moreover since \( \pi \) is self contragredient then
\[
L^{S_0}(s, \pi \times \pi) = L^{S_0}(s, 1_F)L^{S_0}(s, \pi; \text{Ad}).
\]
where the last factor on the right is the partial adjoint \( L \)-function. Thus
\[
\frac{L^{S_0}(s, \Pi, \text{Asai}; \eta)}{L^{S}(s, \Pi \times \Pi)} = \frac{1}{L^{S_0}(s, \eta)L^{S_0}(s, \pi, \text{Ad})}.
\]
If \( \omega = \eta \) then
\[
L^{S_0}(s, \Pi, \text{Asai}; 1) = \frac{L^{S_0}(s, \pi \times \pi)L^{S_0}(s, 1_F)}{L^{S_0}(s, \eta)}.
\]
On the other hand
\[
L^{S_0}(s, \pi \times \pi \otimes \eta) = L^{S_0}(s, \pi \times \tilde{\pi}) = L^{S_0}(s, 1_F)L^{S_0}(s, \pi, \text{Ad}).
\]
Thus
\[
\frac{L^{S_0}(s, \Pi, \text{Asai}; 1)}{L^{S}(s, \Pi \times \Pi)} = \frac{1}{L^{S_0}(s, \eta)L^{S_0}(s, \pi, \text{Ad})}.
\]
Also
\[
\text{vol}(E^1/E^\omega) = L(1, \eta)\text{vol}(F^1/F^\omega).
\]
We can thus simplify our ratio of \( L \)-functions to obtain the following result:

**Proposition 4.** — One has
\[
\Theta_{\Pi}(f) = \prod_{v_0 \in S_0} \Theta_{\Pi, v_0}(f_{v_0}) \frac{L_{S_0}(1, \eta)L^{S}(\frac{1}{2}, \Pi \otimes \Omega^{-1})}{L^{S_0}(1, \pi, \text{Ad})}.
\]

5. Local comparisons

As explained in the introduction, when \( f \) and the family \((f_\epsilon)\) match we have a spectral identity, with a single \( \epsilon \) on the right, which is uniquely determined by \( \Pi \):
\[
\Theta_{\Pi}(f) = \theta_{\sigma_\epsilon}(f_\epsilon) + \theta_{\sigma_\epsilon \otimes \eta}(f_\epsilon).
\]
We recall the expression for \( \theta_{\sigma_\epsilon} \):
\[
\theta_{\sigma_\epsilon}(f_\epsilon) = \sum_{\phi} \int \sigma_\epsilon(f_\epsilon)\phi(t)\Omega(t)^{-1}dt \int \phi(t)\Omega(t)dt.
\]
Let us write \( \sigma \) for \( \sigma_\epsilon \).
5.1. Local comparison: inert case. — Now consider a place $v_0$ of $F$, inert in $E$. Let $v$ be the corresponding place of $E$. If the distribution $\theta_\sigma$ is not identically 0, then $\sigma_{v_0}$ admits non-zero linear forms transforming under the characters $\Omega_v$ and $\Omega_v^{-1}$ of $T_{v_0}$, a group compact modulo the center. Thus there are smooth unit vectors $e_T$ and $e_T'$ such that

$$\sigma_{v_0}(t)e_T = \Omega(t)^{-1}e_T, \quad \sigma_{v_0}(t)e_T' = \Omega(t)e_T'. $$

We may assume that $e_T' = \sigma_{v_0}(w_\epsilon)(e_T)$.

We define a local distribution:

$$\theta_{\sigma_{v_0}}(f_{v_0}) = \sum_{u_i} \langle \sigma_{v_0}(f_{v_0})u_i, e_T' \rangle \langle u_i, e_T \rangle,$$

the sum over an orthonormal basis. This can also be written as

$$\int f(g)\omega_{\sigma_{v_0}}(g)dg,$$

where we have set

$$\omega_{\sigma_{v_0}}(g) = \langle \sigma_{v_0}(g)e_T, e_T' \rangle.$$

Note that $\omega_{\sigma_{v_0}}(w_\epsilon) = 1$. There is a similar distribution $\theta_{\sigma_{v_0} \otimes \eta_{v_0}}$ defined by:

$$\theta_{\sigma_{v_0} \otimes \eta_{v_0}}(f_{v_0}) = \sum_{u_i} \langle \sigma_{v_0} \otimes \eta_{v_0}(f_{v_0})u_i, e_T' \rangle \langle u_i, e_T \rangle,$$

as well as a function

$$\omega_{\sigma_{v_0} \otimes \eta_{v_0}}(g) = \langle \sigma_{v_0} \otimes \eta_{v_0}(g)e_T, e_T' \rangle,$$

where we have set:

$$e_T' = \sigma_{v_0} \otimes \eta_{v_0}(w_\epsilon)(e_T) = \eta_{v_0}(-\epsilon)e_T'. $$

Thus

$$\omega_{\sigma_{v_0} \otimes \eta_{v_0}}(g) = \eta_{v_0}(\det g)\eta_{v_0}(-\epsilon)\omega_{v_0}(g). $$

In particular, if $\det g \in -\epsilon \text{Norm}(E^\times)$ then $\omega_{\sigma_{v_0}}(g) = \omega_{\sigma_{v_0} \otimes \eta_{v_0}}(g)$.

The global distribution $\theta_\sigma$ decomposes into a tensor product

$$\theta_\sigma(f_{v_0} \otimes f_{v_0}) = \theta_{v_0}(f_{v_0}) \theta_{v_0} \sigma_{v_0}(f_{v_0}).$$

 Likewise, we have obtained a decomposition of the global distribution $\Theta_\Pi$ into a tensor product $\Theta^v \otimes \Theta_{\Pi_{v_0}}$. The global spectral identity gives then a linear relation of the form:

$$\Theta_{\Pi_{v_0}}(f_v) = C_1 \theta_{\sigma_{v_0}}(f_{v_0}) + C_2 \theta_{\sigma_{v_0} \otimes \eta_{v_0}}(f_{v_0}).$$

To analyze the situation conveniently, let us go back to a local situation. That is, let $E/F$ be a local quadratic extension and $\Pi$ an irreducible unitary representation of $G(E)$ with trivial central character. We assume that $\Pi$ is the
base change of a representation $\pi$ of $G(F)$ with central character $\omega$, and $\omega = 1$ or $\omega = \eta_{EF}$. We assume that $\epsilon$ is such that the representation $\sigma$ of $G_\epsilon(F)$ corresponding to $\pi$ has a vector transforming under $\Omega$ and $\Omega^{-1}$ with $\Omega_{EF^\epsilon} = \omega$.

We assume that we have a relation of the form:

$$\Theta_{\Pi}(f) = C_1 \theta_{\sigma}(f_\epsilon) + C_2 \theta_{\sigma \otimes \eta}(f_\epsilon)$$

for any pair of functions $(f, f_\epsilon)$ with matching orbital integrals; that is, $f$ correspond to a function $\Phi$ on $S_x$ and $U(x; \Phi) = H(x; f_\epsilon)$ for $x \in \epsilon \text{Norm}(E^\epsilon)$.

In particular, if $f_\epsilon$ is supported on the set of $g \in G_\epsilon$ such that $\det(g) \in -\epsilon \text{Norm}(E^\epsilon)$ then this simplifies to

$$\Theta_{\Pi}(f) = C \int_{G_\epsilon(F)} \omega_{\sigma}(g) f_\epsilon(g) dg, \quad C = C_1 + C_2.$$  

Our goal in this section is to compute the constant $C$.

**Proposition 5.** — Suppose that $f$ and $f_\epsilon$ have matching orbital integrals and $f_\epsilon$ is supported on the set $\{g \in G_\epsilon(F) \mid \det(g) \in -\epsilon \text{Norm}(E^\epsilon)\}$. Then

$$\Theta_{\Pi}(f) = \epsilon(1, \eta, \psi_F) 2\eta(-\epsilon) L(0, \eta) \theta_{\sigma}(f_\epsilon).$$

Moreover, we can choose the pair in such a way that both sides are non-zero.

5.2. Computation of $\Theta_{\Pi}$. — To prove the proposition, we first consider a sequence of functions $(f_j)$ on $G(E)$ which tends to $\delta_\epsilon$. That is, $f_j \geq 0$, $\int f_j dg = 1$ and $\text{supp} f_j \rightarrow \epsilon$. We let $\Phi_j$ be the associated function on $S$. It tends to $\delta_w$ on $S(F)$. Note that we can start with an approximation $\Phi_j$ of $\delta_w$ on $S$ and then choose for the functions $f_j$ an approximation of $\delta_\epsilon$. We also choose a function $\phi \in C_0^\infty (E)$ of small support. We set

$$f^j(g) = \int_E \phi(x) f_j \left[ \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right] dx$$

and we denote by $\Phi^j$ the corresponding function on $S(F)$.

We first compute $\Theta_{\Pi}(f^j)$. Recall that the scalar product is given by:

$$\langle W_1, W_2 \rangle = \int_{E^J} W_1 \overline{W_2} [\text{diag}(a, 1)] d^*a.$$  

and

$$\Theta_{\Pi}(f) = \sum \lambda(\Pi(f) W_i) F(W_i).$$
where $W_i$ is an orthonormal basis. We have:

$$
\lambda(\Pi(f^j)W_i) = \iint W_i \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \Omega^{-1}(a)f^j(g) d^2a dg \\
= \iint W_i \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \left( \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right) \Omega^{-1}(a)f_j(g)\phi(x) d^2a dx dg \\
= \iint \hat{\phi}(a)W_i \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \Omega^{-1}(a)f_j(g) d^2a dg
$$

where $\hat{\phi}$ denotes the Fourier transform of $\phi$.

For now we assume that $E$ is non-Archimedean. We choose $\phi$ in such a way that $\hat{\phi}(0) = 0$. Then there is an element $W^0 \in \mathcal{W}(\Pi, \psi)$ such that

$$
W^0 \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] = \hat{\phi}(a) \Omega^{-1}(a).
$$

Then the above expression is

$$
\langle \Pi(f_j)W_i, W^0 \rangle = \langle W_i, \Pi(f_j^*)W^0 \rangle.
$$

Since

$$
\Pi(f_j^*)W^0 = \sum_i \langle \Pi(f_j^*)W^0, W_i \rangle W_i
$$

we see that:

$$
\Theta_{\Pi}(f^j) = \mathcal{P}(\Pi(f_j^*)W^0).
$$

Since $f_j$ is an approximation of $\delta_c$, for $j$ large enough, $\Pi(f_j^*)W^0 = W^0$ and

$$
\Theta_{\Pi}(f^j) = \mathcal{P}(W^0) = \int W^0 \left[ \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \right] \omega\eta(b) d^2b = \int_{\mathbb{F}_x} \hat{\phi}(b)\eta(b) d^2b.
$$

Clearly, we can choose $\phi$ with $\hat{\phi}(0) = 0$ in such a way that this is non-zero.

We pass to the Archimedean case. We first explain how to choose the function $\phi$. We assume, as we may, that $\psi_F(x) = \exp(2i\pi x)$. Let $\phi_0$ be a smooth function of compact support on $F$ with $\int \phi_0(x) dx \neq 0$. Then the function $\phi_1 = \phi_0 \ast \phi_0^*$ is smooth of compact support. Its Fourier transform is $\hat{\phi}_1 = |\hat{\phi}_0|^2$ and is thus $\geq 0$. Moreover, $\hat{\phi}_1(0) = |\hat{\phi}_0(0)|^2 > 0$. Now consider the function on $E$ defined by:

$$
\phi_2(x + iy) = \phi_1(x)\phi_1(y).
$$

The Fourier transform on $E$ of this function is given by

$$
\hat{\phi}_2(x + iy) = 2\hat{\phi}_1(2x)\hat{\phi}_1(-2y).
$$

Let $Q > 0$ be an integer. There is a smooth function of compact support $\phi$ on $E$, whose Fourier transform is given by:

$$
\hat{\phi}(z) = 2^{2Q+1}(z\overline{z})^Q z\hat{\phi}_1(2x)\hat{\phi}_1(-2y), \quad z = x + iy.
$$
Then
\[
\int_{F^*} \hat{\phi}(x) \eta(x) d^\times x = \hat{\phi}_1(0) \int |x|^{2Q+1} \hat{\phi}_1(x) d^\times x > 0.
\]
This function, which will be also denoted by \( \phi_Q \) to stress the dependence on \( Q \), will have the required properties, provided the integer \( Q \) is sufficiently large.

To continue we recall that the unitary representation \( \Pi \) can be realized in the Hilbert space \( \mathcal{H} \) of square integrable functions on \( E^\times \). Let \( P_1 \) be the group of matrices with last row \((0, 1)\). The representation of \( P_1(E) \) on \( \mathcal{H} \) is the representation induced by the generic character of \( N(E) \) determined by \( \psi_E \). It is (topologically) irreducible. The subspace \( \mathcal{V} \) of smooth vectors may be viewed as a certain space of smooth functions on \( E^\times \) (Kirilov model). If \( W \) is in the Whittaker model of \( \Pi \) then the function \( \Phi \) defined by
\[
\Phi(a) = W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
is in \( \mathcal{V} \) and determines \( W \). Let \( \mathcal{V}' \) be the topological dual of \( \mathcal{V} \) (space of generalized vectors). We have the inclusions: \( \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' \). We extend the scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \) to \( \mathcal{V} \times \mathcal{V}' \) and \( \mathcal{V}' \times \mathcal{V} \). The group and its enveloping algebra operate on \( \mathcal{V} \) and \( \mathcal{V}' \). For instance, the Whittaker linear form from \( \mathcal{W} \) is the element of \( \mathcal{V}' \) such that, for \( \Phi \in \mathcal{V} \):
\[
\langle \Phi, W \rangle = \Phi(e).
\]
Likewise, \( \lambda \) and \( \mathcal{P} \) are elements of \( \mathcal{V}' \) such that
\[
\langle \Pi, \lambda \rangle = \int_{E^\times} \Phi(a) \Omega^{-1}(a) d^\times a, \quad \langle \Phi, \mathcal{P} \rangle = \int_{F^\times} \Phi(b) \omega \eta \eta(b) d^\times b,
\]
and
\[
\Theta_{\Pi}(f) = \langle \Pi(f) \mathcal{P}, \lambda \rangle = \langle \mathcal{P}, \Pi(f^*) \lambda \rangle.
\]
We have already remarked that the convergence of these integrals follows at any rate from the local analog of the integral representation for Whittaker functions given below (see (44)). Moreover, the continuity follows for instance from the global theory (since we assume anyway that \( \Pi \) is a component of a cuspidal representation). The continuity of \( \mathcal{P} \) means that there is \( M \) such that, if \( (X_\alpha) \), \( 1 \leq \alpha \leq N \), is a basis of the space of elements of the enveloping algebra with (filtration) degree \( \leq M \), then, for suitable constants \( c_\alpha > 0 \),
\[
|\langle \Phi, \mathcal{P} \rangle| \leq \sum_\alpha c_\alpha ||\Pi(X_\alpha) \Phi||.
\]
In this sum \( X_1 = 1 \). Let \( \mathcal{H}^N \) be the Hilbert sum of \( N \) copies of \( \mathcal{H} \) and \( \mathcal{V}_0 \) the (non-closed) subspace of \( N \)-tuples of the form \( v = (\Pi(X_\alpha) \phi) \), with \( \phi \in \mathcal{V} \). Let \( \mathcal{P}_0 \) be the linear form on \( \mathcal{V}_0 \) defined by \( \mathcal{P}_0(v) = \langle \phi, \mathcal{P} \rangle \). This linear form is continuous for the topology induced by the topology of \( \mathcal{H}^N \), thus extends.
to the closure of $V_0$ in $\mathcal{H}^N$ and is therefore given by the scalar product with a vector $(\Phi_\alpha) \in \mathcal{H}^N$. In other words,

$$(30) \quad \langle \Phi, \mathcal{P} \rangle = \sum_\alpha \langle \Pi(X_\alpha) \Phi, \Phi_\alpha \rangle$$

with $\Phi_\alpha \in \mathcal{H}$.

Now the scalar product $\langle \Phi, \mathcal{P} \rangle$ is defined if $\Phi$ is a vector of class $C^M$. We will prove the following assertion:

**Lemma 5.** — If $Q$ is sufficiently large, then the function $\Phi^0$ defined

$$\Phi^0(z) = \overline{\phi_Q(z)} \Omega(z)^{-1}$$

is square integrable and is, in fact, a vector of class $C^M$ in the representation. Moreover:

$$\mathcal{P}(\Phi^0) = \int_{F^\times} \Phi^0(x) \omega(x) d^\times x = \int_{F^\times} \hat{\phi_Q}(x) \eta(x) d^\times x.$$

We prove the lemma in the case where $\omega = \eta$. Then the restriction of $\Omega$ to $F^\times$ is $\eta$. Thus $\Omega$ has the form:

$$\Omega(z) = \frac{z^m}{(z \bar{z})^{m/2}},$$

where $m$ is an odd integer. Taking a large enough $Q$, we see that the function $\Phi^0$ of the above lemma has in turn the form

$$\Phi^0(z) = (z \bar{z})^{\frac{m}{2}} \Psi(z)$$

where $P$ is another integer which tends to infinity with $Q$. The function $\Psi$ (which depends on $Q$) is in $S(E)$, the space of Schwartz functions on $E$. Let us denote by $\mathcal{V}(P)$ the space of functions of the above form, with $\Psi \in S(E)$. It is contained in $\mathcal{H}$. It will suffice to prove the following general lemma:

**Lemma 6.** — If $P$ is large enough the space $\mathcal{V}(P)$ is contained in the space of vectors of class $C^M$ and, for $\Phi \in \mathcal{V}(P)$,

$$\mathcal{P}(\Phi) = \int_{F^\times} \Phi(x) \omega(x) d^\times x.$$
\[ H = \frac{1}{2} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\}; \quad T = \frac{1}{2} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix} \right\}. \]

We define similarly conjugate elements \( \bar{X}_+ \) and so on, by replacing \( i \) by \( -i \) in the above formulas. They commute to the previous ones. The Casimir element

\[ \Omega = X_+X_- + X_-X_+ \frac{1}{2} H^2 \]

is in the center of the enveloping algebra and \( T - \frac{1}{2} H \) in the center of the Lie algebra. Thus \( X_+X_- \) can be expressed in terms of \( T \) and elements of the center of the enveloping algebra. As a consequence,

\[ \Pi(X_+)\Pi(X_-) = R(\Pi(T)), \]

where \( R \) is a polynomial (of degree 2). We have then, for \( \Phi \in \mathcal{V} \),

\[ \Pi(X_+)\Phi(z) = 2i\pi z\Phi(z), \quad \Pi(T)\Phi(z) = z \frac{\partial \Phi(z)}{\partial z}, \]

\[ 2i\pi z\Pi(X_-)\Phi(z) = \Pi(X_+)\Pi(X_-)\Phi(z) = R(\Pi(T))\Phi(z). \]

In other words (after a change of notations) we see that there is a polynomial \( R \) such that

\[ \Pi(X_-)\Phi(z) = z^{-1}R\left( z \frac{\partial}{\partial z} \right)\Phi(z). \]

Similarly, there is a polynomial \( R' \) such that

\[ \Pi(X_-)\Phi(z) = z^{-1}R'\left( z \frac{\partial}{\partial z} \right)\Phi(z). \]

Now the space \( \mathcal{V}(\mathcal{P}) \) is invariant under the operators of multiplication by \( z, \bar{z} \) and the differential operators \( z\partial/\partial z, \bar{z}\partial/\partial \bar{z} \). This implies that in fact \( \mathcal{V}(\mathcal{P}) \) is contained in the space of smooth vectors for the group \( \mathcal{P}_1(\mathcal{E}) \). Now multiplication by \( z^{-1} \) sends \( \mathcal{V}(\mathcal{P}) \) to \( \mathcal{V}(\mathcal{P} - 1) \). It follows that if \( P \) is sufficiently large and \( \Phi \in \mathcal{V}(\mathcal{P}) \) the vector \( \Pi(X_-)\Phi \) which, \textit{a priori}, is only a vector in \( \mathcal{V}_0 \) is in fact in \( \mathcal{H} \), more precisely, in \( \mathcal{V}(\mathcal{P} - 1) \). Moreover, it is given by the same formula as above. Likewise for \( \Pi(\bar{X}_-)\Phi \). The first assertion of the lemma follows. To prove the second assertion, we let \( \mathcal{V}_0 \) be the subspace of \( \Phi \in \mathcal{V}(\mathcal{P}) \) for which the function \( \Psi \) is flat at \( 0 \in \mathcal{E} \), that is, all its derivatives vanish at \( 0 \). From the above analysis, it follows that such a vector is in \( \mathcal{V} \). Then \( \mathcal{P}(\Phi) \) is given by the integral over \( F^\infty \). Now recall the decomposition (30). From the above computation, there exist (non commutative) polynomials \( P_{\alpha} \) in \( (z^{-1}, \bar{z}^{-1}, z\partial/\partial z, \bar{z}\partial/\partial \bar{z}) \) such that

\[ \mathcal{P}(\Phi) = \sum_{\alpha} \int (P_{\alpha}\Phi)(z) \bar{\Phi}_{\alpha}(z) d^\infty z. \]
This applies in particular to a $\Phi \in \mathcal{V}(P)$ for $P$ large enough. Thus we have to show that the difference
\[ \int_{F_{x}} |x|^{2P+1} \Psi(x) d^{2}x - \sum_{\alpha} \int_{E_{x}} (P_{\alpha}((z\zeta)^{\frac{1}{2}}P\Psi(z))) \Phi_{\alpha}(z) d^{2}z \]
vanishes for all $\Psi \in \mathcal{S}(E)$, if $P$ is large enough. At any rate, this is true for $\Psi$ flat at 0. Since the difference is a distribution (linear form in $\Psi$), the difference is at most a linear combination of derivatives of $\Psi$ at 0. Thus it will vanish if $\Psi$ vanishes at sufficiently high order at 0. Thus it might not vanish on $\mathcal{V}(P)$ but it will vanish on the smaller space $\mathcal{V}(P_{1})$ for $P_{1} \gg P$. We are done.

In summary then, we have found a smooth function of small support $\phi$ on $E$ and an element $\Phi^{0} \in \mathcal{H}$ of class $C^{M}$ such that
\[ \lambda\left( \int_{N(E)} \phi(z)\Pi(n^{-1})d_{n}n\Phi \right) = \langle \Phi, \Phi_{0} \rangle, \quad n = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right), \quad \Phi \in \mathcal{V}, \]
\[ \mathcal{P}(\Phi^{0}) = \int_{F_{x}} \hat{\phi}(b)\eta(b)d^{2}b \neq 0. \]
Suppose that $f_{j}$ is a smooth approximation of $\delta_{e}$. From (30) we get:
\[ \mathcal{P}(\Pi(f_{j})\Phi^{0}) = \sum_{\alpha} \int \langle \Pi(X_{\alpha})\Pi(g)\Phi^{0}, \Phi_{\alpha} \rangle f_{j}(g)dg \]
If we write
\[ \text{Ad}g^{-1}X_{\alpha} = \sum_{\beta} \lambda_{\alpha,\beta}(g)X_{\beta} \]
this becomes
\[ \sum_{\alpha,\beta} \int \lambda_{\alpha,\beta}(g)\langle \Pi(X_{\beta})\Phi^{0}, \Pi(g)^{-1}\Phi_{\alpha} \rangle f_{j}(g)dg. \]
As $j \to +\infty$ this tends to
\[ \sum_{\alpha,\beta} \lambda_{\alpha,\beta}(e)\langle \Pi(X_{\beta})\Phi^{0}, \Phi_{\alpha} \rangle = \sum_{\alpha} \langle \Pi(X_{\alpha})\Phi_{0}, \Phi_{\alpha} \rangle = \mathcal{P}(\Phi^{0}). \]

We thus consider an approximation $\Phi_{j}$ of $\delta_{w}$ on $S$. We can choose an approximation $f_{j}$ of $\delta_{e}$ on $G(E)$ such that $\Phi_{j}$ is the function associated to $f_{j}$. Just as in the $p$-adic case, we define $f_{j}$ in terms of $\phi$ and $\Phi^{j}$. Then
\[ \lambda(\Pi(f_{j})\Phi) = \lambda\left( \int_{N(E)} \phi(z)\Pi(n^{-1})d_{n}n(\Pi(f_{j})\Phi) \right) = \langle \Phi(f_{j})\Phi, \Phi_{0} \rangle, \]
or
\[ \Pi(f_{j})^{*}\lambda = \Pi(f_{j}^{*})\Phi_{0}. \]
Then
\[ \Theta_{\Pi}(f) = \mathcal{P}(\Pi(f_{j})^{*}\lambda) = \mathcal{P}(\Pi(f_{j}^{*})\Phi_{0}). \]
which tends to \( \langle \mathcal{P}, \Phi_0 \rangle = \mathcal{F}(\Phi_0) \). Thus we find again

\[
\lim_{j \to +\infty} \Theta_\Pi(f^j) = \int_{F^\times} \tilde{\phi}(b)\eta(b)d^\times b.
\]

We introduce the following function on \( F \):

\[
\phi_1(x) = \int \phi(x + y\sqrt{3})|2_f| \cdot |\xi| d^\times y.
\]

Then, denoting by \( \mathcal{F} \) the Fourier transform for a function on \( F \), we get for \( x \in F \):

\[
\tilde{\phi}(x) = \mathcal{F}(\phi_1)(2x).
\]

Thus, in all cases,

\[
\lim_{j \to +\infty} \Theta_\Pi(f^j) = \eta(2) \int_{F^\times} \mathcal{F}(\phi_1)(x)\eta(x)d^\times x.
\]

5.3. Computation on \( G_\epsilon \). — We now compute the orbital integral of \( \Phi^j \).

After a change of variables, we get

\[
U(x; \Phi^j) = \int \int \Phi^j \bigg[ \left( \begin{array}{cc} y\overline{y} & x^{-1} \\ \overline{z} & z \end{array} \right) \bigg] \Omega(y)\eta(z)d^\times ydz.
\]

From (29) we get

\[
\Phi^j(s) = \int \Phi^j \bigg[ \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) s \left( \begin{array}{cc} 1 & 0 \\ \overline{t} & 1 \end{array} \right) \bigg] \phi(t)dt.
\]

After changing \( t \) to \(-t\) we find that the orbital integral of \( \Phi^j \) is equal to

\[
\int \int \Phi^j \bigg[ \left( \begin{array}{cc} y\overline{y} & x^{-1} \\ \overline{z} & z \end{array} \right) - t \mathcal{E} \overline{t} y + t\overline{t}z \quad y - tz \bigg] \phi(-t)\Omega(y)\eta(z)dt dz.
\]

Recall \( \phi \) has compact support and the support of \( \Phi^j \) tends to \( w \). Thus there is a sequence \( \{c_j\} \) of positive real numbers such that \( \lim_{j \to +\infty} c_j = +\infty \) and the non-vanishing of \( U(x; \Phi^j) \) implies \( |x| > c_j \). Now we choose a function \( f^j \) which matches \( f^j \) (or \( \Phi^j \)). For \( x \in \epsilon \text{Norm}(E^\times) \) we write \( x = \epsilon a\pi \) and then

\[
U(x; \Phi^j) = H(x; f^j_t) = \int f^j_t \int t_1 \left( \begin{array}{cc} \epsilon^{-1} & 0 \\ 1 & \epsilon^{-1} \end{array} \right) t_2 \Omega(t_1 t_2^{-1})dt_1 dt_2.
\]

Suppose the integral is non-zero. Then \( |x| > c_j \). If \( j \) is large enough, then in the integrand, the determinant of the matrix \( g = t_1(\ast)t_2 \) belongs to \(-\epsilon \text{Norm}(E^\times)\). Thus we may and will assume that \( f^j_t \) is supported on the set \( \{g \mid \text{det} g \in -\epsilon \text{Norm}(E^\times)\} \). From (15) we get

\[
\int \int \omega_g(g)f^j_t(g)d^\times y = \frac{1}{L(1, \eta)^2} \int_{x = \epsilon a\pi} \omega_g \left[ \left( \begin{array}{cc} a^{-1} & 0 \\ 1 & \epsilon^{-1} \end{array} \right) \right] U(x; \Phi^j) \frac{dx}{|1 - x|}.
\]
Now we define a function $\omega$ on $\epsilon \text{Norm}(E^\infty) - \{1\}$ by

$$
\omega(x) := \omega_\epsilon \left( \left[ \frac{a^{-1}}{1} \right] \right), \quad \text{for } x = \epsilon a\bar{a}.
$$

One can check that indeed the right hand side depends only on $x$. The function $\omega(x^{-1})$ extends to a smooth function near 0 on $\epsilon \text{Norm}(E)$, whose value at 0 is

$$
\omega_\epsilon \left( \left[ \frac{0}{1} \right] \right) = 1.
$$

We denote by $\tilde{\omega}$ a smooth function on $F$, supported on a neighborhood of 0, and equal to $\omega(x^{-1})|1-x|^{-2}$ for $x$ near 0 in $\epsilon \text{Norm}(E)$. In particular $\tilde{\omega}(0) = 1$. Thus, for large enough $j$,

$$
\int \omega_\epsilon(g) f_j^2(g) d\sigma = \frac{1}{L(1,\eta)^2} \int_{\epsilon \text{Norm}(E^\infty)} U(x; \Phi^j) \frac{\omega(x) dx}{|1-x|^2}
$$

$$
= \frac{1}{L(1,\eta)^2} \int_{\epsilon \text{Norm}(E^\infty)} U(x^{-1}; \Phi^j) \tilde{\omega}(x) dx.
$$

Equivalently,

$$
(33) \quad \int \omega_\epsilon(g) f_j^2(g) d\sigma = \frac{1}{2L(1,\eta)^2} [I_1 + \eta(\epsilon)I_0]
$$

where we have set

$$
I_{(1)} := \int_E U(x^{-1}; \Phi^j) \left\{ \frac{1}{\eta(x)} \right\} \tilde{\omega}(x) dx
$$

$$
= \int \Phi_j \left( \left[ \begin{array}{ccc} y & -t\bar{y} - t\bar{z} & y - tz \\ \bar{y} - t\bar{z} & \bar{z} & z \\ 1 \\ \end{array} \right] \right) \phi(-t) \tilde{\omega} \left( \frac{xz}{y\bar{y}} \right)
$$

$$
\times \Omega(y) \left\{ \frac{\eta(z)}{\eta(x)} \right\} dt |y|^{-1} d^r y |z| d^s z dx.
$$

To compute this, we first change $x$ to $xz/y\bar{y}$:

$$
= \int \Phi_j \left( \left[ \begin{array}{ccc} x - t\bar{y} - t\bar{z} & y - tz & 1 \\ \bar{y} - t\bar{z} & \bar{z} & z \\ y & z & 1 \\ \end{array} \right] \right) \phi(-t) \tilde{\omega} \left( \frac{xz}{y\bar{y}} \right)
$$

$$
\times \Omega(y) \left\{ \frac{\eta(z)}{\eta(x)} \right\} dt |y|^{-1} d^r y |z| d^s z dx.
$$

Next we change $y$ to $y + tz$ to get:

$$
\int \Phi_j \left( \left[ \begin{array}{ccc} x - t\bar{y} - t\bar{z} & y & 1 \\ \bar{y} - t\bar{z} & \bar{z} & z \\ y & z & 1 \\ \end{array} \right] \right) \phi(-t) \tilde{\omega} \left( \frac{xz}{(y + tz)(\bar{y} + t\bar{z})} \right)
$$

$$
\times \Omega(y + tz) \left\{ \frac{\eta(z)}{\eta(x)} \right\} dt |y + tz|^{-1} E|E| d^r y |z| d^s z dx.
$$
Now we change $t$ to $t/\gamma$ to get:
\[
\int \Phi_j \left[ \left( \frac{x - (t + t\gamma + t\gamma/\gamma)}{\gamma} \right)^{r/\gamma} \right] d\omega \left( \frac{xz}{y + t\gamma/\gamma + t\gamma/\gamma} \right) \\
\times \Omega \left( y + \frac{t\gamma}{\gamma} \right) \left\{ \frac{\eta(x)}{\eta(x)} \right\} dt |y + t\gamma/\gamma|^{-2} d^\gamma y|F| d^\gamma z d\gamma.
\]
We finally change $z$ to $zy/\gamma$:
\[
(34) \quad I_{1/\gamma} = \int \Phi_j \left[ \left( \frac{x - (t + t\gamma + t\gamma/\gamma)}{\gamma} \right)^{r/\gamma} \right] d\omega \left( \frac{xz}{y + t\gamma/\gamma + t\gamma/\gamma} \right) \\
\times \Omega \left( y + \frac{t\gamma}{\gamma} \right) \left\{ \frac{\eta(x)}{\eta(x)} \right\} dt |y + t\gamma/\gamma|^{-1} d^\gamma y|F| d^\gamma z d\gamma.
\]
To evaluate the limit of the integrals we remind ourselves that $\Phi_j$ is an approximation of $\delta_w$ on $S_\eta$. In particular (see Lemma 4)
\[
L(1, 1_F) \int \Phi_j \left[ \left( \frac{x - (t + t\gamma + t\gamma/\gamma)}{\gamma} \right)^{r/\gamma} \right] d\omega \left( \frac{xz}{y + t\gamma/\gamma + t\gamma/\gamma} \right) \\
\times \Omega \left( y + \frac{t\gamma}{\gamma} \right) \left\{ \frac{\eta(x)}{\eta(x)} \right\} dt |y + t\gamma/\gamma|^{-1} d^\gamma y|F| = 1,
\]
with $x, z \in F$ and $y \in E$. If we change $z$ to $zy/\gamma$ and take into account the relations $L(1, 1_F) = L(1, 1_F) L(1, \eta)$ and $d^\gamma y = L(1, 1_F) d^\gamma y|F|^{-1}$ we get:
\[
(35) \quad \int \Phi_j \left[ \left( \frac{x - (t + t\gamma + t\gamma/\gamma)}{\gamma} \right)^{r/\gamma} \right] d\omega \left( \frac{xz}{y + t\gamma/\gamma + t\gamma/\gamma} \right) \\
\times \Omega \left( y + \frac{t\gamma}{\gamma} \right) \left\{ \frac{\eta(x)}{\eta(x)} \right\} dt |y + t\gamma/\gamma|^{-1} d^\gamma y|F| = L(1, \eta).
\]
We are going to see that the limit of the integral $I_1$ corresponding to the factor $\eta(z)$ in (34) is 0. We prove this in the Archimedean case, leaving the easier non-Archimedean case to the reader. We change $x$ to $x + (t + t\gamma + t\gamma/\gamma)$ to arrive (up to a multiplicative constant) at
\[
\int \Psi_j(x, z, y) A(x, z, y) \eta(z) d\omega d^\gamma y
\]
where we have set
\[
A(x, z, y) = |1 - xz|^{2} \cdot \Omega(y)|y|^{-1} \int \phi \left( \frac{-t}{\gamma} \right) \\
\times \Omega \left( \frac{zx + (t + t\gamma)(1 + t\gamma) - 1}{(1 + t\gamma)(1 + t\gamma)} \right) \frac{1}{|1 + t\gamma|^{2}_F} dt,
\]
and
\[
\Psi_j(x, z, y) = \Phi_j \left[ \left( \frac{x - (t + t\gamma + t\gamma/\gamma)}{\gamma} \right)^{r/\gamma} \right] \left\{ \frac{1}{|1 - xz|^{2}_F} \right\}.
\]
Recall that $\Phi_j$ is an arbitrary smooth approximation of $\delta_w$. Equivalently, $\Psi_j$ is an arbitrary approximation of $L(1, \eta) \delta_{(0, 0, 1)}$ on the space $F \times F \times E$ for the measure $d\omega d^\gamma y$. Thus, there is no harm in assuming that our approximation of $\delta$ satisfies:
\[
\Psi_j(x, z, y) = \Psi_j(x, -z, y).
\]
For $\phi$ with small support, $A$ is continuous near $(0, 0, 1)$ on $F \times F \times E$. We can write $A = A_+ + A_-$ where $A_+(x, -z, y) = \pm A_\pm(x, z, y)$. In view of our assumption on $\Psi_j$ the integral of $A_+(x, z, y)\eta(z)$ against $\Psi_j$ is 0. Now $A_-(x, z, y)\eta(z)$ is continuous and vanishes at $(0, 0, 1)$. We are integrating on $F \times F \times E$ against an approximation of a multiple of $\delta_{(0,0,1)}$. The limit of the integral is thus 0, as claimed.

To compute the integral $I_\eta$ corresponding to the factor $\eta(x)$ in (34), we first replace $t$ by a new variable $s$ such that:

$$s + \bar{t} = t + \bar{t}z, \quad s - \bar{t} = t - \bar{t}.$$

Explicitly, for $\phi$ of small support:

$$t = p + q\sqrt{\delta}, \quad s = p_1 + q_1\sqrt{\delta}, \quad q = q_1,$$

$$p = -1 + \sqrt{1 + 2pz + q_1^2\delta z^2} = \frac{2p_1 + q_1^2\delta z}{1 + \sqrt{1 + 2p_1z + q_1^2\delta z^2}}dt$$

$$= ds|1 + 2p_1z + q_1^2\delta z^2|^{-\frac{1}{2}}.$$

Then the integral becomes:

$$I_\eta = \int \Phi_j\left[\left(\frac{x - 2p_1}{y} \quad \frac{y}{zy}\right)\right]\phi\left(\frac{-t}{y}\right)$$

$$\times \sqrt{\frac{z}{(1 + tz)(1 + t\bar{z})}}\Omega(y)|y|_E^{-1}\Omega(1 + tz)\left|\frac{1}{1 + t\bar{z}}\right|^E$$

$$\times \eta(x)|1 + 2p_1z + q_1^2\delta z^2|^{-\frac{1}{2}}dzdydx,$$

where now

$$t = \frac{-1 + \sqrt{1 + 2p_1z + q_1^2\delta z^2} + q_1\sqrt{\delta}}{z}.$$
Since $|z| F d^* z d x = d |x| F d^* x$, this can be written as
\[
\int \Phi_j \left( \begin{pmatrix} p_1 & y \\ \overline{z} y & \overline{y} \end{pmatrix} \right) A(p_1, z, y) \frac{d p_1 z d^* y}{|1 - p_1 z|^2_F}
\]
with
\[
A(p_1, z, y) := |1 - p_1 z|^2_F \int \phi \left( \frac{-t z}{F} \right) \bar{\omega} \left( \frac{x z}{(1 + t z)(1 + \overline{t} \overline{z})} \right) \times |1 - p_1 z + x z + q_1^2 \delta z^2|^{-\frac{1}{2}} \Omega(y)|y|^{-1} \times \left( \frac{1 + t z}{1 + \overline{t} \overline{z}} \right)^{\frac{1}{2}} \eta(x) |x| F d^* x \delta_{F,1} d q_1.
\]
For $\phi$ of small support, $A$ is continuous on $F \times F \times E$ near $(0,0,1)$ with
\[
A(0,0,1) = \int \phi \left( -\frac{1}{2} x - q_1 \sqrt{\delta} \right) \eta(x) |x| d^* x \delta_{F,1} d q_1
\]
the last equation by definition of $\phi_1$ (see (31)). Thus (see (35))
\[
\lim_{j \to +\infty} I_\eta = L(1, \eta) \eta(-2) \int \phi_1(x) \eta(x) |x| F d^* x
\]
and we find from (33)
\[
\lim_{j \to +\infty} \int \omega_\sigma(g) f_1^j(g) d g = \frac{\eta(2 \epsilon)}{2 L(1, \eta)} \int \phi_1(x) \eta(x) |x| F d^* x.
\]
We compare this with (32):
\[
\lim_{j \to +\infty} \Theta_{\Pi}(f^j) = \eta(2) \int F(\phi_1)(x) \eta(x) d^* x,
\]
which, by the Tate functional equation, is equal to:
\[
= \eta(2) \epsilon(1, \eta, \psi_F) \frac{L(0, \eta)}{L(1, \eta)} \int \phi_1(x) |x| F \eta(x) d^* x
\]
\[
= \epsilon(1, \eta, \psi_F) 2 \eta(-\epsilon) L(0, \eta) \lim_{j \to +\infty} \int \omega_\sigma(g) f_1^j(g) d g.
\]
Proposition 5 follows.

**5.4. Local comparison: split case.** — We now consider the similar decomposition at a split place $v_0$. The isomorphism of (2.2) takes $G_\epsilon(F_{v_0})$ to $GL(2, F_{v_0})$. $T_{v_0}$ to the group of $A_{v_0}$, diagonal matrices and the character $\Omega$ to the character $\text{diag}(a_1, a_2) \mapsto \Omega_{v_0}(a_1 a_2^{-1})$. We may realize the representation $\sigma_{v_0}$ in its Whittaker model. Then we have again a decomposition
\[
\theta_{v_0}(f^\infty \otimes f_{v_0}) = \theta_{v_0}(f^\infty) \theta_{v_0}(f_{v_0}),
\]
where the local distribution at the place $v_0$ is now given by

$$
\theta_{\sigma_{v_0}}(f_{v_0}) = \sum_W \int \sigma_{v_0}(f_{\epsilon,v_0})W(\alpha)\Omega_{v_1}(\alpha)^{-1}d^*\alpha \int W(aw_{v}^{-1})\Omega_{v_1}(a)^{-1}d^*a.
$$

On the other hand, recall the distribution

$$
\Theta_{\Pi,v_0}(f_{v_1} \otimes f_{v_2}) = \sum_{W_1,W_2} \lambda_{v_1}(\Pi_{v_1}(f_{v_1})W_1)\lambda_{v_2}(\Pi_{v_2}(f_{v_2})W_2)\mathcal{P}_{v_0}(W_1 \otimes W_2).
$$

**Proposition 6.** — If $f_{v_1} \otimes f_{v_2}$ and $f_{\epsilon,v_0}$ match then

$$
\Theta_{\Pi,v_0}(f_{v_1} \otimes f_{v_2}) = \theta_{\sigma_{v_0}}(f_{\epsilon,v_0}).
$$

The distribution $\Theta_{\Pi,v_0}$ can be simplified as follows. Let us identify $E_{v_1}$, $E_{v_2}$ and $F_{v_0}$. The representations $\Pi_{v_1}$ and $\Pi_{v_2}$ are in fact identical. The Hermitian form $\mathcal{P}_{v_0}(W_1 \otimes W_2)$ is invariant under the unitary group at $v_0$. It follows that

$$
\Theta_{\Pi,v_0}(f_{v_1} \otimes f_{v_2}) = \sum_{W_1,W_2} \lambda_{v_1}(\Pi_{v_1}(f_{v_1} \ast f_{v_2})W_1)\lambda_{v_2}(W_2)\mathcal{P}_{v_0}(W_1 \otimes W_2)
$$

where we have set

$$
\bar{f}(g) = f(w^tgw).
$$

Since $\Pi_{v_1}$ is unitary and self contragredient, for $W \in \mathcal{W}(\Pi_{v_1}, \psi_{v_0})$ the function $g \mapsto \overline{W}(\text{diag}(-1,1)g)$ is still in the same space. Thus we can choose the functions $W_1$ in some orthonormal basis and choose then for the functions $W_2$ the functions of the form

$$
\overline{g} \mapsto \overline{W}(\text{diag}(-1,1)g\text{diag}(-1,1)).
$$

Since $\Omega_{v_2}\Omega_{v_1} = 1$ our distribution reads then

$$
\sum_{W_1,W_2} \int \Pi_{v_1}(f_{v_1} \ast f_{v_2})W_1(a_1)\Omega_{v_1}^{-1}(a_1)d^*a_1 \int W_2(a_2)\Omega_{v_1}^{-1}(a_2)d^*a_2
$$

$$
\times \int_{F_{v_0}} W_1 \overline{W}(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix})d^*b,
$$

where $W_1, W_2$ vary independently in the same orthonormal basis. The orthogonality relations give

$$
\sum_W \int \Pi_{v_1}(f_{v_1} \ast f_{v_2})W(a_1)\Omega_{v_1}^{-1}(a_1)d^*a_1 \int W(a_2)\Omega_{v_1}^{-1}(a_2)d^*a_2.
$$

Recall that the matching condition (17) amounts to:

$$
f_{v_1} \ast f_{v_2}(g) = f_{\epsilon,v_0}(gw_{v}^{-1}).
$$
Using the fact that we can change the orthonormal basis, we see that the above expression becomes, after a change of variables,

\[
\sum_{W} \int \Pi_{v_1}(f_{\epsilon,v_0}) W(a_1) \Omega_{v_1}^{-1}(a_1) d^* a_1 \int W(a_2 w \epsilon^{-1}) \Omega_{v_1}^{-1}(a_2) d^* a_2
\]

which is equal to \( \theta_{\sigma_{\epsilon,v_0}}(f_{v_0}) \).

6. Global comparison

We go back to the global relation:

\[
\Theta_{\Pi}(f) = \theta_{\sigma}(f_{\epsilon}) + \theta_{\sigma \otimes \eta}(f_{\epsilon}).
\]

We assume \( f \) is a product with \( f^S \) the characteristic function of \( K^S \). For \( v_0 \) inert not in \( S_0 \) the function \( f_{\epsilon,v_0} \) is the characteristic function of \( \text{Gl}(2,\mathcal{O}_E) \cap G_\epsilon(F) \) and in particular supported on the set of \( g \) with \( \det g \in -\epsilon \text{Norm}(E_\epsilon^O) \).

Let us make the assumption that for any \( v_0 \) inert in \( S_0 \) the function \( f_{\epsilon,v_0} \) is also supported on the set of \( g \) with \( \det g \in -\epsilon \text{Norm}(E_\epsilon^O) \). Then the function \( \rho(w_\epsilon)^{-1}f_{\epsilon} \) is supported on the set of \( g \) with \( \det g \in \text{Norm}(E_\epsilon^O) \) so that the distributions \( J_{\sigma} \) and \( J_{\sigma \otimes \eta} \) take the same value on that function. Equivalently, \( \theta_{\sigma}(f_{\epsilon}) = \theta_{\sigma \otimes \eta}(f_{\epsilon}) \) so that

\[
\theta_{\epsilon}(f_{\epsilon}) = \frac{1}{2} \Theta_{\Pi}(f).
\]

At a split place \( v_0 \in S_0 \), we have

\[
\Theta_{\Pi_{v_0}}(f_{v_0}) = \theta_{\sigma_{v_0}}(f_{v_0}).
\]

At a place \( v_0 \in S_0 \) inert in \( E \), we have (Proposition 5)

\[
\Theta_{\Pi_{v_0}}(f_{v_0}) = \epsilon(1, \eta_{v_0}, \psi_{v_0}) 2\eta_{v_0}(-\epsilon)L(0, \eta_{v_0}) \theta_{\sigma_{v_0}}(f_{\epsilon,v_0}).
\]

Then

\[
\theta_{\epsilon}(f_{\epsilon}) = \frac{1}{2} \Theta_{\Pi}(f) = \frac{1}{2} \prod_{v_0 \in S_0, \text{inert}} \epsilon(1, \eta_{v_0}, \psi_{v_0}) 2\eta_{v_0}(-\epsilon)L(0, \eta_{v_0}) \times \frac{L_{S_0}(1, \eta)L_{S_0}(1, \frac{1}{2} \Pi \otimes \Omega^{-1})}{L_{S_0}(1, \pi, \text{Ad})} \prod_{v_0 \in S_0} \theta_{\sigma_{v_0}}(f_{v_0}).
\]

At this point we go back to distributions of positive type:

\[
J_{\sigma}(f) := \theta_{\sigma}(\rho(w_\epsilon)f), \quad \tilde{J}_{\sigma_{v_0}}(f) := \theta_{\sigma_{v_0}}(\rho(w_\epsilon)f).
\]

Explicitly:

\[
J_{\sigma}(f) = \sum \int \sigma(f) \phi(t) \Omega(t)^{-1} \, dt \int \phi(t) \Omega^{-1}(t) \, dt.
\]
For $v_0$ inert:
\[ \tilde{J}_{\sigma v_0}(f_{v_0}) = \int_{G_{v_0}} f_{v_0}(g)\langle \sigma_{v_0}(g)e_T, e_T' \rangle dg. \]

For $v_0$ split:
\[ \tilde{J}_{\sigma v_0}(f) = \sum_W \int \rho(f)W(a)\Omega_{v_1}^{-1}(a) da \int W(a)\Omega_{v_1}^{-1}(a) da. \]

Because $S_0$ is large enough the product of the factors \( \eta_{v_0}(-\epsilon) \) is one.

**Theorem 2.** For any smooth function \( f = \prod_{v_0 \in S_0} f_{v_0} \) of compact support:
\[ J_{\sigma}(f) = \prod_{v_0 \in S_0} \tilde{J}_{\sigma v_0}(f_{v_0}) \times \left( \prod_{v_0 \in S_0, \text{ inert}} \epsilon(1, \eta_{v_0}, \psi_{v_0})^{2L(0, \eta_{v_0})} \right) \times \frac{L_{S_0}(1, \eta)L_S(1, \Pi \otimes \Omega^{-1})}{L_{S_0}(1, \pi, \text{Ad})}. \]

Indeed, we know in advance that the left-hand side decomposes into a product \( C \prod_{v_0 \in S_0} \tilde{J}_{\sigma v_0}(f_{v_0}) \) for a suitable constant $C$. To evaluate $C$ we choose the data as above. The result follows. Moreover, since the factors of $C$ other than the one we are interested in are $>0$, we conclude that $L_S(\frac{1}{2}, \Pi \otimes \Omega^{-1}) > 0$, as was claimed.

**7. Concluding remarks**

The distributions $\tilde{J}_{\sigma v_0}$ have the following property. We let $f_j$ be an approximation of $\delta$ on $G_{\epsilon, v_0}$.

If $v_0$ is split, we choose a Schwartz-Bruhat function $\phi$ on $F_{v_0}$ such that $\hat{\phi}(0) = 0$. We also choose also an isomorphism of $G_{\epsilon}(F_{v_0})$ with $\text{GL}(2, F_{v_0})$ and set:
\[ f^j(g) = \int f_j \left[ \begin{array}{cc} 1 & -x \\ 0 & 1 \end{array} \right] \phi(x) dx \]

We have then
\[ \lim_{j \to +\infty} \tilde{J}_{\sigma v_0}(f^j) = \int \hat{\phi}(x)dx. \]

Now the Tate functional equation gives:
\[ \int \hat{\phi}(x)dx = \epsilon(1, 1_{v_0}, \psi_{v_0}) L(1-s, 1_{v_0}) \int \phi(x)|x|^{s-1}dx \big|_{s=1}. \]

To have our local distribution independent of the choice of the additive character it is reasonable to set
\[ \tilde{J}_{\sigma v_0}(f) = \epsilon(1, 1_{v_0}, \psi_{v_0}) J_{\sigma v_0}(f). \]
This new distribution is still of positive type and now:

$$\lim_{j \to +\infty} J_{\sigma_{v_0}}(f') = L(s, 1_{v_0}) \int \phi(x)|x|^s \, dx \big|_{s=0}.$$ 

Note that the last expression is a multiple of $\int \phi(x) \log(|x|) \, dx$. For instance, in the non-Archimedean case, the expression is $-\int \phi(x) \nu(x) \, dx$ where $\nu$ is the valuation. In the $p$-adic case there is an asymptotic theory of spherical characters (see [RR]). This is worked in [F4] for the case of the space $GL(2, F)/A(F)$.

Let us assume for simplicity that $\Omega = 1$. The result is then that a spherical character like $J_{\sigma_{v_0}}$ can be represented, near the origin, as a unique linear combination

$$c_1 \int f \left( \left( \begin{array}{cc} 1 & u \\ v & 1 \end{array} \right) a \right) \, dadaudv - c_2 \int f \left( \left( \begin{array}{cc} 1 & u \\ v & 1 \end{array} \right) a \right) \, d\nu(\nu^{-1})dadaudv.$$

For the distribution $J_{\sigma_{v_0}}$ we have just defined, the above assertion amounts to saying that the constant $c_2$ is $1$.

On the other hand, if $v_0$ is inert, then $\tilde{J}_{\sigma_{v_0}}$ is a smooth function and we have simply: $\lim_{j \to +\infty} \tilde{J}_{\sigma_{v_0}}(f_j) = 1$.

Finally, the reader can check that the left hand side of the formula in the theorem does not depend on the choice of the Haar measure on the group $G_\epsilon(E_A)$, and the formulas for a finite set of places $S_0$ and a larger set $S'_0$ are compatible.

8. Appendix: Continuous spectrum over $E$

8.1. A diagonal truncation operator. — To prepare for the computation of the integral of the spectral kernel, we introduce another diagonal truncation operator for functions $\phi$ on $G(E_A)$ invariant under $Z(E_A)G(E)$:

$$\Lambda_T^\phi(a) := \phi(a) - \sum_{\gamma \in A(E) \backslash \text{Norm}(A)} \phi_N(\gamma a) \bar{\tau}_p(H(\gamma a) - T).$$

Here $\tau_p(x) = 1$ if $x > 0$ and equal $0$ otherwise. The sum is over the normalizer of $A(E)$ in $G(E)$ modulo $A(E)$, thus, has only two elements, that we can take to be $e$ and $w$. The result of the truncation is a function on $A(E_A)$ invariant under $A(E)Z(E_A)$. It is useful to remember the following facts:

(i) $\tilde{\tau}(H(a) - T) = 1 \iff |a_1/a_2| > e^T$;

(ii) for any function $f$ invariant under $Z(E_A)$, $f(a) = f(wa^{-1}w)$;
(iii) $H(wa) = H(a^{-1}) = -H(a)$. Thus we can write more explicitly

$$
\Lambda_T^a \phi(a) = \phi(a) - \phi_N(a) \hat{\tau}_P(H(a) - T) - \phi_N(a^{-1} w) \hat{\tau}_P(-H(a) - T),
$$

$$
\Lambda_T^a \phi(a) = \phi(a) \quad \text{if } -T < H(a) < T,
$$

$$
\Lambda_T^a \phi(a) = \phi(a) - \phi_N(a) \quad \text{if } H(a) \geq T,
$$

$$
\Lambda_T^a \phi(a) = \phi(a^{-1} w) - \phi_N(a^{-1} w) \quad \text{if } H(a) \leq -T.
$$

If $\phi$ is of slow increase (as well as all its derivatives) then the function $a \mapsto \Lambda_T^a \phi(\text{diag}(a,1))$ is rapidly decreasing for $|a|$ large and $|a|$ small. Furthermore, if $\Lambda_T^c$ denotes the compact truncation operator introduced in (18), then

$$
\lim_{T \to +\infty} \int (\Lambda_T^a - \Lambda_T^c) \phi(\text{diag}(a,1)) d^\times a = 0.
$$

Consider again a smooth function of compact support $f$ on $\text{Gl}(2, E_h)$. We have a standard majorization [Ar, Lemma 4.3]:

$$
|\rho_1(X_1) \rho_2(X_2) K_f(x, y)| \leq C(f) \|x\|.
$$

Here $X_1$ and $X_2$ are elements of the enveloping algebra of $G_\infty$ acting as left invariant differential operators on the first and second variables respectively.

For given $X_1$ and $X_2$ the scalar $C(f)$ is bounded for $f$ in a bounded set. Thus the function

$$
g \mapsto \int K_f(g, h) \omega_N(\kappa(h)) dh
$$

is of slow increase as well as all its derivatives. Thus we arrive at the following lemma:

**Lemma 7.** — One has

$$
\Theta(f) = \lim_{T \to +\infty} \int \Lambda_T^a \phi(a) \Omega^{-1}(a) d^\times a.
$$

In [JL] and [JLRo] we have introduced a mixed truncation operator $\Lambda_m^T$ for the group $\text{Gl}(2, F)$. It satisfies the following easy lemma:

**Lemma 8.** — Suppose that $\phi$ is a smooth function on $\text{Gl}(2, E_h)$ invariant under the center, uniformly bounded with derivatives of slow increase. Then

$$
\int_{Z(F_a) \text{Gl}(2, F) \setminus \text{Gl}(2, F_a)} \phi(h) dh = \lim_{T \to +\infty} \int \Lambda_m^T \phi(h) dh.
$$

**Proof.** — Since $\phi$ is integrable over the quotient we may write

$$
\int \Lambda_m^T \phi(h) dh = \int \phi(h) dh - \int \sum_{\gamma \in P(F) \setminus \text{Gl}(2, F)} \phi_N(\gamma h) \hat{\tau}_P(\gamma h - T) dh.
$$
In this expression the second term can be computed as
\[ \int_{P(F)Z(F_a) \setminus \text{GL}(2,F_a)} \phi_N(h) \tau_P(h - T) \, dh. \]
Since \( \phi_N \) is also uniformly bounded this is majorized by a constant multiple of
\[ \int_{P(F)Z(F_a) \setminus \text{GL}(2,F_a)} \tau_P(h - T) \, dh \]
\[ \text{itself a multiple of } e^{-T}. \]
The lemma follows.

Using (22) we may define a mixed truncation operator \( \Lambda_{T_1}T_{m} \) for the group \( H(F) \) which also satisfies the above lemma. It is easy to conclude from (37) that
\[ K_f(g,h) = \lim_{T_2 \to +\infty} \Lambda_{T_1}T_{2,m}K_f(g,h), \]
uniformly for \( g \) and \( h \) in compact sets. Since the computation of \( \Lambda_{T_1}T_{2} \phi(a) \) depends only on the values of \( \phi \) on a compact set (depending on \( a \)) we get also
\[ \Lambda_{T_1}T_{1}K_f(a,h) = \lim_{T_2 \to +\infty} \Lambda_{T_1}T_{2,m}K_f(a,h), \]
uniformly for \( a \) and \( h \) in compact sets. Our next task is the following lemma:

**Lemma 9.** — Given \( T_1 \):
\[ \int \int \Lambda_{T_1}T_{1,m}K_f(a,h)\Omega^{-1}(a) \, da \omega(\kappa(h)) \, dh = \lim_{T_2 \to +\infty} \int \int \Lambda_{T_1}T_{2,m}K_f(a,h)\Omega^{-1}(a) \, da \omega(\kappa(h)) \, dh. \]
For the proof of the lemma we use again (37) to show that \( |\Lambda_{T_1}T_{2}K_f(a,g)| \) is bounded by \( C|a|^{-N} \) for \( H(a) \geq T_1 \) and \( H(a) \leq -T_1 \) and by a constant for \( -T_1 < H(a) < T_1 \). The derivatives with respect to the second variable satisfy similar majorizations. It follows that the function defined by
\[ m(g) = \int \Lambda_{T_1}T_{1,m}K_f(a,g)\Omega^{-1}(a) \, da \]
satisfy the conditions of the previous lemma. Moreover:
\[ \Lambda_{T_1}T_{1,m}m(g) = \int \Lambda_{T_1}T_{1,m}T_{2,m}K_f(a,g)\Omega^{-1}(a) \, da. \]
Thus the lemma follows from the previous one. In conclusion:

**Lemma 10.** — Set
\[ \Theta_{T_1,T_2}(f) = \int \int \Lambda_{T_1}T_{1,m}T_{2,m}K_f(a,h)\Omega^{-1}(a) \, da \omega(\kappa(h)) \, dh. \]
Then:
\[ \Theta(f) = \lim_{T_1 \to +\infty} \lim_{T_2 \to +\infty} \Theta_{T_1,T_2}(f). \]
8.2. The continuous kernel over $E$. — We let $f$ be a smooth function of compact support on $G(E)$, which, for now, is $K$-finite, where $K$ is the standard maximal compact subgroup. Then $Kf$ has a spectral expression

$$Kf = \sum_{\Pi} K_{f,\Pi},$$

where the sum is over all classes $\Pi$ of cuspidal data for the parabolic subgroups $P$ of $G$. We set

$$\Theta_{\Pi,\tau_1,\tau_2}(f) = \iint \Lambda_{\tau_1} \Lambda_{\tau_2} K_{f,\Pi}(a, h) \Omega^{-1}(a) d\alpha \omega(h) dh.$$  

Standard estimates [Ar, Lemma 4.4] allow us to write

$$\Theta_{\tau_1,\tau_2}(f) = \sum_{\Pi} \Theta_{\Pi,\tau_1,\tau_2}(f).$$

If $P = G$, then $\Pi$ is a cuspidal automorphic representation with trivial central character and we have simply $\Theta_{\Pi,\tau_1,\tau_2}(f) = \Theta_{\Pi}(f)$. If $P$ is the minimal parabolic subgroup, then $\Pi$ is an equivalence class of pairs of (normalized) characters $(\Pi_1, \Pi_2)$ of $E \times A/E \times E$; the equivalence relation is $(\Pi_1, \Pi_2) \sim (\Pi_2, \Pi_1)$. Here we need only consider pairs such that $\Pi_1 \Pi_2 = 1$.

The expression for $K_{f,\Pi}$ is [Ar, p.935]

$$K_{f,\Pi}(x, y) = \frac{1}{2} \sum_{(\Pi_1, \Pi_2) \in \Pi} \int \sum_{\phi} E(x, I_{\lambda,\Pi_1,\Pi_2}(f) \phi; \lambda, \Pi_1, \Pi_2) E(y, \phi; \lambda, \Pi_1, \Pi_2) d\lambda + \ldots$$

where the dots represent residual terms. Thus the outer sum has either two terms for $(\Pi_1, \Pi_2)$, $(\Pi_2, \Pi_1)$ (the terms being in fact equal) or just one term $(\Pi_1, \Pi_1)$ (where $\Pi_1^2 = 1$).

It is a little simpler to sum over all pairs $\Pi = (\Pi_1, \Pi_2)$ with $\Pi_2 = \Pi_1^{-1}$ and for such a pair define $K_{f,\Pi}$ by

$$K_{f,\Pi}(x, y) = \frac{1}{2} \sum_{\phi} E(x, I_{\lambda,\Pi}(f) \phi; \lambda, \Pi) E(y, \phi; \lambda, \Pi) d\lambda + \frac{\delta(\Pi_1 \Pi_2^{-1})}{\text{vol}(G(E) \backslash G)} \int f(g) \Pi_1(\det g) dg \Pi_1(xy^{-1}).$$

We are using standard notations. We use the same notation $\Pi$ for the representation induced by the pair $\Pi$. In particular

$$E(g, \phi; \lambda, \Pi) = \sum_{P(E) \backslash G(E)} \phi(\gamma g) e^{(\lambda + \rho, H(\gamma g))}.$$  

The variable $\lambda$ is identified to a complex number:

$$e^{(\lambda + \rho, H(\gamma g))} = |a_1 a_2^{-1}|_E^{\lambda + \frac{1}{2}}.$$
We often drop $\Pi$ from the notation.
Then, apart from a residual term, $\Theta_{\Pi, T_1, T_2}(f)$ is given by:

$$\frac{1}{2} \sum_{\Pi} \left( \int \Lambda_{\Pi}^{T_1} E(a, I_{\lambda}(f) \phi; \lambda) \Omega^{-1}(a) \, da \right) \times \left( \int \Lambda_{\Pi}^{T_2} E(h, I_{\lambda}(f) \phi; \lambda) \omega_{\eta}(\kappa(h)) \, dh \right) \, d\lambda.$$ 

From now on, we will assume that $f$ is a convolution product of three $K$-finite functions: $f = f_1 \ast f_2^*, f_1 = f_1^1 \ast f_1^2$. Later, we will assume that $f_2$ is itself a convolution product of sufficiently many $K$-finite functions. Then this is:

$$\frac{1}{2} \sum_{\phi, \phi'} \langle I_{\lambda}(f_2^1) \phi, \phi' \rangle \left( \int \Lambda_{\Pi}^{T_1} E(a, I_{\lambda}(f_1^1) \phi'; \lambda) \Omega^{-1}(a) \, da \right) \times \left( \int \Lambda_{\Pi}^{T_2} E(h, I_{\lambda}(f_2) \phi; \lambda) \omega_{\eta}(\kappa(h)) \, dh \right) \, d\lambda.$$ 

Our goal is now to obtain an absolutely convergent expression for the spectral terms.

We will establish convergence in the case $\omega = \eta$. Then, by hypothesis, the restriction of $\Omega$ to $F^\infty$ is $\eta$. The case $\omega = 1$ is similar.

Also, to establish the convergence, it will be more convenient to replace the unitary similitude group by the group $\text{GL}(2, F)$. Then the above expression is (with $h \in Z(F_h)G(F) \backslash G(F_h)$):

$$\frac{1}{2} \sum_{\phi, \phi'} \langle I_{\lambda}(f_2^1) \phi, \phi' \rangle \left( \int \Lambda_{\Pi}^{T_1} E(a, I_{\lambda}(f_1^1) \phi'; \lambda) \Omega^{-1}(a) \, da \right) \times \left( \int \Lambda_{\Pi}^{T_2} E(h, I_{\lambda}(f_2) \phi; \lambda) \omega_{\eta}(\kappa(h)) \, dh \right) \, d\lambda.$$ 

In this part of the paper, we do not pay much attention to the normalization of the Haar measures.

### 8.3. Period integral over $A$.

Let $(\Pi_1, \Pi_2)$ be a pair of idele class characters with $\Pi_2 = \Pi_1^{-1}$. We set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then, in the domain of convergence we get:

$$E(g, \phi; \lambda) = \sum_{\omega \in Z(E) \setminus A(E)} \phi(w \omega g) e^{(\lambda + \nu, H(w \omega g))} + \phi(g) e^{(\lambda + \nu, H(g))} + \phi(wg) e^{(-\lambda + \nu, H(wg))}.$$ 

On the other hand:

$$E_N(g, \phi; \lambda) = \phi(g) e^{(\lambda + \nu, H(g))} + M(w, \lambda) \phi(g) e^{(-\lambda + \nu, H(g))}.$$
Then:

\[ A_d^T E(a, \phi; \lambda) = \sum_{\alpha} \phi(wvaa)e^{(\lambda + \rho, H(wvaa))} + \phi(a)e^{(\lambda + \rho, H(a))}(1 - \tilde{T}_P(H(a) - T)) + \phi(wa)e^{(\lambda + \rho, H(wa))}(1 - \tilde{T}_P(H(wa) - T)) - M(w, \lambda)\phi(a)e^{(\lambda + \rho, H(a))}(\tilde{T}_P(H(a) - T)) - M(w, \lambda)\phi(wa)e^{(-\lambda + \rho, H(wa))}(\tilde{T}_P(H(wa) - T)). \]

A simple computation gives

\[ \int A_d^T E(a, \phi, \lambda, \Pi)\Omega^{-1}(a)\,d^\gamma a = \mu(\phi; \lambda, \Pi_1, \Pi_2) \]

\[ \int (\delta(\Pi_1^{-1})\phi(e) + \delta(\Pi_2^{-1})\phi(w))e^{T(\frac{1}{2} + \lambda)}\]

\[ + (\delta(\Pi_2^{-1})M(w, \lambda)\phi(e) + \delta(\Pi_1^{-1})M(w, \lambda)\phi(w))e^{T(\frac{1}{2} - \lambda)} \]

where we have set \( a = \text{diag}(a_1, 1) \) and

\[ \mu(\phi; \lambda, \Pi_1, \Pi_2) = \int_{E_k^\times} \phi(wvaa)\Omega^{-1}(a_1)e^{(\lambda + \rho, H(wvaa))}\,d^\gamma a_1. \]

This formula gives the analytic continuation of \( \mu \). However, we will need to have more information on \( \mu(I_{\lambda, \Pi}(f)\phi; \lambda, \Pi) \) where \( f \) is a smooth function of compact support on \( G(E_k) \). We write \( f = f_Sf^S \) where \( f_S \) is a smooth function of compact support on \( G(E_S) \) and \( f^S \) is the characteristic function of \( K^S := \prod g \in S K_e \) and \( K_e = \text{Gl}(2, \mathcal{O}_e) \). We may then assume that \( \phi \) is invariant under \( K^S \). We have, for \( g \in G_S \),

\[ I_{\lambda, \Pi}(f)\phi(g)e^{(\lambda + \rho, H(g))} = \int_{G_S} \phi(gx)e^{(\lambda + \rho, H(gx))}f_0(x^{-1})\,dx \]

where we have set \( f_0 = \hat{f} \). This can be written as

\[ \int_{G_S} \phi(x)e^{(\lambda + \rho, H(x))}f_0(x^{-1}g)\,dx \]

or

\[ \int f_1 \left[ \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right]\,dn |a_2|^\lambda|a_1|^{-\frac{1}{2}}\Pi_2(a_2)^{-1}|a_1|^{-\frac{1}{2}}\Pi_1(a_1)^{-1}d^\gamma a_1, \]

where we have set

\[ f_1(g) = \int_K \phi(k)f_0(k^{-1}g)\,dk. \]

At this point we introduce new functions on \( G_S \):

\[ f_1(g; \lambda, \Pi_1) := f_1(g)\Pi_1(\det g)^{-1}\det g|^{-\lambda - \frac{1}{2}}, \]

Bulletin de la Société Mathématique de France
\[ \phi_S[g; \lambda, \Pi_1] := \int_{E_S^*} \int_{N_S} f_1 \left[ \begin{array}{c} a_1 \\ 0 \end{array} \right] g; \lambda, \Pi_1 \right] d\nu^{\times} a_1. \]

We let \( P_1 \) be the group of matrices with second row \((0, 1)\). Clearly \( \phi_S \) is invariant on the left under \( P_1(E_S) \). Thus there is a Schwartz-Bruhat function \( \Phi_S[(x, y); \lambda, \Pi_1] \) on \( E_S \times E_S \) such that, for \( g \in G_S \),

\[ \Phi_S[(0, 1)g; \lambda, \Pi_1] = \phi_S[g; \lambda, \Pi_1]. \]

The support of \( \Phi_S \) is contained in a compact set of \( E_S^2 - \{(0, 0)\} \) which is independent of \( \lambda \). We have then for \( g \in G_S \):

\[ I_{\lambda, \Pi}(f) \phi(g)e^{(\lambda + \rho, H(g))} \]

\[ = \int_{E_S^*} \Phi_S[(0, t)g; \lambda, \Pi_1] |t|^{2\lambda + 1} \Pi_1 \Pi_2^{-1} \Pi_1(g) | \det g |^{\lambda + \frac{1}{2}}. \]

To obtain a formula valid for all \( g \) we introduce

\[ \Phi((x, y); \lambda, \Pi_1) := \Phi_S[(x_S, y_S); \lambda, \Pi_1] \prod_{v \notin S} \Phi_v(x_v, y_v), \]

where \( \Phi_v, v \notin S \), is the characteristic function of \( \mathcal{O}_v^c \). Then, for \( g \in G(E_\lambda) \),

\[ I_{\lambda, \Pi}(f) \phi(g)e^{(\lambda + \rho, H(g))} \]

\[ = \frac{\Pi_1(g) | \det g |^{\lambda + \frac{1}{2}}}{L^S(2\lambda + 1, \Pi_1 \Pi_2^{-1})} \int_{E_S^*} \Phi((0, t)g; \lambda, \Pi_1) |t|^{2\lambda + 1} \Pi_1 \Pi_2^{-1} \Pi_1(g) | \det g |^{\lambda + \frac{1}{2}}. \]

After a change of variables we find:

\[ \mu(I_{\lambda, \Pi}(f) \phi; \lambda, \Pi) \]

\[ = \frac{1}{L^S(2\lambda + 1, \Pi_1 \Pi_2^{-1})} \times \int \int \Phi[a_1, a_2; \lambda, \Pi_1] \Omega^{-1} \Pi_1(a_1) | a_1 |^{\lambda + \frac{1}{2}} \]

\[ \times d^{x} a_1 \Omega \Pi_2^{-1}(a_2) | a_2 |^{\lambda + \frac{1}{2}} d^{x} a_2. \]

The computations are justified for \( \Re \lambda > 0 \). For a given \( \Pi \), a given \( \phi \) and a given \( f \), this expression gives the analytic continuation of \( \mu \) as a meromorphic function of \( \lambda \), which has no singularity on the line \( \Re \lambda = 0 \). It has a zero at \( \lambda = 0 \) if \( \Pi_1 = \Pi_2 \), that is, if \( \Pi^2 \neq 1 \). Moreover, this function of \( \lambda \) is at most of polynomial increase on the line \( \Re \lambda = 0 \). All its derivatives have the same property.

However, for our purposes, we will need to have uniform estimates. We proceed as follows. First we take \( f \) in a fixed bounded set \( \mathcal{B} \) of the space of smooth functions of compact support. This means that \( f = f_S f^S \) as before, with \( S \) fixed; the function \( f_S \) has support in a fixed set \( \Xi \), open and relatively compact; we may as well assume \( K_S \Xi = \Xi K_S = \Xi \); the function is invariant on the right and the left under a compact open normal subgroup \( K' \) of \( \prod_{v \notin S} \text{finite} \ K_v \) (and
thus under the subgroup \( K'' = K'K^S \) of \( K^\infty \). For each element \( X \) of the
enveloping algebra of \( G_\infty \) there is a constant \( C_X \) such that
\[
\sup |\rho(X)f| \leq C_X.
\]
Consider the function:
\[
f_1(g) = \int \phi(k)f_0(k^{-1}g)\phi(k)dk.
\]
The integral does not change if we replace \( \phi \) by \( \phi_1 \) defined by
\[
\phi_1(k) = \text{vol}(K'')^{-1} \int \phi(kk'')dk''.
\]
Then \( \phi_1 \) is invariant on the right and therefore on the left under \( K'' \). Thus the
same is true of \( f_1 \). Moreover \( f_1 = f_1.Sf^S \). For any \( X \),
\[
\sup |\rho(X)f_1| \leq \|\phi\|C_X.
\]
Finally, the support of \( f_1.S \) is still contained in \( \Xi \). If \( I_{1,\Pi}(f) \neq 0 \) for a \( f \in \mathcal{B} \),
then \( \Pi_1 \) is unramified outside \( S \) and, for each finite \( v \in S \), \( \Pi_1|\mathcal{O}^\times_v \) belongs
to a finite set. We now consider the function \( f_1(g; \lambda, \Pi_1) \). It has a support
contained in \( \Xi \). It is invariant under a compact subgroup \( K'_v \) of \( \prod_{v \in \mathcal{S}, \text{finite}} K_v \)
(depending only on \( K' \)). Suppose \( \lambda \) is in a strip \( a \leq \text{Re} \lambda \leq b \). Then, for any \( X \),
there is a polynomial \( P_X(\Pi_1, \lambda) \) such that
\[
\sup |\rho(X)f_1(\bullet; \lambda, \Pi_1)| \leq \|\phi\|P_X(\Pi_1, \lambda).
\]
By a polynomial we means a function \( P(\Pi_1, \lambda) \) of the following form. We may
identify the enveloping algebra of \( E_\infty^\times \) to the space of distributions with sup-
port \( \{e\} \). Thus if \( f \) is smooth function on \( E_\infty^\times \) or \( E_\mathbb{R}^\times \) and \( X \) in the enveloping
algebra we can define \( \langle X, f \rangle \). Then a polynomial is a function of the form
\( P(\Pi_1, \lambda) = \langle X, f \rangle \) where \( f(a) = \Pi_1(a)a^\lambda \) and \( X \) is in the enveloping algebra.
Note that such a function depends only on \( \lambda \) and the infinite components of \( \Pi_1 \).
In fact, if we write, for \( v \) real,
\[
\Pi_{1,v}(x) = |x|_{v}^{iu_v} \left( \frac{x}{|x|} \right)^{\epsilon_v}, \quad \epsilon_v = 0, 1, \quad d_v = 1,
\]
and, for \( v \) complex,
\[
\Pi_{1,v}(z) = (z\overline{z})^{iu_v} \frac{z^{n_v}}{(z\overline{z})^{n_v/2}}, \quad n_v \in \mathbb{Z}, \quad d_v = 2,
\]
with \( \sum_{v \in \mathbb{N}} d_v u_v = 0 \), then a polynomial \( P(\Pi_1, \lambda) \) is just a polynomial in the variables \((n_v, u_v)\) and \( \lambda \). To allow for the case where \( \lambda \) is complex, we will say
that any linear combination of products \( P(\Pi_1, \lambda) Q(\Pi_1, \lambda) \), where \( P \) and \( Q \) are
polynomials, is also a polynomial. For instance, for any integer \( N \), \( (1+\lambda \overline{\lambda})^N \) is a
polynomial in that sense. It will be understood that all polynomials considered
are positive.
This being so, all derivatives of \( f_1(\bullet; \lambda, \Pi_1) \) have the same properties (with different polynomials). A similar assertion is valid for the function \( \phi_S(\bullet; \lambda, \Pi_1) \) except that its support is contained in a fixed set, compact modulo \( P_1(E_S) \).

The Schwartz-Bruhat function \( \Phi_S(\bullet; \lambda, \Pi_1) \) is compactly supported: its support is contained in a fixed open set of \( E_S^2 - 0 \), relatively compact in \( E_S^2 - 0 \). It is invariant on the right under \( K_0' \). Thus its support is a compact set of \( E_S^2 \) and it is invariant under translation by a compact open subgroup of \( E_{S_{\text{finite}}} \).

Take \( \lambda \) in a strip. Then \( \Phi_S(\bullet; \lambda, \Pi_1) \) is bounded by a bound for \( \phi_S(\bullet; \lambda, \Pi_1) \).

Since \( G_\infty \) operates on \( E_S^2 \), every element \( X \) of the enveloping algebra defines a differential operator \( X_a \) on \( E_S^2 \) (with non constant coefficients). We have

\[
(X_a \Phi_S)[(0,1)g; \lambda, \Pi_1] = (\rho(X) \phi_S)(g; \lambda, \Pi_1).
\]

Consider a differential operator \( \xi \) on \( E_S^2 \) with constant coefficients. On the complement of \( (0,0) \) we can write \( \xi = \sum c_i X_{i,a} \) where the \( X_i \) are elements of the enveloping algebra and the functions \( c_i \) are smooth functions on \( E_S^2 - (0,0) \).

Using the majorization for the functions \( \rho(X_i) \phi_S \) we conclude that

\[
|\xi \Phi_S[(x,y); \lambda, \Pi_1]| \leq \|\phi\| P(\Pi_1, \lambda).
\]

Thus the functions \( \Phi_S[(x,y); \lambda, \Pi_1] \) remain in a bounded set of \( \mathcal{C}_0^\infty(E_S^2) \) and \textit{a fortiori} of the Schwartz-Bruhat space \( \mathcal{S}(E_S^2) \).

Note that the functions

\[
\xi \Phi_S[(x,y); \lambda, \Pi_1]
\]

are continuous in \((x,y,\lambda)\) and holomorphic in \( \lambda \). In (43) the analytic continuation of the Tate integral is obtained by using the Poisson summation formula. Thus in a strip, say \(-\frac{1}{4} \leq \text{Re} \lambda \leq \frac{1}{4}\), we have then a uniform estimate:

\[
\left| \int \int \Phi((a_1, a_2); \lambda, \Pi_1) \Omega_{-1} \Omega_{\Pi_1}^{-1} \Omega \right| \leq \|\phi\| P(\Pi_1, \lambda).
\]

We have similar estimates for the derivatives by Cauchy integral formula.

We stress that we need only consider characters \( \Pi_1 \) such that \( I_{\lambda, \Pi}(f) \neq 0 \) for \( f \in \mathcal{B} \). Thus for \( v \notin S \) the character \( \Pi_{1,v} \) is unramified and for \( v \) finite in \( S \) the restriction of \( \Pi_{1,v} \) to \( \mathfrak{O}_v^n \) belongs to a finite set. If furthermore \( \Pi_2 = 1 \) this forces \( \Pi_1 \) to belong to a finite set. There is then a polynomial \( P(\Pi_1, \lambda) \) such that, on the line \( \text{Re} \lambda = 0 \), for characters \( \Pi_1 \) of the above type,

\[
\frac{1}{L^2(2\lambda + 1, \Pi_1^2)}
\]

is bounded by \( P(\Pi_1, \lambda) \). Combining with the previous inequality, we get:

**Lemma 11.** Suppose \( f \) is in a bounded set \( \mathcal{B} \). Then there is a polynomial \( P(\Pi_1, \lambda) \) such that, for \( \text{Re} \lambda = 0 \), and all characters \( \Pi_1 \):

\[
|\mu(I_{\lambda, \Pi, \Pi_1^{-1}}(f); \lambda, \Pi_1^{-1})| \leq P(\Pi_1, \lambda)\|\phi\|.
\]
On the other hand, we can obtain similar estimates for the functions
\[ L^S(2\lambda + 1, \Pi_1^2) \]
in a strip (with modification for a pole at \( \lambda = 0 \) if \( \Pi_1^2 = 1 \)), hence also for their derivatives. Thus we have also similar estimates for the derivative of the expression in the proposition on the line \( \text{Re}\lambda = 0 \). We can use the notations of generalized vectors (dual to the smooth vectors). There is a generalized vector \( \mu_{\lambda, \Pi} \) such that
\[ \mu(I_{\lambda, \Pi}(f) \phi; \lambda, \Pi) = \langle \phi, I_{\lambda, \Pi}(f^*) \mu_{\lambda, \Pi} \rangle \]
and our estimate amounts to
\[ \| I_{\lambda, \Pi}(f^*) \mu_{\lambda, \Pi} \| \leq P(\Pi_1, \lambda). \]

**Remark.** — We define the Whittaker linear form \( W \) on the space of \( I_{\lambda, \Pi} \) by analytic continuation of the integral
\[ W(\phi; \lambda, \Pi) = \int_{E_\lambda} \phi(wn)e^{(\lambda + \rho, H(wn))} \psi_E(-x)dx, \quad n = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}. \]
After a change of variables, we easily find
\[ W\left[ I_{\lambda, \Pi}\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} I_{\lambda, \Pi}(f) \phi \right) \right] \]
\[ = \frac{\Pi_1(a)|a|^{\lambda + \frac{1}{2}}}{L^S(2\lambda + 1, \Pi_1 \Pi_2^{-1})} \int_{E_\lambda} \widetilde{\Phi}(at, t^{-1}; \lambda, \Pi_1) \cdot |t|^{2\lambda \Pi_1 \Pi_2^{-1}(t)} dt \]
where \( \widetilde{\Phi} \) denotes the Fourier transform with respect to the second variable. This integral is absolutely convergent and gives the analytic continuation of \( W \).

### 8.4. Intertwining period and intertwining operator.

We need to consider only pairs \( \Pi = (\Pi_1, \Pi_2) \) (with \( \Pi_2 = \Pi_1^{-1} \)) such that the integral of a truncated Eisenstein series over \( \text{GL}(2, F) \) is non-zero. Then (see [JL], [HLRo]) either \( \Pi_1 \) (and \( \Pi_2 \)) have a trivial restriction to \( F_\lambda^\times \) or \( \Pi_2 = \Pi_1^{-1} \) (or, what amounts to the same, \( \Pi_1 = \Pi_1^\tau \)). Of course the two conditions can be both satisfied.

In this section, we assume that \( \Pi_2 = \Pi_1 \). Then we can define an *intertwining period* as follows (see [JL] and [JLRo]):
\[ \mathcal{P}(\phi; \lambda, \Pi) := \int_{T(F_\lambda) \setminus G(F_\lambda)} \phi(\eta h)e^{(\lambda + \rho, H(\eta h))} d\eta. \]
Here \( \eta \in G(E) \) is an element such that \( \eta \tau^{-1} = w \) and \( T \) is the torus such that \( T(F) = \eta^{-1} A(E) \eta \cap \text{GL}(2, F) \).
The integral converges for \( \Re \lambda \gg 0 \) and can be analytically continued. Recall the formula (42). We let \( \Phi_\eta[m; \lambda, \Pi_1] \) be the Schwartz-Bruhat function on \( M(2 \times 2, F_\lambda) \) defined by

\[
\Phi_\eta[m; \lambda, \Pi_1] := \Phi[(0,1)\eta m; \lambda, \Pi_1].
\]

Then we get:

\[
\mathcal{P}(I_{\lambda, \Pi}(f)\phi; \lambda, \Pi) = \frac{1}{L^S(2\lambda + 1, \Pi_1^2)} \times \int_{\text{GL}(2, F_\lambda)} \Phi_\eta[m; \lambda, \Pi_1](\det m)|\det m|^{2\lambda+1} \, dm.
\]

The integral is a Tate integral for the group \( \text{GL}(2, F) \). Using the Iwasawa decomposition we see that the Tate integral is an entire multiple of \( L(2\lambda + 1, \Pi_1^2) \).

In particular, it has no pole on the line \( \Re \lambda = 0 \) except at \( \lambda = 0 \) when \( \Pi_1^2 \neq 1 \). Because we assume that \( \Pi_1^2 = \Pi_2^{-1} = \Pi_1 \) this implies that \( \Pi_1^2 = 1 \).

Then the above ratio is an entire multiple of

\[
\frac{L_{\text{sc}}(2\lambda + 1, 1_F) L(2\lambda, 1_F)}{L^S(2\lambda + 1, \eta)}.
\]

These observations combined with standard estimates for \( L \)-functions can be used to prove the following lemma.

**Lemma 12.** — Let \( \mathcal{B} \) be a bounded set. Then there is a polynomial \( P(\Pi_1, \lambda) \) such that, for all \( \Pi_1 \) with \( \Pi_1^2 = \Pi_1 \) and \( \Pi_1^2 \neq 1 \), all \( \lambda \) with \( \Re \lambda = 0 \), and all \( f \in \mathcal{B} \):

\[
|\mathcal{P}(I_{\lambda, \Pi}(f)\phi; \lambda, \Pi)| \leq P(\Pi_1, \lambda)\|\phi\|.
\]

This can expressed again in terms of generalized vectors. There is a generalized vector \( \mathcal{P}_{\lambda, \Pi} \) such that

\[
\mathcal{P}(I_{\lambda, \Pi}(f)\phi; \lambda, \Pi) = \langle \phi, I_{\lambda, \Pi}(f^*) \mathcal{P}_{\lambda, \Pi} \rangle.
\]

Then:

\[
\|I_{\lambda, \Pi}(f^*) \mathcal{P}_{\lambda, \Pi}\| \leq P(\Pi_1, \lambda).
\]

Combining with the results of the previous subsection, we arrive at the following result.

**Proposition 7.** — Let \( \mathcal{B} \) be a bounded set of the space of smooth functions of compact support. Then there is a polynomial \( P(\Pi_1, \lambda) > 0 \) such that, for any function \( f \) of the form \( f = f_1 * f_2 * f_3 \) with \( f_1, f_3 \in \mathcal{B} \), any \( \Pi_1 \) with \( \Pi_1^2 = \Pi_1 \), any \( \lambda \) with \( \Re \lambda = 0 \):

\[
|\langle I_{\lambda, \Pi}(f) \mathcal{P}_{\lambda, \Pi}, \mu_{\lambda, \Pi} \rangle| \leq P(\Pi_1, \lambda)\|I_{\lambda, \Pi}(f_3)\|.
\]
For the proof we remark that the expression in the proposition has the form
\[ \langle I_{\Pi}(f_2) I_{\Pi}(f_3) P_{\lambda, \Pi}, I_{\lambda, \Pi}(f_1^*) \mu_{\lambda, \Pi} \rangle \]
so that our assertion follows from the previous lemmas, for the characters \( \Pi \) whose restriction to \( F_\kappa^\times \) is non-trivial. On the other hand, if this restriction is trivial then \( \Pi_1^\star = 1 \). We need only consider those \( \Pi \) such that \( I_{\lambda, \Pi}(f_1) \neq 0 \) for \( f_1 \in B \). There are only finitely many such \( \Pi \). So we do not need to have estimates uniform in this case. We have written
\[ \mu_{\lambda, \Pi}(I_{\lambda, \Pi}(f) \phi) = L^2(2\lambda + 1, 1_E)^{-1} \tilde{\mu}_{\lambda, \Pi}(I_{\lambda, \Pi}(f) \phi) \]
where \( \tilde{\mu}(\cdot) \) has no singularity on \( \Re \lambda = 0 \). Since \( P_{\Pi_1, \lambda}(I_{\lambda, \Pi}(f) \phi) \) has a pole at \( \lambda = 0 \) in this case, we need to modify the previous lemma and replace \( P_{\Pi_1, \lambda}(I_{\lambda, \Pi}(f) \phi) \) by its product with \( L^2(2\lambda + 1, 1_E)^{-1} \) which has a zero at \( \lambda = 0 \). The product has then no singularity at \( \lambda = 0 \). It is then easy to obtain the required estimates in this case.

Using similar arguments, we can obtain estimates for the intertwining operator, more precisely, for expressions of the form
\[ \langle M(w, \lambda) I_{\lambda, \Pi}(f)(\phi), \phi' \rangle. \]

8.5. Continuous contribution. — We study the terms corresponding to pairs \( \Pi \) with \( \Pi_\lambda^\star = \Pi_{\lambda}^{-1} \), that is, \( \Pi_\lambda^\star = \Pi_\lambda \). Each such term will ultimately contribute an integral. We first consider the terms with \( \Pi_\lambda \neq \Omega \), (a condition equivalent to \( \Pi_\lambda \neq \Omega \) since \( \Omega^\tau = \Omega^{-1} \)) and \( \Pi_\lambda | F_\kappa^\times \neq 1 \). Then in fact
\[ \int A_{\lambda_1}^T E(a, \phi, \lambda) \Omega^{-1}(a) da, \quad \int A_{\lambda_2}^T E(h_1, \phi, \lambda) dh_1 \]
are independent of \( T_1 \) and \( T_2 \), being equal respectively, to \( \mu(\phi; \lambda, \Pi) \) and \( P(\phi; \lambda, \Pi) \). Thus:
\[ \Theta_{\Pi_1, T_1, T_2}(f) = \frac{1}{2} \sum_{\phi, \phi'} \langle I_{\lambda}(f_2^\star) \phi, \phi' \rangle \mu(I_{\lambda}(f_1^*) \phi', \lambda) \overline{P(I_{\lambda}(f_2) \phi, \lambda)} d\lambda. \]

Note that if \( \Pi_1 = \Pi_2 \) then the expression for \( K_{f, \Pi}(x, y) \) contains a residual term: \( \int \Pi_1(\det g)f(g)dg \Pi_1(\det x)\Pi_1(\det y) \). However, since \( \Omega \neq \Pi_1 \), the integral of the truncation of this term over \( A \) against \( \Omega^{-1} \) contains as a factor the integral of \( \Pi_1 \Omega^{-1} \) over the quotient of the ideles of absolute value 1 and is thus 0. With the notation of generalized vectors, the above integral becomes:
\[ \int \langle I_{\lambda}(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \rangle d\lambda. \]

For \( f = f_1^1 \ast f_1^2 \ast f_2^2 \):
\[ \langle I_{\lambda}(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \rangle = \langle I_{\lambda}(f_1^2) I_{\lambda}(f_2^2) P_{\lambda, \Pi}, I_{\lambda}(f_1^1)* \mu_{\lambda, \Pi} \rangle. \]
By Proposition 7, there is then a polynomial \( P(\Pi_1, \lambda) \) such that
\[
\| \langle I_\lambda(f) P_{\lambda, \Pi, \mu_{\lambda, \Pi}} \rangle \| \leq P(\Pi_1, \lambda) \| I_\lambda(\Pi_1^1) \| .
\]
In fact, this is true for \( f_1^1, f_2^* \) smooth in a bounded set and any smooth function \( f_1^1 \).

We will make repeated use of the following lemma:

**Lemma 13.** — Suppose that \( \mathcal{B} \) is a bounded set of the space of smooth functions of compact support and \( P(\Pi_1, \lambda) \) a polynomial. Then there is a constant \( C > 0 \) such that, for any \( f \in \mathcal{B} \),
\[
\sum_{\Pi_1} \int P(\Pi_1, \lambda) \| I_{\lambda, \Pi_1, \Pi_1^{-1}}(f) \| d\lambda < C,
\]
the sum over all (normalized) idele class characters of \( E \).

**Proof of the Lemma.** — The operator \( I_{\lambda, \Pi_1, \Pi_1^{-1}}(f) \) is represented by the following kernel function on \( K \times K \):
\[
H_{f, \lambda, \Pi}(k_1, k_2) := \int f[k_1^{-1} \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} b & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ x \end{array} \right) k_2] \Pi_1(a)|a|^{1/2} d^*a d^*b dx.
\]
Thus its operator norm is bounded by the supremum over \( K \times K \) of
\[
\left| \int \phi(a : k_1 : k_2) \Pi_1(a)|a|^{1/2} d^*a \right|
\]
where
\[
\phi(a : k_1 : k_2) := \int f[k_1^{-1} \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} b & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ x \end{array} \right) k_2] |a|^{1/2} d^*b dx.
\]
For \( f \in \mathcal{B}, k_i \in K \), the function \( a \mapsto \phi(a : k_1 : k_2) \) remains in a bounded set \( \mathcal{B}_0 \) of the space of smooth functions of compact support on \( E_\lambda^\times \). Thus it will suffice to show there is \( C \) such that, for \( \phi \in \mathcal{B}_0 \),
\[
\sum_{\Pi_1} \int P(\Pi_1, \lambda) \left| \int \phi(a) \Pi_1(1)|a|^{1/2} d^*a \right| d\lambda < C.
\]
This is clear if \( P = 1 \). In general
\[
P(\Pi_1, \lambda) \left| \int \phi(a) \Pi_1(1)|a|^{1/2} d^*a \right| = \left| \int \phi_1(a) \Pi_1(1)|a|^{1/2} d^*a \right|
\]
where \( \phi_1 = \rho(X)\phi \) for a suitable element \( X \) of the enveloping algebra of \( E_\lambda^\times \).

Since \( \phi_1 \) remains in another bounded set, our assertion follows. \( \square \)
From our observations on the independence of the integrals on the truncation parameters, we see that we can sum over all $\Pi_1$ of the above type and get

$$\lim_{T_1 \to +\infty} \lim_{T_2 \to +\infty} \sum_{\Pi_1} \Theta_{\Pi_1, T_1, T_2}(f) = \sum_{\Pi_1} \frac{1}{2} \int \langle I_{\lambda}(f) \mathcal{P}_{\lambda, \mu, \mu} \rangle d\lambda,$$

with

$$\sum_{\Pi_1} \int |\langle I_{\lambda}(f) \mathcal{P}_{\lambda, \mu, \mu} \rangle| d\lambda < +\infty.$$

Moreover, if $B$ is a bounded set, there is $C > 0$ such that, for $f$ of the form $f = f_1 * f_2 * f_3$, with each $f_i$ in $B$, then

$$\sum_{\Pi_1} \int |\langle I_{\lambda}(f) \mathcal{P}_{\lambda, \mu, \mu} \rangle| d\lambda < C.$$

Next, we discuss the contribution of the term $\Pi_1 = \Omega$, if any. Since we are assuming that $\Omega^* = \Omega^{-1}$ in any case and for now we are assuming that $\Pi_1 = \Pi_2^{-1}$ this implies that $\Pi_2 = \Pi_1 = \Omega$ so that $\Omega$ is a quadratic character satisfying $\Omega|_{F_k}^* = \eta$. Moreover $\Pi_{1|F_k}^*$ is then non-trivial. The expression for $K_2$ contains again a residual term; however its truncated integral over $\text{GL}(2, F_\lambda)$ is zero. Indeed, it contains as a factor:

$$\int_{Z(F_\lambda)G(F)/G(F_\lambda)} \eta(\det h) dh - \int_{Z(F_\lambda)A(F)N(F_\lambda)\backslash G(F_\lambda)} \eta(\det h) \hat{\tau}_\mathcal{P}(H(h) - T) dh$$

which is zero. Then $\int \chi_{T_2}^* E(h_1, \phi, \lambda) dh_1$ is again independent of $T_2$ but

$$\int \chi_{T_2}^* E(a, \phi, \lambda) \Omega^{-1}(a) da$$

does depend on $T_1$. We have then for $f = f_1^* * f_2^* * f_3^*$, all $K$-finite functions:

$$\Theta_{\Pi_1, T_1, T_2}(f) = \frac{1}{2} \int \langle I_{\lambda}(f) \mathcal{P}_{\lambda, \mu, \mu} \rangle d\lambda$$

$$+ \frac{1}{2} \sum_{\phi} \left( I_{\lambda}(f_1^* * f_2^*) \phi(e) + I_{\lambda}(f_1^* * f_2^*) \phi(w) \right)$$

$$\times e^{T_1(\frac{1}{2} + \lambda)} \mathcal{P}(I(f_2) \phi, \lambda) d\lambda$$

$$+ \frac{1}{2} \sum_{\phi} \left( (I_{-\lambda}(f_1^* * f_2^*) M(w, \lambda) \phi(e) + I_{-\lambda}(f_1^* * f_2^*) M(w, \lambda) \phi(w) \right)$$

$$\times e^{T_1(\frac{1}{2} - \lambda)} \mathcal{P}(I_{-\lambda}(f_2) M(w, \lambda) \phi, \lambda) d\lambda$$

where for the last expression we have used the functional equation of the period $\mathcal{P}$ (inherited from the functional equation of the Eisenstein series, see [JLRo]) to insert an intertwining operator. In the last expression, the sum over
φ does not depend on the choice of the basis. Thus we can ignore the intertwining operator. In the second expression, we can also change λ to −λ; we see that the second expression is equal to the third. That is:

\[ \Theta_{\Pi_1, T_1, T_2}(f) = \frac{1}{2} \int \langle I_\lambda(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \rangle d\lambda \]

\[ + \sum_{\phi} \left\{ (I_\lambda(f_1^* f_2^*) \phi(e) + I_\lambda(f_1^* f_2^*) \phi(w)) \frac{e^{T_1(\frac{1}{2} - \lambda)}}{\frac{1}{2} - \lambda} P(I_\lambda(f_2) \phi, \bar{\lambda}) \right\} d\lambda. \]

We will move the integral to the line \text{Re} \lambda = 1 rather than to the line \text{Re} \lambda = \frac{1}{2} as we did in [J2] (a similar contour integration for \text{Gl}(2) is done in [F3]). The term \( P(\ast, \bar{\lambda}) \) is the product of a Tate integral which is entire, hence bounded in vertical strips and the factor \( (L^3(2\lambda + 1, 1_E))^{-1} \). Thus this moving of the contour of integration is legitimate. In doing so we pick up a residue at \( \lambda = \frac{1}{2} \).

The integral on the line \text{Re} \lambda = 1 tends to 0 as \( T_1 \) tends to infinity. So we are left with

\[ \lim_{T_1 \to +\infty} \lim_{T_2 \to +\infty} \Theta_{\Pi_1, T_1, T_2}(f) = \frac{1}{2} \int \langle I_\lambda(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \rangle d\lambda \]

\[ + \sum_{\phi} (I_{-\lambda}(f_1^* f_2^*) \phi(e) + I_{-\lambda}(f_1^* f_2^*) \phi(w)) P(I_{\lambda}(f_2) \phi, \bar{\lambda}). \]

We view again \( P \) as a generalized vector in the (non-unitary) representation \( L^3_{\Omega, \Omega} \). Then \( L^3_{\Omega, \Omega}(f) P \) is a smooth vector in this representation and the second term is the sum \( I_{-\lambda, \Omega, \Omega}(f) P(e) + I_{-\lambda, \Omega, \Omega}(f) P(w) \).

Finally, we study the contribution of the terms where \( \Pi_1^* = \Pi_1 \) and \( \Pi_1 \neq \Pi_2 \). Then \( \Pi_1 \neq \Omega \). Also \( \Pi_2^* = 1 \). Thus, for a given \( f \), or even \( f \) in a given bounded set, there are only finitely many such \( \Pi_1 \) with \( I_{\lambda, \Pi_1, \Pi_2}(f) \neq 0 \). Thus the sum \( \sum_{\phi} \Theta_{\Pi_1, T_1, T_2}(f) \) is finite, and in the sum, each term is itself written as a finite sum of terms. The kernel \( K_{f, \Pi} \) has once more a residual component whose truncated integral over the diagonal is 0. This time \( \int A_{m, \lambda}^f E(a, \phi, \lambda) \Omega^{-1}(a) da \) is independent of \( T_1 \) but

\[ \int A_{m, \lambda}^f E(h_1, \phi, \lambda) dh_1 = \frac{e^{\lambda T_2}}{\lambda} \int_{K \cap H_1} \phi(k) dk \]

\[ + \frac{e^{-\lambda T_2}}{-\lambda} \int_{K \cap H_1} M(w, \lambda) \phi(k) dk + P(\phi, \lambda, \Pi). \]

Since \( \Pi_1 = \Pi_2 \) we have \( M(w, 0) = -1 \) and the two first terms have the same residue at \( \lambda = 0 \). The last term has also a simple pole at \( \lambda = 0 \). However the factor \( (39) \) reduces to \( \mu(I_{\lambda}(f_1) \phi, \lambda, \Pi) \) which has a zero at \( \lambda = 0 \) because in \( (43) \) the reciprocal of the \( L \)-factor does. Thus the poles of the terms in \( (40) \) are compensated by the zero of \( (39) \). As \( T_2 \) tends to infinity the terms containing
\( e^{\pm \lambda T_2} \) tend to 0 and we find the contribution of those finitely many terms to be the same as before (see Proposition 7).

8.6. Discrete terms. — Finally, we discuss the terms whose contributions will be purely discrete. We consider those \( \Pi \) such that \( \Pi_1 L_{\mathcal{F}}^* = 1 \) but \( \Pi_1^2 \neq \Pi_1 \). This condition amounts to \( \Pi_1 \neq \Omega \) and the integral \( \int A_T E(a, \phi, \lambda) \Omega^{-1}(a) \mathrm{d}a \) does not depend on \( T_1 \). However, we have then, in the sense of analytic continuation,

\[
\begin{align*}
\Phi(T) &= \int A_T E(h_1, \phi, \lambda) \mathrm{d}h_1 = \frac{e^{\lambda T_2}}{\lambda} \mathcal{P}_c(\phi) + \frac{e^{-\lambda T_2}}{-\lambda} \mathcal{P}_c(M(\lambda, w) \phi), \\
\mathcal{P}_c(\phi) &= \int_{K_F} \phi(k) \mathrm{d}k,
\end{align*}
\]

where \( K_F \) denotes the standard maximal compact subgroup of \( \text{GL}(2, F) \). Since the left hand side is holomorphic at \( \lambda = 0 \) we get (See also [JL, Lemma 7], for a computational proof.)

\[
\mathcal{P}_c(\phi) = \mathcal{P}_c(M(0, w) \phi).
\]

It follows that

\[
\Theta_{\Pi, T_1, T_2}(f) = \frac{1}{2} \sum \mu(I_{\lambda}(f)) \left( \frac{e^{\lambda T_2}}{\lambda} \mathcal{P}_c(\phi) + \frac{e^{-\lambda T_2}}{-\lambda} \mathcal{P}_c(M(\lambda, w) \phi) \right) \mathrm{d}\lambda.
\]

Again, in terms of generalized vectors, this can be written in the form:

\[
\frac{1}{2} \int \left( \langle I_{\lambda, \Pi}(f) \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle \frac{e^{-\lambda T_2}}{-\lambda} + \langle I_{\lambda, \Pi}(f) M(w, \lambda)^* \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle \frac{e^{\lambda T_2}}{\lambda} \right) \mathrm{d}\lambda.
\]

Consider then the functions:

\[
\Phi_1(\lambda) := \sum_{\Pi} \langle I_{\lambda, \Pi}(f) \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle, \quad \Phi_2(\lambda) := \sum_{\Pi} \langle I_{\lambda, \Pi}(f) M(w, \lambda)^* \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle,
\]

the sums over all the \( \Pi \) of the above type.

Lemme 14. — The above series converge absolutely, uniformly on compact sets, and define continuous functions on the line \( \Re \lambda = 0 \). Moreover \( \Phi_1(0) = \Phi_2(0) \). If \( f_2 \) is the convolution of sufficiently many \( K \)-finite functions, the functions are differentiable on the line \( \Re \lambda = 0 \).

Indeed, for the first assertion, we write as before \( f_1^1 * f_2^2 * f_2^2 \). Then

\[
\langle I_{\lambda, \Pi}(f) \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle = \langle I_{\lambda, \Pi}(f_1^1) I_{\lambda, \Pi}(f_2^2) \mathcal{P}_c, I_{\lambda, \Pi}(f_1^1)^* \mu_{\lambda, \Pi} \rangle.
\]

As before, there is a polynomial \( P(\Pi_1, \lambda) \) such that

\[
\|I_{\lambda, \Pi}(f_1^1)^* \mu_{\lambda, \Pi}\| \leq P(\Pi_1, \lambda).
\]

It is easy to see that

\[
\|I_{\lambda, \Pi}(f_2^2) \mathcal{P}_c\| \leq C
\]
where \( C \) is a constant. Hence
\[
\| \langle I_{\lambda, \Pi}(f) \mathcal{P}_c, \mu_{\lambda, \Pi} \rangle \| \leq C P(\Pi_1, \lambda) \| I_{\lambda, \Pi}(f_1^2) \|.
\]
Since
\[
\sum_{\Pi_1} P(\Pi_1, \lambda) \| I_{\lambda, \Pi}(f_1^2) \| < +\infty
\]
the first assertion follows for \( \Phi_1 \) (with uniform convergence on compact sets, even for \( f_1^2, f_1, f_2 \) smooth in a bounded set). Since the intertwining operator is unitary, the argument for \( \Phi_2 \) is similar. For the differentiability, we proceed as follows. We do assume that the functions are \( K \)-finite. Then we need only consider those \( \Pi \) such that the corresponding operators are non-zero. This means that the restriction of \( \Pi_1 \) to the maximal compact subgroup of \( \mathbb{E}_h^\infty \) takes only finitely values. For the purpose of convergence, we may as well fix this restriction. Then all the representations \( I_{\lambda, \Pi} \) operate on the same Hilbert space of functions on the maximal compact subgroup \( K \). Thus it suffices to show that
\[
\sum_{\phi' \in \mathcal{K}} \sum_{\Pi \subseteq \Pi_1} \langle I_{\lambda, \Pi}(f_1^2) \phi, \phi' \rangle \mu(I_{\lambda, \Pi}(f_1^2)) \phi', \lambda, \Pi \rangle \mathcal{P}_c(I_{\lambda, \Pi}(f_2) \phi)
\]
is differentiable, the sum over \( \phi, \phi' \) being finite. We may as well fix \( \phi \) and \( \phi' \). At this point, we do assume that \( f_2 = f_1^2 + f_2^2 \) where the functions \( f_1^2, f_2 \), \( i = 1, 2 \), are \( K \)-finite. Then the terms \( \langle I_{\lambda, \Pi}(f_1^2) \phi, \phi' \rangle \) and \( \mathcal{P}_c(I_{\lambda, \Pi}(f_2) \phi) \) are bounded by constant multiples of \( \| I_{\lambda, \Pi}(f_1^2) \| \) and \( \| I_{\lambda, \Pi}(f_2) \| \), respectively. Their respective derivatives are bounded, uniformly in \( (\Pi_1, \lambda) \). As for \( \mu(I_{\lambda, \Pi}(f_1^2)) \phi', \lambda, \Pi \) it is bounded as well as its derivatives by a polynomial in \( (\Pi_1, \lambda) \). Our assertion follows for \( \Phi_1 \). For \( \Phi_2 \), we observe that
\[
\sum_{\Pi_1} \langle I_{\lambda, \Pi}(f_1^2) \phi, \phi' \rangle \mu(I_{\lambda, \Pi}(f_1^2)) \phi', \lambda, \Pi \rangle \mathcal{P}_c(I_{\lambda, \Pi}(f_2) \phi) = \sum_{\phi''} \sum_{\Pi_1} \langle I_{\lambda, \Pi}(f_1^2) \phi, \phi' \rangle \mu(I_{\lambda, \Pi}(f_1^2)) \phi', \lambda, \Pi \rangle \mathcal{P}_c(I_{\lambda, \Pi}(f_2) \phi) \cdot \langle I_{\lambda, \Pi}(f_2) \phi, \phi'' \rangle
\]
with a finite sum over \( \phi'' \). We proceed as before, this time assuming that, in turn, \( f_2 \) is the convolution of two \( K \)-finite functions.

We have then, at least if \( f_2 \) is the convolution of sufficiently many \( K \)-finite functions,
\[
\sum_{\Pi} \Theta_{\Pi, T_2}(f) = \int \left( \Phi_1(\lambda) e^{-\lambda T_2} - \Phi_2(\lambda) e^{\lambda T_2} \right) d\lambda.
\]
This expression is independent of \( T_1 \). So we only need to take the limit as \( T_2 \to +\infty \). It is of the form \( e^{c(\Phi_1(0))} \).
8.7. Conclusion. — We are now ready to state the main result of this section:

**Theorem 3.** For any smooth function of compact support $f$:

$$
\Theta(f) = \sum_{\Pi \text{ cuspidal}} \Theta_{\Pi}(f) + \sum_{\Pi_1^0 = \Pi_1} \frac{1}{2} \sum_{\Pi} \left\langle I_{\lambda, \Pi}(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \right\rangle d\lambda
$$

$$
+ c \sum_{\Pi_1^0 \neq \Pi_1} \left\langle I_{0, \Pi}(f) P_c, \mu_{0, \Pi} \right\rangle
$$

$$
+ c_1 \left\{ \left( I_{\frac{1}{2}, \Omega, \Omega}(f) P \right)(e) + \left( I_{-\frac{1}{2}, \Omega, \Omega}(f) P \right)(w) \right\}.
$$

The expression is absolutely convergent:

$$
\sum_{\Pi_1^0 = \Pi_1} \int \left| \left\langle I_{\lambda, \Pi}(f) P_{\lambda, \Pi}, \mu_{\lambda, \Pi} \right\rangle \right| d\lambda < +\infty,
$$

$$
\sum_{\Pi_1|F_0^\circ = 1} \left| \left\langle I_{0, \Pi}(f) P_c, \mu_{0, \Pi} \right\rangle \right| < +\infty.
$$

By [DM], every smooth function of compact support can be written as a finite sum of triple convolution products. Thus it suffices to prove our assertion when $f = f_1 \ast f_2 \ast f_3$. If each $f_i$ remains in a bounded set both sides are well defined and bounded. Since the space of smooth functions of compact support is a bornological space, each side may be regarded as a distribution on the product of three copies of the group. We have shown both sides agree when each $f_i$ is $K$-finite and $f_3$ is itself the convolution of sufficiently many $K$-finite functions. Thus they agree for arbitrary $f_i$ and our assertion follows. Note that the right hand side is in fact a distribution, as follows for instance from the kernel theorem.

9. Appendix: Continuous spectrum over $F$

We pass to the much easier discussion of the convergence of the spectral expression for $G_\epsilon$. For simplicity we only consider the case $\omega = 1$, $\Omega = 1$. The only difficult case is the case where $\epsilon$ is a norm. Of course, no truncation is needed. We then identify $G_\epsilon$ with $\text{Gl}(2, F)$. Consider a pair of characters $\pi = (\pi_1, \pi_2)$ with $\pi_1 \pi_2 = 1$. Since $\text{Gl}(2, F) = P(F)T(F)$ we have:

$$
\int E(t, \phi; \pi, \lambda) dt = \nu(\phi; \lambda, \pi)
$$

**BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE**
where
\[ \nu(\phi; \pi, \lambda) := \int_{T(F)Z(F)} \phi(t) e^{(\lambda + \rho, H(t))} \, dt. \]

Using again the notation of generalized vectors, what we have to show is that
\[ \sum \int \langle I_{\lambda,\pi}(f) \nu_{\lambda,\pi}, \nu_{\lambda,\pi} \rangle \, d\lambda \]
is absolutely convergent. This can be seen directly when \( f \) is \( K \)-finite. For our purposes this is not quite enough since we want to use smooth functions of compact support. We use again an integral representation of the form (42):
\[ I_{\lambda,\Pi}(f) \phi(g) = \pi_1(\lambda) |\det g|^{\lambda + \frac{1}{2}} \int_{E_\lambda} \Phi[(0, t)g; \lambda, \pi_1] \cdot |t|^{2\lambda + 1} \pi_1^{-1} \pi_2^{-1} \, dt, \]
\[ \phi[g; \lambda, \pi_1] := \Phi[(0, 1)t; \lambda, \pi_1] \]
on the simple algebra of which \( T \) is the multiplicative group. Then
\[ \nu(I_{\lambda,\pi}(f) \phi) = \frac{1}{L^\infty(2\lambda + 1, \pi_1^{-1})} \int \phi(t) \pi_1(\det t) \, |\det t|^{-1} \, dt. \]
The integral is then a Tate integral for \( E \). In particular, it is a holomorphic multiple of \( L(\lambda + \frac{1}{2}, \pi_1 \circ \text{Norm}) \). We can then finish the proof as before.

BIBLIOGRAPHY


[BM] Baruch (E.M.), Mao (Z.) – Central critical value of automorphic \( L \)-functions, preprint

[CHM] Casselman (W.), Hecht (H.), Milčić (D.) – Bruhat filtrations and Whittaker vectors for real \( \ell \)-adics, preprint.


