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ON THE $\bar{\partial}$-EQUATION IN A BANACH SPACE

BY IMRE PATYI (*)

ABSTRACT. — We define a separable Banach space X and prove the existence of a $\bar{\partial}$-closed C^∞-smooth $(0,1)$-form f on the unit ball B of X, which is not $\bar{\partial}$-exact on any open subset. Further, we show that the sheaf cohomology groups $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X, and Ω is any pseudoconvex domain in X, e.g., $\Omega = B$. As the Dolbeault group $H^{0,1}_{\bar{\partial}}(B) \neq 0$, the Dolbeault isomorphism theorem does not generalize to arbitrary Banach spaces. Lastly, we construct a C^∞-smooth integrable almost complex structure on $M = B \times \mathbb{C}$ such that no open subset of M is biholomorphic to an open subset of a Banach space. Hence the Newlander–Nirenberg theorem does not generalize to arbitrary Banach manifolds.

RÉSUMÉ. — SUR L'ÉQUATION $\bar{\partial}$ DANS UN ESPACE DE BANACH. — On définit un espace de Banach séparable X et on montre l'existence d'une forme $\bar{\partial}$ fermée du type $(0,1)$ de classe C^∞ sur la boule unité B de X, qui n'est $\bar{\partial}$ exacte dans aucun ouvert. On montre en outre que $H^q(\Omega, \mathcal{O}) = 0$ pour $q \geq 1$ et $\Omega \subset X$ ouvert pseudo-convexe, par exemple, $\Omega = B$. Il s'ensuit que l'isomorphisme de Dolbeault ne se généralise pas aux espaces de Banach quelconques. On montre également que le théorème de Newlander-Nirenberg ne se généralise pas aux variétés de Banach quelconques.

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Introduction

This paper addresses three fundamental problems that arise in complex analysis on Banach spaces and on Banach manifolds.

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† To my mother and father.

The first concerns vanishing of Dolbeault cohomology groups. Presently there is one definitive result on this: the Dolbeault cohomology group $H^{0,1}_{\bar\partial}(\Omega) = 0$ for any pseudoconvex open $\Omega \subset \ell_1$, see [L3, Cor. 0.2]. In no other infinite dimensional Banach space is a similar result available. Here, we shall show that such a vanishing theorem cannot be true in complete generality. In Section 1 we shall define a separable Banach space X and a C^∞-smooth $\bar\partial$-closed $(0,1)$-form f on its unit ball such that on no open set G is the equation $\bar\partial u = f_{|G}$ solvable. This implies $H^{0,1}_{\bar\partial}(\Omega) \neq 0$ for any bounded open set $\Omega \subset X$. We note that globally non-solvable $\bar\partial$-equations in Fréchet spaces were constructed earlier by Dineen [D] and Meise-Vogt [MV].

The second issue to be considered is that of an infinite dimensional version of the Dolbeault isomorphism between the Dolbeault cohomology groups $H^{0,q}_{\bar\partial}(\Omega)$ and the sheaf cohomology groups $H^q(\Omega, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X. Currently no instance of such an isomorphism is known when Ω is open in an infinite dimensional Banach space. We shall show that $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, for all pseudoconvex open subsets Ω of the above space X. In particular, $0 = H^1(\Omega, \mathcal{O}) \not\cong H^{0,1}_{\bar\partial}(\Omega) \neq 0$ for any bounded pseudoconvex open set $\Omega \subset X$. The vanishing of sheaf cohomology follows from a theorem of Lempert [L3, Thm. 0.3] plus a Runge-type approximation theorem to be proved in Section 2.

The last issue to be addressed concerns the extension of the Newlander-Nirenberg theorem on integrating almost complex structures to an infinite dimensional setting. The question is whether a (C^∞-smooth) formally integrable almost complex manifold is locally biholomorphic to a vector space. In finite dimensions it is true, see [NN], while in some Fréchet manifolds it is known to be false, see [LB], [L5]. This failure in itself is perhaps not surprising, as on Fréchet manifolds even real vector fields may not be integrable. However, in Section 3, given any C^∞-smooth $\bar\partial$-closed but nowhere $\bar\partial$-exact $(0,1)$-form f on the unit ball B of a Banach space X, we explicitly construct a C^∞-smooth integrable almost complex structure on $M = B \times \mathbb{C}$ such that no open subset of M is biholomorphic to an open subset of a Banach space, giving a Banach manifold to which the Newlander-Nirenberg theorem does not generalize. The manifold M is a smoothly trivial principal $(\mathbb{C}, +)$ bundle over B and its almost complex structure will be determined by the form f, which we use as a deformation tensor.

Below we shall use freely basic notions of infinite dimensional complex analysis, see [L1], [L2] for the definition and basic properties of the following items: differential calculus on infinite dimensional spaces; smoothness classes $C^m(\Omega)$, $C^m_{p,q}(\Omega)$ of functions and of (p,q)-forms with $m = 0, 1, \ldots, \infty, \omega$; the $\bar\partial$-complex; complex manifolds, almost complex manifolds; pseudoconvexity; holomorphic mappings and integrability of almost complex structures.

Notation

Denote by $B_X(a,r)=\{x\in X\,;\ \|x-a\|<r\}$ the open ball with center $a\in X$ of radius $0<r\le\infty$ in a Banach space $(X,\|\cdot\|)$. Put $B_X(r)=B_X(0,r)$. Denote by $C^m(\Omega)$, $C^m_{0,1}(\Omega)$ the space of complex functions and of $(0,1)$-forms of smoothness class $m=0,1,\dots,\infty,\omega$, and define for $u\in C^m(\Omega)$, $m<\infty$, the $C^m(\Omega)$ norm by

$$\|u\|_{C^m(\Omega)}=\sum_{k\le m}\sup_{x\in\Omega}\|u^{(k)}(x)\|\le\infty,$$

where $\|u^{(k)}(x)\|$ is the operator norm of the k-th Fréchet derivative $u^{(k)}$ of u. The $C^m(\Omega)$ norm of $f\in C^m_{0,1}(\Omega)$ is defined by

$$\|f\|_{C^m(\Omega)}=\|u\|_{C^m(\Omega\times B_X(1))}\le\infty,$$

where $u(x,\xi)=f(x)\xi$ for $x\in\Omega$, $\xi\in B_X(1)$.

1. Non-solvability

We consider the solvability of

$$\bar\partial u=f\quad\text{on}\quad\Omega,\tag{1.1}$$

where $f\in C^m_{0,1}(\Omega)$ is a $\bar\partial$-closed $(0,1)$-form with $m=1,2,\dots,\infty$ on a domain Ω in a Banach space X.

Coeuré in [C] (see also [M]) gave an f on $X=\Omega=\ell_2$ of class C^1 for which (1.1) is not solvable on any open set. Lempert in [L2] extended Coeuré's example and produced, with $p=2,3,\dots$, a $\bar\partial$-closed form $f\in C^{p-1}_{0,1}(\ell_p)$ for which (1.1) is not solvable on any open set. Based on the mere existence of these examples, we prove that there is a form f of class C^∞ on $\Omega=B_X(1)$ in, say, the ℓ_1-sum X of a suitable sequence of $\ell_p(\mathbb{C}^{n(p)})$ spaces with $p\ge 2$ integer, for which (1.1) is not solvable on any open subset of Ω.

Let Y be ℓ_q, $1\le q<\infty$, or c_0. We define the Y-sum X of a sequence of Banach spaces $(X_n,\|\cdot\|_n)_{n=1}^\infty$ by

$$X=\bigl\{x=(x_n)\,;\ x_n\in X_n,\ y=|x|\in Y,\ \|x\|=\|y\|_Y\bigr\},$$

where $|x|=(\|x_1\|_1,\|x_2\|_2,\dots)$.

Then X is a Banach space and we have inclusions $I_n\colon X_n\to X$, $I_n(x_n)=(0,\dots,0,x_n,0,\dots)$ with x_n at the n-th place and projections $\pi_n\colon X\to X_n$, $\pi_n(x)=x_n$, $\pi_{m,n}\colon X\to X$, $\pi_{m,n}(x)=(z_i)$, where $z_i=x_i$ if $m\le i\le n$, $z_i=0$ otherwise; $1\le m\le n\le\infty$, not both ∞. The I_n are isometries onto their image and I_n, π_n, $\pi_{m,n}$ have operator norm 1.

THEOREM 1.1. — *For a suitable sequence of integers $n(p) \geq 1$, $p \geq 2$, and for any Y as above, on the Y-sum X of $(\ell_p(\mathbb{C}^{n(p)}))_{p=2}^{\infty}$ there exists a $\bar\partial$-closed $f \in C^{\infty}_{0,1}(B_X(1))$ for which* (1.1) *is not solvable on any $B_X(a,r) \subset B_X(1)$.*

REMARK. — For $1 \leq p < \infty$ regard $\ell_p(\mathbb{C}^n)$ embedded in ℓ_p via

$$J_n : \ell_p(\mathbb{C}^n) \longrightarrow \ell_p, \quad J_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots),$$

put $B_p(r) = B_{\ell_p}(r)$, $B_{p,n}(r) = B_{\ell_p(\mathbb{C}^n)}(r)$,

$$\varrho_n : \ell_p \longrightarrow \ell_p(\mathbb{C}^n), \quad \varrho_n(x) = (x_1, \ldots, x_n, 0, \ldots),$$

and let $f \in C^m_{0,1}(B_p(1))$ be $\bar\partial$-closed and of finite $C^m(B_p(1))$ norm for some $m \geq 1$. Then, for some $0 < r \leq 1$, $\bar\partial u = f$ has a bounded solution u on $B_p(r)$ if and only if for all $n \geq 1$ there are solutions of $\bar\partial u_n = J_n^* f$ on $B_{p,n}(r)$ such that

$$\sup_{B_{p,n}(r)} |u_n| \leq M \|J_n^* f\|_{C^m(B_{p,n}(1))} \leq M \|f\|_{C^m(B_p(1))},$$

with M independent of the dimension n.

This observation is the pillar of the argument below. Such a reformulation of the solvability of (1.1) was already given for Hilbert space by Mazet in [M], Appendix 3, Section 1, Remark 2.

PROPOSITION 1.2. — *With the notations of the remark above, the following statement (E_p) is false for any integer $p \geq 2$.*

(E_p) *There exist a radius $0 < r \leq 1$, a constant $0 < M < \infty$ such that for all $n = 1,2,\ldots$ and for all $\bar\partial$-closed $f \in C^{\infty}_{0,1}(B_{p,n}(1))$ of finite $C^{p-1}(B_{p,n}(1))$ norm, the equation $\bar\partial u = f$ has a solution on $B_{p,n}(r)$ satisfying*

$$\sup_{B_{p,n}(r)} |u| \leq M \|f\|_{C^{p-1}(B_{p,n}(1))}.$$

Proof. — Denote by (E'_p) the statement (E_p) with "$f \in C^{\infty}_{0,1}(B_{p,n}(1))$" replaced by "$f \in C^{p-1}_{0,1}(B_{p,n}(1))$". Fix p and suppose for a contradiction that (E_p) is true. Since the $\bar\partial$ differential operator has constant coefficients, approximation using convolution shows that (E'_p) is also true.

We claim that (E'_p) implies the solvability on $B_p(r)$ of (1.1) with any $\bar\partial$-closed $f \in C^{p-1}_{0,1}(B_p(1))$ of finite $C^{p-1}(B_p(1))$ norm. Let u_n be a solution of $\bar\partial u_n = f_{|B_{p,n}(r)}$ guaranteed by (E'_p). The functions $v_n = \varrho_n^* u_n$ on $B_p(r)$ satisfy, with a suitable constant N, that $|v_n(x)|$, $|(\bar\partial v_n)(x)\xi| \leq N$ for $x \in B_p(r)$, $\xi \in B_p(1)$.

It follows from the Cauchy-Pompeiu representation formula [H, Thm. 1.2.1] applied to 1-dimensional slices that $(v_n)_1^\infty$ is a locally equicontinuous family on $B_p(r)$. The Arzelà-Ascoli theorem gives a subsequence $v_{n'} \to v$ converging uniformly on compacts in $B_p(r)$. As v is continuous and $\bar\partial v = f$ holds restricted to $B_{p,n}(r)$ for every n in the distributional sense, it follows by approximation that $\bar\partial v = f$ holds in the distributional sense restricted to any finite dimensional slice of $B_p(r)$. The "elliptic regularity of the $\bar\partial$ operator" implies that v is a C^{p-1} solution of $\bar\partial v = f$ on $B_p(r)$. See [L2, Props. 2.3, 2.4].

Now, pull back the form g in Coeuré's or Lempert's example for ℓ_p by $x \mapsto \varepsilon x$ with an $\varepsilon > 0$ so small that the resulting form f has finite $C^{p-1}(B_{\ell_p}(1))$ norm. Then (1.1) is not solvable on any open subset of ℓ_p. This contradiction proves Proposition 1.2. □

Proof of Theorem 1.1. — We shall use the method of "Condensation of Singularities." As (E_p) is false for $p \geq 2$ integer, we have sequences $n(p) \geq 1$ of integers, $r_p \to +0$ of radii, $f_p \in C^\infty_{0,1}(B_{p,n(p)}(1))$ of $\bar\partial$-closed forms with $\|f_p\|_{C^{p-1}(B_{p,n(p)}(1))} = 1$ such that if $\bar\partial u = f_p$ on $B_{p,n(p)}(r_p)$ then $\sup_{B_{p,n(p)}(r_p)} |u| \geq p^{p+1}$.

Let X be the Y-sum of $\ell_p(\mathbb{C}^{n(p)})$, $p = 2, 3, \ldots$. Put

$$f = \sum_{p=2}^{\infty} p^{-p} \pi_p^* f_p.$$

One checks that f is in $C^\infty_{0,1}(B_X(1))$ and is $\bar\partial$-closed.

We claim that $\bar\partial u = f$ cannot be solved on any open subset of $B_X(1)$.

Indeed, suppose for a contradiction that there are a ball $B_X(a,r)$ and a function u with $\bar\partial u = f$ on $B_X(a,r)$. Take r so small that

$$\sup_{B_X(a,r)} |u| = N < \infty.$$

This can be done as u is continuous at a (even C^∞). Choose $q \geq 2$ so large that $\|\pi_{q+1,\infty}(a)\| < \frac{1}{3}r$. Fix $p > q, N$ so large that $r_p < \frac{1}{3}r$. Let

$$v(z) = u\big(\pi_{2,q}(a) + I_p(z)\big)$$

for $z \in B_{p,n(p)}(r_p)$. Then $\bar\partial v = p^{-p} f_p$ on $B_{p,n(p)}(r_p)$, so

$$N \geq \sup_{B_{p,n(p)}(r_p)} |v| \geq p^{-p} p^{p+1} = p > N.$$

This contradiction proves Theorem 1.1. □

Further, we claim that

$$\dim_{\mathbb{C}} H^{0,1}_{\bar\partial}\big(B_X(1)\big) = \infty.$$

We group the indices p into pairwise disjoint infinite sets P_n, $n \geq 1$. Then for the Y-sum $X^{(n)}$ of $\ell_p(\mathbb{C}^{n(p)})$, $p \in P_n$, we have inclusions $J_n : X^{(n)} \to X$ and projections $\varrho_n : X \to X^{(n)}$ both of operator norm 1. Let $g_n \in C^\infty_{0,1}(B_{X^{(n)}}(1))$ be a $\bar\partial$-closed nowhere $\bar\partial$-exact form whose existence is guaranteed by the proof of Theorem 1.1. Then $f_n = \varrho_n^* g_n$ are linearly independent in $H^{0,1}_{\bar\partial}(B_X(1))$. Indeed, suppose that $\lambda_1 f_1 + \cdots + \lambda_n f_n = \bar\partial u$, $\lambda_i \in \mathbb{C}$. Then by restricting to $X^{(i)}$ we see that $\lambda_i g_i$ is $\bar\partial$-exact, hence $\lambda_i = 0$.

Should it turn out (as it is yet unknown) that on the unit ball B of ℓ_2 there are $\bar\partial$-closed $(0,1)$-forms of arbitrarily high finite smoothness that are nowhere $\bar\partial$-exact, then the construction in Section 1 with $Y = \ell_2$ would yield a $\bar\partial$-closed $f \in C^\infty_{0,1}(B)$ which is nowhere $\bar\partial$-exact: a non-solvable $\bar\partial u = f$ in Hilbert space.

2. Approximation

We consider the following kind of approximation in a Banach space X.

$$(A) \qquad \begin{cases} \textit{For any } 0 < r < R,\ \varepsilon > 0, \textit{ and } f : B_X(R) \to \mathbb{C} \textit{ holomorphic, there} \\ \textit{exists an entire function } g : X \to \mathbb{C} \textit{ with } |f - g| < \varepsilon \textit{ on } B_X(r). \end{cases}$$

THEOREM 2.1. — *The statement (A) holds for the ℓ_1-sum X of any sequence of finite dimensional Banach spaces $(X_n, \|\cdot\|_n)$.*

Lempert in [L4] proved (A) for $X = \ell_1$. When this manuscript was first written, Theorem 2.1 was the most general theorem proving (A). Later, however, (A) was proved in [L6] for any X with a countable unconditional basis, *i.e.*, for most classical Banach spaces. It is not clear whether all spaces X in Theorem 2.1 have a countable unconditional basis, or even a Schauder basis.

The proof of Theorem 2.1 is a modification and extension of Lempert's method in [L4]. Lempert's argument is based on the so-called monomial expansion of functions holomorphic on a ball $\|x\| < R \leq \infty$ of ℓ_1 (an analogue of the power series expansion on a finite dimensional space), and on the use of a dominating function $\Delta(q, z)$ defined and continuous on $\mathbb{C} \times B_{\ell_1}(1)$, whose role in the estimation of monomial series is similar to the role of the geometric series in estimating power series.

We replace the monomials by so-called multihomogeneous functions but use the same dominating function Δ of Lempert.

2.1. Multihomogeneous functions.

Let X be the ℓ_1-sum of a sequence of finite dimensional Banach spaces $(X_n, \|\cdot\|_n)$. For $\lambda = (\lambda_n) \in \ell_\infty$ and $x \in X$ put $\lambda x = (\lambda_1 x_1, \lambda_2 x_2, \ldots) \in X$. In the rest of this Section k denotes a multiindex. A multiindex $k = (k_n)$ for us is a sequence of integers $k_n \geq 0$ with $k_n = 0$ for n large enough. The support of k is the set $\operatorname{supp} k = \{n\,;\ k_n \neq 0\}$. We define $\|k\| = \sum |k_n|$, and $\# k$ as the number of elements of the support of k. For a sequence of complex numbers $\lambda = (\lambda_n)$ we put $\lambda^k = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \in \mathbb{C}$, a finite product. For a multiindex k, a holomorphic function $\varphi : B_X(R) \to \mathbb{C}$ is called *k-homogeneous* if $\varphi(\lambda x) = \lambda^k \varphi(x)$ for all $x \in B_X(R)$, $\lambda = (\lambda_n) \in \ell_\infty$ with $|\lambda_n| = 1$.

A k-homogeneous function φ is a homogeneous polynomial of degree $\|k\|$ depending only on those finitely many variables $x_n \in X_n$ for which $n \in \operatorname{supp} k$. In particular, φ extends automatically to an entire function on X, and $\varphi(\lambda x) = \lambda^k \varphi(x)$ holds for all $x \in X$ and $\lambda \in \ell_\infty$.

We define the norm $[\varphi]$ of a k-homogeneous function φ by

$$[\varphi] = \sup_{\|x\| \leq 1} \big|\varphi(x)\big|.$$

The set of all k-homogeneous functions φ for a fixed k is a finite dimensional Banach space with this norm.

Proposition 2.2. — *For φ k-homogeneous and $x \in X$,*

$$\big|\varphi(x)\big| \leq [\varphi]\, |x|^k\, \|k\|^{\|k\|}\, k^{-k}.$$

Proof. — If $x_i = 0$ for some $i \in \operatorname{supp} k$, then $\varphi(x) = 0$ as seen from the definition. So we may suppose that $\operatorname{supp} k = \{1, 2, \ldots, n\}$ and $x_i \neq 0$ for $1 \leq i \leq n$. Put

$$y = \Big(\frac{k_1}{\|k\|}\frac{x_1}{\|x_1\|_1}, \ldots, \frac{k_n}{\|k\|}\frac{x_n}{\|x_n\|_n}, 0, \ldots\Big) \in X.$$

Then $\|y\| = 1$, so $[\varphi] \geq |\varphi(y)| = k^k \|k\|^{-\|k\|} |x|^{-k} |\varphi(x)|$, as claimed. $\square$

2.2. The dominating function of Lempert.

This function is defined by the series

$$\Delta(q, z) = \sum_k \frac{\|k\|^{\|k\|}}{k^k} |q|^{\# k} |z^k|$$

for $(q, z) \in \mathbb{C} \times B_{\ell_1}(1)$. See [L2, Section 4].

THEOREM 2.3. — (a) *The series for* Δ *converges uniformly on compacts in* $\mathbb{C} \times B_{\ell_1}(1)$.

(b) *For each* $0 < \theta < 1$ *there is an* $\varepsilon > 0$ *such that* Δ *is bounded on* $B_{\mathbb{C}}(\varepsilon) \times B_{\ell_1}(\theta)$.

Proof. — See [L4, Thm. 2.1]. ▯

We remark that the norm of a monomial z^k on ℓ_1 is $[z^k] = k^k \|k\|^{-\|k\|}$ as a simple calculation shows. So $\Delta(q,z)$ can be written as

$$\Delta(q,z) = \sum |z^k| \, [z^k]^{-1} \, |q|^{\# k},$$

where we add up normalized monomials with a weight counting the number of variables in the monomials.

2.3. Multihomogeneous expansions.

Let $T = (\mathbb{R}/\mathbb{Z})^\infty = \{t = (t_n) \,;\; 0 \le t_n < 1\}$ be the infinite dimensional torus, a compact topological group with the product topology and with Haar measure $\mathrm{d}t$ of total mass equal to 1.

For a holomorphic function $f \colon B_X(R) \to \mathbb{C}$ we define the multihomogeneous expansion of f by $f \sim \sum f_k$, with

$$f_k(x) = \int_{t \in T} f(\mathrm{e}^{2\pi i\, t} x)\, \mathrm{e}^{-2\pi i\, (k \cdot t)} \, \mathrm{d}t,$$

where k is any multiindex, $\mathrm{e}^{2\pi i\, t} x = (\mathrm{e}^{2\pi i\, t_1} x_1, \mathrm{e}^{2\pi i\, t_2} x_2, \ldots)$ and $(k \cdot t) = \sum k_n t_n$, a finite sum. Then f_k is defined, holomorphic and k-homogeneous on $B_X(R)$. We call f_k the *k-homogeneous component* of the function f. Let

$$S = \{\sigma = (\sigma_n) \,;\; 0 \le \sigma_n \to 0\}, \quad S_1 = \{\sigma \in S \,;\; 0 \le \sigma_n < 1\},$$

$$\sigma A = \{\sigma x \,;\; x \in A\}$$

for $A \subset X$, $\sigma \in S$ as in [L4, Section 2].

PROPOSITION 2.4. — (a) *If* $f \colon B_X(R) \to \mathbb{C}$ *is a holomorphic function, then we have the estimate*

$$M(\sigma) = \sup_k \, [f_k] \, \sigma^k R^{\|k\|} < \infty$$

for all $\sigma \in S_1$.

(b) *If* f_k *is* k*-homogeneous and* $M(\sigma) < \infty$ *for all* $\sigma \in S_1$, *then the series* $g = \sum f_k$ *converges uniformly on compact subsets of* $B_X(R)$, g *is holomorphic on* $B_X(R)$, *and the* k*-homogeneous component* g_k *of* g *is equal to* f_k.

Proof. — We use the following compactness criterion: A subset $K \subset X$ is compact if and only if K is closed, bounded, and the tail sums $R_n(x) = \sum_{\nu \geq n} \|x_\nu\|_\nu \to 0$ uniformly on K as $n \to \infty$.

We outline the proof. If K is compact, then $R_n \to 0$ uniformly on K by Dini's theorem on monotone convergence of continuous functions on a compact space to a continuous limit. In the other direction, fix $\varepsilon > 0$. We produce a finite covering of K by ε-balls. Fix n so large that $R_n < \frac{1}{2}\varepsilon$ on K, and project K onto the space of the first n coordinates, this is a bounded set in a finite dimensional space, so it has a finite covering by balls $B_X(x_i, \frac{1}{2}\varepsilon)$. Now, $B_X(x_i, \varepsilon)$ cover K.

This criterion implies, in particular, that any compact $K \subset B_X(1)$ is contained in $\sigma^2 B_X(1)$ for suitable $\sigma \in S_1$, and all the sets $\sigma B_X(1)$ for $\sigma \in S$ have compact closure. The utility of such a criterion was already observed by Ryan [R] in a similar context.

Proof of (a). — The set $\overline{\sigma B_X(R)}$ being compact,

$$\sup_{\|x\|<1} \big|f(\sigma R x)\big| = M < \infty.$$

Thus, $\varphi(x) = f(\sigma R x)$ for $\|x\| < 1$ is bounded by M on $B_X(1)$. So is its k-homogeneous component $\varphi_k(x) = \sigma^k R^{\|k\|} f_k(x)$, hence $[f_k]\,\sigma^k R^{\|k\|} \leq M$, or $M(\sigma) \leq M < \infty$.

Proof of (b). — Without loss of generality we may suppose that the given compact is σL, where $L \subset B_X(r)$ is compact, $\sigma \in S_1$ and $r < R$. Then putting $x = \sigma y$ for $|y| < r$, $y \in L$, we have that

$$\begin{aligned} |f_k(x)| &\leq [f_k]\, \|k\|^{\|k\|}\, k^{-k}\, |x|^k \\ &= [f_k] \|k\|^{\|k\|}\, k^{-k}\, \sigma^k\, |y|^k \\ &= [f_k]\, \sigma^k\, R^{\|k\|} \cdot \|k\|^{\|k\|}\, k^{-k}\, |y/R|^k. \end{aligned}$$

Summing on k, we get

$$\sum \big|f_k(x)\big| \leq M(\sigma)\Delta(1,z) \leq M < \infty$$

where $z = |y/R|$ ranges in a compact subset of $B_{\ell_1}(1)$, and the series for Δ converges uniformly by Theorem 2.3 (a).

This concludes the proof of Proposition 2.4. $\square$

PROPOSITION 2.5. — *Let f_k be k-homogeneous. If for each multiindex k and for all $\sigma \in S$ (!) we have* $\sup_k [f_k]\,\sigma^k R^{\|k\|} < \infty$, *then $\sum f_k$ is an entire function on X.*

Proof. — If $M(\sigma) < \infty$ for all $\sigma \in S$, then $M(\lambda\sigma) < \infty$ for all $0 < \lambda < \infty$, $\sigma \in S_1$, which has the same effect as changing R to λR in Proposition 2.4 (b). Hence the multihomogeneous series converges on the whole of X. $\square$

We quote two lemmas from [L4].

PROPOSITION 2.6. — *If the numbers* $0 \le c_k < \infty$ *are such that*

$$\sup_k c_k\, \sigma^k < \infty$$

for all $\sigma \in S_1$, *then for any* $Q \ge 1$ *and* $\sigma \in S_1$ *the estimate* $\sup_k c_k\, \sigma^k Q^{\# k} < \infty$ *holds.*

Proof. — See [L4, Prop. 4.2]. ◻

PROPOSITION 2.7. — *Let* $0 < \theta < 1$ *and* $\mathcal{K}$ *a set of multiindices* k. *Then if* $0 < c_k < \infty$, $k \in \mathcal{K}$, *satisfy*

$$\inf_{k\in\mathcal{K}} c_k \theta^{\|k\|} > 0 \quad \textit{and} \quad \sup_{k\in\mathcal{K}} c_k \sigma^k < \infty$$

for all $\sigma \in S_1$, *then* $\sup_{k\in\mathcal{K}} c_k \sigma^k < \infty$ *for all* $\sigma \in S$, *too.*

Proof. — See [L4, Prop. 4.3]. ◻

Proof of Theorem 2.1. — Let us expand f in a multihomogeneous series $\sum f_k$. Fix a number $0 < \theta < 1$ with $r < \theta^2 R$. For any $\delta > 0$, $Q > 1$ (to be suitably chosen below) put

$$c_k = [f_k]\, R^{\|k\|}, \quad c'_k = c_k Q^{\# k},$$

$$\mathcal{K} = \big\{k\,;\ c'_k \theta^{\|k\|} \equiv [f_k](\theta R)^{\|k\|} Q^{\# k} \ge \delta\big\}, \quad g(x) = \sum_{k\in\mathcal{K}} f_k(x).$$

We claim that this g is an entire function on X.

Indeed, by Proposition 2.5 it is enough to show for all $\sigma \in S$ that

$$\sup_{k\in\mathcal{K}} [f_k]\, \sigma^k R^{\|k\|} \equiv \sup_{k\in\mathcal{K}} c_k\, \sigma^k < \infty.$$

As $\inf_{k\in\mathcal{K}} c'_k \theta^{\|k\|} \ge \delta > 0$, and for $\sigma \in S_1$ Proposition 2.4 (a) implies that

$$\sup_{k\in\mathcal{K}} [f_k]\, \sigma^k R^{\|k\|} \equiv \sup_{k\in\mathcal{K}} c_k\, \sigma^k < \infty,$$

so by Prop. 2.6, $\sup_{k\in\mathcal{K}} c'_k \sigma^k < \infty$ holds for all $\sigma \in S_1$. Now both conditions of Prop. 2.7 are verified, hence $\sup_{k\in\mathcal{K}} c_k\, \sigma^k \le \sup_{k\in\mathcal{K}} c'_k\, \sigma^k < \infty$ for all $\sigma \in S$. Therefore, by Proposition 2.5, g is an entire function.

For $k \notin \mathcal{K}$ we have $[f_k](\theta R)^{\|k\|} Q^{\# k} \leq \delta$. We estimate $|f(x) - g(x)|$. For $\|x\| < r$ by Proposition 2.2, we have

$$
\begin{aligned}
|f(x) - g(x)| &\leq \sum_{k \notin \mathcal{K}} |f_k(x)| \leq \sum_{k \notin \mathcal{K}} [f_k] \frac{\|k\|^{\|k\|}}{k^k} |x|^k \\
&\leq \sum_{k \notin \mathcal{K}} \delta Q^{-\# k} (\theta R)^{-\|k\|} \frac{\|k\|^{\|k\|}}{k^k} |x|^k \\
&\leq \delta \sum_{k \notin \mathcal{K}} Q^{-\# k} \frac{\|k\|^{\|k\|}}{k^k} \left| \frac{\theta x}{r} \right|^k \leq \delta \sup_{\|w\| \leq \theta} \Delta(Q^{-1}, w)
\end{aligned}
$$

as $\theta R > r/\theta$ and $|w| = |\theta x / r| \leq \theta$. But the last expression can be made $< \varepsilon$ by choosing first Q large enough to make the sup finite by Theorem 2.3 (b), and then by choosing δ small enough.

Thus, the proof of the approximation Theorem 2.1 is concluded. $\square$

Let $Y = \ell_q$, $1 \leq q < \infty$, or $Y = c_0$. Let e^n_{pi}, $1 \leq i \leq n$, be the standard basis of $\ell_p(\mathbb{C}^n)$. Then the Y-sum X of any sequence $\ell_{p_k}(\mathbb{C}^{n_k})$ spaces, $k \geq 1$, has a countable unconditional basis: $e^{n_1}_{p_1 1}, e^{n_1}_{p_1 2}, \ldots, e^{n_1}_{p_1 n_1}; e^{n_2}_{p_2 1}, \ldots, e^{n_2}_{p_2 n_2}; \ldots$. Now, the approximation theorem of Lempert [L6, Thm. 0.1], or in the case $Y = \ell_1$, Theorem 2.1 above, implies by the vanishing theorem [L3, Thm. 0.3] that the sheaf cohomology groups $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, on any pseudoconvex open set $\Omega \subset X$ for the sheaf $\mathcal{O}$ of germs of holomorphic functions on X. So for any Y, the space X of Theorem 1.1 has the property that $H^{0,1}_{\bar\partial}(\Omega) \neq 0$ (in fact, infinite dimensional) and $H^1(\Omega, \mathcal{O}) = 0$ for any bounded pseudoconvex open set $\Omega \subset X$: the Dolbeault isomorphism theorem does not generalize to arbitrary Banach spaces.

We remark that if the form f is real-analytic and Ω pseudoconvex, then by [L1, Prop. 3.2] the equation (1.1) has real-analytic local solutions; since $H^1(\Omega, \mathcal{O}) = 0$, we get a global real-analytic solution, too.

3. Almost complex manifolds

Theorem 1.1 verifies the hypothesis of Theorem 3.1 below in a case.

THEOREM 3.1. — *Let X be a Banach space and suppose that on $B = B_X(1)$ there exists a $\bar\partial$-closed $f \in C^\infty_{0,1}(B)$ that is not $\bar\partial$-exact on any open subset. Then on $M = B \times \mathbb{C}$ a C^∞-smooth integrable almost complex structure M_f can be constructed in such a way that no open subset of M_f is biholomorphic to an open subset of a Banach space.*

As the referee has kindly pointed it out, the method of this section is analogous to one used earlier to construct nonrealizable CR hypersurfaces by Jacobowitz in [J].

We recall the definition of almost complex structure. An almost complex structure on a C^m-smooth manifold M is a splitting of the complexified tangent bundle $\mathbb{C}\otimes TM = T^{1,0}\oplus T^{0,1}$ into the direct sum of two complex vector bundles of class C^{m-1} with $T^{0,1}=\overline{T^{1,0}}$, $m=1,\ldots,\infty,\omega$ and $m-1=m$ for $m=\infty,\omega$. An almost complex structure is called formally integrable (or just integrable) if $m\geq 2$ and the Lie bracket of two $(1,0)$ vector fields of class C^1 is also a $(1,0)$ vector field; here $(1,0)$ can be changed to $(0,1)$.

The proof of Theorem 3.1 requires a few steps.

3.1. Construction of the almost complex structure on M.

The construction will be described in a setting more general than that of Theorem 3.1, namely, in the context of principal bundles.

Denote by $\zeta^{1,0}$, $\zeta^{0,1}$ the $(1,0)$-part, $(0,1)$-part of a complex tangent vector ζ to an almost complex manifold. Let B be a complex Banach manifold, G a finite dimensional complex Lie group with Lie algebra $\mathfrak{g}=T_eG$, $f\in C^\infty_{0,1}(B,\mathfrak{g}^{1,0})$ a $(0,1)$-form with values in $\mathfrak{g}^{1,0}$, and $L_z\colon G\to G$ the left translation $L_z(s)=zs$, $z,s\in G$. Define the holomorphic Maurer-Cartan form $\mu\in C^\infty_{1,0}(G,\mathfrak{g}^{1,0})$ by

$$\mu(\nu)=(\mathrm{d}L_z)^{-1}\nu^{1,0}=\big((\mathrm{d}L_z)^{-1}\nu\big)^{1,0}$$

for $\nu\in\mathbb{C}\otimes T_zG$. Recall the holomorphic Maurer-Cartan formula $\mathrm{d}\mu+\frac{1}{2}[\mu,\mu]=0$, which can be proved similarly to or deduced from the usual Maurer-Cartan formula. Define on any complex Banach manifold N the Lie bracket $[\varphi,\psi]\in C^\infty_{0,2}(N,\mathfrak{g}^{1,0})$ of forms $\varphi,\psi\in C^\infty_{0,1}(N,\mathfrak{g}^{1,0})$ by the usual formula

$$[\varphi,\psi](\zeta,\zeta')=\big[\varphi(\zeta),\psi(\zeta')\big]-\big[\varphi(\zeta'),\psi(\zeta)\big]$$

for $\zeta,\zeta'\in\mathbb{C}\otimes T_xN$, where the brackets on the right hand side are taken in the Lie algebra $\mathfrak{g}^{1,0}$. In particular, $[f,f](\zeta,\zeta')=2\,[f(\zeta),f(\zeta')]$.

We define an almost complex structure M_f on $M=B\times G$ by putting $(\zeta,\nu)\in\mathbb{C}\otimes T_{(x,z)}M=\mathbb{C}\otimes T_xB\oplus\mathbb{C}\otimes T_zG$ in $T^{0,1}_{(x,z)}M$ if and only if

$$\zeta=\zeta^{0,1}\quad\text{and}\quad\mu(\nu)=f(\zeta). \tag{3.1}$$

In the setting of Theorem 3.1 we identify $G=\mathbb{C}$ and $\mathfrak{g}^{1,0}=\mathbb{C}$ via the correspondence $G\ni s\sim s\,\partial/(\partial z)_{|z=0}\in\mathfrak{g}^{1,0}$, where z is the usual coordinate on $\mathbb{C}$.

We verify below the following: Definition (3.1) gives an almost complex structure M_f on M and makes it into an almost complex principal G bundle; M is formally integrable if and only if $\bar{\partial} f + \frac{1}{2}[f,f] = 0$ holds; if M_f is locally biholomorphic to a Banach space, then $\overline{D}u = f$, where $u: B \to G$ is defined locally, and $\overline{D}$ is defined by

$$\overline{D}u(\zeta) = \mu\big(\mathrm{d}u(\zeta^{0,1})\big)$$

for $\zeta \in T_x B$. In the setting of Theorem 3.1 this $\overline{D}u$ reduces to the usual $\bar{\partial}u$.

3.2. Verification.

To verify that (3.1) defines an almost complex structure on M, we need to check conditions 1)–2).

1) *If* $V = (\zeta, \nu)$, $\overline{V} = (\bar{\zeta}, \bar{\nu})$ *are in* $T^{0,1}_{(x,z)}M$, *then* $V = 0$.

We have $0 = \zeta^{0,1} = \bar{\zeta}^{0,1} \equiv \overline{\zeta^{1,0}}$, or $\zeta = 0$. Similarly $\mu(\nu) = \mu(\bar{\nu}) = 0$ implies $\nu^{1,0} = \bar{\nu}^{1,0} = 0$, or $\nu = 0$.

2) *Given* $V = (\zeta, \nu)$, *decompose it as* $V = V_1 + V_2$ *with* V_1, $\overline{V_2} \in T^{0,1}M$. One checks that

$$V_1 = \big(\zeta^{0,1}, \mathrm{d}L_z\, f(\zeta) - \mathrm{d}L_z\, \overline{f(\bar{\zeta})} + \nu^{0,1}\big),$$
$$V_2 = \big(\zeta^{1,0}, \mathrm{d}L_z\, \overline{f(\bar{\zeta})} - \mathrm{d}L_z\, f(\zeta) + \nu^{1,0}\big)$$

is the unique way of decomposition.

3) Condition of formal integrability: *If* $V = (\zeta, \nu)$, $V' = (\zeta', \nu')$ *are* C^∞ *sections of* $T^{0,1}M$ *over an open subset of* M, *then their Lie bracket* $[V, V']$ *is also a section of* $T^{0,1}M$.

Denote by $\mathcal{L}_Z$ the Lie derivative along a complex vector field Z. We can write $[V, V']$ as

$$[V, V'] = (\zeta^*, \nu^*) = \big([\zeta, \zeta'] + \mathcal{L}_\nu \zeta' - \mathcal{L}_{\nu'} \zeta,\ [\nu, \nu'] + \mathcal{L}_\zeta \nu' - \mathcal{L}_{\zeta'} \nu\big).$$

We work out below the condition of formal integrability for M_f in terms of f.

(a) The first component ζ^* is $(0,1)$ because so are $[\zeta, \zeta']$, $\mathcal{L}_\nu \zeta'$, $\mathcal{L}_{\nu'} \zeta$ since B is a complex manifold.

(b) Taking the $\mathcal{L}_\zeta$, $\mathcal{L}_\nu$ Lie derivatives of the identity $\mu(\nu') - f(\zeta') = 0$ and reversing the roles of V, V' we find the equations

$$\mu(\mathcal{L}_\zeta \nu') - \mathcal{L}_\zeta(f(\zeta')) = 0, \qquad \mathcal{L}_\nu(\mu\nu') - f(\mathcal{L}_\nu \zeta') = 0,$$
$$\mu(\mathcal{L}_{\zeta'} \nu) - \mathcal{L}_{\zeta'}(f(\zeta)) = 0, \qquad \mathcal{L}_{\nu'}(\mu\nu) - f(\mathcal{L}_{\nu'} \zeta) = 0,$$

whose alternating sum is

$$\{\mu\nu^* - f(\zeta^*)\} + \{\mathcal{L}_\nu(\mu\nu') - \mathcal{L}_{\nu'}(\mu\nu) - \mu([\nu,\nu'])\} \\ - \{\mathcal{L}_\zeta(f(\zeta')) - \mathcal{L}_{\zeta'}(f(\zeta)) - f([\zeta,\zeta'])\} = 0.$$

Hence, by Cartan's formula for exterior derivatives, the condition of formal integrability is that

$$(\mathrm{d}\mu)(\nu,\nu') - (\mathrm{d}f)(\zeta,\zeta') = 0.$$

Since $\mu(\nu) = f(\zeta)$, $\mu(\nu') = f(\zeta')$ we get by the holomorphic Maurer-Cartan formula that

$$-\tfrac{1}{2}\,[f,f](\zeta,\zeta') - (\mathrm{d}f)(\zeta,\zeta') = 0$$

for all $(0,1)$ vector fields ζ, ζ' on B. Hence the almost complex manifold M_f is formally integrable if and only if

$$\bar\partial f + \tfrac{1}{2}\,[f,f] = 0,$$

which condition reduces to $\bar\partial f = 0$ when G is commutative as in Theorem 3.1.

3.3. Geometric properties of M.

To check that M is a principal G bundle we need to verify that $\pi: M = B \times G \to B$, $\pi(x,z) = x$ is holomorphic and that G has a simply transitive action on the fibers of M. Indeed, π is holomorphic as $\mathrm{d}\pi(\zeta,\nu) = \zeta$ takes $(0,1)$-vectors to $(0,1)$-vectors. The action of $w \in G$ on M is given by the left translation $\ell_w(x,z) = (x,wz)$ in the fiber direction. This is holomorphic because $(\mathrm{d}\ell_w)(\zeta,\nu) = (\zeta,\, \mathrm{d}L_w\, \nu)$ and $\mu(\mathrm{d}L_w\, \nu) = \mu(\nu)$.

In the setting of Theorem 3.1 a direct verification shows that $\Phi: M_f \to M_g$, $\Phi(x,z) = (x, z + u(x))$ is a bundle biholomorphism, where $g = f + \bar\partial u$ and $u \in C^\infty(B)$ is any function. Hence the bundle biholomorphism type of M_f depends only on the Dolbeault cohomology class of f.

We return now to the general setting.

Proposition 3.2. — *If $p_0 = (x_0,z_0) \in M_f$ has a neighborhood that is C^m-biholomorphic, $m = 1,2,\dots,\infty$, to an open set in a Banach space, then there are a neighborhood $U_0 \subset B$ of x_0 and $u \in C^m(U_0,G)$ such that $\bar D u = f$ on U_0.*

Proof. — The Banach space $T^{0,1} = T^{0,1}_{p_0} M$ has a natural splitting as a direct sum $T^{0,1} = V^{0,1} \oplus H^{0,1}$ of vertical and horizontal closed subspaces

$$V^{0,1} = \{(0,\nu) \in T^{0,1}\,;\ \nu \in T^{0,1}_{z_0} G\},$$
$$H^{0,1} = \{(\zeta,\nu) \in T^{0,1}\,;\ \nu \in T^{1,0}_{z_0} G\}.$$

Suppose now that $\Phi: U \to V$ is biholomorphism of a neighborhood U of p_0 in M_f onto a neighborhood V of 0 in a Banach space W. Then the splitting $T^{0,1} = V^{0,1} \oplus H^{0,1}$ induces via $(\mathrm{d}\Phi)(p_0)$ a splitting

$$T_0^{0,1}W \equiv W = V_W^{0,1} \oplus H_W^{0,1}.$$

Since $N = \Phi^{-1}(V \cap H_W^{0,1})$ is an almost complex C^m-submanifold of M passing through p_0 transversely to $V^{0,1}$, hence to $\{x_0\} \times G$, N is the image near p_0 of a holomorphic section $s: U_0 \to G$ on a neighborhood U_0 of x_0. Then writing $s(x) = (x, u(x))$ and applying (3.1) we obtain that $\mu(\mathrm{d}u(\zeta)) = f(\zeta)$ for all $\zeta \in T_x^{0,1}U_0$, but this is the same as saying $\overline{D}u = f$ on U_0; thus concluding the proof of Proposition 3.2 and hence that of Theorem 3.1. $\square$

We have seen that the Newlander-Nirenberg theorem does not generalize to arbitrary integrable almost complex Banach manifolds. It is unknown if it generalizes to Hilbert manifolds.

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