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ON THE $\bar{\partial}$ -EQUATION IN A BANACH SPACE

BY IMRE PATYI (*)

ABSTRACT. — We define a separable Banach space X and prove the existence of a $\bar{\partial}$ -closed C^∞ -smooth $(0, 1)$ -form f on the unit ball B of X , which is not $\bar{\partial}$ -exact on any open subset. Further, we show that the sheaf cohomology groups $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X , and Ω is any pseudoconvex domain in X , e.g., $\Omega = B$. As the Dolbeault group $H_{\bar{\partial}}^{0,1}(B) \neq 0$, the Dolbeault isomorphism theorem does not generalize to arbitrary Banach spaces. Lastly, we construct a C^∞ -smooth integrable almost complex structure on $M = B \times \mathbb{C}$ such that no open subset of M is biholomorphic to an open subset of a Banach space. Hence the Newlander–Nirenberg theorem does not generalize to arbitrary Banach manifolds.

RÉSUMÉ. — SUR L'ÉQUATION $\bar{\partial}$ DANS UN ESPACE DE BANACH. — On définit un espace de Banach séparable X et on montre l'existence d'une forme $\bar{\partial}$ fermée du type $(0, 1)$ de classe C^∞ sur la boule unité B de X , qui n'est $\bar{\partial}$ exacte dans aucun ouvert. On montre en outre que $H^q(\Omega, \mathcal{O}) = 0$ pour $q \geq 1$ et $\Omega \subset X$ ouvert pseudo-convexe, par exemple, $\Omega = B$. Il s'ensuit que l'isomorphisme de Dolbeault ne se généralise pas aux espaces de Banach quelconques. On montre également que le théorème de Newlander–Nirenberg ne se généralise pas aux variétés de Banach quelconques.

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Introduction

This paper addresses three fundamental problems that arise in complex analysis on Banach spaces and on Banach manifolds.

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[†] To my mother and father.

The first concerns vanishing of Dolbeault cohomology groups. Presently there is one definitive result on this: the Dolbeault cohomology group $H_{\bar{\partial}}^{0,1}(\Omega) = 0$ for any pseudoconvex open $\Omega \subset \ell_1$, see [L3, Cor.0.2]. In no other infinite dimensional Banach space is a similar result available. Here, we shall show that such a vanishing theorem cannot be true in complete generality. In Section 1 we shall define a separable Banach space X and a C^∞ -smooth $\bar{\partial}$ -closed $(0,1)$ -form f on its unit ball such that on no open set G is the equation $\bar{\partial}u = f|_G$ solvable. This implies $H_{\bar{\partial}}^{0,1}(\Omega) \neq 0$ for any bounded open set $\Omega \subset X$. We note that globally non-solvable $\bar{\partial}$ -equations in Fréchet spaces were constructed earlier by Dineen [D] and Meise-Vogt [MV].

The second issue to be considered is that of an infinite dimensional version of the Dolbeault isomorphism between the Dolbeault cohomology groups $H_{\bar{\partial}}^{0,q}(\Omega)$ and the sheaf cohomology groups $H^q(\Omega, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X . Currently no instance of such an isomorphism is known when Ω is open in an infinite dimensional Banach space. We shall show that $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, for all pseudoconvex open subsets Ω of the above space X . In particular, $0 = H^1(\Omega, \mathcal{O}) \neq H_{\bar{\partial}}^{0,1}(\Omega) \neq 0$ for any bounded pseudoconvex open set $\Omega \subset X$. The vanishing of sheaf cohomology follows from a theorem of Lempert [L3, Thm. 0.3] plus a Runge-type approximation theorem to be proved in Section 2.

The last issue to be addressed concerns the extension of the Newlander-Nirenberg theorem on integrating almost complex structures to an infinite dimensional setting. The question is whether a (C^∞ -smooth) formally integrable almost complex manifold is locally biholomorphic to a vector space. In finite dimensions it is true, see [NN], while in some Fréchet manifolds it is known to be false, see [LB], [L5]. This failure in itself is perhaps not surprising, as on Fréchet manifolds even real vector fields may not be integrable. However, in Section 3, given any C^∞ -smooth $\bar{\partial}$ -closed but nowhere $\bar{\partial}$ -exact $(0,1)$ -form f on the unit ball B of a Banach space X , we explicitly construct a C^∞ -smooth integrable almost complex structure on $M = B \times \mathbb{C}$ such that no open subset of M is biholomorphic to an open subset of a Banach space, giving a Banach manifold to which the Newlander-Nirenberg theorem does not generalize. The manifold M is a smoothly trivial principal $(\mathbb{C}, +)$ bundle over B and its almost complex structure will be determined by the form f , which we use as a deformation tensor.

Below we shall use freely basic notions of infinite dimensional complex analysis, see [L1], [L2] for the definition and basic properties of the following items: differential calculus on infinite dimensional spaces; smoothness classes $C^m(\Omega)$, $C_{p,q}^m(\Omega)$ of functions and of (p,q) -forms with $m = 0, 1, \dots, \infty, \omega$; the $\bar{\partial}$ -complex; complex manifolds, almost complex manifolds; pseudoconvexity; holomorphic mappings and integrability of almost complex structures.

Notation

Denote by $B_X(a, r) = \{x \in X; \|x - a\| < r\}$ the open ball with center $a \in X$ of radius $0 < r \leq \infty$ in a Banach space $(X, \|\cdot\|)$. Put $B_X(r) = B_X(0, r)$. Denote by $C^m(\Omega)$, $C_{0,1}^m(\Omega)$ the space of complex functions and of $(0, 1)$ -forms of smoothness class $m = 0, 1, \dots, \infty, \omega$, and define for $u \in C^m(\Omega)$, $m < \infty$, the $C^m(\Omega)$ norm by

$$\|u\|_{C^m(\Omega)} = \sum_{k \leq m} \sup_{x \in \Omega} \|u^{(k)}(x)\| \leq \infty,$$

where $\|u^{(k)}(x)\|$ is the operator norm of the k -th Fréchet derivative $u^{(k)}$ of u . The $C^m(\Omega)$ norm of $f \in C_{0,1}^m(\Omega)$ is defined by

$$\|f\|_{C^m(\Omega)} = \|u\|_{C^m(\Omega \times B_X(1))} \leq \infty,$$

where $u(x, \xi) = f(x)\xi$ for $x \in \Omega$, $\xi \in B_X(1)$.

1. Non-solvability

We consider the solvability of

$$(1.1) \quad \bar{\partial}u = f \quad \text{on } \Omega,$$

where $f \in C_{0,1}^m(\Omega)$ is a $\bar{\partial}$ -closed $(0, 1)$ -form with $m = 1, 2, \dots, \infty$ on a domain Ω in a Banach space X .

Coeuré in [C] (see also [M]) gave an f on $X = \Omega = \ell_2$ of class C^1 for which (1.1) is not solvable on any open set. Lempert in [L2] extended Coeuré's example and produced, with $p = 2, 3, \dots$, a $\bar{\partial}$ -closed form $f \in C_{0,1}^{p-1}(\ell_p)$ for which (1.1) is not solvable on any open set. Based on the mere existence of these examples, we prove that there is a form f of class C^∞ on $\Omega = B_X(1)$ in, say, the ℓ_1 -sum X of a suitable sequence of $\ell_p(\mathbb{C}^{n(p)})$ spaces with $p \geq 2$ integer, for which (1.1) is not solvable on any open subset of Ω .

Let Y be ℓ_q , $1 \leq q < \infty$, or c_0 . We define the Y -sum X of a sequence of Banach spaces $(X_n, \|\cdot\|_n)_{n=1}^\infty$ by

$$X = \{x = (x_n); x_n \in X_n, y = |x| \in Y, \|x\| = \|y\|_Y\},$$

where $|x| = (\|x_1\|_1, \|x_2\|_2, \dots)$.

Then X is a Banach space and we have inclusions $I_n: X_n \rightarrow X$, $I_n(x_n) = (0, \dots, 0, x_n, 0, \dots)$ with x_n at the n -th place and projections $\pi_n: X \rightarrow X_n$, $\pi_n(x) = x_n$, $\pi_{m,n}: X \rightarrow X$, $\pi_{m,n}(x) = (z_i)$, where $z_i = x_i$ if $m \leq i \leq n$, $z_i = 0$ otherwise; $1 \leq m \leq n \leq \infty$, not both ∞ . The I_n are isometries onto their image and $I_n, \pi_n, \pi_{m,n}$ have operator norm 1.

THEOREM 1.1. — *For a suitable sequence of integers $n(p) \geq 1$, $p \geq 2$, and for any Y as above, on the Y -sum X of $(\ell_p(\mathbb{C}^{n(p)}))_{p=2}^\infty$ there exists a $\bar{\partial}$ -closed $f \in C_{0,1}^\infty(B_X(1))$ for which (1.1) is not solvable on any $B_X(a, r) \subset B_X(1)$.*

REMARK. — For $1 \leq p < \infty$ regard $\ell_p(\mathbb{C}^n)$ embedded in ℓ_p via

$$J_n: \ell_p(\mathbb{C}^n) \longrightarrow \ell_p, \quad J_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots),$$

put $B_p(r) = B_{\ell_p}(r)$, $B_{p,n}(r) = B_{\ell_p(\mathbb{C}^n)}(r)$,

$$\varrho_n: \ell_p \longrightarrow \ell_p(\mathbb{C}^n), \quad \varrho_n(x) = (x_1, \dots, x_n, 0, \dots),$$

and let $f \in C_{0,1}^m(B_p(1))$ be $\bar{\partial}$ -closed and of finite $C^m(B_p(1))$ norm for some $m \geq 1$. Then, for some $0 < r \leq 1$, $\bar{\partial}u = f$ has a bounded solution u on $B_p(r)$ if and only if for all $n \geq 1$ there are solutions of $\bar{\partial}u_n = J_n^* f$ on $B_{p,n}(r)$ such that

$$\sup_{B_{p,n}(r)} |u_n| \leq M \|J_n^* f\|_{C^m(B_{p,n}(1))} \leq M \|f\|_{C^m(B_p(1))},$$

with M independent of the dimension n .

This observation is the pillar of the argument below. Such a reformulation of the solvability of (1.1) was already given for Hilbert space by Mazet in [M], Appendix 3, Section 1, Remark 2.

PROPOSITION 1.2. — *With the notations of the remark above, the following statement (E_p) is false for any integer $p \geq 2$.*

$$(E_p) \quad \left\{ \begin{array}{l} \text{There exist a radius } 0 < r \leq 1, \text{ a constant } 0 < M < \infty \text{ such that} \\ \text{for all } n = 1, 2, \dots \text{ and for all } \bar{\partial}\text{-closed } f \in C_{0,1}^\infty(B_{p,n}(1)) \text{ of finite} \\ C^{p-1}(B_{p,n}(1)) \text{ norm, the equation } \bar{\partial}u = f \text{ has a solution on } B_{p,n}(r) \\ \text{satisfying} \\ \sup_{B_{p,n}(r)} |u| \leq M \|f\|_{C^{p-1}(B_{p,n}(1))}. \end{array} \right.$$

Proof. — Denote by (E'_p) the statement (E_p) with “ $f \in C_{0,1}^\infty(B_{p,n}(1))$ ” replaced by “ $f \in C_{0,1}^{p-1}(B_{p,n}(1))$ ”. Fix p and suppose for a contradiction that (E_p) is true. Since the $\bar{\partial}$ differential operator has constant coefficients, approximation using convolution shows that (E'_p) is also true.

We claim that (E'_p) implies the solvability on $B_p(r)$ of (1.1) with any $\bar{\partial}$ -closed $f \in C_{0,1}^{p-1}(B_p(1))$ of finite $C^{p-1}(B_p(1))$ norm. Let u_n be a solution of $\bar{\partial}u_n = f|_{B_{p,n}(r)}$ guaranteed by (E'_p) . The functions $v_n = \varrho_n^* u_n$ on $B_p(r)$ satisfy, with a suitable constant N , that $|v_n(x)|, |(\bar{\partial}v_n)(x)\xi| \leq N$ for $x \in B_p(r)$, $\xi \in B_p(1)$.

It follows from the Cauchy-Pompeiu representation formula [H, Thm. 1.2.1] applied to 1-dimensional slices that $(v_n)_1^\infty$ is a locally equicontinuous family on $B_p(r)$. The Arzelà-Ascoli theorem gives a subsequence $v_{n'} \rightarrow v$ converging uniformly on compacts in $B_p(r)$. As v is continuous and $\bar{\partial}v = f$ holds restricted to $B_{p,n}(r)$ for every n in the distributional sense, it follows by approximation that $\bar{\partial}v = f$ holds in the distributional sense restricted to any finite dimensional slice of $B_p(r)$. The “elliptic regularity of the $\bar{\partial}$ operator” implies that v is a C^{p-1} solution of $\bar{\partial}v = f$ on $B_p(r)$. See [L2, Props. 2.3, 2.4].

Now, pull back the form g in Coeuré’s or Lempert’s example for ℓ_p by $x \mapsto \varepsilon x$ with an $\varepsilon > 0$ so small that the resulting form f has finite $C^{p-1}(B_{\ell_p}(1))$ norm. Then (1.1) is not solvable on any open subset of ℓ_p . This contradiction proves Proposition 1.2. $\square$

Proof of Theorem 1.1. — We shall use the method of “Condensation of Singularities.” As (E_p) is false for $p \geq 2$ integer, we have sequences $n(p) \geq 1$ of integers, $r_p \rightarrow +0$ of radii, $f_p \in C_{0,1}^\infty(B_{p,n(p)}(1))$ of $\bar{\partial}$ -closed forms with $\|f_p\|_{C^{p-1}(B_{p,n(p)}(1))} = 1$ such that if $\bar{\partial}u = f_p$ on $B_{p,n(p)}(r_p)$ then $\sup_{B_{p,n(p)}(r_p)} |u| \geq p^{p+1}$.

Let X be the Y -sum of $\ell_p(\mathbb{C}^{n(p)})$, $p = 2, 3, \dots$. Put

$$f = \sum_{p=2}^{\infty} p^{-p} \pi_p^* f_p.$$

One checks that f is in $C_{0,1}^\infty(B_X(1))$ and is $\bar{\partial}$ -closed.

We claim that $\bar{\partial}u = f$ cannot be solved on any open subset of $B_X(1)$.

Indeed, suppose for a contradiction that there are a ball $B_X(a, r)$ and a function u with $\bar{\partial}u = f$ on $B_X(a, r)$. Take r so small that

$$\sup_{B_X(a, r)} |u| = N < \infty.$$

This can be done as u is continuous at a (even C^∞). Choose $q \geq 2$ so large that $\|\pi_{q+1, \infty}(a)\| < \frac{1}{3}r$. Fix $p > q, N$ so large that $r_p < \frac{1}{3}r$. Let

$$v(z) = u(\pi_{2,q}(a) + I_p(z))$$

for $z \in B_{p,n(p)}(r_p)$. Then $\bar{\partial}v = p^{-p}f_p$ on $B_{p,n(p)}(r_p)$, so

$$N \geq \sup_{B_{p,n(p)}(r_p)} |v| \geq p^{-p}p^{p+1} = p > N.$$

This contradiction proves Theorem 1.1. $\square$

Further, we claim that

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(B_X(1)) = \infty.$$

We group the indices p into pairwise disjoint infinite sets P_n , $n \geq 1$. Then for the Y -sum $X^{(n)}$ of $\ell_p(\mathbb{C}^{n(p)})$, $p \in P_n$, we have inclusions $J_n: X^{(n)} \rightarrow X$ and projections $\varrho_n: X \rightarrow X^{(n)}$ both of operator norm 1. Let $g_n \in C_{0,1}^\infty(B_{X^{(n)}}(1))$ be a $\bar{\partial}$ -closed nowhere $\bar{\partial}$ -exact form whose existence is guaranteed by the proof of Theorem 1.1. Then $f_n = \varrho_n^* g_n$ are linearly independent in $H_{\bar{\partial}}^{0,1}(B_X(1))$. Indeed, suppose that $\lambda_1 f_1 + \cdots + \lambda_n f_n = \bar{\partial}u$, $\lambda_i \in \mathbb{C}$. Then by restricting to $X^{(i)}$ we see that $\lambda_i g_i$ is $\bar{\partial}$ -exact, hence $\lambda_i = 0$.

Should it turn out (as it is yet unknown) that on the unit ball B of ℓ_2 there are $\bar{\partial}$ -closed $(0,1)$ -forms of arbitrarily high finite smoothness that are nowhere $\bar{\partial}$ -exact, then the construction in Section 1 with $Y = \ell_2$ would yield a $\bar{\partial}$ -closed $f \in C_{0,1}^\infty(B)$ which is nowhere $\bar{\partial}$ -exact: a non-solvable $\bar{\partial}u = f$ in Hilbert space.

2. Approximation

We consider the following kind of approximation in a Banach space X .

$$(A) \quad \left\{ \begin{array}{l} \text{For any } 0 < r < R, \varepsilon > 0, \text{ and } f: B_X(R) \rightarrow \mathbb{C} \text{ holomorphic, there} \\ \text{exists an entire function } g: X \rightarrow \mathbb{C} \text{ with } |f - g| < \varepsilon \text{ on } B_X(r). \end{array} \right.$$

THEOREM 2.1. — *The statement (A) holds for the ℓ_1 -sum X of any sequence of finite dimensional Banach spaces $(X_n, \|\cdot\|_n)$.*

Lempert in [L4] proved (A) for $X = \ell_1$. When this manuscript was first written, Theorem 2.1 was the most general theorem proving (A). Later, however, (A) was proved in [L6] for any X with a countable unconditional basis, *i.e.*, for most classical Banach spaces. It is not clear whether all spaces X in Theorem 2.1 have a countable unconditional basis, or even a Schauder basis.

The proof of Theorem 2.1 is a modification and extension of Lempert's method in [L4]. Lempert's argument is based on the so-called monomial expansion of functions holomorphic on a ball $\|x\| < R \leq \infty$ of ℓ_1 (an analogue of the power series expansion on a finite dimensional space), and on the use of a dominating function $\Delta(q, z)$ defined and continuous on $\mathbb{C} \times B_{\ell_1}(1)$, whose role in the estimation of monomial series is similar to the role of the geometric series in estimating power series.

We replace the monomials by so-called multihomogeneous functions but use the same dominating function Δ of Lempert.

2.1. Multihomogeneous functions.

Let X be the ℓ_1 -sum of a sequence of finite dimensional Banach spaces $(X_n, \|\cdot\|_n)$. For $\lambda = (\lambda_n) \in \ell_\infty$ and $x \in X$ put $\lambda x = (\lambda_1 x_1, \lambda_2 x_2, \dots) \in X$. In the rest of this Section k denotes a multiindex. A multiindex $k = (k_n)$ for us is a sequence of integers $k_n \geq 0$ with $k_n = 0$ for n large enough. The support of k is the set $\text{supp } k = \{n; k_n \neq 0\}$. We define $\|k\| = \sum |k_n|$, and $\#k$ as the number of elements of the support of k . For a sequence of complex numbers $\lambda = (\lambda_n)$ we put $\lambda^k = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \in \mathbb{C}$, a finite product. For a multiindex k , a holomorphic function $\varphi: B_X(R) \rightarrow \mathbb{C}$ is called *k-homogeneous* if $\varphi(\lambda x) = \lambda^k \varphi(x)$ for all $x \in B_X(R)$, $\lambda = (\lambda_n) \in \ell_\infty$ with $|\lambda_n| = 1$.

A *k-homogeneous* function φ is a homogeneous polynomial of degree $\|k\|$ depending only on those finitely many variables $x_n \in X_n$ for which $n \in \text{supp } k$. In particular, φ extends automatically to an entire function on X , and $\varphi(\lambda x) = \lambda^k \varphi(x)$ holds for all $x \in X$ and $\lambda \in \ell_\infty$.

We define the norm $[\varphi]$ of a *k-homogeneous* function φ by

$$[\varphi] = \sup_{\|x\| \leq 1} |\varphi(x)|.$$

The set of all *k-homogeneous* functions φ for a fixed k is a finite dimensional Banach space with this norm.

PROPOSITION 2.2. — For φ *k-homogeneous* and $x \in X$,

$$|\varphi(x)| \leq [\varphi] |x|^k \|k\|^{\|k\|} k^{-k}.$$

Proof. — If $x_i = 0$ for some $i \in \text{supp } k$, then $\varphi(x) = 0$ as seen from the definition. So we may suppose that $\text{supp } k = \{1, 2, \dots, n\}$ and $x_i \neq 0$ for $1 \leq i \leq n$. Put

$$y = \left(\frac{k_1}{\|k\|} \frac{x_1}{\|x_1\|_1}, \dots, \frac{k_n}{\|k\|} \frac{x_n}{\|x_n\|_n}, 0, \dots \right) \in X.$$

Then $\|y\| = 1$, so $[\varphi] \geq |\varphi(y)| = k^k \|k\|^{-\|k\|} |x|^{-k} |\varphi(x)|$, as claimed. $\square$

2.2. The dominating function of Lempert.

This function is defined by the series

$$\Delta(q, z) = \sum_k \frac{\|k\|^{\|k\|}}{k^k} |q|^{\#k} |z^k|$$

for $(q, z) \in \mathbb{C} \times B_{\ell_1}(1)$. See [L2, Section 4].

THEOREM 2.3. — (a) *The series for Δ converges uniformly on compacts in $\mathbb{C} \times B_{\ell_1}(1)$.*

(b) *For each $0 < \theta < 1$ there is an $\varepsilon > 0$ such that Δ is bounded on $B_{\mathbb{C}}(\varepsilon) \times B_{\ell_1}(\theta)$.*

Proof. — See [L4, Thm. 2.1]. $\square$

We remark that the norm of a monomial z^k on ℓ_1 is $[z^k] = k^k \|k\|^{-\|k\|}$ as a simple calculation shows. So $\Delta(q, z)$ can be written as

$$\Delta(q, z) = \sum |z^k| [z^k]^{-1} |q|^{\#k},$$

where we add up normalized monomials with a weight counting the number of variables in the monomials.

2.3. Multihomogeneous expansions.

Let $T = (\mathbb{R}/\mathbb{Z})^\infty = \{t = (t_n); 0 \leq t_n < 1\}$ be the infinite dimensional torus, a compact topological group with the product topology and with Haar measure dt of total mass equal to 1.

For a holomorphic function $f: B_X(R) \rightarrow \mathbb{C}$ we define the multihomogeneous expansion of f by $f \sim \sum f_k$, with

$$f_k(x) = \int_{t \in T} f(e^{2\pi i t} x) e^{-2\pi i (k \cdot t)} dt,$$

where k is any multiindex, $e^{2\pi i t} x = (e^{2\pi i t_1} x_1, e^{2\pi i t_2} x_2, \dots)$ and $(k \cdot t) = \sum k_n t_n$, a finite sum. Then f_k is defined, holomorphic and k -homogeneous on $B_X(R)$. We call f_k the k -homogeneous component of the function f . Let

$$S = \{\sigma = (\sigma_n); 0 \leq \sigma_n \rightarrow 0\}, \quad S_1 = \{\sigma \in S; 0 \leq \sigma_n < 1\},$$

$$\sigma A = \{\sigma x; x \in A\}$$

for $A \subset X$, $\sigma \in S$ as in [L4, Section 2].

PROPOSITION 2.4. — (a) *If $f: B_X(R) \rightarrow \mathbb{C}$ is a holomorphic function, then we have the estimate*

$$M(\sigma) = \sup_k [f_k] \sigma^k R^{\|k\|} < \infty$$

for all $\sigma \in S_1$.

(b) *If f_k is k -homogeneous and $M(\sigma) < \infty$ for all $\sigma \in S_1$, then the series $g = \sum f_k$ converges uniformly on compact subsets of $B_X(R)$, g is holomorphic on $B_X(R)$, and the k -homogeneous component g_k of g is equal to f_k .*

Proof. — We use the following compactness criterion: A subset $K \subset X$ is compact if and only if K is closed, bounded, and the tail sums $R_n(x) = \sum_{\nu \geq n} \|x_\nu\|_\nu \rightarrow 0$ uniformly on K as $n \rightarrow \infty$.

We outline the proof. If K is compact, then $R_n \rightarrow 0$ uniformly on K by Dini's theorem on monotone convergence of continuous functions on a compact space to a continuous limit. In the other direction, fix $\varepsilon > 0$. We produce a finite covering of K by ε -balls. Fix n so large that $R_n < \frac{1}{2}\varepsilon$ on K , and project K onto the space of the first n coordinates, this is a bounded set in a finite dimensional space, so it has a finite covering by balls $B_X(x_i, \frac{1}{2}\varepsilon)$. Now, $B_X(x_i, \varepsilon)$ cover K .

This criterion implies, in particular, that any compact $K \subset B_X(1)$ is contained in $\sigma^2 B_X(1)$ for suitable $\sigma \in S_1$, and all the sets $\sigma B_X(1)$ for $\sigma \in S$ have compact closure. The utility of such a criterion was already observed by Ryan [R] in a similar context.

Proof of (a). — The set $\overline{\sigma B_X(R)}$ being compact,

$$\sup_{\|x\| < 1} |f(\sigma R x)| = M < \infty.$$

Thus, $\varphi(x) = f(\sigma R x)$ for $\|x\| < 1$ is bounded by M on $B_X(1)$. So is its k -homogeneous component $\varphi_k(x) = \sigma^k R^{\|k\|} f_k(x)$, hence $[f_k] \sigma^k R^{\|k\|} \leq M$, or $M(\sigma) \leq M < \infty$.

Proof of (b). — Without loss of generality we may suppose that the given compact is σL , where $L \subset B_X(r)$ is compact, $\sigma \in S_1$ and $r < R$. Then putting $x = \sigma y$ for $|y| < r$, $y \in L$, we have that

$$\begin{aligned} |f_k(x)| &\leq [f_k] \|k\|^{\|k\|} k^{-k} |x|^k \\ &= [f_k] \|k\|^{\|k\|} k^{-k} \sigma^k |y|^k \\ &= [f_k] \sigma^k R^{\|k\|} \cdot \|k\|^{\|k\|} k^{-k} |y/R|^k. \end{aligned}$$

Summing on k , we get

$$\sum |f_k(x)| \leq M(\sigma) \Delta(1, z) \leq M < \infty$$

where $z = |y/R|$ ranges in a compact subset of $B_{\ell_1}(1)$, and the series for Δ converges uniformly by Theorem 2.3 (a).

This concludes the proof of Proposition 2.4. $\square$

PROPOSITION 2.5. — *Let f_k be k -homogeneous. If for each multiindex k and for all $\sigma \in S$ (!) we have $\sup_k [f_k] \sigma^k R^{\|k\|} < \infty$, then $\sum f_k$ is an entire function on X .*

Proof. — If $M(\sigma) < \infty$ for all $\sigma \in S$, then $M(\lambda\sigma) < \infty$ for all $0 < \lambda < \infty$, $\sigma \in S_1$, which has the same effect as changing R to λR in Proposition 2.4 (b). Hence the multihomogeneous series converges on the whole of X . $\square$

We quote two lemmas from [L4].

PROPOSITION 2.6. — *If the numbers $0 \leq c_k < \infty$ are such that*

$$\sup_k c_k \sigma^k < \infty$$

for all $\sigma \in S_1$, then for any $Q \geq 1$ and $\sigma \in S_1$ the estimate $\sup_k c_k \sigma^k Q^{\#k} < \infty$ holds.

Proof. — See [L4, Prop. 4.2]. $\square$

PROPOSITION 2.7. — *Let $0 < \theta < 1$ and $\mathcal{K}$ a set of multiindices k . Then if $0 < c_k < \infty$, $k \in \mathcal{K}$, satisfy*

$$\inf_{k \in \mathcal{K}} c_k \theta^{\|k\|} > 0 \quad \text{and} \quad \sup_{k \in \mathcal{K}} c_k \sigma^k < \infty$$

for all $\sigma \in S_1$, then $\sup_{k \in \mathcal{K}} c_k \sigma^k < \infty$ for all $\sigma \in S$, too.

Proof. — See [L4, Prop. 4.3]. $\square$

Proof of Theorem 2.1. — Let us expand f in a multihomogeneous series $\sum f_k$. Fix a number $0 < \theta < 1$ with $r < \theta^2 R$. For any $\delta > 0$, $Q > 1$ (to be suitably chosen below) put

$$c_k = [f_k] R^{\|k\|}, \quad c'_k = c_k Q^{\#k},$$

$$\mathcal{K} = \{k; c'_k \theta^{\|k\|} \equiv [f_k](\theta R)^{\|k\|} Q^{\#k} \geq \delta\}, \quad g(x) = \sum_{k \in \mathcal{K}} f_k(x).$$

We claim that this g is an entire function on X .

Indeed, by Proposition 2.5 it is enough to show for all $\sigma \in S$ that

$$\sup_{k \in \mathcal{K}} [f_k] \sigma^k R^{\|k\|} \equiv \sup_{k \in \mathcal{K}} c_k \sigma^k < \infty.$$

As $\inf_{k \in \mathcal{K}} c'_k \theta^{\|k\|} \geq \delta > 0$, and for $\sigma \in S_1$ Proposition 2.4 (a) implies that

$$\sup_{k \in \mathcal{K}} [f_k] \sigma^k R^{\|k\|} \equiv \sup_{k \in \mathcal{K}} c_k \sigma^k < \infty,$$

so by Prop. 2.6, $\sup_{k \in \mathcal{K}} c'_k \sigma^k < \infty$ holds for all $\sigma \in S_1$. Now both conditions of Prop. 2.7 are verified, hence $\sup_{k \in \mathcal{K}} c_k \sigma^k \leq \sup_{k \in \mathcal{K}} c'_k \sigma^k < \infty$ for all $\sigma \in S$. Therefore, by Proposition 2.5, g is an entire function.

For $k \notin \mathcal{K}$ we have $[f_k](\theta R)^{\|k\|} Q^{\#k} \leq \delta$. We estimate $|f(x) - g(x)|$. For $\|x\| < r$ by Proposition 2.2, we have

$$\begin{aligned} |f(x) - g(x)| &\leq \sum_{k \notin \mathcal{K}} |f_k(x)| \leq \sum_{k \notin \mathcal{K}} [f_k] \frac{\|k\|^{\|k\|}}{k^k} |x|^k \\ &\leq \sum_{k \notin \mathcal{K}} \delta Q^{-\#k} (\theta R)^{-\|k\|} \frac{\|k\|^{\|k\|}}{k^k} |x|^k \\ &\leq \delta \sum_{k \notin \mathcal{K}} Q^{-\#k} \frac{\|k\|^{\|k\|}}{k^k} \left| \frac{\theta x}{r} \right|^k \leq \delta \sup_{\|w\| \leq \theta} \Delta(Q^{-1}, w) \end{aligned}$$

as $\theta R > r/\theta$ and $|w| = |\theta x/r| \leq \theta$. But the last expression can be made $< \varepsilon$ by choosing first Q large enough to make the sup finite by Theorem 2.3 (b), and then by choosing δ small enough.

Thus, the proof of the approximation Theorem 2.1 is concluded. $\square$

Let $Y = \ell_q$, $1 \leq q < \infty$, or $Y = c_0$. Let e_{pi}^n , $1 \leq i \leq n$, be the standard basis of $\ell_p(\mathbb{C}^n)$. Then the Y -sum X of any sequence $\ell_{p_k}(\mathbb{C}^{n_k})$ spaces, $k \geq 1$, has a countable unconditional basis: $e_{p_1 1}^{n_1}, e_{p_1 2}^{n_1}, \dots, e_{p_1 n_1}^{n_1}; e_{p_2 1}^{n_2}, \dots, e_{p_2 n_2}^{n_2}; \dots$. Now, the approximation theorem of Lempert [L6, Thm. 0.1], or in the case $Y = \ell_1$, Theorem 2.1 above, implies by the vanishing theorem [L3, Thm. 0.3] that the sheaf cohomology groups $H^q(\Omega, \mathcal{O}) = 0$, $q \geq 1$, on any pseudoconvex open set $\Omega \subset X$ for the sheaf $\mathcal{O}$ of germs of holomorphic functions on X . So for any Y , the space X of Theorem 1.1 has the property that $H_{\bar{\partial}}^{0,1}(\Omega) \neq 0$ (in fact, infinite dimensional) and $H^1(\Omega, \mathcal{O}) = 0$ for any bounded pseudoconvex open set $\Omega \subset X$: the Dolbeault isomorphism theorem does not generalize to arbitrary Banach spaces.

We remark that if the form f is real-analytic and Ω pseudoconvex, then by [L1, Prop. 3.2] the equation (1.1) has real-analytic local solutions; since $H^1(\Omega, \mathcal{O}) = 0$, we get a global real-analytic solution, too.

3. Almost complex manifolds

Theorem 1.1 verifies the hypothesis of Theorem 3.1 below in a case.

THEOREM 3.1. — *Let X be a Banach space and suppose that on $B = B_X(1)$ there exists a $\bar{\partial}$ -closed $f \in C_{0,1}^\infty(B)$ that is not $\bar{\partial}$ -exact on any open subset. Then on $M = B \times \mathbb{C}$ a C^∞ -smooth integrable almost complex structure M_f can be constructed in such a way that no open subset of M_f is biholomorphic to an open subset of a Banach space.*

As the referee has kindly pointed it out, the method of this section is analogous to one used earlier to construct nonrealizable CR hypersurfaces by Jacobowitz in [J].

We recall the definition of almost complex structure. An almost complex structure on a C^m -smooth manifold M is a splitting of the complexified tangent bundle $\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1}$ into the direct sum of two complex vector bundles of class C^{m-1} with $T^{0,1} = \overline{T^{1,0}}$, $m = 1, \dots, \infty, \omega$ and $m-1 = m$ for $m = \infty, \omega$. An almost complex structure is called formally integrable (or just integrable) if $m \geq 2$ and the Lie bracket of two $(1,0)$ vector fields of class C^1 is also a $(1,0)$ vector field; here $(1,0)$ can be changed to $(0,1)$.

The proof of Theorem 3.1 requires a few steps.

3.1. Construction of the almost complex structure on M .

The construction will be described in a setting more general than that of Theorem 3.1, namely, in the context of principal bundles.

Denote by $\zeta^{1,0}$, $\zeta^{0,1}$ the $(1,0)$ -part, $(0,1)$ -part of a complex tangent vector ζ to an almost complex manifold. Let B be a complex Banach manifold, G a finite dimensional complex Lie group with Lie algebra $\mathfrak{g} = T_e G$, $f \in C_{0,1}^\infty(B, \mathfrak{g}^{1,0})$ a $(0,1)$ -form with values in $\mathfrak{g}^{1,0}$, and $L_z: G \rightarrow G$ the left translation $L_z(s) = zs$, $z, s \in G$. Define the holomorphic Maurer-Cartan form $\mu \in C_{1,0}^\infty(G, \mathfrak{g}^{1,0})$ by

$$\mu(\nu) = (dL_z)^{-1} \nu^{1,0} = ((dL_z)^{-1} \nu)^{1,0}$$

for $\nu \in \mathbb{C} \otimes T_z G$. Recall the holomorphic Maurer-Cartan formula $d\mu + \frac{1}{2}[\mu, \mu] = 0$, which can be proved similarly to or deduced from the usual Maurer-Cartan formula. Define on any complex Banach manifold N the Lie bracket $[\varphi, \psi] \in C_{0,2}^\infty(N, \mathfrak{g}^{1,0})$ of forms $\varphi, \psi \in C_{0,1}^\infty(N, \mathfrak{g}^{1,0})$ by the usual formula

$$[\varphi, \psi](\zeta, \zeta') = [\varphi(\zeta), \psi(\zeta')] - [\varphi(\zeta'), \psi(\zeta)]$$

for $\zeta, \zeta' \in \mathbb{C} \otimes T_x N$, where the brackets on the right hand side are taken in the Lie algebra $\mathfrak{g}^{1,0}$. In particular, $[f, f](\zeta, \zeta') = 2[f(\zeta), f(\zeta')]$.

We define an almost complex structure M_f on $M = B \times G$ by putting $(\zeta, \nu) \in \mathbb{C} \otimes T_{(x,z)} M = \mathbb{C} \otimes T_x B \oplus \mathbb{C} \otimes T_z G$ in $T_{(x,z)}^{0,1} M$ if and only if

$$(3.1) \quad \zeta = \zeta^{0,1} \quad \text{and} \quad \mu(\nu) = f(\zeta).$$

In the setting of Theorem 3.1 we identify $G = \mathbb{C}$ and $\mathfrak{g}^{1,0} = \mathbb{C}$ via the correspondence $G \ni s \sim s \partial / (\partial z)|_{z=0} \in \mathfrak{g}^{1,0}$, where z is the usual coordinate on $\mathbb{C}$.

We verify below the following: Definition (3.1) gives an almost complex structure M_f on M and makes it into an almost complex principal G bundle; M is formally integrable if and only if $\bar{\partial}f + \frac{1}{2}[f, f] = 0$ holds; if M_f is locally biholomorphic to a Banach space, then $\bar{D}u = f$, where $u: B \rightarrow G$ is defined locally, and $\bar{D}$ is defined by

$$\bar{D}u(\zeta) = \mu(\mathrm{d}u(\zeta^{0,1}))$$

for $\zeta \in T_x B$. In the setting of Theorem 3.1 this $\bar{D}u$ reduces to the usual $\bar{\partial}u$.

3.2. Verification.

To verify that (3.1) defines an almost complex structure on M , we need to check conditions 1)–2).

1) If $V = (\zeta, \nu)$, $\bar{V} = (\bar{\zeta}, \bar{\nu})$ are in $T_{(x,z)}^{0,1}M$, then $V = 0$.

We have $0 = \zeta^{0,1} = \bar{\zeta}^{0,1} \equiv \bar{\zeta}^{1,0}$, or $\zeta = 0$. Similarly $\mu(\nu) = \mu(\bar{\nu}) = 0$ implies $\nu^{1,0} = \bar{\nu}^{1,0} = 0$, or $\nu = 0$.

2) Given $V = (\zeta, \nu)$, decompose it as $V = V_1 + V_2$ with $V_1, \bar{V}_2 \in T^{0,1}M$. One checks that

$$\begin{aligned} V_1 &= (\zeta^{0,1}, \mathrm{d}L_z f(\zeta) - \mathrm{d}L_z \overline{f(\bar{\zeta})} + \nu^{0,1}), \\ V_2 &= (\zeta^{1,0}, \mathrm{d}L_z \overline{f(\bar{\zeta})} - \mathrm{d}L_z f(\zeta) + \nu^{1,0}) \end{aligned}$$

is the unique way of decomposition.

3) Condition of formal integrability: If $V = (\zeta, \nu)$, $V' = (\zeta', \nu')$ are C^∞ sections of $T^{0,1}M$ over an open subset of M , then their Lie bracket $[V, V']$ is also a section of $T^{0,1}M$.

Denote by $\mathcal{L}_Z$ the Lie derivative along a complex vector field Z . We can write $[V, V']$ as

$$[V, V'] = (\zeta^*, \nu^*) = ([\zeta, \zeta'] + \mathcal{L}_\nu \zeta' - \mathcal{L}_{\nu'} \zeta, [\nu, \nu'] + \mathcal{L}_\zeta \nu' - \mathcal{L}_{\zeta'} \nu).$$

We work out below the condition of formal integrability for M_f in terms of f .

(a) The first component ζ^* is $(0, 1)$ because so are $[\zeta, \zeta']$, $\mathcal{L}_\nu \zeta'$, $\mathcal{L}_{\nu'} \zeta$ since B is a complex manifold.

(b) Taking the $\mathcal{L}_\zeta, \mathcal{L}_\nu$ Lie derivatives of the identity $\mu(\nu') - f(\zeta') = 0$ and reversing the roles of V, V' we find the equations

$$\begin{aligned} \mu(\mathcal{L}_\zeta \nu') - \mathcal{L}_\zeta(f(\zeta')) &= 0, & \mathcal{L}_\nu(\mu \nu') - f(\mathcal{L}_\nu \zeta') &= 0, \\ \mu(\mathcal{L}_{\zeta'} \nu) - \mathcal{L}_{\zeta'}(f(\zeta)) &= 0, & \mathcal{L}_{\nu'}(\mu \nu) - f(\mathcal{L}_{\nu'} \zeta) &= 0, \end{aligned}$$

whose alternating sum is

$$\begin{aligned} \{ \mu \nu^* - f(\zeta^*) \} + \{ \mathcal{L}_\nu(\mu \nu') - \mathcal{L}_{\nu'}(\mu \nu) - \mu([\nu, \nu']) \} \\ - \{ \mathcal{L}_\zeta(f(\zeta')) - \mathcal{L}_{\zeta'}(f(\zeta)) - f([\zeta, \zeta']) \} = 0. \end{aligned}$$

Hence, by Cartan's formula for exterior derivatives, the condition of formal integrability is that

$$(\mathrm{d}\mu)(\nu, \nu') - (\mathrm{d}f)(\zeta, \zeta') = 0.$$

Since $\mu(\nu) = f(\zeta)$, $\mu(\nu') = f(\zeta')$ we get by the holomorphic Maurer-Cartan formula that

$$-\frac{1}{2}[f, f](\zeta, \zeta') - (\mathrm{d}f)(\zeta, \zeta') = 0$$

for all $(0, 1)$ vector fields ζ, ζ' on B . Hence the almost complex manifold M_f is formally integrable if and only if

$$\bar{\partial}f + \frac{1}{2}[f, f] = 0,$$

which condition reduces to $\bar{\partial}f = 0$ when G is commutative as in Theorem 3.1.

3.3. Geometric properties of M .

To check that M is a principal G bundle we need to verify that $\pi: M = B \times G \rightarrow B$, $\pi(x, z) = x$ is holomorphic and that G has a simply transitive action on the fibers of M . Indeed, π is holomorphic as $\mathrm{d}\pi(\zeta, \nu) = \zeta$ takes $(0, 1)$ -vectors to $(0, 1)$ -vectors. The action of $w \in G$ on M is given by the left translation $\ell_w(x, z) = (x, wz)$ in the fiber direction. This is holomorphic because $(\mathrm{d}\ell_w)(\zeta, \nu) = (\zeta, \mathrm{d}L_w \nu)$ and $\mu(\mathrm{d}L_w \nu) = \mu(\nu)$.

In the setting of Theorem 3.1 a direct verification shows that $\Phi: M_f \rightarrow M_g$, $\Phi(x, z) = (x, z + u(x))$ is a bundle biholomorphism, where $g = f + \bar{\partial}u$ and $u \in C^\infty(B)$ is any function. Hence the bundle biholomorphism type of M_f depends only on the Dolbeault cohomology class of f .

We return now to the general setting.

PROPOSITION 3.2. — *If $p_0 = (x_0, z_0) \in M_f$ has a neighborhood that is C^m -biholomorphic, $m = 1, 2, \dots, \infty$, to an open set in a Banach space, then there are a neighborhood $U_0 \subset B$ of x_0 and $u \in C^m(U_0, G)$ such that $\bar{\partial}u = f$ on U_0 .*

Proof. — The Banach space $T^{0,1} = T_{p_0}^{0,1}M$ has a natural splitting as a direct sum $T^{0,1} = V^{0,1} \oplus H^{0,1}$ of vertical and horizontal closed subspaces

$$\begin{aligned} V^{0,1} &= \{ (0, \nu) \in T^{0,1}; \nu \in T_{z_0}^{0,1}G \}, \\ H^{0,1} &= \{ (\zeta, \nu) \in T^{0,1}; \nu \in T_{z_0}^{1,0}G \}. \end{aligned}$$

Suppose now that $\Phi:U \rightarrow V$ is biholomorphism of a neighborhood U of p_0 in M_f onto a neighborhood V of 0 in a Banach space W . Then the splitting $T^{0,1} = V^{0,1} \oplus H^{0,1}$ induces via $(d\Phi)(p_0)$ a splitting

$$T_0^{0,1}W \equiv W = V_W^{0,1} \oplus H_W^{0,1}.$$

Since $N = \Phi^{-1}(V \cap H_W^{0,1})$ is an almost complex C^m -submanifold of M passing through p_0 transversely to $V^{0,1}$, hence to $\{x_0\} \times G$, N is the image near p_0 of a holomorphic section $s:U_0 \rightarrow G$ on a neighborhood U_0 of x_0 . Then writing $s(x) = (x, u(x))$ and applying (3.1) we obtain that $\mu(du(\zeta)) = f(\zeta)$ for all $\zeta \in T_x^{0,1}U_0$, but this is the same as saying $\bar{D}u = f$ on U_0 ; thus concluding the proof of Proposition 3.2 and hence that of Theorem 3.1. $\square$

We have seen that the Newlander-Nirenberg theorem does not generalize to arbitrary integrable almost complex Banach manifolds. It is unknown if it generalizes to Hilbert manifolds.

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