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THE 3D NAVIER-STOKES EQUATIONS SEEN AS A PERTURBATION OF THE 2D NAVIER-STOKES EQUATIONS

PAR DRAGOŞ IFTIMIE (*)

ABSTRACT. — We consider the periodic 3D Navier-Stokes equations and we take the initial data of the form $u_0 = v_0 + w_0$, where v_0 does not depend on the third variable. We prove that, in order to obtain global existence and uniqueness, it suffices to assume that $\|w_0\|_X \exp(\|v_0\|^2_{L^2(\mathbb{T}^2)}/C\nu^2) \leq C\nu$, where X is a space with a regularity H^δ in the first two directions and $H^{\frac{1}{2}-\delta}$ in the third direction or, if $\delta = 0$, a space which is L^2 in the first two directions and $B^{\frac{1}{2}}_{2,1}$ in the third direction. We also consider the same equations on the torus with the thickness in the third direction equal to ε and we study the dependence on ε of the constant C above. We show that if v_0 is the projection of the initial data on the space of functions independent of the third variable, then the constant C can be chosen independent of ε.

RÉSUMÉ. — LES ÉQUATIONS DE NAVIER-STOKES 3D VUES COMME UNE PERTURBATION DES ÉQUATIONS DE NAVIER-STOKES 2D. — On considère les équations de Navier-Stokes périodiques 3D et on prend la donnée initiale de la forme $u_0 = v_0 + w_0$, où v_0 ne dépend pas de la troisième variable. On démontre que, afin d'obtenir l'existence et l'unicité globale, il suffit de supposer que $\|w_0\|_X \exp(\|v_0\|^2_{L^2(\mathbb{T}^2)}/C\nu^2) \leq C\nu$, où X est un espace avec une régularité H^δ dans les deux premières directions et $H^{\frac{1}{2}-\delta}$ dans la troisième direction ou, si $\delta = 0$, un espace qui est L^2 dans les deux premières directions et $B^{\frac{1}{2}}_{2,1}$ dans la troisième direction. On considère aussi le même système sur le tore avec une épaisseur ε dans la troisième direction et on étudie la dépendance de ε de la constante C ci-dessus. On trouve que, si v_0 est la projection de la donnée initiale sur l'espace des fonctions indépendantes de la troisième variable, alors la constante C peut être choisie indépendante de ε.

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Introduction

The periodic 3D Navier-Stokes equations are the following:

$$\text{(N-S)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u(t,\cdot) = 0 \quad \text{for all } t \geq 0, \\ u_{|t=0} = u_0. \end{cases}$$

Here, $u(t,x)$ is a periodic time-dependent 3-dimensional vector-field. For the sake of simplicity, we assume that the force is vanishing. This is not a serious restriction, it is clear that the difficulty in solving these equations comes from the non linear term. Similar results may be proved in the same way with a force square-integrable in time with values in the right space. The choice of periodic boundary conditions comes from the need to use the Fourier transform; for this reason our methods do not trivially extend to other classical boundary conditions.

It is well-known that in 2D, there exists a global unique solution for square-integrable initial velocity. In larger dimensions, unless some symmetry is assumed, global existence and uniqueness of solutions is known to hold only for small and more regular initial velocities. The goal of this paper is to prove global existence and uniqueness results by considering the 3D Navier-Stokes system as a perturbation of the 2D system. To do that, we write the initial data as the sum of a 2-dimensional initial part and a remainder. The main theorem says that, in order to obtain global existence, it suffices to assume the remainder small, and small compared to the 2-dimensional part.

Some stability results are already proved by G. Ponce, R. Racke, T.C. Sideris and E.S. Titi in [9] but the norm of the remainder is not estimated and the 2-dimensional part of the initial data is assumed to be in $H^1 \cap L^1$ and not in L^2, the optimal assumption. This loss of regularity appears when they take the product of a 2-dimensional function with a 3-dimensional function. This difficulty is overwhelmed here by introducing anisotropic spaces, where the variables are "separated". The loss of regularity is then optimal. Another advantage of these spaces is that they are larger than the usual Sobolev spaces, hence we obtain in the same time more general theorems.

It is natural to ask if the 3D Navier-Stokes equations on thin domains are close to the 2D Navier-Stokes equations from the point of view of global existence and uniqueness of solutions. A second aim of this work is to do the asymptotic study of the Navier-Stokes equations on $\mathbb{T}_\varepsilon = [0, 2\pi a] \times [0, 2\pi b] \times [0, 2\pi\varepsilon]$ when $\varepsilon \to 0$, as was first considered by G. Raugel and G.R. Sell [11], [10] and, afterwards, by J.D. Avrin [1], R. Temam and M. Ziane [12], [13] and I. Moise, R. Temam and M. Ziane [8]. By asymptotic study, we mean proving global existence and uniqueness of solutions for initial data in optimal sets, whose diameters should go to infinity when the slenderness of the domain goes to 0. To do that, it is natural to work in spaces where the third variable is distinguished. It appears that the anisotropic spaces are again well adapted to this study.

In an earlier paper [7], we proved global existence and uniqueness of solutions for (N-S) in $\mathbb{R}^3$ with small initial data in

$$\mathcal{H}^{\delta_1,\delta_2,\delta_3}, \quad \delta_1+\delta_2+\delta_3=\frac{1}{2}, \quad -\frac{1}{2}<\delta_i<\frac{1}{2},$$

a space which is H^{δ_i} in the i-th direction. Here we apply in the periodic case the work we have done there. The precise result is that there exists a positive constant C, independent of ν, such that if $0<\delta<1$ and the initial data is v_0+w_0 with v_0 independent of the third variable, then, in order to obtain global existence and uniqueness of solutions, it suffices to assume that

$$\|w_0\|_X \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \leq C\nu, \tag{0.1}$$

where X is a space which is H^δ in the first two variables and $H^{\frac{1}{2}-\delta}$ in the third variable, or, if $\delta=0$, a space which is L^2 in the first two variables and $B^{\frac{1}{2}}_{2,1}$ in the third variable, where $B^s_{p,q}$ is the usual Besov space given by

$$B^s_{p,q}=\big\{u\in\mathcal{S}' \text{ such that } \big\|\, 2^{is}\|\Delta_i u\|_{L^p}\big\|_{\ell^q}<\infty\big\},$$

where $\Delta_i u$ is defined in (1.1). We shall also prove local existence and uniqueness of solutions for arbitrary initial data in the spaces above.

In the third paragraph we work in $\mathbb{T}_\varepsilon$ and we study the dependence on ε of the constant of inequality (0.1). We shall prove that if v_0 is the projection of the initial data on the space of functions independent of x_3 and $0<\delta\leq\frac{1}{2}$, then the constant C can be chosen independent of ε. This will imply that global existence and uniqueness is achieved as long as

$$\|w_0\|_{H^{\frac{1}{2}}(\mathbb{T}_\varepsilon)} \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \leq C\nu. \tag{0.2}$$

The inequality above can be read in various ways. For instance, it is implied by

$$\|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \leq C\nu\varepsilon^{-\frac{1}{2}},$$

or, for all $\alpha\geq 0$, by

$$\|v_0\|_{L^2(\mathbb{T}^2)} \leq C\nu\big(1+\sqrt{-\alpha\log\varepsilon}\,\big) \quad\text{and}\quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)}\leq C\nu\varepsilon^{-\frac{1}{2}+\alpha}.$$

Finally, if one needs to have a larger v_0, one can take v_0 arbitrarily large, the price to pay is that w_0 has to be assumed exponentially small with respect to that v_0.

Let us compare this theorem with the previous results.

The precise results of G. Raugel and G.R. Sell [11], [10] are rather complicated so we give only an approximation: they consider various boundary conditions and obtain global existence and uniqueness of solutions as long as

$$\|v_0\|_{H^1(\mathbb{T}^2)} \leq C\varepsilon^{-5/24} \quad \text{and} \quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \leq C\varepsilon^{-5/48}$$

or

$$\|v_0\|_{H^1(\mathbb{T}^2)} \leq C\varepsilon^{-17/32}, \ \|v_0^3\|_{L^2(\mathbb{T}^2)} \leq C\varepsilon^{\frac{1}{2}} \ \text{and} \ \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \leq C\varepsilon^{-1/8},$$

where v_0^3 is the third component of v_0.

In the paper of J.D. Avrin [1] it is shown that $\|u_0\|_{H^1} \leq C\lambda_1^{-1/4}$ suffices in the case of homogeneous Dirichlet boundary conditions; we denoted by λ_1 the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions. In the case of a thin domain, the equivalent of Avrin's result would be:

$$\|u_0\|_{H^1} \leq C\varepsilon^{-\frac{1}{2}}.$$

Let us note that in the case of homogeneous Dirichlet boundary conditions the 2-dimensional part can not be defined, so one of the major difficulties of the problem, mixture of 2D functions with 3D functions, does not appear.

I. Moise, R. Temam and M. Ziane [8] prove that it is sufficient to assume that

$$\|v_0\|_{H^1(\mathbb{T}^2)} \leq C\varepsilon^{-\frac{1}{3}+\delta} \quad \text{and} \quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \leq C\varepsilon^{-\frac{1}{6}+\delta},$$

where δ is a positive number.

Finally we mention that spherical domains are considered by R. Temam and M. Ziane [13].

1. Notations and preliminary results

Many of the notations and the results from [7] remain valid here with minor modifications; for those results, we shall only sketch the proofs. The main differences are that we use the Littlewood-Paley theory in two variables instead of three and we have to adjust to the periodic case the definition of the Δ_q operators. We work in

$$\mathbb{T}^3 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$$

and we denote by $(x_1, x_2, x_3) = (x', x_3)$ the variable in $\mathbb{T}^3$. All the functions are assumed to have vanishing integral on $\mathbb{T}^3$. Let

$$L^{p,q} = \big\{u \text{ such that } \|u\|_{L^{p,q}} \stackrel{\text{def}}{=} \big\|\|u(x)\|_{L^q_{x_3}}\big\|_{L^p_{x'}} < \infty\big\},$$

and $\ell^{p,q}$ be the similar space for sequences. Obviously, when $p = q$, the spaces $\ell^{p,p}$ and $L^{p,p}$ are nothing else but the usual ℓ^p and L^p spaces. The order of integrations is important, as shown by the following remark (see [7]):

REMARK 1.1. — Let (X_1,μ_1), (X_2,μ_2) be two measure spaces, $1 \le p \le q$ and $f: X\times Y \to \mathbb{R}$. Then

$$\big\| \|f(\cdot\,,x_2)\|_{L^p(X_1,\mu_1)}\big\|_{L^q(X_2,\mu_2)} \le \big\| \|f(x_1,\cdot\,)\|_{L^q(X_2,\mu_2)}\big\|_{L^p(X_1,\mu_1)}.$$

The Hölder and Young inequalities for the $L^{p,q}$ spaces take the form:

$$\|fg\|_{L^{p,q}} \le \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},$$

where $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$, $\dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_2}$,

$$\|f * g\|_{L^{a,b}} \le \|f\|_{L^{a_1,b_1}}\|g\|_{L^{a_2,b_2}},$$

where $1+\dfrac{1}{a} = \dfrac{1}{a_1} + \dfrac{1}{a_2}$, $1+\dfrac{1}{b} = \dfrac{1}{b_1} + \dfrac{1}{b_2}\cdot$

We denote by $h*$ the operator of convolution with h.

If u is periodic, then it has a Fourier series

$$u(x) = \sum_{n\in\mathbb{Z}^3} u_n \exp(in\cdot x), \quad u_n \in \mathbb{C}.$$

For $q \ge 0$ and $q' \ge 0$, we define

$$(1.1)\qquad \begin{cases} S'_q u = \displaystyle\sum_{n\in\mathbb{Z}^3} u_n \exp(in\cdot x)\chi\Big(\frac{|n'|}{2^q}\Big),\\ S''_q u = \displaystyle\sum_{n\in\mathbb{Z}^3} u_n \exp(in\cdot x)\chi\Big(\frac{|n_3|}{2^q}\Big),\\ \Delta'_q = S'_q - S'_{q-1} = \displaystyle\sum_{n\in\mathbb{Z}^3} u_n \exp(in\cdot x)\varphi\Big(\frac{|n'|}{2^q}\Big) \quad \forall q\ge 1,\\ \Delta'_0 = S'_0 = \displaystyle\sum_{n_3\in\mathbb{Z}} u_{(0,0,n_3)} \exp(in_3x_3),\\ \Delta''_q = S''_q - S''_{q-1} = \displaystyle\sum_{n\in\mathbb{Z}^3} u_n \exp(in\cdot x)\varphi\Big(\frac{|n_3|}{2^q}\Big) \quad \forall q\ge 1,\\ \Delta''_0 = S''_0 = \displaystyle\sum_{n'\in\mathbb{Z}^2} u_{(n',0)} \exp(in'x'),\\ S_{q,q'} = S'_q S''_{q'}, \quad \Delta_{q,q'} = \Delta'_q\Delta''_{q'},\\ S_q = S_{q,q}, \quad \Delta_q = S_q - S_{q-1}, \quad \Delta_0 = S_0, \end{cases}$$

where $\chi:\mathbb{R}\to[0,1]$ is a smooth function such that $\operatorname{supp}\chi \subset\,]-1,1[$, $\chi\equiv 1$ on $[0,\frac12]$, χ is decreasing on $[0,\infty[$, $\chi(\frac34) = \frac12$ and $\varphi(x) = \chi(x) - \chi(2x)$.

Note that $\operatorname{supp}\varphi \subset \left]\frac{1}{4}, 1\right[$ and $\varphi(x) \geq \frac{1}{2}$ for all $x \in [\frac{3}{8}, \frac{3}{4}]$. With these notations, the next inequality stems from Lemma 1.1 below:

$$\|\varphi_{q,q'}\|_{L^{p_1,p_2}} \leq C\, 2^{2q(1-1/p_1)+q'(1-1/p_2)}, \tag{1.2}$$

where $\varphi_{q,q'}$ is given by $\Delta_{q,q'} = \varphi_{q,q'}*$. The same holds for $S_{q,q'}$. Note that this inequality is an extension of the classical equality

$$\|\varphi_q\|_{L^p(\mathbb{R}^d)} = C\, 2^{dq(1-1/p)},$$

where φ_q is given by $\Delta_q = \varphi_q*$, Δ_q being the usual localization operator in $\mathbb{R}^d$ (see [2], [4]). It is important to use smooth cut-off functions; if we would use characteristic functions of dyadic intervals, then inequality (1.2) would not hold in the L^1 case. For further details on the subject we refer to [6, Chap. 7].

LEMMA 1.1. — *Let ϕ be a compactly supported smooth function, $\lambda > 1/(2\pi)$ and*

$$f(x) = \sum_{n\in\mathbb{Z}} \phi\Big(\frac{n}{\lambda}\Big) \exp(inx).$$

Then, for all $1 \leq p \leq \infty$ and $k \in \mathbb{N}$ there exist a constant $C = C(\phi,k)$ such that

$$\|f^{(k)}\|_{L^p} \leq C\lambda^{k+1-1/p},$$

where $f^{(k)}$ is the k-th derivative of f.

Proof. — First we remark that we can restrict ourselves to the case $k = 0$. Indeed, we have $f^{(k)} = \lambda^k g_k$, where

$$g_k(x) = \sum_n \psi_k\Big(\frac{n}{\lambda}\Big) \exp(inx) \quad \text{and} \quad \psi_k(x) = (ix)^k\phi(x).$$

Interpolating L^p between L^1 and L^∞ shows that it suffices to consider the cases $p = 1$ and $p = \infty$. We have

$$|f(x)| \leq \sum_{n\in\lambda\,\operatorname{supp}\phi} \Big|\phi\Big(\frac{n}{\lambda}\Big)\Big| \leq C\|\phi\|_{L^\infty}\lambda,$$

thus the case $p = \infty$ is proven.

Before going any further let us note that if $\lambda \leq 1/(2\pi)$ then $\|f\|_{L^\infty}$ is bounded independently of λ, hence so is $\|f\|_{L^1}$. To estimate $\|f\|_{L^1}$ for $\lambda > 1/(2\pi)$ we write

$$\|f\|_{L^1} = \int_0^{2\pi} |f(x)|\,\mathrm{d}x = \int_0^{1/\lambda} |f(x)|\,\mathrm{d}x + \int_{1/\lambda}^{2\pi} |f(x)|\,\mathrm{d}x.$$

To estimate the first integral we use the bound on the sup norm of f:

$$\int_0^{1/\lambda} \big|f(x)\big|\,\mathrm{d}x \leq \frac{1}{\lambda}\|f\|_{L^\infty} \leq C\|\phi\|_{L^\infty}.$$

In order to bound the second integral we use Abel's summation formula to deduce that

$$\begin{aligned} f(x) &= \sum_n \frac{\exp(i(n+1)x) - \exp(inx)}{\exp(ix)-1}\phi\Big(\frac{n}{\lambda}\Big) \\ &= \sum_n \frac{\exp(inx)}{\exp(ix)-1}\Big\{\phi\Big(\frac{n-1}{\lambda}\Big) - \phi\Big(\frac{n}{\lambda}\Big)\Big\} \\ &= \sum_n \frac{\exp(inx)}{(\exp(ix)-1)^2}\Big\{\phi\Big(\frac{n-2}{\lambda}\Big) - 2\phi\Big(\frac{n-1}{\lambda}\Big) + \phi\Big(\frac{n}{\lambda}\Big)\Big\}. \end{aligned}$$

Taylor's formula gives

$$\Big|\phi\Big(\frac{n-2}{\lambda}\Big) - 2\phi\Big(\frac{n-1}{\lambda}\Big) + \phi\Big(\frac{n}{\lambda}\Big)\Big| \leq \frac{C}{\lambda^2},$$

for some constant $C = C(\phi)$. Thus

$$\int_{1/\lambda}^{2\pi} \big|f(x)\big|\,\mathrm{d}x \leq \int_{1/\lambda}^{2\pi} \sum_{|n|<C\lambda} \frac{C\,\mathrm{d}x}{\lambda^2 x^2} \leq \frac{C}{\lambda}\int_{1/\lambda}^{2\pi}\frac{\mathrm{d}x}{x^2} \leq C.$$

This completes the proof. □

As a corollary we find a Littlewood-Paley lemma in two variables:

LEMMA 1.2. — *If u is a periodic function on $\mathbb{T}^3$ such that*

$$\operatorname{supp}\widehat{u} \subset B(0,\lambda_1,\lambda_2) \overset{\text{def}}{=\!=} \Big\{\xi \in \mathbb{R}^3 \text{ such that } |\xi'| < \lambda_1,\ |\xi_3| < \lambda_2\Big\},$$

$1 \leq a_1 \leq b_1 \leq \infty$, $1 \leq a_2 \leq b_2 \leq \infty$ and $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$ is a multi-index, then

$$\|\partial^\alpha u\|_{L^{b_1,b_2}} \leq C\lambda_1^{\alpha_1+\alpha_2+2(1/a_1-1/b_1)}\lambda_2^{\alpha_3+(1/a_2-1/b_2)}\|u\|_{L^{a_1,a_2}}.$$

Proof. — Recall that

$$\widehat{u} = (2\pi)^3\sum_{n\in\mathbb{Z}^3} u_{-n}\delta_n.$$

Let

$$\phi_{\lambda_1,\lambda_2}(x) = \frac{1}{(2\pi)^3} \sum_{n\in\mathbb{Z}^3} \exp(in\cdot x)\chi\Big(\frac{|n'|}{2\lambda_1}\Big)\chi\Big(\frac{|n_3|}{2\lambda_2}\Big),$$

where χ is defined immediately after relation (1.1). The localization of $\widehat{\phi}_{\lambda_1,\lambda_2}$ and $\widehat{u}$ implies that $\widehat{\phi}_{\lambda_1,\lambda_2}\widehat{u} = \widehat{u}$, so

$$u = \phi_{\lambda_1,\lambda_2} * u.$$

Since

$$\phi_{\lambda_1,\lambda_2}(x) = \phi_{\lambda_1}(x')\phi_{\lambda_2}(x_3)$$

with

$$\phi_{\lambda_1}(x') = \frac{1}{(2\pi)^2} \sum_{n'\in\mathbb{Z}^2} \exp(in'\cdot x')\chi\Big(\frac{|n'|}{2\lambda_1}\Big)$$

and

$$\phi_{\lambda_2}(x_3) = \frac{1}{2\pi} \sum_{n_3\in\mathbb{Z}} \exp(in_3\cdot x_3)\chi\Big(\frac{|n_3|}{2\lambda_2}\Big),$$

applying Young's inequality and Lemma 1.1 yields

$$\begin{aligned}\|\partial^\alpha u\|_{L^{b_1,b_2}} &\le \|\partial^\alpha \phi_{\lambda_1,\lambda_2}\|_{L^{\frac{a_1b_1}{a_1b_1+a_1-b_1},\frac{a_2b_2}{a_2b_2+a_2-b_2}}} \|u\|_{L^{a_1,a_2}}\\ &\le C\lambda_1^{\alpha_1+\alpha_2+2(1/a_1-1/b_1)}\lambda_2^{\alpha_3+(1/a_2-1/b_2)}\|u\|_{L^{a_1,a_2}}.\end{aligned}$$

The proof is completed. ∎

DEFINITION 1.1. — *We denote by M the operator given by*

$$Mu(x_1,x_2) = \frac{1}{2\pi}\int_0^{2\pi} u(x)\,\mathrm{d}x_3 = \sum_{n'\in\mathbb{Z}^2} u_{(n',0)}\exp(in'\cdot x').$$

It is easy to check that M, defined as a Fourier series, is the orthogonal projection on the space of functions not depending on the third variable in every Sobolev space H^s.

When we will say that a possibly non-integrable function u *has vanishing* mean we understand that $u_{(0,0,0)} = 0$. Similarly, *vanishing mean in the third direction* refers to $u_{(n',0)} = 0$ for all $n' \in \mathbb{Z}^2$. Let us now introduce the first class of spaces we shall use:

DEFINITION 1.2. — *We denote by* $H^{s,s'}$ *the space*

$$H^{s,s'} = \{u \in \mathcal{D}'(\mathbb{T}^3) \text{ such that } |u|_{s,s'} < \infty\},$$

where

$$|u|_{s,s'} = \left\|(1+|n'|^2)^{s/2}(1+n_3^2)^{s'/2}u_n\right\|_{\ell^2},$$

in which u_n *are the Fourier coefficients of the function* u*. The homogeneous variant of this space is*

$$\dot{H}^{s,s'} = \{u \in H^{s,s'} \text{ and } Mu = 0\}.$$

The following two lemmas are similar to Lemmas 1.2 and 1.3 from [7] and give a characterization of $H^{s,s'}$ in terms of dyadic decomposition.

LEMMA 1.3. — *If* $u \in H^{s,s'}$ *then*

$$|u|_{s,s'} \simeq \left\|2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2}\right\|_{\ell^2}.$$

Proof. — Definition 1.1 implies that for all $q, q' \geq 1$

$$\|\Delta_{q,q'}u\|_{L^2}^2 = (2\pi)^3 \sum_n |u_n|^2 \varphi^2\Big(\frac{|n'|}{2^q}\Big)\varphi^2\Big(\frac{|n_3|}{2^{q'}}\Big).$$

Using the localization of φ we obtain

$$\begin{aligned}(1.3)\qquad & C_1 \sum_{\substack{3\cdot 2^{q-3}\leq|n'|\leq 3\cdot 2^{q-2}\\ 3\cdot 2^{q'-3}\leq|n_3|\leq 3\cdot 2^{q'-2}}} |u_n|^2\big(1+|n'|^2\big)^s\big(1+|n_3|^2\big)^{s'}\\ &\leq 2^{2qs+2q's'}\|\Delta_{q,q'}u\|_{L^2}^2\\ &\leq C_2 \sum_{\substack{2^{q-2}\leq|n'|\leq 2^q\\ 2^{q'-2}\leq|n_3|\leq 2^{q'}}} |u_n|^2\big(1+|n'|^2\big)^s\big(1+|n_3|^2\big)^{s'},\end{aligned}$$

for some constants C_1 and C_2. Similarly,

$$\begin{aligned}(1.4)\qquad & C_1 \sum_{3\,2^{q-3}\leq|n'|\leq 3\cdot 2^{q-2}} |u_{(n',0)}|^2\big(1+|n'|^2\big)^s \leq 2^{2qs}\|\Delta_{q,0}u\|_{L^2}^2\\ &\leq C_2 \sum_{2^{q-2}\leq|n'|\leq 2^q} |u_{(n',0)}|^2\big(1+|n'|^2\big)^s \quad \forall q \geq 1,\end{aligned}$$

and

$$\begin{aligned}(1.5)\qquad & C_1 \sum_{3\,2^{q'-3}\leq|n_3|\leq 3\,2^{q'-2}} |u_{(0,n_3)}|^2\big(1+|n_3|^2\big)^{s'} \leq 2^{2q's'}\|\Delta_{0,q'}u\|_{L^2}^2\\ &\leq C_2 \sum_{2^{q'-2}\leq|n_3|\leq 2^{q'}} |u_{(0,n_3)}|^2\big(1+|n_3|^2\big)^{s'} \quad \forall q' \geq 1.\end{aligned}$$

Using that $\Delta_{0,0}u = u_{0,0}$ and summing relations (1.3), (1.4) and (1.5) gives the desired conclusion. □

LEMMA 1.4. — *If $u_{p,p'}$ is a sequence of square integrable functions such that*

$$\begin{aligned}
&\operatorname{supp}\widehat{u}_{p,p'} \subset \{1/\gamma 2^p \le |\xi'| \le \gamma 2^p,\ 1/\gamma 2^{p'} \le |\xi_3| \le \gamma 2^{p'}\} && \textit{for } p,p' \ge 1,\\
&\operatorname{supp}\widehat{u}_{p,0} \subset \{1/\gamma 2^p \le |\xi'| \le \gamma 2^p,\ |\xi_3| \le \gamma\} && \textit{for } p \ge 1,\\
&\operatorname{supp}\widehat{u}_{0,p'} \subset \{|\xi'| \le \gamma,\ 1/\gamma 2^{p'} \le |\xi_3| \le \gamma 2^{p'}\} && \textit{for } p' \ge 1,\\
&\operatorname{supp}\widehat{u}_{0,0} \subset \{|\xi'| \le \gamma, |\xi_3| \le \gamma\},
\end{aligned}$$

for some constant $\gamma > 1$ and

$$\big\|2^{ps+p's'}\|u_{p,p'}\|_{L^2}\big\|_{\ell^2} < \infty,$$

then

$$u = \sum_{p,p'} u_{p,p'} \in H^{s,s'} \quad \text{and} \quad |u|_{s,s'} \le C\|2^{ps+p's'}\|u_{p,p'}\|_{L^2}\|_{\ell^2}.$$

- *If $s > 0$ it suffices to assume that*
$$\operatorname{supp}\widehat{u}_{p,p'} \subset \{|\xi'| \le \gamma 2^p,\ 1/\gamma 2^{p'} \le |\xi_3| \le \gamma 2^{p'}\}.$$
- *If $s' > 0$ it suffices to assume that*
$$\operatorname{supp}\widehat{u}_{p,p'} \subset \{1/\gamma 2^p \le |\xi'| \le \gamma 2^p,\ |\xi_3| \le \gamma 2^{p'}\}.$$
- *If $s > 0$ and $s' > 0$ it suffices to assume that*
$$\operatorname{supp}\widehat{u}_{p,p'} \subset \{|\xi'| \le \gamma 2^p,\ |\xi_3| \le \gamma 2^{p'}\}.$$

Proof. — We prove the relevant case $s > 0$. Similar proofs work for the other situations. We use that the operators $\Delta_{q,q'}$ are bounded in L^2 independently of q and q', and the localization of $\Delta_{q,q'}$ and $u_{p,p'}$ to deduce the existence of an integer N such that

$$\begin{aligned}
2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2} &\le 2^{qs+q's'}\sum_{p,p'}\|\Delta_{q,q'}u_{p,p'}\|_{L^2}\\
&\le \sum_{\substack{p \ge q-N\\ |p'-q'| \le N}} 2^{(q-p)s+(q'-p')s'}2^{ps+p's'}\|u_{p,p'}\|_{L^2}\\
&= a_{q,q'} * b_{q,q'},
\end{aligned}$$

where

$$a_{q,q'} = \begin{cases} 2^{qs+q's'} & \text{if } q \le N,\ |q'| \le N,\\ 0 & \text{otherwise,}\end{cases}$$

$$b_{q,q'} = 2^{qs+q's'}\|u_{q,q'}\|_{L^2}.$$

Young's inequality yields

$$\big\|2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2}\big\|_{\ell^2} \le \|a_{q,q'}\|_{\ell^1}\cdot\|b_{q,q'}\|_{\ell^2}.$$

Since $s > 0$ one has $\|a_{q,q'}\|_{\ell^1} < \infty$. Applying Lemma 1.3 completes the proof. $\square$

The next theorem as well as its proof is a variant of the product Theorem 1.1 from [7] which states that the product of a function from $\mathcal{H}^{s_1,s_2,s_3}$ with a function from $\mathcal{H}^{t_1,t_2,t_3}$ lies in $\mathcal{H}^{s_1+t_1-\frac{1}{2},s_2+t_2-\frac{1}{2},s_3+t_3-\frac{1}{2}}$ provided that $s_i < \frac{1}{2}$, $t_i < \frac{1}{2}$, $s_i + t_i > 0$, $i \in \{1,2,3\}$.

THEOREM 1.1. — *Let $u \in H^{s,s'}$, $v \in H^{t,t'}$ such that $s,t < 1$, $s+t > 0$, $s',t' < \frac{1}{2}$ and $s' + t' > 0$. Then $uv \in H^{s+t-1,s'+t'-\frac{1}{2}}$ and there exists a constant C such that*

$$|uv|_{s+t-1,s'+t'-\frac{1}{2}} \leq C \cdot |u|_{s,s'} \cdot |v|_{t,t'}. \tag{1.6}$$

Sketch of the proof. — We use the following anisotropic equivalent of Bony's decomposition:

$$uv = (T' + R' + \widetilde{T}')(T'' + R'' + \widetilde{T}''),$$

where T' and $\widetilde{T}'$ correspond to the 2-dimensional paraproducts, R' corresponds to the 2-dimensional remainder and the double prime refers to the third variable. For instance, the definition of the term $T'R''$ is

$$T'R''(u,v) = \sum_{i=-1}^{1} \sum_{p,p'} S'_{p-2}\Delta''_{p'}u\Delta'_p\Delta''_{p'-i}v.$$

The theorem holds for each of these operators under weaker assumptions. If a term contains T' then we have to assume that $s < 1$, if it contains R' then $s + t > 0$ and if it contains $\widetilde{T}'$ then $t < 1$. A similar rule holds for T'', R'' and $\widetilde{T}''$. Let us prove that if $s < 1$ and $s' + t' > 0$ then $T'R''(u,v) \in H^{s+t-1,s'+t'-\frac{1}{2}}$. We follow the proof of Theorem 1.1 from [7]. Let

$$w^i_{p,p'} = S'_{p-2}\Delta''_{p'}u\Delta'_p\Delta''_{p'-i}v.$$

Using several times the anisotropic form of Hölder's inequality, the definition of the operator S'_q as well as the anisotropic Littlewood-Paley Lemma 1.2 one can show that

$$\begin{aligned} \|\Delta_{q,q'}w^i_{p,p'}\|_{L^2} &\leq 2^{q'/2}\|\Delta_{q,q'}w^i_{p,p'}\|_{L^{2,1}} \\ &\leq 2^{q'/2} \sum_{r\leq p-2} 2^r\|\Delta'_r\Delta''_{p'}u\|_{L^2} \cdot \|\Delta'_p\Delta''_{p'-i}v\|_{L^2}, \end{aligned} \tag{1.7}$$

(see [7]). Defining

$$a_{q,q'} = 2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2}, \quad b_{q,q'} = 2^{qt+q't'}\|\Delta_{q,q'}v\|_{L^2}$$

and using that $s<1$ yields

$$\|\Delta_{q,q'} w^i_{p,p'}\|_{L^2} \le C 2^{q'/2}\, 2^{p(1-s-t)}\, 2^{-p'(s'+t')}\, \|a_{p,p'}\|_{\ell^\infty_p} \cdot b_{p,p'-i},$$

whence

$$2^{q(s+t-1)+q'(s'+t'-\frac12)}\|\Delta_{q,q'} w^i_{p,p'}\|_{L^2} \le C 2^{(q-p)(s+t-1)+(q'-p')(s'+t')}\|a_{p,p'}\|_{\ell^2_p} \cdot b_{p,p'-i}.$$

The localization of $w^i_{p,p'}$ shows that an integer N exists so that $|p-q|\le N$ and $q'<p'+N$, so

$$2^{q(s+t-1)+q'(s'+t'-\frac12)}\big\|\Delta_{q,q'} T'R''(u,v)\big\|_{L^2} \le C\sum_{i=-1}^{1}\sum_{\substack{|p-q|\le N\\ p'>q'-N}} 2^{(q'-p')(s'+t')}\|a_{p,p'}\|_{\ell^2_p}\cdot b_{p,p'-i}.$$

Taking the ℓ^2_q norm gives

$$\big\|2^{q(s+t-1)+q'(s'+t'-\frac12)}\|\Delta_{q,q'} T'R''(u,v)\|_{L^2}\big\|_{\ell^2_q} \le C\sum_{i=-1}^{1}\sum_{p'>q'-N} 2^{(q'-p')(s'+t')}\|a_{p,p'}\|_{\ell^2_p}\cdot \|b_{p,p'-i}\|_{\ell^2_p}.$$

Taking the $\ell^2_{q'}$ norm, applying Young's inequality and using that $s'+t'>0$ yields

$$\big\|2^{q(s+t-1)+q'(s'+t'-\frac12)}\|\Delta_{q,q'} T'R''(u,v)\|_{L^2}\big\|_{\ell^2} \le C\sum_{i=-1}^{1}\big\|\|a_{p,p'}\|_{\ell^2_p}\cdot\|b_{p,p'-i}\|_{\ell^2_p}\big\|_{\ell^1_{p'}}.$$

Finally, Hölder's inequality implies

$$\big\|2^{q(s+t-1)+q'(s'+t'-\frac12)}\|\Delta_{q,q'} T'R''(u,v)\|_{L^2}\big\|_{\ell^2} \le C\cdot\|a_{p,p'}\|_{\ell^2}\cdot\|b_{p,p'}\|_{\ell^2},$$

that is

$$\big|T'R''(u,v)\big|_{s+t-1,s'+t'-\frac12} \le C\cdot |u|_{s,s'}\cdot|v|_{t,t'}.$$

This completes the proof for $T'R''$. The other terms can be bounded in the same way. $\square$

We now add an interpolation property for these spaces:

PROPOSITION 1.1. — *Let* s,t,s',t' *be four real numbers,* $\alpha \in [0,1]$ *and* $u \in H^{s,s'} \cap H^{t,t'}$. *Then* $u \in H^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}$ *and*

$$|u|_{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'} \leq |u|^{\alpha}_{s,s'} \cdot |u|^{1-\alpha}_{t,t'}.$$

Proof. — We have from Hölder's inequality that

$$\begin{aligned}
&|u|_{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}\\
&\quad = \big\|(1+|n'|^2)^{\frac{1}{2}(\alpha s+(1-\alpha)t)}(1+n_3^2)^{\frac{1}{2}(\alpha s'+(1-\alpha)t')}u_n\big\|_{\ell^2}\\
&\quad \leq \big\|((1+|n'|^2)^{\frac{1}{2}s}(1+n_3^2)^{\frac{1}{2}s'}u_n)^{\alpha}\big\|_{\ell^{2/\alpha}}\\
&\qquad\qquad \cdot \|((1+|n'|^2)^{\frac{1}{2}t}(1+n_3^2)^{t'/2}u_n)^{1-\alpha}\|_{\ell^{2/(1-\alpha)}}\\
&\quad = |u|^{\alpha}_{s,s'} \cdot |v|^{1-\alpha}_{t,t'}.
\end{aligned}$$

This completes the proof. □

We will need to estimate $|\nabla u|_{s,s'}$ in terms of norms of u. The coming proposition gives an useful equivalence.

PROPOSITION 1.2. — *Let* u *be a periodic function on the three dimensional torus with vanishing mean. The following norms are equivalent:*

$$|\nabla u|_{s,s'}, \quad |u|_{s+1,s'} + |u|_{s,s'+1}, \quad \sup_{\alpha\in[0,1]} |u|_{s+\alpha,s'+1-\alpha}.$$

Proof. — Using the interpolation property, one sees that the norm

$$\sup_{\alpha\in[0,1]} |u|_{s+\alpha,s'+1-\alpha}$$

is equivalent to the norm

$$|u|_{s+1,s'} + |u|_{s,s'+1}.$$

On the other hand, we have by definition that

$$\begin{aligned}
|\nabla u|^2_{s,s'} &= |\partial_1 u|^2_{s,s'} + |\partial_2 u|^2_{s,s'} + |\partial_3 u|^2_{s,s'}\\
&= \sum_{n\in\mathbb{Z}^3} (1+|n'|^2)^s(1+n_3^2)^{s'}(n_1^2+n_2^2+n_3^2)|u_n|^2
\end{aligned}$$

and that

$$\begin{aligned}
&|u|^2_{s+1,s'} + |u|^2_{s,s'+1}\\
&= \sum_{n\in\mathbb{Z}^3} \big\{(1+|n'|^2)^{s+1}(1+n_3^2)^{s'} + (1+|n'|^2)^s(1+n_3^2)^{s'+1}\big\}|u_n|^2\\
&= \sum_{n\in\mathbb{Z}^3} (1+|n'|^2)^s(1+n_3^2)^{s'}(2+n_1^2+n_2^2+n_3^2)|u_n|^2
\end{aligned}$$

Since $u_{(0,0,0)} = 0$, the conclusion follows. □

If $v \in L^2(\mathbb{T}^2)$ then one can write $v \in L^2(\mathbb{T}^3)$ by defining

$$v(x_1, x_2, x_3) = v(x_1, x_2).$$

It is obvious that

$$\Delta_{q,0} v = \Delta_q v, \quad \Delta_{q,q'} v = 0 \quad \text{if} \quad q' \geq 1.$$

It follows that, in the proof of Theorem 1.1 there is no loss on q'. This enables us to modify that theorem as follows:

THEOREM 1.2. — *Let $v \in H^s(\mathbb{T}^2)$, $w \in H^{t,t'}$ such that $s < 1$, $t < 1$ and $s + t > 0$. Then*

$$vw \in H^{s+t-1,t'} \quad \textit{and} \quad |vw|_{s+t-1,t'} \leq C \cdot |v|_s \cdot |w|_{t,t'}.$$

Proof. — We treat x_3 as a parameter and we use the decomposition of the product vw as the sum of two-dimensional paraproducts and remainder:

$$vw = T_v w + R(v, w) + \widetilde{T}_v w, \tag{1.8}$$

where

$$\begin{cases} T_v w = \displaystyle\sum_p S'_{p-2} v \Delta'_p w, \\ R(v, w) = \displaystyle\sum_{i=-1}^{1} \sum_p \Delta'_p v \Delta'_{p-i} w, \\ \widetilde{T}_v w = T_w v. \end{cases} \tag{1.9}$$

We prove that the theorem holds under weaker assumptions for each of these operators. More precisely, we have the following

LEMMA 1.5. — *There exists a constant C such that if T, R and $\widetilde{T}$ are the operators defined above, then for all $v \in H^s(\mathbb{T}^2)$ and $w \in H^{t,t'}$ we have*

$$\begin{aligned} |T_v w|_{s+t-1,t'} &\leq C \cdot |v|_s \cdot |w|_{t,t'} && \textit{if } s < 1, \\ |\widetilde{T}_v w|_{s+t-1,t'} &\leq C \cdot |v|_s \cdot |w|_{t,t'} && \textit{if } t < 1, \\ \big|R(v,w)\big|_{s+t-1,t'} &\leq C \cdot |v|_s \cdot |w|_{t,t'} && \textit{if } s + t > 0. \end{aligned}$$

Proof. — Let us prove the assertion on T. We have

$$\begin{aligned} \|\Delta_{q,q'} T_v w\|_{L^2} &\leq \sum_{|p-q| \leq 1} \big\|\Delta_{q,q'}(S'_{p-2} v \Delta'_p w)\big\|_{L^2} \\ &= \sum_{|p-q| \leq 1} \big\|\Delta'_q(S'_{p-2} v \Delta_{p,q'} w)\big\|_{L^2} \\ &\leq C \sum_{|p-q| \leq 1} \big\|S'_{p-2} v \Delta_{p,q'} w\big\|_{L^2} \\ &\leq C \sum_{|p-q| \leq 1} \|S'_{p-2} v\|_{L^\infty} \cdot \|\Delta_{p,q'} w\|_{L^2}. \end{aligned}$$

Since v is two-dimensional and $s < 1$, we infer

$$\|S'_{p-2}v\|_{L^\infty} \leq C2^{p(1-s)}|v|_s.$$

Therefore

$$\|\Delta_{q,q'}T_v w\|_{L^2} \leq C2^{q(1-s)}|v|_s \sum_{|p-q|\leq 1} \|\Delta_{p,q'}w\|_{L^2}. \tag{1.10}$$

It remains to multiply by $2^{q(s+t-1)+q't'}$ and to take the ℓ^2 norm to obtain the result on T.

We consider now the $\widetilde{T}$ term. The following sequence of inequalities holds:

$$\begin{aligned}
\|\Delta_{q,q'}\widetilde{T}_v w\|_{L^2} &\leq \sum_{|p-q|\leq 1} \big\|\Delta_{q,q'}(\Delta'_p v\ S'_{p-2}w)\big\|_{L^2} \\
&= \sum_{|p-q|\leq 1} \big\|\Delta'_q(\Delta'_p v\ S'_{p-2}\Delta''_{q'}w)\big\|_{L^2} \\
&\leq \sum_{|p-q|\leq 1} \big\|\Delta'_p v\ S'_{p-2}\Delta''_{q'}w\big\|_{L^2} \\
&\leq C \sum_{|p-q|\leq 1} \|\Delta'_p v\|_{L^2} \cdot \|S'_{p-2}\Delta''_{q'}w\|_{L^{\infty,2}}.
\end{aligned} \tag{1.11}$$

One can estimate

$$\begin{aligned}
\|S'_{p-2}\Delta''_{q'}w\|_{L^{\infty,2}} &\leq \sum_{r\leq p-2} \|\Delta_{r,q'}w\|_{L^{\infty,2}} \\
&\leq C \sum_{r\leq p-2} 2^r \|\Delta_{r,q'}w\|_{L^2} \\
&\leq C2^{-q't'} \sum_{r\leq p-2} 2^{r(1-t)} \|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\|_{\ell^2_r} \\
&\leq C2^{-q't'-p(t-1)} \|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\|_{\ell^2_r}.
\end{aligned}$$

Thus

$$\begin{aligned}
&2^{q(s+t-1)+q't'}\|\Delta_{q,q'}\widetilde{T}_v w\|_{L^2} \\
&\qquad\qquad \leq C \sum_{|p-q|\leq 1} 2^{ps}\|\Delta'_p v\|_{L^2} \|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\|_{\ell^2_r}.
\end{aligned}$$

The conclusion for $\widetilde{T}$ now follows by taking the ℓ^2 norm.

Finally, we prove the assertion on R. One has

$$\begin{aligned}\|\Delta_{q,q'}R(v,w)\|_{L^2} &\leq \sum_{i=-1}^{1}\sum_{p\geq q-2}\big\|\Delta_{q,q'}(\Delta'_p v\Delta'_{p-i}w)\big\|_{L^2}\\ &= \sum_{i=-1}^{1}\sum_{p\geq q-2}\|\Delta'_q(\Delta'_p v\Delta_{p-i,q'}w)\|_{L^2}\\ &\leq C\sum_{i=-1}^{1}\sum_{p\geq q-2}2^q\|\Delta'_p v\Delta_{p-i,q'}w\|_{L^{1,2}}\\ &\leq C\sum_{i=-1}^{1}\sum_{p\geq q-2}2^q\|\Delta'_p v\|_{L^2}\|\Delta_{p-i,q'}w\|_{L^2}.\end{aligned}$$

It follows that

$$\begin{aligned}(1.12)\quad & 2^{q(s+t-1)+q't'}\big\|\Delta_{q,q'}R(v,w)\big\|_{L^2}\\ &\leq C\sum_{i=-1}^{1}\sum_{p\geq q-2}2^{(q-p)(s+t)}2^{ps}\|\Delta'_p v\|_{L^2}\cdot 2^{(p-i)t+q't'}\|\Delta_{p-i,q'}w\|_{L^2}.\end{aligned}$$

Applying Young's inequality completes the proof of Lemma 1.5. □

The decomposition (1.8) and Lemma 1.5 implies Theorem 1.2. □

In Section 2 we shall need to apply Theorem 1.2 in the case $s>1$. The coming inequality is a variant of an inequality proved by J.-Y. Chemin and N. Lerner in [5]. It shows how to avoid this difficulty in some cases.

PROPOSITION 1.3. — *There exists a constant C such that for all $v\in H^s(\mathbb{T}^2)$ and w such that* $\operatorname{div} v=0$, $\nabla w\in H^{t,t'}$, $s<2$, $t<1$ *and* $s+t>0$ *there exists a sequence* $(a_{q,q'})$ *such that*

$$\begin{aligned}&\big|\langle\Delta_{q,q'}(v\cdot\nabla w)\mid\Delta_{q,q'}w\rangle\big|\\ &\qquad\leq Ca_{q,q'}\,2^{-q(s+t-1)-q't'}|v|_s\cdot|\nabla w|_{t,t'}\cdot\|\Delta_{q,q'}w\|_{L^2},\end{aligned}$$

and $\|a_{q,q'}\|_{\ell^2}=1$.

Proof. — We write

$$\begin{aligned}\big|\langle\Delta_{q,q'}(v\cdot\nabla w)\mid\Delta_{q,q'}w\rangle\big| &= \big|\langle\Delta_{q,q'}(T_v\nabla w)\mid\Delta_{q,q'}w\rangle\big|\\ &\quad+\big|\langle\Delta_{q,q'}(R(v,\nabla w))\mid\Delta_{q,q'}w\rangle\big|\\ &\quad+\big|\langle\Delta_{q,q'}(T_{\nabla w}v)\mid\Delta_{q,q'}w\rangle\big|,\end{aligned}$$

where T and R are the two-dimensional paraproduct and remainder defined in the last theorem. The hypothesis on s, t and Lemma 1.5 imply that the terms

$$\big|\langle\Delta_{q,q'}(R(v,\nabla w))\mid\Delta_{q,q'}w\rangle\big|, \tag{1.13}$$

$$\big|\langle\Delta_{q,q'}(T_{\nabla w}v)\mid\Delta_{q,q'}w\rangle\big| \tag{1.14}$$

are well estimated. One has to bound

$$\big|\langle\Delta_{q,q'}(T_v\nabla w)\mid\Delta_{q,q'}w\rangle\big|. \tag{1.15}$$

Some simple computations and the localization of the terms of $T_v\nabla w$ show that

$$\begin{aligned}\big\langle\Delta_{q,q'}(T_v\nabla w)\mid\Delta_{q,q'}w\big\rangle\\ &= \sum_{j,|p-q|\le 4}\big\langle[\Delta_{q,q'},S_{p-2}v^j]\partial_j\Delta_p w\mid\Delta_{q,q'}w\big\rangle\\ &\qquad+\tfrac12\sum_{\substack{j,\,|p-q|\le 4\\|p'-q|\le 4}}\big\langle(S_{p'-2}-S_{p-2})v^j\partial_j\Delta_{q,q'}\Delta'_{p'}w\mid\Delta_{q,q'}\Delta'_p w\big\rangle\end{aligned}$$

(see [3], [5]). Therefore, it suffices to estimate the model terms

$$I_1=\big|\langle[\Delta_{q,q'},S_qv^j]\partial_j\Delta_{q,q'}w\mid\Delta_{q,q'}w\rangle\big|,$$

$$I_2=\big|\langle\Delta_qv^j\partial_j\Delta_{q,q'}\Delta'_{p'}w\mid\Delta_{q,q'}\Delta'_p w\rangle\big|.$$

The last term is bounded as follows

$$\|\Delta_qv^j\|_{L^\infty}\le C2^q\|\Delta_qv^j\|_{L^2}\le C2^{q(1-s)}|v|_s, \tag{1.16}$$

$$\|\partial_j\Delta_{q,q'}w\|_{L^2}\le Cb_{q,q'}2^{-qt-q't'}|\nabla w|_{t,t'}, \tag{1.17}$$

where $\|b_{q,q'}\|_{\ell^2}=1$. As for I_1 we remark that

$$[h\,*,f]\,b(x)=\int h(y)\big(f(x-y)-f(x)\big)b(x-y)\,\mathrm{d}y,$$

thus

$$\big\|[h\,*,f]\,b\big\|_{L^2}\le C\|\nabla f\|_{L^\infty}\|b\|_{L^2}\|xh\|_{L^1}. \tag{1.18}$$

Applying this inequality with $f=S_qv^j$, $b=\partial_j\Delta_{q,q'}w$ and $h*=\Delta_{q,q'}$ it comes

$$\big\|[\Delta_{q,q'},S_qv^j]\,\partial_j\Delta_{q,q'}w\big\|_{L^2}\le Ca_{q,q'}2^{-q(s+t-1)-q't'}|v|_s\cdot|\nabla w|_{t,t'},$$

where $\|a_{q,q'}\|_{\ell^2}=1$. This completes the proof. $\square$

We now introduce the second class of spaces we will use:

DEFINITION 1.3. — *We denote by* $HB^{s,s'}$ *the space defined by*

$$HB^{s,s'} = \{u \in \mathcal{D}'(\mathbb{T}^3) \text{ such that } |u|_{HB^{s,s'}} < \infty\},$$

where $q,q' \geq 0$ *and*

$$|u|_{HB^{s,s'}} \stackrel{\text{def}}{=} \left\| 2^{qs+q's'} \|\Delta_{q,q'} u\|_{L^2} \right\|_{\ell^{2,1}}.$$

The homogeneous version is

$$H\dot{B}^{s,s'} = \{u \in HB^{s,s'} \text{ and } Mu = 0\}.$$

REMARK 1.2. — Since $B^{\frac{1}{2}}_{2,1}(\mathbb{T}) \hookrightarrow \mathcal{C}(\mathbb{T})$, it follows that $HB^{s,\frac{1}{2}}$ is embedded in the space of functions continuous in x_3 with values in $H^s(\mathbb{T}^2)$.

The last defined class of spaces is similar to the class HB^{s_1,s_2,s_3} introduced in the case of the entire space in [7], the purpose being the same, that is, avoiding the critical case $\delta = 0$. The study of these spaces is similar to those ones and with the study of the $H^{s,s'}$. More precisely, all the assertions valid for the $H^{s,s'}$ spaces are valid for the $HB^{s,s'}$ spaces if we replace the ℓ^2 norms with the $\ell^{2,1}$ norms. The following proposition as well as its proof is similar to Theorem 1.2 from [7] which states that the product of a function from HB^{s_1,s_2,s_3} with a function from HB^{t_1,t_2,t_3} lies in $HB^{s_1+t_1-\frac{1}{2},s_2+t_2-\frac{1}{2},s_3+t_3-\frac{1}{2}}$ provided that $s_i,t_i < \frac{1}{2}$, $s_i+t_i > 0$, $i \in \{1,2\}$ and $s_3,t_3 \leq \frac{1}{2}$, and $s_3+t_3 > 0$.

PROPOSITION 1.4. — *Let* $u \in HB^{s,s'}$, $v \in HB^{t,t'}$ *such that* $s,t < 1$ *and* $s+t > 0$; $s',t' \leq \frac{1}{2}$, *and* $s'+t' > 0$. *Then* $uv \in HB^{s+t-1,s'+t'-\frac{1}{2}}$ *and*

$$|uv|_{HB^{s+t-1,s'+t'-\frac{1}{2}}} \leq C|u|_{HB^{s,s'}} \cdot |v|_{HB^{t,t'}}.$$

Sketch of the proof. — The proof is almost identical to the one of Theorem 1.1, the modification which enables us to take the case $s' = \frac{1}{2}$ or $t' = \frac{1}{2}$ into account is that the classical paraproduct

$$T: B^s_{2,1}(\mathbb{R}) \times B^t_{2,1}(\mathbb{R}) \longrightarrow B^{s+t-\frac{1}{2}}_{2,1}(\mathbb{R})$$

is well-defined and continuous if $s \leq \frac{1}{2}$. We shall prove that each of the operators from (1.6) is continuous under weaker assumptions. The only problem in the proof is that at the end we have to commute some norms which give raise to the wrong inequality. To show that the other terms can be handled in the same way, we prove the assertion for some other term, say $R'\widetilde{T}''$. By definition

$$R'\widetilde{T}''(u,v) = \sum_{i=-1}^{1} \sum_{p,p'} z^i_{p,p'} \quad \text{with} \quad z^i_{p,p'} = \Delta'_p \Delta''_{p'} u \; \Delta'_{p-i} S''_{p'-2} v.$$

We will prove that $R'\widetilde{T}''(u,v) \in HB^{s+t-1,s'+t'-\frac{1}{2}}$ provided that $s+t>0$ and $t' \leq \frac{1}{2}$. As in inequality (1.7) one obtains

$$\|\Delta_{q,q'} z^i_{p,p'}\|_{L^2} \leq 2^q \sum_{r' \leq p'-2} 2^{r'/2} \|\Delta'_p \Delta''_{p'} u\|_{L^2} \cdot \|\Delta'_{p-i} \Delta''_{r'} v\|_{L^2}.$$

Recall that $a_{q,q'} = 2^{qs+q's'} \|\Delta_{q,q'} u\|_{L^2}$ and $b_{q,q'} = 2^{qt+q't'} \|\Delta_{q,q'} v\|_{L^2}$. There exists an integer N such that $|p'-q'| \leq N$ and $p > q-N$. We have

$$\begin{aligned}(1.19) \quad 2^{q(s+t-1)+q'(s'+t'-\frac{1}{2})} & \|\Delta_{q,q'} z^i_{p,p'}\|_{L^2} \\ & \leq C 2^{(s+t)(q-p)} \sum_{r' \leq p'-2} 2^{(r'-p')(\frac{1}{2}-t')} a_{p,p'} b_{p-i,r'}.\end{aligned}$$

We now sum on i, p, p' and q' to obtain

$$\begin{aligned}\sum_{q'} & 2^{q(s+t-1)+q'(s'+t'-\frac{1}{2})} \big\|\Delta_{q,q'} R'\widetilde{T}''(u,v)\big\|_{L^2} \\ & \leq C \sum_{i=-1}^{1} \sum_{p>q-N} 2^{(s+t)(q-p)} \sum_{p'} \sum_{r' \leq p'-2} 2^{(r'-p')(\frac{1}{2}-t')} a_{p,p'} b_{p-i,r'} \\ & \leq C \sum_{i=-1}^{1} \sum_{p>q-N} 2^{(s+t)(q-p)} \|a_{p,p'}\|_{\ell^1_{p'}} \cdot \|b_{p-i,r'}\|_{\ell^1_{r'}}\end{aligned}$$

Using that $q < p+N$ and $s+t>0$ and applying Young's inequality yields

$$\begin{aligned}\big\|2^{q(s+t-1)+q'(s'+t'-\frac{1}{2})} & \|\Delta_{q,q'} R'\widetilde{T}''(u,v)\|_{L^2}\big\|_{\ell^{2,1}} \\ & \leq C \sum_{i=-1}^{1} \big\|\|a_{p,p'}\|_{\ell^1_{p'}} \cdot \|b_{p-i,p'}\|_{\ell^1_{p'}}\big\|_{\ell^1_p}.\end{aligned}$$

Finally, we apply Hölder's inequality to obtain

$$\big\|2^{q(s+t-1)+q'(s'+t'-\frac{1}{2})} \|\Delta_{q,q'} R'\widetilde{T}''(u,v)\|_{L^2}\big\|_{\ell^{2,1}} \leq C \|a_{p,p'}\|_{\ell^{2,1}} \cdot \|b_{p,p'}\|_{\ell^{2,1}},$$

which implies

$$\big|R'\widetilde{T}''(u,v)\big|_{HB^{s+t-1,s'+t'-\frac{1}{2}}} \leq C |u|_{HB^{s,s'}} \cdot |v|_{HB^{t,t'}}.$$

This completes the proof for $R'\widetilde{T}''$. ∎

We also need to know what happens when we multiply a 2-dimensional function with a 3-dimensional one. The result is

PROPOSITION 1.5. — *Let $v \in H^s(\mathbb{T}^2)$, $w \in HB^{t,t'}$ such that $s,t < 1$ and $s+t > 0$. Then*

$$vw \in HB^{s+t-1,t'} \quad \textit{and} \quad |vw|_{HB^{s+t-1,t'}} \leq C|v|_s \cdot |w|_{HB^{t,t'}}.$$

In equation (1.9) we defined two-dimensional paraproduct and remainder for three-dimensional functions. We prove that the proposition holds under weaker hypothesis for each of these operators. More precisely, we have the following

LEMMA 1.6. — *There exists a constant C such that if T, R and $\widetilde{T}$ are the operators introduced in equation* (1.9), *then for all $v \in H^s(\mathbb{T}^2)$ and $w \in HB^{t,t'}$ we have*

$$\begin{aligned}
&|T_v w|_{HB^{s+t-1,t'}} \leq C|v|_s \cdot |w|_{HB^{t,t'}} && \textit{if } s < 1,\\
&|\widetilde{T}_v w|_{HB^{s+t-1,t'}} \leq C|v|_s \cdot |w|_{HB^{t,t'}} && \textit{if } t < 1,\\
&|R(v,w)|_{HB^{s+t-1,t'}} \leq C|v|_s \cdot |w|_{HB^{t,t'}} && \textit{if } s+t > 0.
\end{aligned}$$

Proof. — For T we start again from inequality (1.10), we multiply by $2^{q(s+t-1)+q't'}$ and we take the $\ell^{2,1}$ norm to obtain

$$\big\|2^{q(s+t-1)+q't'}\|\Delta_{q,q'}T_v w\|_{L^2}\big\|_{\ell^{2,1}} \leq C|v|_s \cdot \big\|2^{qt+q't'}\|\Delta_{p,q'}w\|_{L^2}\big\|_{\ell^{2,1}}.$$

We now consider the $\widetilde{T}$ term. Starting again from inequality (1.11), multiplying by $2^{q(s+t-1)+q't'}$ and summing on q' gives

$$\begin{aligned}
\sum_{q'} &2^{q(s+t-1)+q't'}\|\Delta_{q,q'}\widetilde{T}_v w\|_{L^2}\\
&\leq C2^{q(s+t-1)}\sum_{|p-q|\leq 1}\|\Delta'_p v\|_{L^2}\sum_{q'}2^{q't'}\|S'_{p-2}\Delta''_{q'}w\|_{L^{\infty,2}}.
\end{aligned}$$

Furthermore, one can bound

$$\begin{aligned}
\sum_{q'}2^{q't'}\|S'_{p-2}\Delta''_{q'}w\|_{L^{\infty,2}} &\leq \sum_{r\leq p-2}\sum_{q'}2^{q't'}\|\Delta_{r,q'}w\|_{L^{\infty,2}}\\
&\leq C\sum_{r\leq p-2}2^r\sum_{q'}2^{q't'}\|\Delta_{r,q'}w\|_{L^2}\\
&\leq C\sum_{r\leq p-2}2^{r(1-t)}\big\|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\big\|_{\ell^{2,1}}\\
&\leq C2^{-p(t-1)}\big\|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\big\|_{\ell^{2,1}}.
\end{aligned}$$

Thus

$$\sum_{q'} 2^{q(s+t-1)+q't'} \|\Delta_{q,q'}\widetilde{T}_v w\|_{L^2} \leq C \sum_{|p-q|\leq 1} 2^{ps}\|\Delta'_p v\|_{L^2}\big\|2^{rt+q't'}\|\Delta_{r,q'}w\|_{L^2}\big\|_{\ell^{2,1}}.$$

The conclusion for $\widetilde{T}$ now follows by taking the ℓ^2_q norm.

Finally, we prove the assertion on R. Starting from inequality (1.12) and summing on q' yields

$$\sum_{q'} 2^{q(s+t-1)+q't'} \|\Delta_{q,q'}R(v,w)\|_{L^2} \leq C \sum_{i=-1}^{1}\sum_{p\geq q-2} 2^{(q-p)(s+t)}2^{ps}\|\Delta'_p v\|_{L^2}\cdot \sum_{q'} 2^{(p-i)t+q't'}\|\Delta_{p-i,q'}w\|_{L^2}.$$

Applying Young's inequality completes the proof. □

We now prove an interpolation property for the *HB* spaces:

PROPOSITION 1.6. — *Let* s,t,s',t' *be four real numbers,* $\alpha \in [0,1]$ *and* $u \in HB^{s,s'} \cap HB^{t,t'}$. *Then* $u \in HB^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}$ *and*

$$|u|_{HB^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}} \leq |u|^{\alpha}_{HB^{s,s'}}\cdot |u|^{1-\alpha}_{HB^{t,t'}}.$$

Proof. — From the definition of the *HB* spaces and using Hölder's inequality, we infer that

$$\begin{aligned}
&|u|_{HB^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}}\\
&\quad = \big\|2^{q(\alpha s+(1-\alpha)t)+q'(\alpha s'+(1-\alpha)t')}\|\Delta_{q,q'}u\|_{L^2}\big\|_{\ell^{2,1}}\\
&\quad = \big\|(2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2})^{\alpha}(2^{qt+q't'}\|\Delta_{q,q'}u\|_{L^2})^{1-\alpha}\big\|_{\ell^{2,1}}\\
&\quad \leq \big\|(2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2})^{\alpha}\big\|_{\ell^{2/\alpha,1/\alpha}}\\
&\qquad\qquad \cdot \big\|(2^{qt+q't'}\|\Delta_{q,q'}u\|_{L^2})^{1-\alpha}\big\|_{\ell^{2/(1-\alpha),1/(1-\alpha)}}\\
&\quad = \big\|2^{qs+q's'}\|\Delta_{q,q'}u\|_{L^2}\big\|^{\alpha}_{\ell^{2,1}}\cdot\big\|2^{qt+q't'}\|\Delta_{q,q'}u\|_{L^2}\big\|^{1-\alpha}_{\ell^{2,1}}\\
&\quad = |u|^{\alpha}_{HB^{s,s'}}\cdot|v|^{1-\alpha}_{HB^{t,t'}}.
\end{aligned}$$

This completes the proof. □

As for the anisotropic Sobolev spaces, we now give an estimate for the HB norm of a gradient.

PROPOSITION 1.7. — *Let u be a periodic function on the three dimensional torus with vanishing mean. Then the following norms are equivalent:*

$$|\nabla u|_{HB^{s,s'}}, \quad |u|_{HB^{s+1,s'}} + |u|_{HB^{s,s'+1}}, \quad \sup_{\alpha\in[0,1]} |u|_{HB^{s+\alpha,s'+1-\alpha}}.$$

Proof. — Using the previous proposition proves that the norm

$$\sup_{\alpha\in[0,1]} |u|_{HB^{s+\alpha,s'+1-\alpha}}$$

is equivalent to the norm

$$|u|_{HB^{s+1,s'}} + |u|_{HB^{s,s'+1}}.$$

To show the other equivalence, we first prove the following inequality:

$$\|\nabla\Delta_{q,q'}u\|_{L^2} \geq C(2^q + 2^{q'})\|\Delta_{q,q'}u\|_{L^2}. \tag{1.20}$$

The localization of $\Delta_{q,q'}u$ clearly implies this relation for $q \geq 1$ and $q' \geq 1$. Since u has vanishing mean, one has that $\Delta_{0,0}u = 0$, so the case $q = q' = 0$ is trivial. Assume now that $q = 0$ and $q' > 0$. Since $\Delta_{0,q'}$ depends only on x_3 we have

$$\begin{aligned}\|\nabla\Delta_{0,q'}u\|_{L^2} = \|\partial_3\Delta_{0,q'}u\|_{L^2} &\geq C2^{q'}\|\Delta_{0,q'}u\|_{L^2}\\ &\geq \tfrac{1}{2}C(1+2^{q'})\|\Delta_{0,q'}u\|_{L^2}.\end{aligned}$$

The case $q = 0$ and $q' > 0$ is similar so relation (1.20) is proved.

The localization of $\Delta_{q,q'}u$ implies that

$$\|\nabla\Delta_{q,q'}u\|_{L^2} \leq C'(2^q + 2^{q'})\|\Delta_{q,q'}u\|_{L^2}.$$

Using this relation together with (1.20) we infer that

$$\begin{aligned}|\nabla u|_{HB^{s,s'}} &= \big\|2^{qs+q's'}\|\Delta_{q,q'}\nabla u\|_{L^2}\big\|_{\ell^{2,1}}\\ &\simeq \big\|2^{qs+q's'}(2^q+2^{q'})\|\Delta_{q,q'}u\|_{L^2}\big\|_{\ell^{2,1}}\\ &= \big\|(2^{(q+1)s+q's'} + 2^{qs+q'(s'+1)})\|\Delta_{q,q'}u\|_{L^2}\big\|_{\ell^{2,1}}\\ &\simeq |u|_{HB^{s+1,s'}} + |u|_{HB^{s,s'+1}}.\end{aligned}$$

The proof is completed. □

Finally, we show how the statement and the proof of Proposition 1.3 can be modified in the case of the *HB* spaces.

PROPOSITION 1.8. — *There exists a constant C such that for all $v \in H^s(\mathbb{T}^2)$ and w such that* $\operatorname{div} v = 0$, $\nabla w \in HB^{t,t'}$, $s < 2$, $t < 1$ *and* $s + t > 0$ *there exists a sequence* $(a_{q,q'})$ *such that*

$$\begin{aligned} &\left|\langle \Delta_{q,q'}(v \cdot \nabla w) \mid \Delta_{q,q'} w\rangle\right| \\ &\qquad\qquad \leq C a_{q,q'} 2^{-q(s+t-1)-q't'} |v|_s \cdot |\nabla w|_{HB^{t,t'}} \cdot \|\Delta_{q,q'} w\|_{L^2}, \end{aligned}$$

and $\|a_{q,q'}\|_{\ell^{2,1}} = 1$.

Proof. — As in the proof of Proposition 1.3 we write

$$\begin{aligned} \left|\langle \Delta_{q,q'}(v \cdot \nabla w) \mid \Delta_{q,q'} w\rangle\right| = \left|\langle \Delta_{q,q'}(T_v \nabla w) \mid \Delta_{q,q'} w\rangle\right| & \\ + \left|\langle \Delta_{q,q'}(R(v, \nabla w)) \mid \Delta_{q,q'} w\rangle\right| & \\ + \left|\langle \Delta_{q,q'}(T_{\nabla w} v) \mid \Delta_{q,q'} w\rangle\right|, & \end{aligned}$$

where T and R are the two-dimensional paraproduct and remainder defined in relation (1.9). The hypothesis on s, t and Lemma 1.6 imply that

$$\left|\langle \Delta_{q,q'}(R(v, \nabla w)) \mid \Delta_{q,q'} w\rangle\right| \quad \text{and} \quad \left|\langle \Delta_{q,q'}(T_{\nabla w} v) \mid \Delta_{q,q'} w\rangle\right|$$

are well estimated. It remains to estimate

$$\left|\langle \Delta_{q,q'}(T_v \nabla w) \mid \Delta_{q,q'} w\rangle\right|.$$

As in Proposition 1.3, we see that it suffices to bound

$$I_1 = \left|\langle [\Delta_{q,q'}, S_q v^j]\, \partial_j \Delta_{q,q'} w \mid \Delta_{q,q'} w\rangle\right|,$$

$$I_2 = \left|\langle \Delta_q v^j \partial_j \Delta_{q,q'} \Delta'_{p'} w \mid \Delta_{q,q'} \Delta'_p w\rangle\right|,$$

under the assumptions $|p - q| \leq 4$, and $|p' - q| \leq 4$. To estimate the last term we write

$$\left\{ \begin{aligned} &\|\Delta_q v^j\|_{L^\infty} \leq C 2^q \|\Delta_q v^j\|_{L^2} \leq C 2^{q(1-s)} |v|_s, \\ &\|\partial_j \Delta_{q,q'} w\|_{L^2} \leq C b_{q,q'} 2^{-qt-q't'} |\nabla w|_{HB^{t,t'}}, \end{aligned} \right. \tag{1.21}$$

where $\|b_{q,q'}\|_{\ell^{2,1}} = 1$. For I_1 we remark again that

$$\left\|[h*, f]\, b\right\|_{L^2} \leq C \|\nabla f\|_{L^\infty} \cdot \|b\|_{L^2} \cdot \|xh\|_{L^1}.$$

Applying this inequality with $f = S_q v^j$, $b = \partial_j \Delta_{q,q'} w$ and $h* = \Delta_{q,q'}$ it comes

$$\left\|[\Delta_{q,q'}, S_q v^j] \partial_j \Delta_{q,q'} w\right\|_{L^2} \leq C a_{q,q'} 2^{-q(s+t-1)-q't'} |v|_s \cdot |\nabla w|_{HB^{t,t'}},$$

where $\|a_{q,q'}\|_{\ell^{2,1}} = 1$. The conclusion follows. □

We now write the 3D Navier-Stokes equations as a perturbation of the 2D Navier-Stokes equations. Let us define $v = Mu$ and $w = (I - M)u$. Applying the projections M and $I - M$ to *(N-S)* it is not difficult to see that the Navier-Stokes equations

$$\text{(N-S)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u(t, \cdot) = 0 \quad \text{for all} \;\; t \geq 0, \\ u_{|\,t=0} = u_0, \end{cases}$$

are equivalent to the following coupled systems

$$(1.22) \qquad \begin{cases} \partial_t v + v \nabla v - \nu \Delta v = -M(w \nabla w) - \nabla p_1, \\ \operatorname{div} v = 0, \\ v_{|\,t=0} = v_0 \;\; (= M u_0), \end{cases}$$

for some p_1 independent of x_3 and

$$(1.23) \qquad \begin{cases} \partial_t w + v \nabla w + w \nabla v + (I - M)(w \nabla w) - \nu \Delta w = -\nabla p_2, \\ \operatorname{div} w = 0, \\ w_{|\,t=0} = w_0 \;\; \big(= (I - M) u_0\big). \end{cases}$$

As far as v is concerned, only classical L^2 energy estimates are needed; indeed, in dimension two the regularity obtained via L^2 energy estimates suffices to ensure global existence and uniqueness. The problem is to derive estimates on w. Since M and $I - M$ are projections, their norms are equal to 1, so the estimates below shall not involve these operators.

We shall also consider the case when $u_0 = v_0 + w_0$ where v_0 is not necessarily the projection of u_0, hence it is not possible to write the same equations for v and w. We will replace them with some simpler ones:

$$(1.24) \qquad \begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v = -\nabla p', \\ \operatorname{div} v(t, \cdot) = 0 \quad \text{for all} \;\; t \geq 0, \\ v_{|\,t=0} = v_0, \end{cases}$$

for some p' independent of x_3 and

$$(1.25) \qquad \begin{cases} \partial_t w + w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w - \nu \Delta w = -\nabla p'', \\ \operatorname{div} w(t, \cdot) = 0 \quad \text{for all} \;\; t \geq 0, \\ w_{|\,t=0} = w_0. \end{cases}$$

2. The case of the $H^{s,s'}$ spaces

Let $0 < \delta < 1$. We shall prove the following theorems:

THEOREM 2.1 (global existence and uniqueness). — *There exists a positive constant $C = C(\delta)$ such that if the initial data u_0 has vanishing mean over the three-dimensional torus,* $\operatorname{div} u_0 = 0$, $v_0 = Mu_0 \in L^2(\mathbb{T}^2)$, $w_0 = (I - M)u_0 \in H^{\delta,\frac{1}{2}-\delta}$ *and*

$$|w_0|_{\delta,\frac{1}{2}-\delta} \exp\Big(\frac{\|v_0\|_{L^2}^2}{C\nu^2}\Big) < C\nu,$$

then the (N-S) *equations have a unique global solution such that*

$$w = (I - M)u \in L^4\big(]0,\infty[; H^{(1+\delta)/2,(1-\delta)/2}\big) \cap L^\infty\big(]0,\infty[; H^{\delta,\frac{1}{2}-\delta}\big) \tag{2.1}$$

and

$$v = Mu \in L^2\big(]0,\infty[; H^1\big) \cap L^\infty\big(]0,\infty[; L^2\big).$$

THEOREM 2.2 (global existence and uniqueness). — *There exists a positive constant $C = C(\delta)$ such that if the initial data verifies $u_0 = v_0 + w_0$, where v_0 and w_0 have vanishing mean over the three-dimensional torus,* $\operatorname{div} v_0 = \operatorname{div} w_0 = 0$, $v_0 \in L^2(\mathbb{T}^2)$, $w_0 \in H^{\delta,\frac{1}{2}-\delta}$ *and*

$$|w_0|_{\delta,\frac{1}{2}-\delta} \exp\Big(\frac{\|v_0\|_{L^2}^2}{C\nu^2}\Big) < C\nu,$$

then the (N-S) *equations have a unique global solution such that, if v is the unique solution of the 2D* (*with three components*) *Navier-Stokes equations* (1.24) *with*

$$v \in L^2\big(]0,\infty[; H^1\big) \cap L^\infty\big(]0,\infty[; L^2\big),$$

then

$$w = u - v \in L^4\big(]0,\infty[; H^{(1+\delta)/2,(1-\delta)/2}\big) \cap L^\infty\big(]0,\infty[; H^{\delta,\frac{1}{2}-\delta}\big)$$

and is a solution of system (1.25).

THEOREM 2.3 (local existence and uniqueness). — *If the initial data verifies $u_0 = v_0 + w_0$, where v_0 and w_0 have vanishing mean over the three-dimensional torus,* $\operatorname{div} v_0 = \operatorname{div} w_0 = 0$, $v_0 \in L^2(\mathbb{T}^2)$ *and* $w_0 \in H^{\delta,\frac{1}{2}-\delta}$ *then there exist $T > 0$ and a unique solution of* (N-S) *on* $[0,T]$ *such that if v is the unique solution of the 2D* (*with three components*) *Navier-Stokes equations* (1.24) *with*

$$v \in L^2\big(]0,\infty[; H^1\big) \cap L^\infty\big(]0,\infty[; L^2\big),$$

then

$$w = u - v \in L^4\big(]0,T[; H^{(1+\delta)/2,(1-\delta)/2}\big) \cap L^\infty\big(]0,T[; H^{\delta,\frac{1}{2}-\delta}\big)$$

and is a solution of system (1.25).

The smallness assumption of Theorem 2.1 is a particular case of the one of Theorem 2.2. We give two different theorems because v and w are not defined in the same way in the two theorems (see relations (1.22), (1.23), (1.24) and (1.25)). Moreover, we will need to make the asymptotic study, that is we will consider the Navier-Stokes equations in $\mathbb{T}_\varepsilon$ and we will study the dependence on ε of the constant C. In order to obtain optimal results, we will need to assume that w is "homogeneous" in the third variable, which corresponds to the case of Theorem 2.1. In short, Theorem 2.1 is a particular case of Theorem 2.2 when ε is fixed, but this changes when $\varepsilon \to 0$. That is why we prefer to prove Theorem 2.1, even though systems (1.22) and (1.23) are more complicated than systems (1.24) and (1.25). The proof of Theorem 2.2 is similar to that of Theorem 2.1; it suffices to replace the system for $(I-M)u$ with system (1.25), the estimates are simpler.

Sketch of the proof of local existence. — We proved in Section 2 of [7] a local existence and uniqueness theorem (Theorem 2.2) for solutions of the Navier-Stokes equations with initial data in a space $\mathcal{H}^{\delta_1,\delta_2,\delta_3}$. The proofs given there can be adjusted to the case of the initial data in the space $H^{\delta,\frac{1}{2}-\delta}$. Let us show that those arguments can be modified to allow the presence of a two-dimensional term, the v term. The proof will consist of some a priori estimates. As usual, the existence can be rigorously justified by an approximation procedure.

Applying the operator $\Delta_{q,q'}$ to the equation (1.25) of w, taking the scalar product with $\Delta_{q,q'}w$ and using inequality (1.20) as well as the product Theorems 1.1, 1.2 and Lemma 1.3 yields:

$$\begin{aligned}
&\partial_t \|\Delta_{q,q'}w\|_{L^2}^2 + C\nu(4^q + 4^{q'})\|\Delta_{q,q'}w\|_{L^2}^2 \\
&\quad \le C\big(2^{q(1-\delta)+q'(\delta-\frac{1}{2})} + 2^{-q\delta+q'(\frac{1}{2}+\delta)}\big) a_{q,q'} |w|^2_{(1+\delta)/2,(1-\delta)/2} \|\Delta_{q,q'}w\|_{L^2} \\
&\quad + C\big(2^{q(1-\delta/2)+q'(\delta-1)/2} + 2^{-q\delta/2+q'(1+\delta)/2}\big) \\
&\qquad\qquad\qquad \cdot a_{q,q'} |v|_{\frac{1}{2}} |w|_{(1+\delta)/2,(1-\delta)/2} \|\Delta_{q,q'}w\|_{L^2},
\end{aligned}$$

where $\sum\limits_{q,q'} a_{q,q'}^2(t) = 1$ for all t. Gronwall's Lemma gives

$$
\begin{aligned}
&\|\Delta_{q,q'} w(t)\|_{L^2} \\
&\quad \leq \|\Delta_{q,q'} w_0\|_{L^2} \exp\bigl(-C\nu(4^q + 4^{q'})t\bigr) \\
&\qquad + C\bigl(2^{q(1-\delta)+q'(\delta-\frac{1}{2})} + 2^{-q\delta+q'(\frac{1}{2}+\delta)}\bigr) \\
&\qquad\qquad \cdot a_{q,q'} |w|^2_{(1+\delta)/2,(1-\delta)/2} * \exp\bigl(-C\nu(4^q + 4^{q'})t\bigr) \\
&\qquad + C(2^{q(1-\delta/2)+q'(\delta-1)/2} + 2^{-q\delta/2+q'(1+\delta)/2}) \\
&\qquad\qquad \cdot a_{q,q'} |v|_{\frac{1}{2}} |w|_{(1+\delta)/2,(1-\delta)/2} * \exp\bigl(-C\nu(4^q + 4^{q'})t\bigr).
\end{aligned}
$$

Taking the $L^4(0,T)$ norm and using Young's inequality yields:

$$
\begin{aligned}
&\|\Delta_{q,q'} w\|_{L^4(0,T;L^2)} \\
&\quad \leq C\nu^{-1/4}(4^q + 4^{q'})^{-1/4} \|\Delta_{q,q'} w_0\|_{L^2} \bigl(1 - \exp(-C\nu(4^q + 4^{q'})T)\bigr)^{1/4} \\
&\qquad + C\nu^{-3/4}\bigl(2^{q(1-\delta)+q'(\delta-\frac{1}{2})} + 2^{-q\delta+q'(\frac{1}{2}+\delta)}\bigr)(4^q + 4^{q'})^{-3/4} \\
&\qquad\qquad\qquad \cdot \|a_{q,q'} |w|^2_{(1+\delta)/2,(1-\delta)/2}\|_{L^2(0,T)} \\
&\qquad + C\nu^{-3/4}\bigl(2^{q(1-\delta/2)+q'(\delta-1)/2} + 2^{-q\delta/2+q'(1+\delta)/2}\bigr)(4^q + 4^{q'})^{-3/4} \\
&\qquad\qquad\qquad \cdot \|a_{q,q'} |v|_{\frac{1}{2}} |w|_{(1+\delta)/2,(1-\delta)/2}\|_{L^2(0,T)}.
\end{aligned}
$$

It is easy to check that multiplying by $2^{q(1+\delta)/2+q'(1-\delta)/2}$, taking the ℓ^2 norm and using Hölder's inequality as well as Remark 1.1 implies

$$
\begin{aligned}
\text{(2.2)}\quad &\|w\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\leq A(T) + C\nu^{-3/4}\|v\|_{L^4(0,T;H^{\frac{1}{2}})} \cdot \|w\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad\qquad + C\nu^{-3/4}\|w\|^2_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})},
\end{aligned}
$$

where

$$
A(T) = C\nu^{-1/4}\bigl\|2^{q\delta+q'(\frac{1}{2}-\delta)}\|\Delta_{q,q'} w_0\|_{L^2}(1 - \exp(-C\nu(4^q+4^{q'})T))^{1/4}\bigr\|_{\ell^2}.
$$

The Lebesgue dominated convergence theorem shows that $\lim_{T\to 0} A(T) = 0$. On the other hand, we know that $v \in L^\infty(0,\infty;L^2) \cap L^2(0,\infty;H^1)$. Since $|v|^2_{\frac{1}{2}} \leq C\|v\|_{L^2} \cdot |v|_1$, it follows that $v \in L^4(0,\infty;H^{\frac{1}{2}})$. Let $T^\star$ be such that $A(T^\star) < \nu^{3/4}/(16C)$ and $\|v\|_{L^4(0,T^\star;H^{\frac{1}{2}})} < \nu^{3/4}/(2C)$. Then, one has from (2.2) and for all $0 \leq t \leq T^\star$

$$
\|w\|_{L^4(0,t;H^{(1+\delta)/2,(1-\delta)/2})} < \frac{\nu^{3/4}}{8C} + \frac{2C}{\nu^{3/4}}\|w\|^2_{L^4(0,t;H^{(1+\delta)/2,(1-\delta)/2})}.
$$

But the quantity $\|w\|_{L^4(0,t;H^{(1+\delta)/2,(1-\delta)/2})}$ is continuous in time and vanishes for $t=0$. We infer that

$$\|w\|_{L^4(0,t;H^{1+\delta/2,1-\delta/2})} \leq \frac{\nu^{3/4}}{4C}$$

for all $0 \leq t \leq T^\star$. One can deduce from relation (2.7) that

$$\partial_t |w|^2_{\delta,\frac{1}{2}-\delta} \leq \frac{C}{\nu}|v|^2_1 \cdot |w|^2_{\delta,\frac{1}{2}-\delta} + \frac{C}{\nu}|w|^4_{(1+\delta)/2,(1-\delta)/2}.$$

Gronwall's lemma implies that $w \in L^\infty(0,T^\star;H^{\delta,\frac{1}{2}-\delta})$. This completes the proof. ∎

Proof of global existence. — We apply $\Delta_{q,q'}$ to the equation verified by w and we multiply by $\Delta_{q,q'}w$ to obtain:

$$(2.3)\quad \begin{aligned} \partial_t\|\Delta_{q,q'}w\|^2_{L^2} + \nu\|\Delta_{q,q'}\nabla w\|^2_{L^2} &\leq C\big|\langle\Delta_{q,q'}(I-M)(w\cdot\nabla w) \mid \Delta_{q,q'}w\rangle\big| \\ &\quad + C\big|\langle\Delta_{q,q'}(v\cdot\nabla w) \mid \Delta_{q,q'}w\rangle\big| \\ &\quad + C\big|\langle\Delta_{q,q'}(w\cdot\nabla v) \mid \Delta_{q,q'}w\rangle\big|. \end{aligned}$$

Since w is divergence free an integration by parts shows that

$$\big|\langle\Delta_{q,q'}(I-M)(w\cdot\nabla w) \mid \Delta_{q,q'}w\rangle\big| = \big|\langle\Delta_{q,q'}(I-M)(w\otimes w) \mid \Delta_{q,q'}\nabla w\rangle\big|$$

and we can use the product Theorem 1.1 to deduce that

$$(2.4)\quad \begin{aligned} &\big|\langle\Delta_{q,q'}(I-M)(w\cdot\nabla w) \mid \Delta_{q,q'}w\rangle\big| \\ &\qquad \leq Cb_{q,q'}2^{-q\delta-q'(\frac{1}{2}-\delta)}\cdot|w|^2_{(1+\delta)/2,(1-\delta)/2}\cdot\|\Delta_{q,q'}\nabla w\|_{L^2} \end{aligned}$$

where $\sum\limits_{q,q'} b^2_{q,q'} = 1$. Next we use Proposition 1.3 to obtain that

$$(2.5)\quad \begin{aligned} &\big|\langle\Delta_{q,q'}(v\cdot\nabla w) \mid \Delta_{q,q'}w\rangle\big| \\ &\qquad \leq Ca_{q,q'}2^{-q\delta-q'(\frac{1}{2}-\delta)}|v|_1\cdot|\nabla w|_{\delta,\frac{1}{2}-\delta}\cdot\|\Delta_{q,q'}w\|_{L^2}, \end{aligned}$$

where $\sum\limits_{q,q'} a^2_{q,q'} = 1$.

Applying Theorem 1.2 and Lemma 1.3 gives

$$\begin{aligned}(2.6)\qquad &\big|\langle \Delta_{q,q'}(w\cdot\nabla v)\mid \Delta_{q,q'}w\rangle\big|\\ &\qquad\qquad \le Cc_{q,q'}2^{q(1-\delta)-q'(\frac12-\delta)}|v|_1\cdot|w|_{\delta,\frac12-\delta}\cdot\|\Delta_{q,q'}w\|_{L^2},\end{aligned}$$

where $\sum\limits_{q,q'} c_{q,q'}^2=1$.

Using relations (2.4), (2.5) and (2.6) in (2.3) yields

$$\begin{aligned}\partial_t\|\Delta_{q,q'}w\|_{L^2}^2 &+ 2\nu\|\Delta_{q,q'}\nabla w\|_{L^2}^2\\ &\le Ca_{q,q'}2^{-q\delta-q'(\frac12-\delta)}|v|_1\cdot|\nabla w|_{\delta,\frac12-\delta}\cdot\|\Delta_{q,q'}w\|_{L^2}\\ &\quad + Cb_{q,q'}2^{-q\delta-q'(\frac12-\delta)}|w|^2_{(1+\delta)/2,(1-\delta)/2}\cdot\|\Delta_{q,q'}\nabla w\|_{L^2}\\ &\quad + Cc_{q,q'}2^{q(1-\delta)-q'(\frac12-\delta)}|v|_1\cdot|w|_{\delta,\frac12-\delta}\cdot\|\Delta_{q,q'}w\|_{L^2}.\end{aligned}$$

Multiplying both sides by $4^{q\delta+q'(\frac12-\delta)}$, using Schwarz's inequality, summing and using Proposition 1.2 implies

$$\begin{aligned}(2.7)\quad &\partial_t|w|^2_{\delta,\frac12-\delta}+2\nu|\nabla w|^2_{\delta,\frac12-\delta}\\ &\qquad \le C|v|_1\cdot|\nabla w|_{\delta,\frac12-\delta}\cdot|w|_{\delta,\frac12-\delta}+C|w|^2_{(1+\delta)/2,(1-\delta)/2}\cdot|\nabla w|_{\delta,\frac12-\delta}.\end{aligned}$$

Interpolating $H^{(1+\delta)/2,(1-\delta)/2}$ between $H^{\delta,\frac12-\delta}$ and $H^{1,\frac12}$ and using again Proposition 1.2 we find

$$(2.8)\qquad |w|^2_{(1+\delta)/2,(1-\delta)/2}\le|w|_{\delta,\frac12-\delta}\cdot|w|_{1,\frac12}\le|w|_{\delta,\frac12-\delta}\cdot|\nabla w|_{\delta,\frac12-\delta}.$$

Therefore

$$\begin{aligned}\partial_t|w|^2_{\delta,\frac12-\delta}&+2\nu|\nabla w|^2_{\delta,\frac12-\delta}\\ &\le C|v|_1\cdot|\nabla w|_{\delta,\frac12-\delta}\cdot|w|_{\delta,\frac12-\delta}+C|w|_{\delta,\frac12-\delta}\cdot|\nabla w|^2_{\delta,\frac12-\delta}\\ &\le \frac{C}{\nu}|v|_1^2\cdot|w|^2_{\delta,\frac12-\delta}+C|w|_{\delta,\frac12-\delta}\cdot|\nabla w|^2_{\delta,\frac12-\delta}+\frac12\nu|\nabla w|^2_{\delta,\frac12-\delta}.\end{aligned}$$

One deduces

$$\begin{aligned}(2.9)\quad &\partial_t|w|^2_{\delta,\frac12-\delta}+3\nu/2|\nabla w|^2_{\delta,\frac12-\delta}\\ &\qquad\qquad \le \frac{C}{\nu}|v|_1^2\cdot|w|^2_{\delta,\frac12-\delta}+C|w|_{\delta,\frac12-\delta}\cdot|\nabla w|^2_{\delta,\frac12-\delta}.\end{aligned}$$

Let us assume that $C_1 > C$ and

$$|w|_{\delta,\frac{1}{2}-\delta} \le \frac{\nu}{2C_1}\cdot \tag{2.10}$$

It follows that

$$\partial_t |w|^2_{\delta,\frac{1}{2}-\delta} + \nu|\nabla w|^2_{\delta,\frac{1}{2}-\delta} \le \frac{C}{\nu}|v|_1^2 \cdot |w|^2_{\delta,\frac{1}{2}-\delta}. \tag{2.11}$$

Gronwall's inequality then implies

$$|w(t)|^2_{\delta,\frac{1}{2}-\delta} \le |w_0|^2_{\delta,\frac{1}{2}-\delta} \exp\Big(\int_0^t \frac{C}{\nu}|v(\tau)|_1^2 \,\mathrm{d}\tau\Big). \tag{2.12}$$

We have to estimate $\int_0^t (C/\nu)|v(\tau)|_1^2\,\mathrm{d}\tau$ in terms of $\|v_0\|_{L^2}$. To do that we take the product of equation (1.22) with v and we integrate by parts to obtain that

$$\begin{aligned}
\partial_t \|v\|^2_{L^2} + 2\nu|v|_1^2 &\le \big|\langle M(w\nabla w) \mid v\rangle\big| \\
&\le \big|\langle M(w\otimes w) \mid \nabla v\rangle\big| \\
&\le C|v|_1 \cdot \big|M(w\otimes w)\big|_{L^2(\mathbb{T}^2)} \\
&\le C|v|_1 \cdot \big|M(w\otimes w)\big|_{H^\delta(\mathbb{T}^2)} \\
&= C|v|_1 \cdot \big|M(w\otimes w)\big|_{\delta,\frac{1}{2}-\delta} \\
&\le C|v|_1 \cdot |w\otimes w|_{\delta,\frac{1}{2}-\delta}.
\end{aligned} \tag{2.13}$$

Using the product Theorem 1.1 and inequalities (2.8), (2.10) yields

$$\begin{aligned}
\partial_t \|v\|^2_{L^2} + 2\nu|v|_1^2 &\le C|v|_1 \cdot |w|^2_{(1+\delta)/2,(1-\delta)/2} \\
&\le C|v|_1 \cdot |w|_{\delta,\frac{1}{2}-\delta} |\nabla w|_{\delta,\frac{1}{2}-\delta} \\
&\le \frac{C\nu}{C_1}|v|_1 \cdot |\nabla w|_{\delta,\frac{1}{2}-\delta} \\
&\le \nu|v|_1^2 + \frac{C^2\nu}{C_1^2}|\nabla w|^2_{\delta,\frac{1}{2}-\delta}.
\end{aligned} \tag{2.14}$$

Hence

$$\partial_t \|v\|^2_{L^2} + \nu|v|_1^2 \le \frac{C\nu}{C_1^2}|\nabla w|^2_{\delta,\frac{1}{2}-\delta}. \tag{2.15}$$

Integrating this inequality gives

$$\int_0^t \big|v(\tau)\big|_1^2 \,\mathrm{d}\tau \le \frac{C}{C_1^2}\int_0^t \big|\nabla w(\tau)\big|^2_{\delta,\frac{1}{2}-\delta}\,\mathrm{d}\tau + \frac{1}{\nu}\|v_0\|^2_{L^2}. \tag{2.16}$$

We go back to inequality (2.11) and we integrate to obtain

$$\begin{aligned}\int_0^t \big|\nabla w(\tau)\big|^2_{\delta,\frac12-\delta}\,\mathrm{d}\tau &\le \frac{1}{\nu}|w_0|^2_{\delta,\frac12-\delta} + \frac{C}{\nu^2}\int_0^t \big|v(\tau)\big|_1^2\cdot\big|w(\tau)\big|^2_{\delta,\frac12-\delta}\,\mathrm{d}\tau\\ &\le \frac{1}{\nu}|w_0|^2_{\delta,\frac12-\delta} + \frac{C}{C_1^2}\int_0^t \big|v(\tau)\big|_1^2\,\mathrm{d}\tau.\end{aligned}$$

The inequality above along with relation (2.16) yields for large enough C_1

$$\int_0^t \big|v(\tau)\big|_1^2\,\mathrm{d}\tau \le \frac{2}{\nu}\|v_0\|^2_{L^2} + \frac{C}{\nu}|w_0|^2_{\delta,\frac12-\delta}.$$

Now, we use this inequality in (2.12) and we find

$$\big|w(t)\big|^2_{\delta,\frac12-\delta} \le |w_0|^2_{\delta,\frac12-\delta}\exp\Big(\frac{C}{\nu^2}\big(|w_0|^2_{\delta,\frac12-\delta} + \|v_0\|^2_{L^2}\big)\Big).$$

Recall that this holds only as long as

$$|w|_{\delta,\frac12-\delta} \le \frac{\nu}{2C_1}\cdot$$

Hence the condition to assume initially is

$$|w_0|^2_{\delta,\frac12-\delta}\exp\Big(\frac{C}{\nu^2}\big(|w_0|^2_{\delta,\frac12-\delta} + \|v_0\|^2_{L^2}\big)\Big) \le \frac{\nu^2}{4C_1^2}\cdot$$

This is implied by a condition of the type

$$|w_0|_{\delta,\frac12-\delta}\exp\Big(\frac{\|v_0\|^2_{L^2}}{C'\nu^2}\Big) \le C'\nu.$$

Indeed, if the latter holds, we have

$$|w_0|_{\delta,\frac12-\delta} \le C'\nu,$$

which gives

$$\begin{aligned}|w_0|^2_{\delta,\frac12-\delta}\exp\Big(\frac{1}{C'\nu^2}\big(|w_0|^2_{\delta,\frac12-\delta} + \|v_0\|^2_{L^2}\big)\Big) &\le \exp(C')|w_0|^2_{\delta,\frac12-\delta}\exp\Big(\frac{\|v_0\|^2_{L^2}}{C'\nu^2}\Big)\\ &\le \exp(C')C'\nu.\end{aligned}$$

We proved that $\nabla w \in L^2(]0,\infty[;H^{\delta,\frac12-\delta})$. From inequalities (2.12) and (2.8) we deduce that

$$w \in L^\infty\big(]0,\infty[;H^{\delta,\frac12-\delta}\big) \cap L^4\big(]0,\infty[;H^{(1+\delta)/2,(1-\delta)/2}\big).$$

Finally, integrating relation (2.15) shows that

$$v \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1).$$

This completes the proof of global existence. $\square$

Proof of uniqueness. — Let u_1 and u_2 be two solutions with the same initial data such that for $i = 1, 2$

$$w_i = (I - M)u_i \in L^4\big(0, T; H^{(1+\delta)/2,(1-\delta)/2}\big) \cap L^\infty\big(0, T; H^{\delta,\frac{1}{2}-\delta}\big),$$
$$v_i = Mu_i \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

We deduce by interpolation that $v_i \in L^4(0, T; H^{\frac{1}{2}})$. The difference $v_1 - v_2$ verifies the equation

$$\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2) + v_1\nabla(v_1 - v_2) + (v_1 - v_2)\nabla v_2 + \operatorname{div} M(w_1 \otimes w_1 - w_2 \otimes w_2) = \nabla p_1,$$

for some p_1. The usual L^2 energy estimates give

$$\partial_t\|v_1 - v_2\|_{L^2}^2 + 2\nu|v_1 - v_2|_1^2 \leq C\|v_1 - v_2\|_{L^2} \cdot |v_1 - v_2|_1 \cdot |v_2|_1 + 2\big\|M(w_1 \otimes w_1 - w_2 \otimes w_2)\big\|_{L^2} \cdot |v_1 - v_2|_1.$$

We infer that

$$(2.17)\qquad \partial_t\|v_1 - v_2\|_{L^2}^2 + \nu|v_1 - v_2|_1^2 \leq C\|v_1 - v_2\|_{L^2}^2 \cdot |v_2|_1^2 + C\big\|M(w_1 \otimes w_1 - w_2 \otimes w_2)\big\|_{L^2}^2.$$

But

$$\begin{aligned}\big\|M(w_1 \otimes w_1 - w_2 \otimes w_2)\big\|_{L^2} &\leq \big|M(w_1 \otimes w_1 - w_2 \otimes w_2)\big|_{\delta,\frac{1}{2}-\delta}\\ &\leq \big|(w_1 - w_2) \otimes w_1\big|_{\delta,\frac{1}{2}-\delta} + \big|w_2 \otimes (w_1 - w_2)\big|_{\delta,\frac{1}{2}-\delta}\\ &\leq C|w_1 - w_2|_{(1+\delta)/2,(1-\delta)/2} \cdot \big(|w_1|_{(1+\delta)/2,(1-\delta)/2} + |w_2|_{(1+\delta)/2,(1-\delta)/2}\big).\end{aligned}$$

Applying Gronwall's lemma in (2.17) now yields

$$\begin{aligned}\big\|(v_1 - v_2)(t)\big\|_{L^2}^2 + \nu\int_0^t |v_1 - v_2|_1^2\,\mathrm{d}\tau &\leq C\exp\Big(C\int_0^t |v_2|_1^2\,\mathrm{d}\tau\Big)\int_0^t \big|(w_1 - w_2)\big|_{(1+\delta)/2,(1-\delta)/2}^2\\ &\qquad \cdot \big(|w_1|_{(1+\delta)/2,(1-\delta)/2}^2 + |w_2|_{(1+\delta)/2,(1-\delta)/2}^2\big)\,\mathrm{d}\tau\\ &\leq C\exp\Big(C\int_0^t |v_2|_1^2\,\mathrm{d}\tau\Big)\|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})}^2\\ &\qquad \cdot \big(\|w_1\|_{L^4(0,t;H^{(1+\delta)/2,(1-\delta)/2})}^2 + \|w_2\|_{L^4(0,t;H^{(1+\delta)/2,(1-\delta)/2})}^2\big).\end{aligned}$$

Since $|v_1 - v_2|_{\frac{1}{2}} \leq \|v_1 - v_2\|_{L^2}^{\frac{1}{2}} \cdot |v_1 - v_2|_1^{\frac{1}{2}}$ we infer that

$$
\begin{aligned}
(2.18)\quad & \|v_1 - v_2\|_{L^4(0,T;H^{\frac{1}{2}})} \\
&\leq C \exp\Big(C\int_0^T |v_2|_1^2 \,\mathrm{d}\tau\Big) \|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad \cdot \big(\|w_1\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} + \|w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})}\big).
\end{aligned}
$$

We turn to the estimate of $w_1 - w_2$. Its equation is

$$
\begin{aligned}
\partial_t(w_1 - w_2) &- \nu\Delta(w_1 - w_2) + (I - M)w_1\nabla(w_1 - w_2) \\
&+ (I - M)(w_1 - w_2)\nabla w_2 + v_1\nabla(w_1 - w_2) + (v_1 - v_2)\nabla w_2 \\
&\qquad + w_1\nabla(v_1 - v_2) + (w_1 - w_2)\nabla v_2 = \nabla p_2,
\end{aligned}
$$

for some p_2. As in the proof of local existence, one can deduce that

$$
\begin{aligned}
&\|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\quad \leq C\|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad \cdot \big(\|w_1\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} + \|w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad\qquad + \|v_1\|_{L^4(0,T;H^{\frac{1}{2}})} + \|v_2\|_{L^4(0,T;H^{\frac{1}{2}})}\big) \\
&\qquad + C\|v_1 - v_2\|_{L^4(0,T;H^{\frac{1}{2}})}\big(\|w_1\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad\qquad + \|w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})}\big).
\end{aligned}
$$

In view of (2.18) we obtain

$$
\begin{aligned}
(2.19)\quad & \|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&\qquad \leq \|w_1 - w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} B(T),
\end{aligned}
$$

where

$$
\begin{aligned}
B(T) = C\big\{&\|w_1\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})} \\
&+ \|w_2\|_{L^4(0,T;H^{(1+\delta)/2,(1-\delta)/2})}\big\} \exp\Big(C\int_0^T |v_2|_1^2 \,\mathrm{d}\tau\Big) \\
&\qquad + C\big(\|v_1\|_{L^4(0,T;H^{\frac{1}{2}})} + \|v_2\|_{L^4(0,T;H^{\frac{1}{2}})}\big).
\end{aligned}
$$

Since B is continuous and $B(0) = 0$ we obtain from (2.19) and (2.18) local uniqueness, that is global uniqueness. $\square$

3. The case of the $HB^{s,s'}$ spaces

We shall prove the following theorems:

THEOREM 3.1 (global existence and uniqueness). — *There exists $C>0$ such that if the initial data u_0 has vanishing mean over the three-dimensional torus,*

$$\operatorname{div} u_0 = 0, \quad v_0 = Mu_0 \in L^2(\mathbb{T}^2), \quad w_0 = (I-M)u_0 \in HB^{0,\frac12}$$

and

$$|w_0|_{HB^{0,\frac12}} \exp\Big(\frac{\|v_0\|_{L^2}^2}{C\nu^2}\Big) < C\nu,$$

then the (N-S) *equations have a unique global solution such that*

$$w = (I-M)u \in L^4\big(]0,\infty[;HB^{\frac12,\frac12}\big) \cap L^\infty\big(]0,\infty[;HB^{0,\frac12}\big) \tag{3.1}$$

and

$$v = Mu \in L^2\big(]0,\infty[;H^1\big) \cap L^\infty\big(]0,\infty[;L^2\big).$$

THEOREM 3.2 (global existence and uniqueness). — *There exists $C>0$ such that if the initial data verifies $u_0 = v_0 + w_0$, where v_0 and w_0 have vanishing mean over the three-dimensional torus,*

$$\operatorname{div} v_0 = \operatorname{div} w_0 = 0, \quad v_0 \in L^2(\mathbb{T}^2), \quad w_0 \in HB^{0,\frac12}$$

and

$$|w_0|_{HB^{0,\frac12}} \exp\Big(\frac{\|v_0\|_{L^2}^2}{C\nu^2}\Big) < C\nu,$$

then the (N-S) *equations have a unique global solution such that, if v is the unique solution of the 2D* (*with three components*) *Navier-Stokes equations* (1.24) *with*

$$v \in L^2\big(]0,\infty[;H^1\big) \cap L^\infty\big(]0,\infty[;L^2\big),$$

then

$$w = u - v \in L^4\big(]0,\infty[;HB^{\frac12,\frac12}\big) \cap L^\infty\big(]0,\infty[;HB^{0,\frac12}\big)$$

and is a solution of system (1.25).

As far as local existence is concerned, the 2-dimensional part v is not important. Indeed, a square integrable 2D function belongs to $HB^{0,\frac12}$ as a 3D function, so $u_0 \in HB^{0,\frac12}$. It is proved in [7] in a more difficult setting the local existence of a solution $u \in L^4(0,T;HB^{\frac12,\frac12})$. But $v \in L^4(0,T;H^{\frac12})$ so $v \in L^4(0,T;HB^{\frac12,\frac12})$ which implies that

$$w = u - v \in L^4(0,T;HB^{\frac12,\frac12}).$$

As for the case of $H^{s,s'}$ spaces, Theorem 3.1 is a particular case of Theorem 3.2, the reason of its presence is that in the asymptotic study we have to work in homogeneous spaces in order to obtain optimal results. Let us remark that the space $HB^{0,\frac12}$ is invariant for the scaling $x_3 \mapsto \lambda x_3$, as well as for the usual scaling of the Navier-Stokes equations.

Proof of global existence. — As in Theorem 2.1 we may find the inequality

$$\begin{aligned}(3.2)\quad \partial_t\|\Delta_{q,q'}w\|_{L^2}^2 &+ \nu\|\Delta_{q,q'}\nabla w\|_{L^2}^2 \\ &\le CF_{q,q'}\|\Delta_{q,q'}w\|_{L^2} + CG_{q,q'}\|\Delta_{q,q'}w\|_{L^2} + CH_{q,q'}\|\Delta_{q,q'}w\|_{L^2}\end{aligned}$$

where

$$\begin{aligned} F_{q,q'} &= \frac{|\langle \Delta_{q,q'}(v\cdot\nabla w) \mid \Delta_{q,q'}w\rangle|}{\|\Delta_{q,q'}w\|_{L^2}}, \\ G_{q,q'} &= \frac{|\langle \Delta_{q,q'}(w\cdot\nabla v) \mid \Delta_{q,q'}w\rangle|}{\|\Delta_{q,q'}w\|_{L^2}}, \\ H_{q,q'} &= \frac{|\langle \Delta_{q,q'}(I-M)(w\cdot\nabla w) \mid \Delta_{q,q'}w\rangle|}{\|\Delta_{q,q'}w\|_{L^2}}, \end{aligned}$$

if $\|\Delta_{q,q'}w\|_{L^2}\neq 0$ and 0 otherwise. The function $t\mapsto \|\Delta_{q,q'}w\|_{L^2}$ is a Lipschitz function, hence its derivative exists almost everywhere. A variant of Gronwall's inequality and inequality (3.2) now implies that

$$\partial_t\|\Delta_{q,q'}w\|_{L^2} + \nu(4^q+4^{q'})\|\Delta_{q,q'}w\|_{L^2} \le CF_{q,q'} + CG_{q,q'} + CH_{q,q'}.$$

Multiplying by $2^{q'/2}$ and summing on q' yields

$$\begin{aligned}\partial_t\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2} &+ \nu\sum_{q'}(4^q+4^{q'})2^{q'/2}\|\Delta_{q,q'}w\|_{L^2} \\ &\le C\sum_{q'}2^{q'/2}F_{q,q'} + C\sum_{q'}2^{q'/2}G_{q,q'} + C\sum_{q'}2^{q'/2}H_{q,q'}.\end{aligned}$$

Now we multiply by $\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}$ and we sum on q to obtain

$$\begin{aligned}\partial_t\sum_q\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)^2 \\ + 2\nu\sum_q\Big(\Big(\sum_{q'}(4^q+4^{q'})2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big) \\ \le C\sum_q\Big(\Big(\sum_{q'}2^{q'/2}F_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big) \\ + C\sum_q\Big(\Big(\sum_{q'}2^{q'/2}G_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big) \\ + C\sum_q\Big(\Big(\sum_{q'}2^{q'/2}H_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big).\end{aligned}$$

From Schwarz's inequality we get

$$\sum_{q'}(4^q+4^{q'})2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\\ \geq\Big(\sum_{q'}2^{q'/2}(2^q+2^{q'})\|\Delta_{q,q'}w\|_{L^2}\Big)^2,$$

$$\sum_q\Big(\Big(\sum_{q'}2^{q'/2}F_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big)\\ \leq\Big\{\sum_q\Big(\sum_{q'}2^{q'/2}F_{q,q'}\Big)^2\Big\}^{\frac12}\Big\{\sum_q\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)^2\Big\}^{\frac12},$$

$$\sum_q\Big(\Big(\sum_{q'}2^{q'/2}G_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big)\\ =\sum_q\Big(\Big(\sum_{q'}2^{-q/2+q'/2}G_{q,q'}\Big)\Big(\sum_{q'}2^{q/2+q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big)\\ \leq\Big\{\sum_q\Big(\sum_{q'}2^{-q/2+q'/2}G_{q,q'}\Big)^2\Big\}^{\frac12}\\ \cdot\Big\{\sum_q\Big(\sum_{q'}2^{q/2+q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)^2\Big\}^{\frac12}.$$

and the same inequality for the H-term

$$\sum_q\Big(\Big(\sum_{q'}2^{q'/2}H_{q,q'}\Big)\Big(\sum_{q'}2^{q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)\Big)\\ \leq\Big\{\sum_q\Big(\sum_{q'}2^{-q/2+q'/2}H_{q,q'}\Big)^2\Big\}^{\frac12}\\ \cdot\Big\{\sum_q\Big(\sum_{q'}2^{q/2+q'/2}\|\Delta_{q,q'}w\|_{L^2}\Big)^2\Big\}^{\frac12}.$$

It follows that

$$\partial_t|w|^2_{HB^{0,\frac12}}+\frac{\nu}{C}|\nabla w|^2_{HB^{0,\frac12}}\\ \leq C|w|_{HB^{0,\frac12}}\|2^{q'/2}F_{q,q'}\|_{\ell^{2,1}}\\ +C|w|_{HB^{\frac12,\frac12}}\big(\|2^{-q/2+q'/2}G_{q,q'}\|_{\ell^{2,1}}+\|2^{-q/2+q'/2}H_{q,q'}\|_{\ell^{2,1}}\big).$$

Using Propositions 1.4 and 1.5 yields

$$\|2^{-q/2+q'/2}G_{q,q'}\|_{\ell^{2,1}}\leq C|v|_1\cdot|w|_{HB^{\frac12,\frac12}},$$

$$\|2^{-q/2+q'/2}H_{q,q'}\|_{\ell^{2,1}}\leq C|w|_{HB^{\frac12,\frac12}}|\nabla w|_{HB^{0,\frac12}}.$$

Proposition 1.8 gives

$$\|2^{q'/2}F_{q,q'}\|_{\ell^{2,1}} \leq C|v|_1 \cdot |\nabla w|_{HB^{0,\frac{1}{2}}}.$$

Furthermore, applying Proposition 1.7 and interpolating $HB^{\frac{1}{2},\frac{1}{2}}$ between $HB^{0,\frac{1}{2}}$ and $HB^{1,\frac{1}{2}}$ yields

$$\begin{aligned}
(3.3)\qquad \partial_t|w|^2_{HB^{0,\frac{1}{2}}} + \frac{\nu}{C}|\nabla w|^2_{HB^{0,\frac{1}{2}}}
&\leq C|w|^2_{HB^{\frac{1}{2},\frac{1}{2}}}\big(|v|_1 + |\nabla w|_{HB^{0,\frac{1}{2}}}\big)\\
&\qquad + C|w|_{HB^{0,\frac{1}{2}}}|v|_1 \cdot |\nabla w|_{HB^{0,\frac{1}{2}}}\\
&\leq C|w|_{HB^{0,\frac{1}{2}}}|w|_{HB^{1,\frac{1}{2}}}\big(|v|_1 + |\nabla w|_{HB^{0,\frac{1}{2}}}\big)\\
&\qquad + C|w|_{HB^{0,\frac{1}{2}}}|v|_1 \cdot |\nabla w|_{HB^{0,\frac{1}{2}}}\\
&\leq C|w|_{HB^{0,\frac{1}{2}}}|\nabla w|_{HB^{0,\frac{1}{2}}}\big(|v|_1 + |\nabla w|_{HB^{0,\frac{1}{2}}}\big)\\
&\leq \frac{\nu}{2C}|\nabla w|^2_{HB^{0,\frac{1}{2}}} + \frac{C}{\nu}|v|_1^2 \cdot |w|^2_{HB^{0,\frac{1}{2}}}\\
&\qquad + C|\nabla w|^2_{HB^{0,\frac{1}{2}}}|w|_{HB^{0,\frac{1}{2}}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(3.4)\qquad \partial_t|w|^2_{HB^{0,\frac{1}{2}}} + \frac{\nu}{C}|\nabla w|^2_{HB^{0,\frac{1}{2}}}
&\leq \frac{C}{\nu}|v|_1^2 \cdot |w|^2_{HB^{0,\frac{1}{2}}} + C|\nabla w|^2_{HB^{0,\frac{1}{2}}} \cdot |w|_{HB^{0,\frac{1}{2}}}.
\end{aligned}$$

This inequality is entirely similar to inequality (2.9), so we can repeat the argument valid in the Sobolev spaces case to obtain the existence of a solution such that

$$w \in L^\infty\big([0,\infty]; HB^{0,\frac{1}{2}}\big), \quad \nabla w \in L^2\big([0,\infty]; HB^{0,\frac{1}{2}}\big).$$

We use again Proposition 1.7 and the interpolation to deduce that

$$w \in L^4\big([0,\infty]; HB^{\frac{1}{2},\frac{1}{2}}\big).$$

This completes the proof of the global existence. $\square$

Proof of uniqueness. — An uniqueness result is proved in [7] but in a space smaller than the one we consider here. Therefore, we have to give another proof. Let $T > 0$. We prove that a solution with w in

$$L^4\big([0,T]; HB^{\frac{1}{2},\frac{1}{2}}\big) \cap L^\infty\big([0,T]; HB^{0,\frac{1}{2}}\big)$$

and with initial data u_0 is unique in this class. We saw in inequality (3.3) that

$$\partial_t |w|^2_{HB^{0,\frac12}} + \frac{\nu}{C}|\nabla w|^2_{HB^{0,\frac12}} \le C|w|^2_{HB^{\frac12,\frac12}}\big(|v|_1 + |\nabla w|_{HB^{0,\frac12}}\big) + C|w|_{HB^{0,\frac12}} \cdot |v|_1 \cdot |\nabla w|_{HB^{0,\frac12}}.$$

Furthermore, we deduce

$$\partial_t |w|^2_{HB^{0,\frac12}} + \frac{\nu}{2C}|\nabla w|^2_{HB^{0,\frac12}} \le \frac{C}{\nu}|w|^4_{HB^{\frac12,\frac12}} + \frac{\nu}{C}|v|_1^2 + \frac{C}{\nu}|w|^2_{HB^{0,\frac12}} \cdot |v|_1^2.$$

Integrating yields

$$\nabla w \in L^2\big([0,T]; HB^{0,\frac12}\big).$$

Moreover, since $|v|_{HB^{1,\frac12}} = |v|_1$, the standard energy estimates for the Navier-Stokes equations imply that

$$u \in L^2\big([0,T]; HB^{1,\frac12}\big), \quad \nabla u \in L^2\big([0,T]; HB^{0,\frac12}\big).$$

Let u_1 and u_2 be two such solutions. Subtracting the equations verified by u_1 and u_2 yields

$$\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2) + u_1 \cdot \nabla(u_1 - u_2) + (u_1 - u_2)\cdot\nabla u_2 = \nabla(p_1 - p_2).$$

Making similar computations as in the proof of the global existence we find the inequality

$$\begin{aligned}(3.5)\quad \partial_t|u_1 - u_2|^2_{HB^{0,\frac12}} &+ \frac{\nu}{C}\big|\nabla(u_1-u_2)\big|^2_{HB^{0,\frac12}} \\ &\le C|u_1-u_2|^2_{HB^{\frac12,\frac12}} \cdot |\nabla u_2|_{HB^{0,\frac12}} \\ &\quad + C|u_1|_{HB^{\frac12,\frac12}} \cdot |u_1-u_2|_{HB^{\frac12,\frac12}}\big|\cdot\nabla(u_1-u_2)\big|_{HB^{0,\frac12}}\end{aligned}$$

Let

$$A \stackrel{\text{def}}{=} C|u_1-u_2|^2_{HB^{\frac12,\frac12}} \cdot |\nabla u_2|_{HB^{0,\frac12}},$$

$$B \stackrel{\text{def}}{=} C|u_1|_{HB^{\frac12,\frac12}} \cdot |u_1-u_2|_{HB^{\frac12,\frac12}} \cdot \big|\nabla(u_1-u_2)\big|_{HB^{0,\frac12}}.$$

Using the interpolation, Schwarz's inequality and Proposition 1.2 we get

$$\begin{aligned} B &\le |u_1|_{HB^{\frac12,\frac12}} \cdot |u_1-u_2|^{\frac12}_{HB^{0,\frac12}} \cdot |u_1-u_2|^{\frac12}_{HB^{1,\frac12}} \cdot \big|\nabla(u_1-u_2)\big|_{HB^{0,\frac12}} \\ &\le |u_1|_{HB^{\frac12,\frac12}} \cdot |u_1-u_2|^{\frac12}_{HB^{0,\frac12}} \cdot \big|\nabla(u_1-u_2)\big|^{3/2}_{HB^{0,\frac12}} \\ &\le \frac{C}{\nu}|u_1|^4_{HB^{\frac12,\frac12}} \cdot |u_1-u_2|^2_{HB^{0,\frac12}} + \frac{\nu}{4C}|\nabla(u_1-u_2)|^2_{HB^{0,\frac12}},\end{aligned}$$

and

$$\begin{aligned} A &\le |u_1-u_2|_{HB^{1,\frac12}}\cdot|u_1-u_2|_{HB^{0,\frac12}}\cdot|\nabla u_2|_{HB^{0,\frac12}}\\ &\le \big|\nabla(u_1-u_2)\big|_{HB^{0,\frac12}}\cdot|u_1-u_2|_{HB^{0,\frac12}}\cdot|\nabla u_2|_{HB^{0,\frac12}}\\ &\le \frac{\nu}{4C}\big|\nabla(u_1-u_2)\big|^2_{HB^{0,\frac12}}+\frac{C}{\nu}|u_1-u_2|^2_{HB^{0,\frac12}}\cdot|\nabla u_2|^2_{HB^{0,\frac12}}. \end{aligned}$$

The two inequalities above along with relation (3.5) imply

$$\partial_t|u_1-u_2|^2_{HB^{0,\frac12}}\le\frac{C}{\nu}|u_1-u_2|^2_{HB^{0,\frac12}}\big(|u_1|^4_{HB^{\frac12,\frac12}}+|\nabla u_2|^2_{HB^{0,\frac12}}\big).$$

Uniqueness now follows from a simple application of Gronwall's lemma. ∎

4. Asymptotic study

In this section we work in

$$\mathbb{T}_\varepsilon=]0,2\pi[\times]0,2\pi[\times]0,2\pi\varepsilon[,\quad \varepsilon\le1$$

and we study the dependence on ε of the constant of Theorem 2.1. All the norms of the 2-dimensional functions are understood to be taken in $\mathbb{T}^2$. We shall prove that the constant from Theorem 2.1 can be chosen independent of ε. This follows from the simple remark that the classical product theorem for the Sobolev spaces is valid for the homogeneous Sobolev spaces, so the constant involved should be scale-invariant; it follows that in the periodic case the constant involved should not depend on the period, hence all the constants appearing in the proof of Theorem 2.1 should not depend on ε. However, the spaces should be "homogeneous" in the third variable, and that is why we have to assume that $Mw=0$. We now redefine in a "natural" way some of the quantities we are working with. From now on, all constants are assumed to be independent of ε. Let u be periodic on $\mathbb{T}_\varepsilon$ and u_n be such that

$$u(x)=\frac{1}{\sqrt{\varepsilon}}\sum_{n\in\mathbb{Z}^3}u_n\exp\Big(i\Big(n_1x_1+n_2x_2+\frac{n_3}{\varepsilon}x_3\Big)\Big).$$

Note that

$$\big\|\varepsilon^{-\frac12}\exp\big(i(n_1x_1+n_2x_2+n_3/\varepsilon x_3)\big)\big\|_{L^2}=(2\pi)^{3/2}.$$

We redefine

$$\|u|_{s,s'}=\big\|u_n(1+|n'|^2)^{s/2}(n_3/\varepsilon)^{s'}\big\|_{\ell^2},$$

$$Mu(x_1, x_2) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi\varepsilon} u(x)\,\mathrm{d}x_3,$$

$$\Delta_{q,q'} u(x) = \varepsilon^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}^3} u_n \exp\big(i(n_1x_1 + n_2x_2 + n_3/\varepsilon x_3)\big) \\ \varphi\Big(\frac{|n'|}{2^q}\Big)\varphi\Big(\frac{|n_3|}{\varepsilon 2^{q'}}\Big).$$

We need to redefine the $|\cdot|_{s,s'}$ norm because in the asymptotic study the proofs will be based on a dilatation in the third variable so we need a norm which is homogeneous. It is obvious that the two norms are equivalent if $Mu = 0$. Furthermore, the $\|\cdot|_{s,s'}$ norm is equivalent to the norm defined by dyadic decomposition:

$$\big\|2^{qs+q's'} \|\Delta_{q,q'} u\|_{L^2}\big\|_{\ell^2},$$

and the constants in this equivalence are independent of ε.

We are ready to prove the following theorem.

THEOREM 4.1. — *Consider the Navier-Stokes equations on the thin three dimensional torus $\mathbb{T}_\varepsilon$ and $0 < \delta \leq \frac{1}{2}$. There exists a positive constant $C = C(\delta)$ independent of ε such that if the initial data u_0 has vanishing mean over T_ε,*

$$\operatorname{div} u_0 = 0, \quad v_0 = Mu_0 \in L^2(\mathbb{T}^2), \quad w_0 = (I - M)u_0 \in H^{\delta, \frac{1}{2}-\delta}$$

and

$$\|w_0|_{\delta,\frac{1}{2}-\delta} \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) < C\nu,$$

then the (N-S) *equations have a unique global solution such that*

$$w = (I - M)u \in L^4\big(]0,\infty[; H^{(1+\delta)/2,(1-\delta)/2}\big) \cap L^\infty\big(]0,\infty[; H^{\delta,\frac{1}{2}-\delta}\big) \tag{4.1}$$

and

$$v = Mu \in L^2\big(]0,\infty[; H^1\big) \cap L^\infty\big(]0,\infty[; L^2\big)$$

Proof. — It suffices to prove that the constants from Lemma 1.3, from Propositions 1.1, 1.3, 1.2, from Theorems 1.1, 1.2 and from relation (2.15) can be chosen independent of ε if the 3D functions are assumed to have vanishing mean in the third direction and the $|\cdot|_{s,s'}$ norm is replaced with the $\|\cdot|_{s,s'}$ norm. We define

$$u_\varepsilon(x_1, x_2, x_3) = \sqrt{\varepsilon}\, u(x_1, x_2, \varepsilon x_3)$$

or, if u is not depending on x_3, $u_\varepsilon = u$. Next we compute the $\|\cdot|_{s,s'}$ norm of u in terms of the $\|\cdot|_{s,s'}$ norm of u_ε. We have

$$\begin{aligned}\|u|_{s,s'} &= \big\|u_n(1 + |n'|^2)^{s/2}(n_3/\varepsilon)^{s'}\big\|_{\ell^2} \\ &= \varepsilon^{-s'}\big\|u_n(1 + |n'|^2)^{s/2} n_3^{s'}\big\|_{\ell^2} = \varepsilon^{-s'}\|u_\varepsilon|_{s,s'}.\end{aligned}$$

We start with Theorem 1.1.

THEOREM 4.2. — *Let u and v two periodic functions on the thin three dimensional torus T_ε such that $u \in \dot{H}^{s,s'}$, $v \in \dot{H}^{t,t'}$, $s,t < 1$ and $s+t > 0$, $s',t' < \frac{1}{2}$ and $s'+t' > 0$. Then $uv \in H^{s+t-1,s'+t'-\frac{1}{2}}$ and*

$$\|uv|_{s+t-1,s'+t'-\frac{1}{2}} \leq C\|u|_{s,s'} \cdot \|v|_{t,t'},$$

with a constant C independent of ε.

Proof. — We have

$$\begin{aligned}\|uv|_{s+t-1,s'+t'-\frac{1}{2}} &= \varepsilon^{\frac{1}{2}-s'-t'}\|(uv)_\varepsilon|_{s+t-1,s'+t'-\frac{1}{2}} \\ &= \varepsilon^{-s'-t'}\|u_\varepsilon v_\varepsilon|_{s+t-1,s'+t'-\frac{1}{2}},\end{aligned}$$

$$\|u|_{s,s'} \cdot \|v|_{t,t'} = \varepsilon^{-s'}\varepsilon^{-t'}\|u_\varepsilon|_{s,s'} \cdot \|v_\varepsilon|_{t,t'}.$$

Applying now Theorem 1.1 for u_ε and v_ε gives the conclusion. □

We now state the variant of Theorem 1.2 on $\mathbb{T}_\varepsilon$.

THEOREM 4.3. — *Let $v \in H^s(\mathbb{T}^2)$ and w be a periodic function on the thin three dimensional torus T_ε such that $w \in \dot{H}^{t,t'}$, $s,t < 1$ and $s+t > 0$. Then*

$$vw \in \dot{H}^{s+t-1,t'}, \quad \|vw|_{s+t-1,t'} \leq C|v|_{H^s(\mathbb{T}^2)} \cdot \|w|_{t,t'},$$

where the constant C is independent of ε.

Proof. — The same proof as above holds, all we have to do is to remark that

$$(vw)_\varepsilon = v(w_\varepsilon). \quad \square$$

Next we consider the case of the Proposition 1.3.

PROPOSITION 4.1. — *There exists a constant C independent of ε such that for all $v \in H^s(\mathbb{T}^2)$ and every w periodic on the thin three dimensional torus T_ε such that $\operatorname{div} v = 0$, $\nabla w \in \dot{H}^{t,t'}$, $s < 2$, $t < 1$ and $s+t > 0$ there exists a sequence $(a_{q,q'})$ such that*

$$\begin{aligned}&\big|\langle \Delta_{q,q'}(v \cdot \nabla w) \mid \Delta_{q,q'} w\rangle\big| \\ &\qquad \leq C a_{q,q'} 2^{-q(s+t-1)-q't'} |v|_{H^s(\mathbb{T}^2)} \cdot \|\nabla w|_{t,t'} \cdot \|\Delta_{q,q'} w\|_{L^2},\end{aligned}$$

and $\|a_{q,q'}\|_{\ell^2} = 1$.

Proof. — We remark that, in fact, the whole argument takes place on $\mathbb{T}^2$, so ε should not affect the inequalities proved there. Let us prove it rigorously. First, we look at the terms given in (1.13) and (1.14). We saw above that in the product Theorem 4.3, the constant C does not depend on ε. Now we need to prove that in the inequalities from Lemma 1.5, the constant C does not depend on ε. This is proved by remarking that $(\Delta'_p w)_\varepsilon = \Delta'_p(w_\varepsilon)$ and $(S'_p w)_\varepsilon = S'_p(w_\varepsilon)$, hence, definitions (1.9) imply that

$$(T_v w)_\varepsilon = T_v(w_\varepsilon), \quad \big(R(v,w)\big)_\varepsilon = R(v, w_\varepsilon), \quad (\widetilde{T}_v w)_\varepsilon = \widetilde{T}_v(w_\varepsilon),$$

thus we can conclude as above. It remains to study the estimate on (1.15). The estimate (1.16) is independent of ε since v is independent of the third variable. Finally, the last place where ε might have an influence is inequality (1.18), more precisely, when we estimate $\|xh\|_{L^1}$. In fact, since f does not depend on the third variable, a closer look to the proof of inequality (1.18) shows that it suffices to estimate $\|x'h\|_{L^1}$, thus it suffices to estimate $\|h\|_{L^1}$. But $h* = \Delta_{q,q'}$, hence

$$h = \frac{1}{2\pi\varepsilon} \sum_{n\in\mathbb{Z}^3} \exp\Big(i\big(n_1x_1 + n_2x_2 + \frac{n_3}{\varepsilon}x_3\big)\Big)\varphi\Big(\frac{|n'|}{2^q}\Big)\varphi\Big(\frac{|n_3|}{\varepsilon 2^{q'}}\Big).$$

It follows that

$$\|h\|_{L^1} = \Big\| \sum_{n\in\mathbb{Z}^3} \exp\big(i(n_1x_1 + n_2x_2 + n_3x_3)\big)\varphi\Big(\frac{|n'|}{2^q}\Big)\varphi\Big(\frac{|n_3|}{\varepsilon 2^{q'}}\Big)\Big\|_{L^1},$$

and this is independent of ε as a consequence of the proof of Lemma 1.1. This completes the proof. □

Finally, it is clear that the proofs of Propositions 1.1 and 1.2 hold for homogeneous norms and with constants independent of ε.

It remains to look at the proof of relation (2.15). As in (2.13), we have

$$\partial_t \|v\|_{L^2}^2 + 2\nu |v|_1^2 \le \big|\langle M(w\otimes w) \mid \nabla v\rangle\big|.$$

Furthermore, the definition of the projection M implies

$$\big\langle M(w\otimes w) \mid \nabla v\big\rangle = \frac{1}{2\pi\varepsilon} \sum_{i,j=1}^{3} \int_{\mathbb{T}_\varepsilon} w_i w_j \partial_i v_j.$$

The product Theorem 4.3 now gives

$$\begin{aligned}\big|\langle M(w\otimes w) \mid \nabla v\rangle\big| &\le \frac{C}{\varepsilon} \sum_{i,j=1}^{3} \|w_i|_{\delta,\frac12-\delta} \cdot \|w_j\partial_i v_j|_{-\delta,\delta-\frac12} \\ &\le C/\varepsilon \|w|_{\delta,\frac12-\delta} \cdot \|w|_{1-\delta,\delta-\frac12} |v|_1.\end{aligned}$$

The definition of the norm $\|\cdot|_{s,s'}$, the hypothesis $\delta \leq \frac{1}{2}$ along with Proposition 1.2 for homogeneous norms yield

$$\|w|_{1-\delta,\delta-\frac{1}{2}} \leq \varepsilon \|w|_{1-\delta,\delta+\frac{1}{2}} \leq C\varepsilon \|\nabla w|_{\delta,\frac{1}{2}-\delta}.$$

The inequalities above imply

$$\partial_t \|v\|_{L^2}^2 + 2\nu |v|_1^2 \leq \big|\langle M(w\otimes w)|\nabla v\rangle\big| \leq C\|w|_{\delta,\frac{1}{2}-\delta}\cdot \|\nabla w|_{\delta,\frac{1}{2}-\delta}\cdot |v|_1.$$

One obtains inequality (2.15) as in relation (2.14). This completes the proof of Theorem 4.1. ∎

As an immediate corollary we find

COROLLARY 4.1. — *There exists a constant $C>0$ independent of ε such that if u_0 has vanishing mean over the three dimensional torus, $v_0 = Mu_0 \in L^2(\mathbb{T}^2)$, $w_0 = (I-M)u_0 \in H^1(\mathbb{T}_\varepsilon)$ and*

$$|w_0|_{H^1(\mathbb{T}_\varepsilon)} \exp\Big(\frac{\|v_0\|_{L^2(\mathbb{T}^2)}^2}{C\nu^2}\Big) < C\nu\varepsilon^{-\frac{1}{2}},$$

then the (N-S) *equations have a unique global solution with initial data u_0.*

Proof. — It suffices to remark that

$$\begin{aligned}\|w\|_{H^1(\mathbb{T}_\varepsilon)} &= \big\|w_n(|n'|^2 + n_3^2/\varepsilon^2)^{\frac{1}{2}}\big\|_{\ell^2}\\ &\geq \tfrac{1}{2}\varepsilon^{-\frac{1}{2}}\big\|w_n(1+|n'|^2)^{\delta/2}(n_3^2/\varepsilon^2)^{(\frac{1}{2}-\delta)/2}\big\|_{\ell^2}\\ &= \tfrac{1}{2}\varepsilon^{-\frac{1}{2}}\|w|_{\delta,\frac{1}{2}-\delta},\end{aligned}$$

and to use Theorem 4.1. ∎

The same method may be used to prove that the constant from Theorem 3.1 is independent of ε. The most delicate argument is the equivalence between

$$|u|_{H\dot{B}^{s,s'}} \quad\text{and}\quad \varepsilon^{-s'}|u_\varepsilon|_{H\dot{B}^{s,s'}},$$

which is the behavior of the Besov spaces with respect to dilatations. In the following we give the proof of this equivalence.

Since we want to reduce the problem to an equivalent one but on the torus not depending on ε, we need to define a dyadic decomposition which depends on ε but is for functions on $\mathbb{T}^3$. This new decomposition is given by

$$\Delta^{\varepsilon}_{q,q'}u(x) = \sum_{n\in\mathbb{Z}^3} u_n \exp\big(i(n_1x_1+n_2x_2+n_3x_3)\big)\chi\Big(\frac{|n'|}{2^q}\Big)\chi\Big(\frac{|n_3|}{\varepsilon 2^{q'}}\Big).$$

Then it is easy to see that

$$|u|_{H\dot{B}^{s,s'}} = \big\|2^{qs+q's'}\|\Delta^{\varepsilon}_{q,q'}u_\varepsilon\|_{L^2}\big\|_{\ell^{2,1}}.$$

We have

$$2^{qs+q's'}\|\Delta^{\varepsilon}_{q,q'}u_\varepsilon\|_{L^2} \le \sum_{q''} 2^{qs+q's'}\|\Delta^{\varepsilon}_{q,q'}\Delta''_{q''}u_\varepsilon\|_{L^2}.$$

But $\Delta^{\varepsilon}_{q,q'}\Delta''_{q''}u_\varepsilon \neq 0$ only when $1/(C\varepsilon)\le 2^{q'-q''}\le C/\varepsilon$, that is when

$$C_1+\frac{1}{\ln 2}\ln\frac{1}{\varepsilon}\le q'-q''\le C_2+\frac{1}{\ln 2}\ln\frac{1}{\varepsilon}\cdot$$

We deduce

$$2^{qs+q's'}\|\Delta^{\varepsilon}_{q,q'}u_\varepsilon\|_{L^2} \;\le\; C\varepsilon^{-s'} \sum_{\substack{C_1+\frac{1}{\ln 2}\ln\frac{1}{\varepsilon}\le q'-q''\\ q'-q''\le C_2+\frac{1}{\ln 2}\ln\frac{1}{\varepsilon}}} 2^{qs+q''s'}\|\Delta_{q,q''}u_\varepsilon\|_{L^2}.$$

Taking the $\ell^{2,1}$ norm and applying Young's inequality we find

$$|u|_{H\dot{B}^{s,s'}} \le C\varepsilon^{-s'}|u_\varepsilon|_{H\dot{B}^{s,s'}}.$$

The reverse inequality may be proved in the same way. This completes the proof.

We end up this section with the remark that all the results above are valid for the domain $\mathbb{R}^2\times\,]0,2\pi\varepsilon[$. The same proofs apply if, for

$$u(x) = \frac{1}{\sqrt{\varepsilon}}\sum_{n\in\mathbb{Z}} u_n(x_1,x_2)\exp\Big(\frac{in}{\varepsilon}x_3\Big),$$

we define

$$S_{q,q'}u(x) = \frac{1}{\sqrt{\varepsilon}}\sum_{n\in\mathbb{Z}} S_q u_n(x_1,x_2)\chi\Big(\frac{n}{2^{q'}}\Big)\exp\Big(\frac{in}{\varepsilon}x_3\Big).$$

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