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Note on pull-back and Lelong number of currents


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NOTE ON PULL-BACK AND
LELONG NUMBER OF CURRENTS

by CHARLES FAVRE (*)

ABSTRACT. — We prove a uniform estimate of the Lelong number of the pull-back of a plurisubharmonic function by a holomorphic map in term of the original Lelong number of this function.

RESUME. — NOTE SUR LE NOMBRE DE LELONG DES PULL-BACK DE COURANTS. Cet article est consacre a l’étude du nombre de Lelong \( \nu(f^*u, 0) \) du pull-back d’une fonction plurisousharmonique \( u \) par une application holomorphe \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0) \) generiquement de rang maximal. Nous prouvons l’estimee \( \nu(f^*u, 0) \leq C_f \times \nu(u, 0) \) avec une constante \( C_f \) uniforme en \( u \).

1. Statement of the main result

Fix \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0) \) a holomorphic germ, and \( T \) a positive closed current of bidegree \((1, 1)\) defined in a neighborhood of the origin in \((\mathbb{C}^n, 0)\). Let \( u \in \text{PSH}(\mathbb{C}^n, 0) \) be a plurisubharmonic (psh) potential for \( T \) such that \( T = \ddc u \). One can set

\[
f^*T := \ddc (u \circ f)
\]

as soon as the psh function \( u \circ f \) is not identically \(-\infty\).

DEFINITION 1 (Lelong number, see [LG86]). — Let \( u \in \text{PSH}(\mathbb{C}^n, 0) \). The function \( r \mapsto \sup_{|z|=r} u(z) \) is an increasing convex function of \( \log r \).
• We can hence define the Leelong number of \( u \) at 0 by setting
\[
\nu(u,0) := \max\{c \geq 0; \text{such that } u(z) < c \log |z| + O(1)\}
\]
which is a finite non-negative real number.

• For a positive closed \((1,1)\) current \( T \) in \((\mathbb{C}^n,0)\), the Leelong number of \( T \) at 0 is
\[
\nu(T,0) := \nu(u,0)
\]
for any psh potential \( T = \ddc u \).

For a given positive closed current \( T \) of bidegree \((1,1)\) so that \( f^*T \) exists, we are interested in estimating the Leelong number of the pull-back \( \nu(f^*T,0) \) in terms of \( \nu(T,0) \). Our theorem can be stated as follows.

**Theorem 2.** — Let \( f: (\mathbb{C}^m,0) \to (\mathbb{C}^n,0) \) be a holomorphic map. Then the following conditions are equivalent:

1) the map \( f \) has generic (maximal) rank equal to \( n \);
2) for any positive closed current \( T \) of bidegree \((1,1)\) \( f^*T \) is well defined, and the operator \( f^* \) is continuous for the weak topology of currents;
3) the range of \( f \) is not pluripolar;
4) for any positive closed current \( T \) of bidegree \((1,1)\) \( f^*T \) is well defined, and there exists a constant \( C > 0 \) (depending only on \( f \)) such that one has the inequality
\[
\nu(T,0) \leq \nu(f^*T,0) \leq C \cdot \nu(T,0)
\]
between Leelong numbers of \( T \) and \( f^*T \) at the origin.

**Remark 3.** — The proof gives an estimate for the constant \( C \) above. Assume \( n = m \) and 1) is satisfied. Then 4) holds with
\[
C = 1 + 2\mu(Jf,0) + n - 1,
\]
where \( \mu(Jf,0) \) is the order of vanishing of the Jacobian determinant of \( f \) at 0.

Using this remark, we also have a semi local version of Theorem 2.

**Corollary 4.** — Let \( X \) and \( Y \) be two connected complex manifolds, and \( f: X \to Y \) be a holomorphic map whose generic rank is maximal equal to \( \dim(Y) \). Then for any compact set \( K \subset X \), there exists a constant \( C_K > 0 \) such that for all positive closed current \( T \) of bidegree \((1,1)\) and all \( p \in K \), one has the inequality
\[
\nu(T,p) \leq \nu(f^*T,p) \leq C_K \cdot \nu(T,f(p))
\]
between Leelong numbers.
Before giving a proof of this theorem and of its corollary, we will make some remarks about the stated results.

The main result of Theorem 2 is contained in the implication 1) \( \Rightarrow \) 4). All the others are either obvious, or were known before.

The second assertion is contained in [M96]. We also refer the reader to this article for more general problems concerning pull-back of positive closed currents by holomorphic mappings.

The upper estimate given in 4) was already known in several different cases (the other inequality is easy to prove).

**Proposition 5** (see [De93]). — Let \( f \) be a finite holomorphic germ \((\mathbb{C}^n,0) \rightarrow (\mathbb{C}^n,0)\) of local degree \( d \) and \( T \) a positive closed current (of any bidegree). Then

\[
\nu(f^*T,0) \leq d \times \nu(T,0).
\]

C. Kiselman also proved 1) \( \Rightarrow \) 4) for monomial morphisms.

**Proposition 6** (see [K87]). — Let \( M = \{ a_{ij} \} \in M(n,\mathbb{N}) \) be an \( n \times n \) matrix with non-negative integer coefficients. We assume that \( \det M \neq 0 \). If

\[
f(z) = \left( \prod_{j=1}^{n} z_j^{a_{1j}}, \ldots, \prod_{j=1}^{n} z_j^{a_{nj}} \right)
\]

then for any positive closed \((1,1)\) current \( T \)

\[
\nu(f^*T,0) \leq \max_i \left\{ \sum_j a_{ij} \right\} \cdot \nu(T,0).
\]

Diller in [D98] also proved the main estimate 4) for birational mappings of \( \mathbb{P}^2 \).

A warning concerning the implication 1) \( \Rightarrow \) 3). When \( f \) does not have generic maximal rank, it is not true in general that the image of \( f \) is contained in a countable union of hypersurfaces. It is contained in a countable union of polydisks of dimension strictly less than \( n \).

**Example 7** (see [H73, 4.2]). — Define \( f: (\mathbb{C}^3,0) \rightarrow (\mathbb{C}^3,0) \) by

\[
f(z,w,t) = (z,ze^w,ze^{e^w}).
\]

Note that \( f \) is independent of the last variable \( t \). Then the set \( f(\mathbb{C}^3,0) \) is pluripolar, but it is not included in a countable union of hypersurfaces.
Proof. — We give a short proof of these facts. We begin proving that 
\(f(\mathbb{C}^3, 0)\) is pluripolar. Decompose the mapping \(f = \pi \circ g \circ p\) with

\[
p(z, w, t) = (z, w),
g(x, y) = (y, e^x, e^{e^x}),
\pi(z, w, t) = (z, zw, zt).
\]

The range of \(g\) is included in the hypersurface \(g(\mathbb{C}^2, 0) \subset \{e^w = t\}\), hence is pluripolar. The morphism \(\pi\) is an isomorphism outside \(\{z = 0\}\). As countable union of pluripolar sets remains pluripolar, we see that the image

\[
f(\mathbb{C}^3, 0) = \pi \circ g \circ p(\mathbb{C}^3, 0) = \pi(g(\mathbb{C}^2, 0))
= \{0\} \bigcup_{k \geq 0} \pi(g(\mathbb{C}^2, 0) \cap \{|z| > 1/k\})
\]

is also pluripolar.

For the second fact, we proceed as follows. Assume first that \(f(\mathbb{C}^3, 0)\) is included in an hypersurface defined by a non identically zero holomorphic map \(h\). We thus have the identity

\[
h(z, z e^w, z e^{e^w}) = 0
\]

for every \(z, w\) in a neighborhood of \(0 \in \mathbb{C}\). Expand \(h\) in power series

\[h = \sum_{k \geq 0} h_k\]

where \(h_k\) is a homogeneous polynomial of degree \(k\) in three variables. Take an index \(k_0 \in \mathbb{N}\) such that \(h_{k_0} \neq 0\). Then

\[h_{k_0}(z, z e^w, z e^{e^w}) = z^{k_0} h_{k_0}(1, e^w, e^{e^w}) \equiv 0.
\]

This would contradict the fact that the three functions \((1, e^w, e^{e^w})\) are algebraically independent.

Now assume \(f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} H_n\) is included in a countable union of hypersurfaces. For each \(n \in \mathbb{N}\), the complex space \(f^{-1} H_n\) is also an hypersurface by what proceeds. But we have

\[(\mathbb{C}^3, 0) \subset f^{-1} f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} f^{-1} H_n,
\]

which can not contain any open subset of \((\mathbb{C}^3, 0)\). \(\square\)
Finally, a word about the motivations of this article. The author came to the problem of estimating Lelong numbers of pull-back of positive closed \((1,1)\) current while working on dynamics of rational maps of the projective space \(f: \mathbb{P}^k \rightarrow \mathbb{P}^k\) with maximal generic rank. Let us give a simple application of Theorem 2 in this context. We first recall some well-known facts which can be found for instance in [Si99].

We let \(\pi: \mathbb{C}^{k+1} - \{0\} \rightarrow \mathbb{P}^k\) be the natural projection onto \(\mathbb{P}^k\), and take \(F = (F_0, \cdots, F_k): \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1}\) a polynomial lift of \(f\) so that
\[
F \circ \pi = \pi \circ f.
\]

We assume that the \(k+1\) polynomials \(\{F_i\}_{0 \leq i \leq k}\) do not contain any common factors. The indeterminacy set of \(f\) is equal to
\[
I(f) := \pi \left( \bigcap_{i=0}^{k} F_i^{-1}(\{0\}) \right).
\]

Given any positive closed current \(T\) of bidegree \((1,1)\) on \(\mathbb{P}^k\), one can find a psh function \(G\) on \(\mathbb{C}^{k+1}\), called its potential, such that

1) there exists a constant \(c > 0\) for which for all \(Z \in \mathbb{C}^{k+1}\) and for all \(\lambda \in \mathbb{C}\),
\[
G(\lambda Z) = c \log |\lambda| + G(Z);
\]

2) \(\pi^*T = dd^cG\).

Conversely, given a psh function \(G\) on \(\mathbb{C}^{k+1}\) satisfying the homogeneity condition 1), one can find a unique positive closed current \(T\) of bidegree \((1,1)\) on \(\mathbb{P}^k\) such that 2) holds.

**Definition 8.** — Let \(f: \mathbb{P}^k \rightarrow \mathbb{P}^k\) be a rational map of maximal generic rank \(k\), and \(T\) be a positive current of bidegree \((1,1)\) with potential \(G\). We define \(f^*T\) to be the positive closed current of bidegree \((1,1)\) whose potential is \(G \circ F\).

The study of the operator \(f^*\) turns out to give many interesting informations on \(f\) and on its dynamics (see [Si99]). When \(f\) is not holomorphic, for any positive closed current \(T\) of bidegree \((1,1)\), the current \(f^*T\) admits singularity points even if \(T\) has a smooth potential. The computation of Lelong numbers of \(f^*T\) can be viewed as a quantitative measure of how bad the singularities of this current are. The estimate 4) allows us to extend a result of [D98].
PROPOSITION 9. — Let \( f: \mathbb{P}^k \to \mathbb{P}^k \) be a rational map with maximal generic rank and \( T \) be a positive closed current of bidegree \((1,1)\). Then \( \nu(f^*T,p) > 0 \) if and only if either \( p \in I(f) \) or \( \nu(T,f(p)) > 0 \).

Proof. — Assume that \( p \notin I(f) \). As \( f \) has generic maximal rank, we can apply Theorem 2. This yields a constant \( C_f > 0 \) such that

\[
\nu(T,f(p)) \leq \nu(f^*T,p) \leq C_f \cdot \nu(T,f(p)).
\]

And it follows that \( \nu(f^*T,p) > 0 \) if and only if \( \nu(T,f(p)) > 0 \). It remains to check that if \( p \) belongs to \( I(f) \), then \( \nu(f^*T,p) > 0 \). Choose \( \sigma \) a local section of \( \pi \) around \( p \), and \( G \in \text{PSH}(\mathbb{C}^{k+1}) \) a potential for \( T \). One can find a constant \( A > 0 \) so that

\[
|F(\sigma(z))| \leq A|z-p|
\]

for points \( z \) near \( p \). As the function \( G \) satisfies an homogeneity relation, one can bound it by

\[
G(Z) \leq B \log |Z| + O(1),
\]

with \( B > 0 \). We thus have

\[
G(F(\sigma(z)) \leq B \log |z-p| + O(1) \quad \text{and} \quad \nu(f^*T,p) \geq B > 0,
\]

which concludes the proof. \( \square \)

Note. — The main theorem has been proved independently by C.Kiselman (see [K99]) with a different method. His proof relies on volume estimates of sublevel sets of psh functions.

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2. Proof of the main theorem

We shall first prove the equivalence between the first three assertions. We conclude by proving $4) \Rightarrow 3)$, and $1) \Rightarrow 4)$.

1) $\Rightarrow$ 2). — We assume that $f$ has generic maximal rank equal to $n$. If $u \in \text{PSH}(\mathbb{C}^n, 0)$ is non-degenerate, the psh function $u \circ f$ can not be identically $-\infty$ as the range of $f$ contains some open ball. Hence $f^*T$ is well-defined for any closed positive current $T$ of bidegree $(1,1)$. For a sequence of positive closed $(1,1)$ current $T_j \to T$ converging weakly towards $T$, one can find a sequence $u_j$ of psh potential of $T_j$ converging in $L^1_\text{loc}$ to $u$ a psh potential for $T$. It remains to check that $u_j \circ f \to u \circ f$ in $L^1_\text{loc}$.

As $f$ has maximal generic rank, $u_j \circ f \to u \circ f$ almost everywhere. Now one can extract a subsequence $u_{j_k} \circ f$ converging in $L^1_\text{loc}$ to a psh function (see [Ho83] p.94). As any such limit should be equal to $u \circ f$, we infer $u_j \circ f \to u \circ f$ in $L^1_\text{loc}$, thus $f^*T_j \to f^*T$ in the weak topology.

2) $\Rightarrow$ 3). — If the range $f(\mathbb{C}^n, 0)$ is pluripolar, one can find $u \in \text{PSH}(\mathbb{C}^n, 0)$ non-degenerate such that $u \circ f \equiv -\infty$. In that case, if $T := dd^c u$, $f^*T$ is not defined.

We also give an example of a sequence of positive closed currents of bidegree $(1,1)$ so that $T_n \to T$, $f^*T_n$ and $f^*T$ are all well-defined, but for which the sequence $f^*T_n$ fails to converge to $f^*T$. For this, work in the unit ball, and take $f(z, w) = (0, w)$, $T_n = dd^c u_n$, with

$$u_n(z, w) = \max\{n^{-1} \log |z|, -2 + |w|^2\}.$$  

Then $T_n \to 0$ but $f^*T_n = dd^c |w|^2$.

3) $\Rightarrow$ 1). — We only sketch the proof. We proceed by induction on $m$. Assume $f: (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic germ such that $\text{rk } Df_z$ the rank of $Df_z$ is smaller than $n - 1$ for any $z \in (\mathbb{C}^m, 0)$. Set

$$N := \max\{\text{rk } Df_z\} \leq n - 1,$$

and define for each $k \leq N$,

$$V_k := \{z \in (\mathbb{C}^m, 0); \text{rk } Df_z \leq k\}.$$

By assumption, $V_N$ contains an open neighborhood of the origin. Define

$$W := V_N - V_{N-1}.$$

The set $V_k$ is the set where all minors of $Df_z$ of size $k + 1$ have zero determinant, and hence defines a closed analytic subspace of $(\mathbb{C}^m, 0)$.
Hence $W$ is a Zariski open set of $V_N$. Now, on $W$ the rank of the differential of $f$ is constant equal to $N$. We can thus apply locally the constant rank theorem. Take any countable covering $\{U_i\}_{i \in I}$ of $W$ by open subsets such that for each $i \in I$, the set $f(U_i)$ is a (non-closed) analytic subset of $(\mathbb{C}^n, 0)$ of dimension $N < n$. For any $i \in I$ $f(U_i)$ is pluripolar. A countable union of pluripolar sets remains pluripolar, hence $f(W) = \bigcup_{i \in I} f(U_i)$ is pluripolar.

As $\dim(V_{N-1}) < m$, we can apply the induction hypothesis to conclude that

$$f(\mathbb{C}^m, 0) = f(W) \cup f(V_{N-1})$$

is pluripolar too.

The implication $4) \Rightarrow 3)$ follows from $2) \Rightarrow 3)$.

In fact, we even have that when the range of $f$ is pluripolar, the supremum of $(\nu(T, 0))^{-1} \nu(f^*T, 0)$ over all positive closed current $T$ of bidegree $(1,1)$ for which $f^*T$ is well-defined, is not finite.

Take $u \in \mathcal{PSH}(\mathbb{C}^n, 0)$ non-degenerate such that $u \circ f \equiv -\infty$. For any $\alpha > 0$, define

$$v_\alpha(z) := \max\{\alpha \log |z|, u(z) + \log |z|\}.$$  

Then

$$\nu(f^*v_\alpha, 0) = \alpha \cdot \nu(\log |f|, 0), \quad \nu(v_\alpha, 0) = \min\{\alpha, \nu(u, 0) + 1\}.$$  

Hence for $\alpha \geq \nu(u, 0) + 1$,

$$\left(\nu(dd^c v_\alpha, 0)\right)^{-1} \nu(f^*dd^c v_\alpha, 0) = C\alpha$$  

with $C = (\nu(u, 0) + 1)^{-1} \nu(\log |f|, 0)$.

1) $\Rightarrow$ 4). — Let us first prove the following general result.

**Lemma 10.** — If $f: (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is an arbitrary holomorphic germ, and $T$ is a positive closed current of bidegree $(1,1)$ so that $f^*T$ is well-defined, one has the inequality

$$\nu(f^*T, 0) \geq \nu(T, 0)$$

between Lelong numbers.

**Proof.** — We fix $u \in \mathcal{PSH}(\mathbb{C}^n, 0)$ a local potential for $T$. We always have $|f(Z)| \leq A|Z|$ for some constant, so that the estimate

$$u(Z) \leq \nu(T, 0) \log |Z| + O(1)$$

implies

$$u(f(Z)) \leq \nu(T, 0) \log |Z| + O(1)$$

which gives us the stated inequality.  \[\square\]
We now proceed with the proof of the upper bound for $\nu(f^*T, 0)$ given in 4). As before, $u$ will denote a local potential for $T$.

Let us show how to reduce the proof of this estimate to the equidimensional case \textit{i.e.} when $n = m$.

We assume the estimate has already been proved for $n = m$. By assumption, the rank of the Jacobian derivative of $f$ is generically equal to $n$. We can therefore find a closed embedding

$$i: L = (\mathbb{C}^n, 0) \hookrightarrow (\mathbb{C}^m, 0)$$

of a piece of $n$-plane into $(\mathbb{C}^m, 0)$ such that the rank of the Jacobian derivative of the restriction

$$\tilde{f} := f \circ i$$

to $(\mathbb{C}^n, 0)$ is also generically equal to $n$. We can now apply the estimate to $\tilde{f}$ and use Lemma 10. We get

$$\nu(f^*T, 0) \leq \nu(i^* \circ f^*T, 0) \leq \nu(\tilde{f}^*T, 0) \leq C_f \cdot \nu(T, 0).$$

Let us deal now with the equidimensional case. The assumption on $f$ can be rewritten as its Jacobian derivative does not vanish identically on a neighborhood of the origin.

Take a line $L$ passing through $0$ intersecting $\text{Crit}(f)$ the critical set of $f$ only at $0$, and not tangent to any irreducible component of $\text{Crit}(f)$. We can assume it is given in coordinates $z = (z_1, \cdots, z_n)$ by

$$L := \{z_2 = \cdots = z_n = 0\}.$$  

We can find an open cone around this line $L$

$$\mathcal{C} := \{z \in U; \ \text{dist}(z, L) < \varepsilon |z|\}$$

such that $\mathcal{C} \cap \text{Crit}(f) = \emptyset$.

Instead of working in this cone, it is more convenient to work on an open set. We thus consider the blow-up $\pi$ of the origin $0$, and replace the germ $f$ by the composition $g := f \circ \pi$. In coordinates,

$$\pi(z) = (z_1, z_1 z_2, \ldots, z_1 z_n).$$

We look at $g$ in the open set $\pi^{-1}(\mathcal{C})$. Define

$$E = \pi^{-1}\{0\} = \{z_1 = 0\}.$$  

Let us point out some special properties of the map $g$.

1) $\text{Crit}(g) = E$.
2) $g^{-1}\{0\} = E$.  

\textit{Bulletin de la Société Mathématique de France}
We can thus write the Jacobian determinant of $g$ under the form

$$Jg(z) = z_N^1 \psi(z)$$

for some integer $N \in \mathbb{N}$ and some holomorphic function $\psi$ which does not vanish at any point of $E$. In a sufficiently small neighborhood $V$ of the origin, we can find a constant $C > 0$ such that for all $z \in V$

(1) \quad |Jg(z)| \geq C|z|^N.

For the proof of Remark 3 and Corollary 4, we will need the following estimation on the integer $N$. It gives precisely a control on the constant $C$ of assertion 4) of the theorem.

**Lemma 11.** — *The integer $N$ introduced above can be chosen as*

$$N = \mu(Jf,0) + n - 1,$$

*where $\mu(Jf,0)$ is the order of vanishing of the holomorphic function $Jf$ at the point 0.*

*Proof.* — Set $N_0 := \mu(Jf,0)$. We first check that for a (generic) suitable choice of line $L$, one has in a small cone $C$ around $L$ as above

(2) \quad |Jf(z)| \geq C|z|^{N_0}.

Expand the holomorphic jacobian determinant $Jf$ in power series

$$Jf = \sum_{k \geq N_0} h_k$$

where $h_k$ is a homogeneous polynomial of degree $k$ and $h_{N_0}$ is not identically zero. Let $\mathbb{P}^{n-1}$ be the set of complex lines in $\mathbb{C}^n$ passing through the origin, and for a point $z \in \mathbb{C}^n$ set $L_z = Cz$. By homogeneity of $h_{N_0}$, one can define the continuous function $H: \mathbb{P}^{n-1} \to \mathbb{R}_+$ by

$$H(L_z) = |z|^{-N_0}|h_{N_0}(z)|.$$

Take a generic line $L$ such that $H(L) > 0$. Then for all lines $L'$ close to $L$, one has $H(L') \geq \frac{1}{2} H(L)$. Hence in a small cone $C$ around $L$, one has $H(L_z) \geq \frac{1}{2} H(L)$. 

*TOME 127 — 1999 — N° 3*
We infer for all \( z \in \mathbb{C} \),
\[
|f(z)| \geq |h_{N_0}(z) - \sum_{k \geq N_0 + 1} h_k(z)|
\geq |h_{N_0}(z)| - \left| \sum_{k \geq N_0 + 1} h_k(z) \right|
\geq 2^{-1} H(L)|z|^{N_0} - C'|z|^{N_0+1} \geq C|z|^{N_0},
\]
for some constants \( C, C' > 0 \).

Now a direct computation yields
\[
\det(D\pi_z) = z_1^{n-1}.
\]

Therefore, if we have chosen a line \( L \) so that equation (2) applies, we get
for all \( z \in \mathbb{C} \),
\[
|\det(Dg_z)| = |\det(D\pi_z) \cdot \det(DF_{\pi(z)})| \\
\geq |z_1|^{n-1} \cdot C|z_1|^{\mu(\mathcal{J}f, 0)},
\]
which concludes the proof of Lemma 11. \( \square \)

In the sequel, we will assume that \( V \) is a small ball in \( \mathbb{C}^n \) endowed with
the usual euclidean metric. If \( r > 0 \) and \( K \) is a compact set, we set
\[
B(K, r) := \{ z \mid \text{dist}(z, K) < r \}.
\]
The key lemma is:
LEMMA 12. — There exists two integers $N_0, N_1 \in \mathbb{N}^*$, and two positive constants $C_0, C_1 > 0$ such that for all $z \in V$,

$$g(B(z,C_0|z_1|^N)) \supset B(g(z), C_1|z_1|^N).$$

Moreover, we can choose $N_0 = N + 1$, and $N_1 = 2N + 1$ (with the above notations).

Proof. — The idea is to approximate the range of $g(B(z,r))$ by $Dg_z(B(z,r))$ and estimate the size of the latter.

We have $|Jg(z)| \geq C|z_1|^N$ for all $z \in V$. In $V$, all eigenvalues of $Dg_z$ are uniformly bounded by some constant $D > 0$. Therefore for all $z \in V - E$,

$$|Dg_z^{-1}|^{-1} \geq \inf\{|\lambda| ; \lambda \in \text{Spec}(Dg_z)| \geq \frac{C}{D^{n-1}}|z_1|^N.$$ 

And for all $z \in V$, for all $r > 0$,

$$Dg_z(B(z,r)) \supset B(g(z), C'|z_1|^N r),$$

for some constant $C' > 0$. Now by Taylor's formula, there exists another constant $C'' > 0$ such that for all $z, w \in V$,

$$|g(w) - g(z) - Dg_z \cdot (w - z)| \leq C''|w - z|^2.$$ 

If we choose $M > N$ and take $r = |z_1|^M$, we infer for $z$ sufficiently small

$$g(B(z, |z_1|^M)) \supset B(g(z), C'|z_1|^{N+M} - C''|z_1|^{2M}),$$

which gives the desired result with $N_1 = N + M$. \(\square\)

To conclude, we follow Diller [D98]. Define

$$\Delta_r := L \cap \{|z| \leq r\}.$$ 

We first apply Lemma 12 to each point of the set $\partial \Delta_r$. We obtain

(3) $$g(B(\partial \Delta_r, C_0 r^{N_0})) \supset B(\partial g(\Delta_r), C_1 r^{N_1}).$$

We consider now translated of $g(\Delta_r)$ by vectors $z$ of norm $|z| < C_1 r^{N_1}$. The estimate (3) tells us that $\partial(z + g(\Delta_r))$ is still included in the range of $g$. We have more precisely for all $|z| \leq C_1 r^{N_1}$,

1) $z \in z + g(\Delta_r),$

2) $\partial(z + g(\Delta_r)) \subset g(B(\partial \Delta_r, C_0 r^{N_0})).$
We are now in position to prove the desired inequality. We start with

\[ u(g(z)) \leq \nu(g^*u, 0) \log |z| + D \]

for some constant \( D \in \mathbb{R} \). We want to prove an analog estimate for \( u \). Fix \( z \in V \) and \( r > 0 \) such that \( |z| < C_1 r^{N_1} \). Then the maximum principle applied to \( u \) on the analytic disk \( z + g(\Delta_r) \) yields

\[
\begin{align*}
    u(z) &\leq \max_{z+g(\Delta_r)} u \leq \max_{\partial (z+g(\Delta_r))} u \\
   &\leq \max_{g(B(\partial \Delta_r, C_0 r^{N_0}))} u \\
   &\leq \max_{w \in B(\partial \Delta_r, C_0 r^{N_0})} u(g(w)) \\
   &\leq \max_{w \in B(\partial \Delta_r, C_0 r^{N_0})} \nu(g^*u, 0) \log |w| + D \\
   &\leq \nu(g^*u, 0) \log r + D' 
\end{align*}
\]

for \( D' := D + \nu(g^*u, 0) \log(\frac{3}{2}) \) (by possibly reducing \( C_0 \) we can assume that \( C_0 r^{N_0} - 1 < \frac{1}{2} \)). As this is true for any \( r \) satisfying \( |z| \leq C_1 r^{N_1} \), we obtain

\[ u(z) \leq \frac{1}{N_1} \nu(g^*u, 0) \log |z| + D''. \]

Thus \( \nu(u, 0) \geq N_1^{-1} \nu(g^*u, 0) \). To conclude the proof we use the general inequality in Lemma 10

\[ \nu(u, 0) \geq \frac{1}{N_1} \nu(g^*u, 0) \geq \frac{1}{N_1} \nu((f \circ \pi)^*u, 0) \geq \frac{1}{N_1} \nu(f^*u, 0). \]

The proof combined with Lemmas 11 and 12 gives more precisely (see Remark 3):

**Lemma 13.** — If \( f: (\mathbb{C}^n, z) \to (\mathbb{C}^n, f(z)) \) is a germ of holomorphic map of maximal generic rank, then for any positive closed current \( T \) of bidegree \( (1,1) \), one has the inequality

\[ \nu(f^*T, z) \leq (2(n - 1 + \mu(Jf, 0)) + 1) \cdot \nu(T, f(z)) \]

between Lelong numbers.

**Proof of Corollary 4.** — We localize first the problem and assume that \( X = B^m(0,1), Y = B^n(0,1) \) are unit balls respectively in \( \mathbb{C}^m \) and \( \mathbb{C}^n \). As before, it is sufficient to prove it in the equidimensional case *i.e.* \( X = Y = B^n(0,1) \).
As $f$ has maximal generic rank, we can apply Lemma 13 at each point $z \in K$. Now on the compact set $K$, the function $z \mapsto \mu(Jf, z)$ is upper semi continuous, hence bounded above by a constant $C_K$. This yields Corollary 4.

BIBLIOGRAPHY