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MINIMAL MODELS OF FOLIATED ALGEBRAIC SURFACES

BY MARCO BRUNELLA

ABSTRACT. — We study holomorphic foliations on algebraic surfaces from the birational point of view. We introduce a notion of *minimal model* and we classify those foliations which do not have such a minimal model in their birational class. An application to the dynamical study of polynomial diffeomorphisms is given.

RÉSUMÉ. — MODÈLES MINIMAUX DES SURFACES ALGÉBRIQUES FEUILLETÉES. — Nous étudions les feuilletages holomorphes sur les surfaces algébriques du point de vue de la géométrie birationnelle. Après avoir introduit une notion de *modèle minimal*, nous classifions les feuilletages qui n'ont pas de modèle minimal dans leur classe d'isomorphisme birationnel. Comme corollaire on obtient un résultat concernant la dynamique des difféomorphismes polynomiaux.

A classical and important result in the birational theory of algebraic surfaces (over $\mathbb{C}$) says that every algebraic surface X which is not (birationally) ruled has a *minimal model*, that is there exists a (necessarily unique) smooth algebraic surface X_0, birational to X, with the following property: if Y is any smooth algebraic surface and if $f:Y\cdots\to X_0$ is a birational map then f is in fact a morphism, that is there are no indeterminacy points. This means, in particular, that the birational classification of nonruled surfaces is reduced to the biregular classification of their minimal models. See [BPV], or [MP] for a wider perspective (but take care that "minimal model" in [BPV] has a different meaning, allowing nonuniqueness and classically denoted as "relative minimal model"; an algebraic surface has a minimal model in the classical sense if and only if it has a unique minimal model in the sense of [BPV]).

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Our aim is to give a "foliated" version of this result. We shall work in the class of holomorphic foliations (with singularities, which can be supposed isolated without loss of generality) on smooth algebraic surfaces. Birational maps act in a natural way on these objects: if $\mathcal{F}$ is a holomorphic foliation on X and if $f: Y \cdots \rightarrow X$ is a birational map, realizing an isomorphism between Zariski open sets Y_0 and X_0, then $f^*(\mathcal{F})$ is the unique foliation on Y coinciding with $f^*(\mathcal{F}_{|X_0})$ on Y_0. Hence one can study birational equivalence classes of foliations and look for special representatives of these classes; these will be our foliated minimal models.

In order to give a precise definition, we have to translate in our context the smoothness condition which appears in the absolute (that is, non foliated) theory. The singularities of an algebraic surface can be removed by a suitable sequence of blow-ups (Zariski, Hironaka), producing a smooth algebraic surface [BPV]. The same is *not* true for the singularities of a foliation on a smooth algebraic surface: the best that we can do by a sequence of blow-ups is to obtain a foliation with *reduced* singularities (Bendixson, Seidenberg), that is isolated singularities generated (locally) by a vector field whose linear part has eigenvalues 1 and λ with $\lambda \notin \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the set of strictly positive rational numbers. If $\lambda = 0$ the singularity is called a *saddle-node*, otherwise *nondegenerate*. Such a foliation will be called *reduced*. Remark that the blow-up of a reduced foliation at any point (singular or not) is still reduced. More precisely, the blow-up at a reduced singular point produces a foliation tangent to the exceptional divisor and having there two reduced singular points. See [CS] (and references therein) for these basic results, and [M] for the relevance of reduced foliations in the birational theory of foliations.

DEFINITION. — *Let $\mathcal{F}$ be a foliation on a smooth algebraic surface X. A minimal model of $(X,\mathcal{F})$ is a reduced foliation $\mathcal{F}_0$ on a smooth algebraic surface X_0 such that:*

(i) *$(X_0,\mathcal{F}_0)$ is birational to $(X,\mathcal{F})$;*

(ii) *if $\mathcal{G}$ is any reduced foliation on a smooth algebraic surface Y and if $f:(Y,\mathcal{G}) \cdots \rightarrow (X_0,\mathcal{F}_0)$ is a birational map then f is in fact a morphism.*

We observe that, as in the absolute case, if a minimal model exists then it is unique: if f is a birational map and f, f^{-1} are morphisms then f is a biregular map. We also observe that if $(X,\mathcal{F})$ is reduced and has a minimal model $(X_0,\mathcal{F}_0)$ then the birational map $(X,\mathcal{F}) \cdots \rightarrow (X_0,\mathcal{F}_0)$ is a morphism (by ii)). In other words, the minimal model (when it exists) can be found by firstly reducing the singularities of the foliation and secondly contracting a curve.

Let us note that "most" foliations have a minimal model. For example, as the proof of the theorem below will clarify, a reduced foliation with no invariant rational curves is a minimal model (in the same way as a smooth algebraic surface with no rational curves is a minimal model). Also, a foliation defined on a surface which is not birationally ruled has a minimal model. On the other hand there are, of course, foliations without minimal model, for instance linear foliations in $\mathbb{C}P^2$. To fix ideas, suppose that such a linear foliation $\mathcal{F}$ is reduced. Then if a minimal model exists it must be obtained by contracting a curve in $\mathbb{C}P^2$; but $\mathbb{C}P^2$ does not contain contractible curves and therefore the minimal model must coincide with $(\mathbb{C}P^2, \mathcal{F})$. Take now two singular points p, q of $\mathcal{F}$ and let L be the line of $\mathbb{C}P^2$ through p and q. If we blow-up p and q and then collapse the strict transform of L (which is an exceptional curve, *i.e.* [BPV] a smooth rational curve with selfintersection -1) we obtain a foliation $\mathcal{G}$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and a birational map $f: \mathbb{C}P^1 \times \mathbb{C}P^1 \cdots\!\to \mathbb{C}P^2$ sending $\mathcal{G}$ to $\mathcal{F}$. The foliation $\mathcal{G}$ is still reduced and f is not a morphism, hence $(\mathbb{C}P^2, \mathcal{F})$ cannot be a minimal model. An easy variation of this argument shows that every linear foliation, reduced or not, has no minimal model. Our intention is to give a classification of these exceptional cases.

We shall say that a foliation on a smooth algebraic surface is *ruled* if it coincides with a rational fibration, that is a fibration whose generic fibres are rational curves (there can be singular fibres, trees of rational curves). It is a *Riccati* foliation if it is transverse to the generic fibre of a rational fibration. Moreover, a Riccati foliation is *nontrivial* if there is a regular fibre which is invariant by the foliation and contains exactly two singularities, both reduced. It is quite clear that ruled and nontrivial Riccati foliations do not have minimal models, by a construction similar to that previously used to show that linear foliations do not have minimal models; but there is also a third, very special, exception. Take the foliation $\mathcal{H}_0$ in $\mathbb{C}P^2$ generated, in an affine chart, by the vector field

$$z\frac{\partial}{\partial z} + \frac{1}{2}\big(1 + i\sqrt{3}\big)w\frac{\partial}{\partial w}\cdot$$

There are three singular points, the origin P_0 and two points "at infinity" P_1, P_2. Take a projective transformation T of $\mathbb{C}P^2$ which cyclically permutes P_0, P_1, P_2. One necessarily has $T^3 = \mathrm{id}$, and moreover T preserves $\mathcal{H}_0$. The quotient $\mathbb{C}P^2/T$ has some (mild) singularities, which can be resolved giving a smooth rational surface H_T, equipped with a foliation $\mathcal{H}_T$ arising from the projection of $\mathcal{H}_0$ on $\mathbb{C}P^2/T$. In fact the choice of T is inessential (two such Ts are conjugate by a projective transformation preserving $\mathcal{H}_0$), hence the final result will be simply noted $(H, \mathcal{H})$. Observe that H contains an $\mathcal{H}$-invariant rational curve with a node L

(arising from the three $\mathcal{H}_0$-invariant lines). The node is a reduced and nondegenerate singularity of $\mathcal{H}$, and it is the only singularity of $\mathcal{H}$ on L. If we blow-up the node, the strict transform of L is an exceptional curve and its contraction gives a still reduced singularity. This is the reason for the absence of a minimal model of $(H,\mathcal{H})$.

THEOREM. — *Let $\mathcal{F}$ be a foliation on a smooth algebraic surface X. Suppose that $\mathcal{F}$ is not birational to a ruled foliation or a nontrivial Riccati foliation or $(H,\mathcal{H})$. Then $(X,\mathcal{F})$ has a minimal model.*

The proof is parallel to that of the analogous absolute result. Firstly, a sequence of blow-ups reduces the singularities of $\mathcal{F}$. The foliated surface so obtained may contain some exceptional curves, whose contraction may still give a foliation with reduced singularities. To obtain the minimal model $(X_0,\mathcal{F}_0)$ we have to contract all these exceptional curves, but some ambiguity appears if these curves are not pairwise disjoint. Hence the main step of the proof will be to show that if such an ambiguity appears then $(X,\mathcal{F})$ is in the forbidden list. Then it will be easy to prove that when the ambiguity does not appear the contraction process leads to the desired minimal model. The arguments are mostly taken from classical birational geometry.

Minimal models of algebraic surfaces are related to "positivity" properties of the canonical line bundle [BPV], [MP]: a smooth algebraic surface X has a minimal model if and only if K_X has nonnegative Kodaira dimension, and the minimal model X_0 is then characterized by the nefness of K_{X_0}. One may ask for similar relations in the foliated case, where the rôle of K_X is played by $T^*_{\mathcal{F}}$, the cotangent line bundle of a reduced foliation [B], [M]; but the example of linear reduced foliations, for which $T^*_{\mathcal{F}}$ is trivial, shows that one has perhaps to work with stronger positivity conditions. The difference between the absolute and the foliated case is, essentially, the following one: in the absolute case the obstruction to minimality is represented by curves over which K_X has negative degree, whereas in the foliated case the same obstruction is also represented by (some) curves over which $T^*_{\mathcal{F}}$ has zero degree. On the other side, it is possible that a minimal foliated surface contains a curve over which $T^*_{\mathcal{F}}$ has negative degree (*e.g.* a smooth rational curve of self-intersection -2, invariant by $\mathcal{F}$ and containing only one reduced nondegenerate singularity); in order to obtain a coherent theory one has probably to contract these curves and to work with singular surfaces, as in the absolute higher dimensional case [MP]. A remarkable theorem of Miyaoka [MP], [SB], stating the pseudoeffectivity of $T^*_{\mathcal{F}}$ for nonruled $\mathcal{F}$, should be of valuable help in this type of questions.

As a corollary to our theorem (and [BS]) we will prove in the last section the following result, answering a question of D. Cerveau [C]. Recall that a polynomial diffeomorphism of $\mathbb{C}^2$ is said to be *nonelementary* if it is *not* conjugate to a diffeomorphism of the type

$$(z,w) \longmapsto (az+P(w), bw+c), \quad a,b \in \mathbb{C}^*,\ c \in \mathbb{C},\ P \in \mathbb{C}[w].$$

Nonelementary diffeomorphisms are precisely those polynomial diffeomorphisms exhibiting a complicated dynamics [BS].

COROLLARY. — *A nonelementary polynomial diffeomorphism of $\mathbb{C}^2$ cannot preserve a holomorphic foliation generated by a polynomial vector field.*

In this paper we have choosen to work in the algebraic setting. We note however that our theorem applies also to foliations on compact complex analytic surfaces: a surface over which there exists a foliation without minimal model is necessarily bimeromorphically ruled, and therefore algebraic.

1. Characterization of the exceptions

Let X be a smooth algebraic surface and let $\mathcal{F}$ be a foliation on X. It is defined by an open cover $\{U_j\}$ of X and holomorphic vector fields with isolated singularities v_j on U_j, satisfying $v_i = g_{ij}v_j$ on $U_i \cap U_j$, with $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. The cocycle $\{g_{ij}^{-1}\}$ defines a line bundle $T_{\mathcal{F}}$ on X, called *tangent bundle of $\mathcal{F}$*. We shall need the following two formulae [B] concerning the computation of the degree of $T_{\mathcal{F}}$ over an algebraic irreducible curve C. If C is invariant by $\mathcal{F}$ then

$$c_1(T_{\mathcal{F}}) \cdot C = \chi(C) - Z(\mathcal{F}, C)$$

where $\chi(C)$ is the virtual Euler characteristic of C and $Z(\mathcal{F}, C)$ is the sum of the multiplicities of the singularities of $\mathcal{F}$ along C [B, lemme 3]. If C is not invariant by $\mathcal{F}$ then

$$c_1(T_{\mathcal{F}}) \cdot C = C^2 - \operatorname{tang}(\mathcal{F}, C)$$

where C^2 is the selfintersection of C and $\operatorname{tang}(\mathcal{F}, C)$ is the sum of the multiplicities of the tangency points between $\mathcal{F}$ and C [B, lemme 2].

Suppose now that $C \subset X$ is a smooth rational curve with vanishing selfintersection: $C^2 = 0$. It is well known [BPV, p. 142] that C is a regular fibre of a (unique) rational fibration $\pi : X \to \Sigma$, where Σ is an algebraic curve; of course, π may have singular fibres, which are trees of rational curves.

LEMMA 1. — *Suppose that C is a smooth $\mathcal{F}$-invariant rational curve with $C^2 = 0$, then:*

(i) *if $Z(\mathcal{F},C) = 0$ then $\mathcal{F}$ coincides with the rational fibration π;*

(ii) *if $Z(\mathcal{F},C) = 2$ then $\mathcal{F}$ is a Riccati foliation with respect to π.*

Proof. — If C is $\mathcal{F}$-invariant, the previous formula gives

$$c_1(T_{\mathcal{F}}) \cdot C = 2 - Z(\mathcal{F}, C).$$

Let $C' \subset X$ be another regular fibre of π, and suppose that it is not $\mathcal{F}$-invariant. Then

$$c_1(T_{\mathcal{F}}) \cdot C' = -\operatorname{tang}(\mathcal{F}, C').$$

But we have

$$c_1(T_{\mathcal{F}}) \cdot C = c_1(T_{\mathcal{F}}) \cdot C',$$

because C and C' are homologous, and so

$$\operatorname{tang}(\mathcal{F}, C') = Z(\mathcal{F}, C) - 2.$$

It follows that:

(i) if $Z(\mathcal{F}, C) = 0$ then every (regular) fibre of π is $\mathcal{F}$-invariant, because $\operatorname{tang}(\mathcal{F}, C')$ cannot be negative, and so $\mathcal{F}$ coincides with the rational fibration;

(ii) if $Z(\mathcal{F}, C) = 2$ then every regular fibre which is not $\mathcal{F}$-invariant is transverse to $\mathcal{F}$, which consequently is a Riccati foliation. $\square$

The next lemma provides a characterization of $(H, \mathcal{H})$ along the same lines.

LEMMA 2. — *Suppose that there exist two $\mathcal{F}$-invariant exceptional curves $C_1, C_2 \subset X$ intersecting transversely at two points p,q. Suppose that p,q are reduced nondegenerate singularities of $\mathcal{F}$, and that they are the only singularities of $\mathcal{F}$ on $C_1 \cup C_2$. Then $(X, \mathcal{F})$ is birational to $(H, \mathcal{H})$.*

Proof. — We shall use some basic facts from the deformation theory of (rational) curves on algebraic manifolds, which can be found, for instance, in [MP, lect. I]. The main fact is the following: the space of irreducible rational curves of a fixed degree (with respect to a fixed ample divisor) on a projective variety is a quasi-projective variety, which can be compactified to a projective variety by adding some chains of rational curves.

Let us analyze the structure of this projective variety near the chain $C_1 \cup C_2$. If we "forget" the intersection point p, we may see C_1 and C_2 as two exceptional curves intersecting only at q.

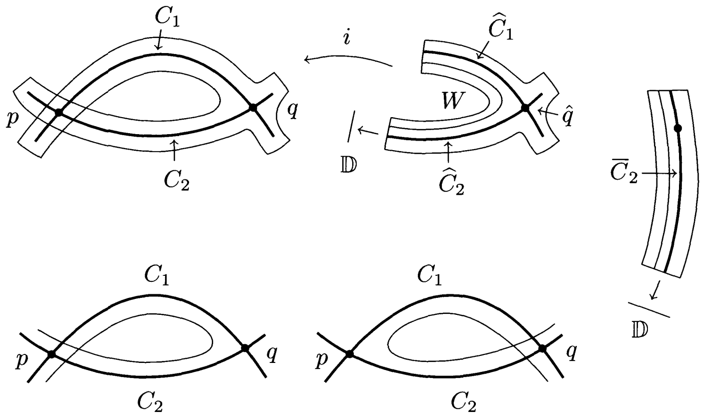

Figure 1

More precisely, see Fig. 1, glueing two tubular neighbourhoods of C_1 and C_2 along their intersection around q (but not along their intersection around p) we obtain an open complex surface W with a natural holomorphic immersion $W \xrightarrow{i} X$, and W contains two exceptional curves $\widehat{C}_1$, $\widehat{C}_2$ with

$$i(\widehat{C}_j) = C_j \ (j = 1,2), \quad \widehat{C}_1 \cap \widehat{C}_2 = \{\hat{q}\}, \quad i(\hat{q}) = q.$$

After contracting $\widehat{C}_1$, the curve $\widehat{C}_2$ becomes a smooth rational curve $\overline{C}_2$ with zero selfintersection, and on a neighbourhood of such a curve (which is trivial [U]) we may find a submersion onto the disc $\mathbb{D}$ whose fibres are smooth rational curves parallel to $\overline{C}_2$. Therefore we may find on a neighbourhood of $\widehat{C}_1 \cup \widehat{C}_2$ in W a holomorphic map onto the disc whose fibres are $\widehat{C}_1 \cup \widehat{C}_2$ and smooth rational curves parallel to $\widehat{C}_1 \cup \widehat{C}_2$. This one-parameter family of rational curves exhausts all the rational curves on a neighbourhood of $\widehat{C}_1 \cup \widehat{C}_2$, by the maximum principle. The projection on X by i gives a one-parameter family of rational curves which are close to $C_1 \cup C_2$ and which have a node near p. In a similar way we see that $C_1 \cup C_2$ deforms to a one-parameter family of rational curves with a node near q, see Fig. 1. These two one-parameter families of rational curves contain all the rational curves close to $C_1 \cup C_2$: if in the deformation of $C_1 \cup C_2$ we smooth the *two* nodes p and q we obtain smooth elliptic curves, not rational ones.

This shows that the compactified space of rational curves is one-dimensional near $C_1 \cup C_2$, and $C_1 \cup C_2$ is a nodal point of that space belonging to two local components: one which parametrizes rational curves with a node near p and the other which parametrizes rational curves with a node near q. Of course, these two local components of the compactified space of rational curves may belong to the same global component, and in fact we will see that they actually do.

Let Σ be the global component containing the rational curves with a node near q and let $\widehat{\Sigma}$ be its normalization. If $t \in \widehat{\Sigma}$ is a point corresponding to a curve D_t, we may decompose

$$D_t = m_1 C_1 \cup m_2 C_2 \cup \widehat{D}_t$$

where m_1, m_2 are nonnegative integer numbers and $\widehat{D}_t$ does not contain C_1 nor C_2. Because D_t is homologous to $C_1 \cup C_2$, we have

$$D_t \cdot C_1 = C_1^2 + C_2 \cdot C_1 = 1 = D_t \cdot C_2$$

and therefore

$$\widehat{D}_t \cdot C_1 = 1 + m_1 - 2m_2, \quad \widehat{D}_t \cdot C_2 = 1 + m_2 - 2m_1.$$

But $\widehat{D}_t \cdot C_j \geq 0$, hence either $m_1 = m_2 = 0$ or $m_1 = m_2 = 1$. In the latter case $\widehat{D}_t$ is homologous to zero and therefore empty, that is $D_t = C_1 \cup C_2$. In the former case D_t does not contain C_1 nor C_2 and therefore the equality $D_t \cdot C_j = 1$ implies that D_t cuts C_j in exactly one point, transversely. Hence we have two maps $f_i : \widehat{\Sigma} \to C_i$, $i = 1, 2$, defined by $f_i(t) = D_t \cap C_i$ if $D_t \neq C_1 \cup C_2$ and extended to all of $\widehat{\Sigma}$ by continuity.

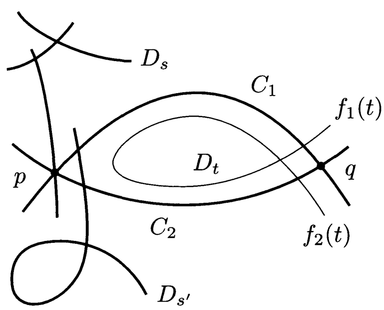

Figure 2

By compactness of $\widehat{\Sigma}$, there exists $s \in \widehat{\Sigma}$ such that $f_1(s) = p$. We claim that $D_s = C_1 \cup C_2$. If $D_s \neq C_1 \cup C_2$ then D_s cuts transversely C_1 and C_2 at p and we can find $s' \in \widehat{\Sigma}$ near s such that $D_{s'}$ is irreducible and cuts C_1 and C_2 in two points near p, see Fig. 2.

This curve $D_{s'}$ is not invariant by $\mathcal{F}$, and $D_{s'}^2 = (C_1 + C_2)^2 = 2$, hence

$$c_1(T_{\mathcal{F}}) \cdot D_{s'} = 2 - tang(\mathcal{F}, D_{s'}).$$

On the other hand,

$$c_1(T_{\mathcal{F}}) \cdot C_i = 2 - Z(\mathcal{F}, C_i) = 0,$$

and from the cohomological equality $D_{s'} = C_1 + C_2$ we obtain

$$\operatorname{tang}(\mathcal{F}, D_{s'}) = 2.$$

The nodal point of $D_{s'}$ is a point of non transversality with $\mathcal{F}$, and one verifies immediately, by applying the definition of tang given in [B], that its contribution to $\operatorname{tang}(\mathcal{F}, D_{s'})$ is at least 2. Moreover, if s' is sufficiently close to s then there will be near p an additional tangency point between $D_{s'}$ and $\mathcal{F}$. Hence

$$\operatorname{tang}(\mathcal{F}, D_{s'}) \geq 3,$$

a contradiction.

This proves that if $f_1(s) = p$ (or $f_2(s) = p$) then necessarily

$$D_s = C_1 \cup C_2;$$

in particular $\sharp f_i^{-1}(p) = 1$ and $f_i : \widehat{\Sigma} \to C_i$ are diffeomorphisms.

In other words, if we take a rational curve with a node near q and we "move" it in such a way that its intersection point with C_1 (or C_2) approaches p, then we finally obtain the rational curves with a node near p. As a consequence of this, the curve Σ is in fact a rational curve with a node, corresponding to $C_1 \cup C_2$, the two local branches of which are the parameter spaces of rational curves with a node near q or p.

Glueing together these (chains of) rational curves, we finally obtain an algebraic surface Y, ruled over $\widehat{\Sigma} \simeq \mathbb{C}P^1$ (notation $\pi : Y \to \widehat{\Sigma}$), and a rational morphism

$$f : Y \to X$$

with the following properties (see Fig. 3):

(i) $\pi^{-1}(0)$ (resp. $\pi^{-1}(1)$) is a nonmultiple fibre composed by two exceptional curves R_1, R_2 (resp. S_1, S_2) intersecting at a point r (resp. s);

(ii) f is an immersion near $\pi^{-1}(0)$ and $\pi^{-1}(1)$, and it maps R_1, S_1 to C_1; R_2, S_2 to C_2; r to p and s to q.

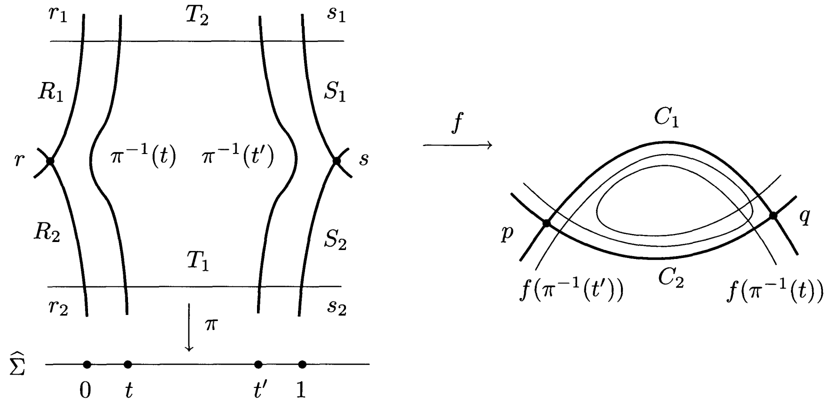

Figure 3

Let us consider $f^{-1}(C_1)$: it contains R_1, S_1, and a curve T_1 which intersects R_2 at a point r_2 mapped by f to q. This curve cuts a generic fibre at a single point, hence it is a section of π and cuts every fibre at a single point. In particular it cuts S_2 at a point s_2 mapped by f to p. Similarly, $f^{-1}(C_2)$ contains a section T_2 cutting R_1 at a point r_1 mapped to q and S_1 at a point s_1 mapped to p. We may assume that f is an immersion near T_1 and T_2. Observe that $f_{|T_i} = f_i$, modulo the identification of the section T_i with the base $\widehat{\Sigma}$.

The restriction of f to a suitable tubular neighbourhood U of the cycle

$$\bigcup_{i=1,2} R_i \cup S_i \cup T_i = Q$$

is a regular covering, of order 3, of a neighbourhood V of the cycle $C_1 \cup C_2$. The covering action of $\mathbb{Z}_3$ on U extends (birationally) to all of Y: the open surface $Y \setminus Q$ is pseudoconvex, because $Q^2 > 0$, and therefore every biholomorphism defined outside a compact set extends to a bimeromorphism of the full $Y \setminus Q$ (Levi's theorem). In order to complete the proof we need to show that the lifted foliation $\mathcal{G} = f^*\mathcal{F}$ is birational to $\mathcal{H}_0$ and that the $\mathbb{Z}_3$-action is birational to that generated by the projective transformation $T \in \mathrm{Aut}(\mathbb{C}P^2)$.

Remark that $\mathcal{G}$ is tangent to R_i, S_i, T_i and its only singularities on these curves are r, r_1, r_2, s, s_1, s_2, all reduced and nondegenerate. By contracting R_2, S_2, T_2, which are permuted by the $\mathbb{Z}_3$-action, the curves R_1, S_1, T_1 become a cycle of 3 rational curves R', S', T' of selfintersection $+1$, permuted by the $\mathbb{Z}_3$-action. The foliation $\mathcal{G}$ becomes a foliation $\mathcal{G}'$ tangent to R', S', T' and having reduced nondegenerate singularities at the 3 crossing points. It is well known that such a configuration of rational curves is locally isomorphic to a configuration of 3 lines in $\mathbb{C}P^2$: that is, there exists a birational map to $\mathbb{C}P^2$, biregular near R', S', T', mapping R', S', T' to 3 lines. This birational map is constructed as follows. Firstly, there exists a rational function g such that

$$g^{-1}(0) = R', \quad g^{-1}(\infty) = T'$$

and $g_{|S'}$ has degree one (remark that if we blow-up $R' \cap T'$ the strict transforms of R' and T' will be disjoint rational curves with zero selfintersection, therefore they will be fibres of a rational fibration and S' will be a section of such a fibration). Similarly, there exists a rational function h such that

$$h^{-1}(0) = S', \quad h^{-1}(\infty) = T'$$

and $h_{|R'}$ has degree one. The quotient $k = h/g$ satisfies

$$k^{-1}(0) = S', \quad k^{-1}(\infty) = R',$$

$k_{|T'}$ has degree one. Then (g, h), as a map to $\mathbb{C}P^2$, is the required birational map, mapping S' to the x-axis, R' to the y-axis and T' to the line at infinity.

Under this map $\mathcal{G}'$ becomes a linear foliation (because $\mathcal{G}'$ has on the line at infinity only two singularities, both reduced and nondegenerate) and the $\mathbb{Z}_3$-action becomes a linear one. Finally, one easily verifies that $\mathcal{H}_0$ is the only linear foliation invariant by a $\mathbb{Z}_3$-action which permutes its 3 singularities. $\square$

2. Proof of the theorem

Let $\mathcal{F}$ be a foliation on a smooth algebraic surface X. By Seidenberg's theorem we may assume, up to birational morphisms, that the singularities of $\mathcal{F}$ are reduced. We shall say that an exceptional curve $C \subset X$ is *$\mathcal{F}$-exceptional* if its contraction produces a foliation which still has only reduced singularities. By looking at blow-ups of reduced singularities, we see that this is equivalent to the following two conditions (see Fig. 4):

1) C is $\mathcal{F}$-invariant

2) either C contains only one singularity of $\mathcal{F}$, of the type $z\partial/\partial z - w\partial/\partial w$ (so that the contraction of C gives a nonsingular point), or C contains two nondegenerate reduced singularities of $\mathcal{F}$ (so that the contraction of C gives a nondegenerate reduced singularity), or C contains one nondegenerate reduced singularity and one saddle-node whose strong separatrix [CS] is in C (so that the contraction of C gives a saddle-node).

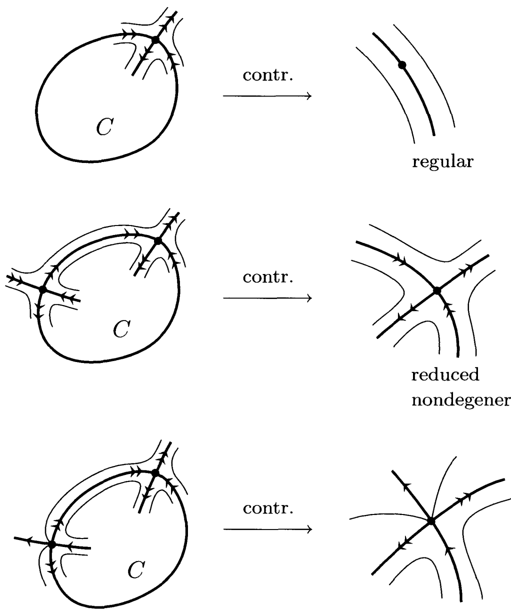

Figure 4

LEMMA 3. — *If X contains two $\mathcal{F}$-exceptional curves C_1,C_2 whose intersection is nonempty, then $\mathcal{F}$ is birational to a ruled foliation or a nontrivial Riccati foliation or $(H,\mathcal{H})$.*

Proof. — Intersection points between C_1 and C_2 are singularities of $\mathcal{F}$, for which C_1 and C_2 are local separatrices [CS]. A reduced singularity has either a single separatrix or a pair of two transverse separatrices [CS], so these intersection points are transverse. Because each $\mathcal{F}$-exceptional curve contains at most two singularities we have only two possibilities:

(i) $C_1 \cap C_2 = \{p\}$. The singular point p has two transverse separatrices, one in C_1 and the other in C_2, and hence it is nondegenerate (because an $\mathcal{F}$-exceptional curve never contains the weak separatrix [CS] of a saddle-node). If C_1 does not contain other singularities of $\mathcal{F}$ then p is of the type $z\partial/\partial z - w\partial/\partial w$, and so also C_2 does not contain other singularities of $\mathcal{F}$ (the singularities that appear by blowing-up a reduced singular point are never of the type $z\partial/\partial z - w\partial/\partial w$). If C_1 contains a saddle-node then C_2 does the same (because the nondegenerate singularity that appears by blowing-up a saddle-node is never conjugate to one of the two singularities that appear by blowing-up a nondegenerate reduced singularity). Hence we are left with three cases (Fig. 5).

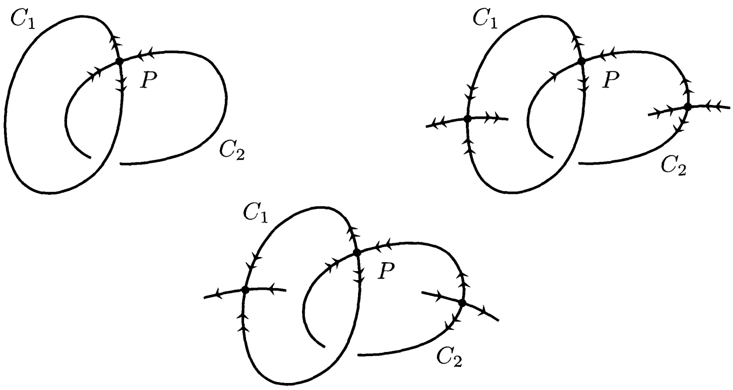

Figure 5

By contracting C_1 the curve C_2 becomes a smooth rational curve C of zero selfintersection which either is free of singularities or contains two reduced singularities which are both nondegenerate or both saddle-nodes with strong separatrices contained in the curve. Hence $Z(\mathcal{F}, C) = 0$ or 2, and lemma 1 shows that $\mathcal{F}$ is (birational to) a ruled foliation or a nontrivial Riccati foliation.

(ii) $C_1 \cap C_2 = \{p, q\}$. As before, p and q are nondegenerate, and we are exactly in the situation considered in lemma 2: $\mathcal{F}$ is birational to $(H, \mathcal{H})$. □

As a consequence of lemma 3, if $(X, \mathcal{F})$ satisfies the hypotheses of the theorem then all its $\mathcal{F}$-exceptional curves are pairwise disjoint. We contract all these curves, but it may happen that the new reduced foliation $\mathcal{F}'$ so obtained contains new $\mathcal{F}'$-exceptional curves. These curves will be again pairwise disjoint, we contract them, and so on. This process stops after a finite number of contractions, because each contraction reduces by one the rank of the second homology group of the surface. Hence we finally have a birational morphism

$$(X, \mathcal{F}) \longrightarrow (X_0, \mathcal{F}_0)$$

(a composition of blow-ups) with $(X_0, \mathcal{F}_0)$ free of $\mathcal{F}_0$-exceptional curves. We claim that $(X_0, \mathcal{F}_0)$ is the desired foliated minimal model.

By the factorization theorem of birational maps [BPV, p. 86] it is sufficient to look at the following situation. Let $\mathcal{G}$ be a reduced foliation on a smooth surface Y, and let $f:(Y,\mathcal{G}) \cdots \to (X_0, \mathcal{F}_0)$ be a birational map which becomes a morphism after only one blow-up at some point $p \in Y$; we have to prove that f itself is a morphism. If $\Pi: Z \to Y$ is the blow-up at p, with exceptional divisor $E \subset Z$, and if $g: Z \to X_0$ is the morphism covering f, then this is equivalent to prove that g contracts E. By contradiction, suppose that this is not true. Let $\widetilde{\mathcal{G}}$ be the reduced foliation on Z derived from $\mathcal{G}$ via Π (or, equivalently, from $\mathcal{F}_0$ via g).

Firstly, observe that E is $\widetilde{\mathcal{G}}$-exceptional, because $\mathcal{G}$ is reduced. The birational morphism g is a composition of blow-downs, and it contracts a curve $C \subset Z$ whose connected components are trees of rational curves. This curve C is completely $\widetilde{\mathcal{G}}$-invariant, because $\mathcal{F}_0$ has reduced singularities. The curve E does not belong to C, and so either it is disjoint from it or it cuts C transversely at singular points of $\widetilde{\mathcal{G}}$ (and in particular $\sharp(E \cap C) \leq 2$). If $E \cap C = \emptyset$ then g is biregular near E and then $\mathcal{F}_0$ would contain the $\mathcal{F}_0$-exceptional curve $g(E)$, contradiction. If $E \cap C \neq \emptyset$ then $g(E)$ is either smooth rational (if $\sharp(E \cap C) = 1$) or rational with a node (if $\sharp(E \cap C) = 2$). In the first case $g(E)^2 \geq 0$ and $\mathcal{F}_0$ has on $g(E)$ at most two singularities; perhaps saddle-nodes, but with strong separatrices in $g(E)$. If $\mathrm{Sing}(\mathcal{F}_0) \cap g(E) = \emptyset$ then necessarily $g(E)^2 = 0$ and $\mathcal{F}_0$ is ruled (lemma 1), contradiction. If $\mathrm{Sing}(\mathcal{F}_0) \cap g(E) \neq \emptyset$ we may blow-up one of the singular points until we obtain a smooth rational curve of zero selfintersection, and this will show that $\mathcal{F}_0$ is birational to a nontrivial Riccati foliation (lemma 1), contradiction. Finally, a similar argument (see Fig. 6) proves that if $g(E)$ is rational with a node (this node will be the only singularity of $\mathcal{F}_0$ on $g(E)$, of nondegenerate type) then $\mathcal{F}_0$ would be birational to a nontrivial Riccati foliation or to $\mathcal{H}$, still a contradiction.

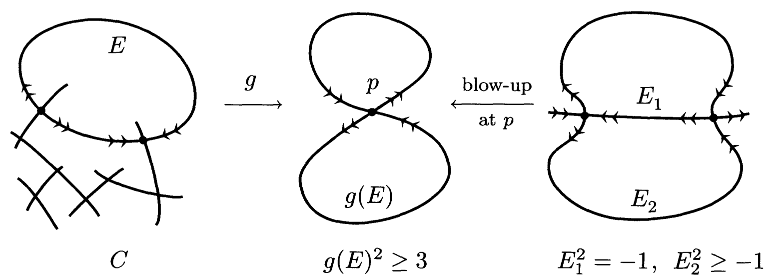

Figure 6

This completes the proof. □

3. Proof of the corollary

Suppose, by contradiction, that the nonelementary polynomial diffeomorphism $F:\mathbb{C}^2 \to \mathbb{C}^2$ leaves invariant a polynomial foliation $\mathcal{F}$. The diffeomorphism F induces a birational map $\widehat{F}:\mathbb{C}P^2 \cdots \to \mathbb{C}P^2$ and the foliation $\mathcal{F}$ extends to $\mathbb{C}P^2$ as a holomorphic foliation $\widehat{\mathcal{F}}$. Of course, we still have $\widehat{F}^*(\widehat{\mathcal{F}}) = \widehat{\mathcal{F}}$.

Recall that [BS] given a nonelementary diffeomorphism F one can construct two closed positive $(1,1)$-currents μ^+, μ^- which are projectively invariant by F, *i.e.*

$$F_*\mu^\pm = \mathrm{Cte}^\pm\mu^\pm, \quad 0 < \mathrm{Cte}^+ < 1, \quad \mathrm{Cte}^- > 1.$$

The supports of these two currents are two F-invariant closed subsets J^+, J^-, which are (roughly speaking) "laminated subsets" so that $\mu^\pm$ can be described as "laminar currents". The diffeomorphism has infinitely many hyperbolic saddle periodic points, and J^+ (resp. J^-) coincides with the closure of the stable (resp. unstable) manifold of any such periodic point. Such a manifold is abstractly isomorphic to $\mathbb{C}$, but its immersion in $\mathbb{C}^2$ is rather wild, and in particular it is recurrent, in the sense that it accumulates onto itself. One of the subsets J^+, J^- must be invariant by $\mathcal{F}$, that is its "leaves" (or, more precisely, the (un)stable manifolds it contains) must be leaves of $\mathcal{F}$, by the following argument of [C]. The foliation $\mathcal{F}$ has finitely many singularities whereas the diffeomorphism F has infinitely many hyperbolic saddle periodic points, hence we may find such a periodic point p, of period m, with $\mathcal{F}$ regular at p. The leaf of $\mathcal{F}$ through p is then invariant by F^m and therefore it coincides either with the stable manifold of F at p or with the unstable one.

Now the existence of a recurrent immersion $\mathbb{C} \to \mathbb{C}^2$ tangent to $\mathcal{F}$ (and hence to $\widehat{\mathcal{F}}$) shows that $\widehat{\mathcal{F}}$ cannot be birationally ruled nor birational to $\mathcal{H}$. Similarly, if $\widehat{\mathcal{F}}$ is birational to a Riccati foliation $\mathcal{G}$ then $\mathcal{G}$ must have exactly two invariant fibres and the monodromy of $\mathcal{G}$ around these two fibres must be an irrational rotation of $\mathbb{C}P^1$. One then easily verifies that $\widehat{\mathcal{F}}$ must have some invariant rational curve different from the line at infinity, and so $\mathcal{F}$ has some invariant affine curve $C \subset \mathbb{C}^2$. That curve must be preserved by (a power of) F, contradicting its nonelementarity [BS].

These arguments and our theorem imply that $\widehat{\mathcal{F}}$ has a minimal model $\widehat{\mathcal{F}}_0$, on some rational surface X. If $G:\mathbb{C}P^2 \cdots \to X$ is a birational conjugation between $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}_0$, then $G \circ \widehat{F} \circ G^{-1}$ is a birational automorphism of X which preserves $\widehat{\mathcal{F}}_0$. By definition of minimal model, $G \circ \widehat{F} \circ G^{-1}$ is in fact a *biregular* automorphism of X. The proof is then completed by the following lemma, of independent interest.

LEMMA 4. — *If $F:\mathbb{C}^2 \to \mathbb{C}^2$ is a nonelementary polynomial diffeomorphism and $\widehat{F}:\mathbb{C}P^2 \cdots\to \mathbb{C}P^2$ is its birational extension to $\mathbb{C}P^2$, then $\widehat{F}$ is not birationally conjugate to a biregular map.*

Proof. — Let F be a polynomial diffeomorphism and let $G:\mathbb{C}P^2 \cdots\to X$ be a birational map such that $G \circ \widehat{F} \circ G^{-1}$ is biregular. Let

$$P = \{p_1, \dots, p_n\} \subset \mathbb{C}P^2$$

be the indeterminacy points of G and let

$$\Sigma = \{R_1, \dots, R_m\} \subset \mathbb{C}P^2$$

be the collection of rational curves contracted by G (more precisely, every $R_j \setminus (R_j \cap P)$ is contracted by G to a point). It is easy to see that the biregularity of F and $G \circ \widehat{F} \circ G^{-1}$ implies the F-invariance of the affine curve $\Sigma \cap \mathbb{C}^2$. As remarked before, if $\Sigma \cap \mathbb{C}^2 \neq \emptyset$ then F is elementary. On the other hand, if $\Sigma \cap \mathbb{C}^2 = \emptyset$ (*i.e.* Σ coincides with the line at infinity) then G is the composition of two birational maps (Fig. 7): a first map $G_1:\mathbb{C}P^2 \cdots\to Y$ which is biregular on $\mathbb{C}^2$ and a second map $G_2: Y \cdots\to X$ which blows-up points in $G_1(P \cap \mathbb{C}^2) \subset Y$ and is biregular near $T = Y \setminus G_1(\mathbb{C}^2)$. Clearly, $\widetilde{F} = G_1 \circ \widehat{F} \circ G_1^{-1}$ is a biregular map of Y, biregularly conjugate to F on $G_1(\mathbb{C}^2)$ and to $G \circ \widehat{F} \circ G^{-1}$ on a neighbourhood of T.

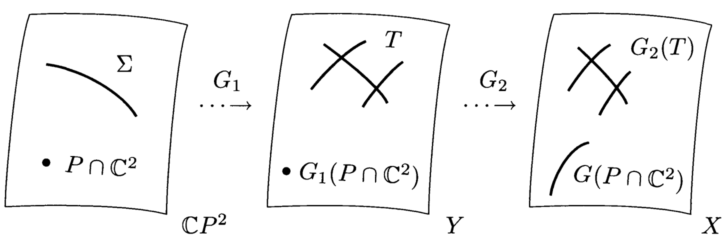

Figure 7

In other words, there exists a rational compactification Y of $\mathbb{C}^2$ over which F extends as a biregular map $\widetilde{F}$. The curve $T \subset Y$ is preserved by $\widetilde{F}$, and up to taking a power of F (this does not change its elementary or nonelementary character) we may suppose that $\widetilde{F}$ preserves each irreducible component of T. These irreducible components generate $H^2(Y, \mathbb{Z})$, therefore $\widetilde{F}$ acts trivially on that cohomology group. A classical result of Blanchard [K, p. 107] says that $\widetilde{F}$ is projectively induced: there exists an embedding of Y in $\mathbb{C}P^N$ such that $\widetilde{F}$ is realized as the restriction to Y of a (projective) automorphism of $\mathbb{C}P^N$. Hence the dynamics of $\widetilde{F}$ is extremely simple and so F is an elementary diffeomorphism. □

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