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LINEARIZATION OF GROUP STACK ACTIONS AND
THE PICARD GROUP OF THE MODULI OF
$\text{SL}_r/\mu_s$-BUNDLES ON A CURVE

PAR YVES LASZLO (*)

ABSTRACT. — In this paper, we consider morphisms of algebraic stacks $X \to Y$ which are torsors under a group stack $G$. We show that line bundles on $Y$ correspond exactly with $G$-linearized line bundles on $X$ (with a suitable definition of a $G$-linearization). We use this fact to determine the precise structure of the Picard group of the moduli stack of $G$-bundles on an algebraic curve when $G$ is any group of type $A_n$.

RESUME. — Dans cet article, on considère les morphismes de champs algébriques $X \to Y$ qui sont des torseurs sous un champ en groupes $G$. Nous prouvons que les fibrés en droites sur $Y$ correspondent exactement aux fibrés en droites sur $X$ munis d’une $G$-linéarisation (avec une définition convenable d’une $G$-linéarisation). Nous utilisons ceci pour déterminer la structure exacte du groupe de Picard du champ des $G$-fibres sur une courbe algébrique lorsque $G$ est un groupe algébrique (non nécessairement simplement connexe) de type $A_n$.

1. Introduction

Let $G$ be a complex simple group and $\tilde{G} \to G$ the universal covering. For simplicity, let us consider the moduli stack $M_G$ (resp. $M_{\tilde{G}}$) of degree $1 \in \pi_1(G)$ principal $G$-bundles (resp. $\tilde{G}$-bundles) over a curve $X$. In [B-L-S], we have studied the natural morphism

$$\iota : \text{Pic}(M_G) \to \text{Pic}(M_{\tilde{G}}),$$

the group $\text{Pic}(M_{\tilde{G}})$ being infinite cyclic by [L-S]. It is proved in [B-L-S] that the kernel of $\iota$ is naturally identified with the finite group $H^1_{\text{ét}}(X, \pi_1(G)^\vee)$ reducing the study of $\text{Pic}(M_G)$ to the computation of the cardinality of $\text{Coker}(\iota)$. Among other things, it has been possible to

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perform this computation in the case where \( G = \text{PSL}_r \) but not in the case where \( G = \text{SL}_r/\mu_s \), where \( s \mid r \), although we were able to give partial results. The reason was that the technical background to study the descent of modules through the morphism \( p : \mathcal{M}_G \to \mathcal{M}_G \) wasn't at our disposal.

The aim of this paper is to compute \( \text{card Coker}(\iota) \) when \( G = \text{SL}_r/\mu_s \).

It turns out to be that \( p \) is a torsor under some group stack, not far from a Galois étale cover in the usual schematic picture. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a torsor under a group scheme \( \mathcal{G} \). We know that a line bundle on \( \mathcal{X} \) descends if and only if it has a \( \mathcal{G} \)-linearization (easy consequence of descent theory). Now, the descent theory of Grothendieck has been adapted to the set-up of fpqc morphisms of stacks in [L-M]. If \( \mathcal{G} \) is now only assumed to be a group stack, there is a notion of \( \mathcal{G} \)-linearization of line bundles on \( \mathcal{X} \) (see section 2). One obtains (theorem 4.1) that a line bundle on \( \mathcal{X} \) descends if and only if it admits a linearization.

We then use this technical result to compute \( \text{card Coker}(\iota) \) when \( G = \text{SL}_r/\mu_s \) (theorem 5.7 and its corollary).

I would like to thank L. Breen for having taught me both the notions of torsor and of linearization of a vector bundle in the set-up of group-stack actions and for his comments on a preliminary version of this paper.

Notations.

Throughout this paper, all the stacks will be implicitly assumed to be algebraic over an algebraically closed field \( \mathbb{k} \) and the morphisms locally of finite type. We fix once and for all a projective, smooth, connected genus \( g \) curve \( X \) and a closed point \( x \) of \( X \). For simplicity, we assume \( g > 0 \) (see remarks 5.6 and 5.10 for the case of \( \mathbb{P}^1 \)). The Picard stack parametrizing families of line bundles of degree 0 on \( X \) will be denoted by \( \mathcal{J}(X) \) and the jacobian variety of \( X \) by \( JX \). If \( G \) is an algebraic group over \( \mathbb{k} \), the quotient stack \( \text{Spec}(\mathbb{k})/G \) (where \( G \) acts trivially on \( \text{Spec}(\mathbb{k}) \)) whose category over a \( \mathbb{k} \)-scheme \( S \) is the category of \( G \)-torsors (or \( G \)-bundles) over \( S \) will be denoted by \( BG \). If \( n \) is an integer and \( A = \mathcal{J}(X) \), \( JX \) or \( BG_m \) we denote by \( n_A \) the \( n^{\text{th}} \)-power morphism \( a \mapsto a^n \). We denote by \( J_n \) (resp. \( J_n \)) the 0-fiber \( A \times_A \text{Spec}(\mathbb{k}) \) of \( n_A \) when \( A = \mathcal{J}(X) \) (resp. \( A = JX \)).

1. Generalities. — Following [Br], for any diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{g} & & \uparrow{f} \\
C & \xrightarrow{\ell} & D
\end{array}
\]
of 2-categories, we will denote by
\[ \ell \ast \lambda : \ell \circ f \Rightarrow \ell \circ g \quad (\text{resp. } \lambda \ast h : f \circ h \Rightarrow g \circ h) \]
the 2-morphism deduced from \( \lambda \).

1.1. — For the convenience of the reader, let us prove a simple formal lemma which will be useful in section 4. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be three 2-categories. Let diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\downarrow_{\delta_0} & & \downarrow_{d_0} \\
\mathcal{C} & \xleftarrow{\delta_1} & \mathcal{B} \\
\downarrow_{d_1} & & \\
\end{array}
\]

be a 2-commutative diagram and let \( \mu : \delta_0 \Rightarrow \delta_1 \) be a 2-morphism.

**Lemma 1.2.** — Assume that \( f \) is an equivalence. There exists a unique 2-morphism \( \mu \ast f^{-1} : d_0 \Rightarrow d_1 \) such that \( (\mu \ast f^{-1}) \ast f = \mu \).

**Proof.** — Let \( \epsilon_k \), for \( k = 0, 1 \) be the 2-morphism \( d_k \circ f \Rightarrow \delta_k \). Let \( b \) be an object of \( \mathcal{B} \). Pick an object \( a \) of \( \mathcal{A} \) and an isomorphism \( \alpha : f(a) \xrightarrow{\sim} b \). Let \( \varphi_\alpha : d_0(b) \xrightarrow{\sim} d_1(b) \) be the unique isomorphism making the diagram

\[
\begin{array}{ccc}
\delta_0(a) & \xrightarrow{\epsilon_0(a)} & d_0 \circ f(a) & \xrightarrow{d_0(\alpha)} & d_0(b) \\
\mu_\alpha & & & & \varphi_\alpha \\
\delta_1(a) & \xrightarrow{\epsilon_1(a)} & d_1 \circ f(a) & \xrightarrow{d_1(\alpha)} & d_1(b)
\end{array}
\]

commutative. We have to show that \( \varphi_\alpha \) does not depend on \( \alpha \) but only on \( b \). Let \( \alpha' : f(a') \xrightarrow{\sim} b \) be another isomorphism. There exists a unique isomorphism \( \iota : a' \xrightarrow{\sim} a \) such that \( \alpha \circ f(\iota) = \alpha' \). Then one has the equality \( \varphi_{\alpha'} = d_1(\alpha) \circ \Phi \circ d_0(\alpha)^{-1} \) where

\[ \Phi = [d_1 \circ f(\iota)] \circ \epsilon_1(a') \circ \mu_{a'} \circ \epsilon_0(a')^{-1} \circ [d_0 \circ f(\iota)]^{-1}. \]

The functoriality of \( \epsilon_i \) and \( \mu \) ensures that one has the equalities

\[ d_k \circ f(\iota) \circ \epsilon_k(a') = \epsilon_k(a) \circ \delta_k(\iota) \]
and
\[ \mu_a = \delta_1(i) \circ \mu_{a'} \circ \delta_0(i)^{-1}. \]
This shows the equality
\[ \Phi = \epsilon_1(a) \circ \mu_a \circ \epsilon_0(a)^{-1} \]
which proves the equality \( \varphi_a = \varphi_{a'} \). We can therefore define \( \mu_b \) to be the isomorphism \( \varphi_a \) for one isomorphism \( \alpha : f(a) \sim b \). We check that the construction is functorial in \( b \) and the lemma follows.

2. Linearizations of line bundles on stacks.

Let us first recall the notion of torsor in the stack context according to [Br].

2.1. — Let \( f : \mathcal{X} \to \mathcal{Y} \) be a faithfully flat morphism of stacks. Let us assume that an algebraic \( \mathbf{g} \text{-} \text{stack} \) \( \mathcal{G} \) acts on \( f \) (the product of \( \mathcal{G} \) is denoted by \( m_{\mathcal{G}} \) and the unit object by 1). Following [Br], this means that there exists a 1-morphism of \( \mathcal{G} \text{-} \text{stacks} \) \( m : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \) and a 2-morphism \( \mu : m \circ (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) \to m \circ (\text{Id}_{\mathcal{G}} \times m) \) such that the obvious associativity condition (see diagram (6.1.3) in [Br]) is satisfied and such that there exists a 2-morphism \( \epsilon : m \circ (1 \times \text{Id}_{\mathcal{X}}) \to \text{Id}_{\mathcal{X}} \) which is compatible to \( \mu \) in the obvious sense (see (6.1.4) of [Br]).

Remark 2.2. — To say that \( m \) is a morphism of \( \mathcal{Y} \text{-} \text{stacks} \) means that the diagram
\[ \begin{array}{ccc} \mathcal{G} \times \mathcal{X} & \xrightarrow{m} & \mathcal{X} \\ \downarrow & & \nearrow \\ \mathcal{Y} & & \mathcal{Y} \end{array} \]
is 2-commutative. In other words, if we denote for simplicity the image of a pair of objects \( m(g, x) \) by \( g \cdot x \). This means that there exists a functorial isomorphism \( t_{g,x} : f(g \cdot x) \to f(x) \).

2.3. — Suppose that \( \mathcal{G} \) acts on such another morphism \( f' : \mathcal{X}' \to \mathcal{Y} \). A morphism \( p : \mathcal{X}' \to \mathcal{X} \) will be said equivariant if there exists a 2-morphism
\[ q : m \circ (\text{Id} \times p) \Rightarrow p \circ m' \]
which is compatible to \( \mu \) (as in [Br, (6.1.6)]) and \( \epsilon \) (which is implicit in [Br]) in the obvious sense.

Definition 2.4. — With the above notations, we say that \( f \) (or \( \mathcal{X} \)) is a \( \mathcal{G} \text{-} \text{torsor} \) over \( \mathcal{Y} \) if the morphism \( pr_\mathcal{X} \times m : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{Y} \mathcal{X} \) is an isomorphism (of stacks) and the geometrical fibers of \( f \) are not empty.
REMARK 2.5. — In down to earth terms, this means that if
\[ \iota : f(x) \rightarrow f(x') \]
is an isomorphism in \( \mathcal{Y} (x, x' \text{ being objects of } \mathcal{X}) \), there exists an object \( g \) of \( \mathcal{G} \) and a unique isomorphism \( (x, g \cdot x) \sim (x, x') \) which induces \( \iota \) by way of \( \iota_{g, x} \) (cf. 2.2).

EXAMPLE 2.6. — Let \( \mathcal{M}_X(G_m) \) be the Picard stack of \( X \). Then, the morphism
\[ \mathcal{M}_X(G_m) \rightarrow \mathcal{M}_X(G_m) \]
of multiplication by \( n \in \mathbb{Z} \) is a torsor under \( B\mu_n \times J_n(X) \) (cf. (3.1)).

2.7. — Let \( \mathcal{L} \) be a line bundle on \( \mathcal{X} \). By definition, the datum \( \mathcal{L} \) is equivalent to the datum of a morphism \( \ell : \mathcal{X} \rightarrow B\mathbb{G}_m \) (see [L-M, prop. 6.15]). If \( \mathcal{L}, \mathcal{L}' \) are two line bundles on \( \mathcal{X} \) defined by \( \ell, \ell' \), we will view an isomorphism \( \mathcal{L} \sim \mathcal{L}' \) as a 2-morphism \( \ell \Rightarrow \ell' \).

DEFINITION 2.8. — A \( \mathcal{G} \)-linearization of \( \mathcal{L} \) is a 2-morphism
\[ \lambda : \ell \circ m \Rightarrow \ell \circ \text{pr}_2 \]
such that the two diagrams of 2-morphisms
\[
\begin{array}{c}
\ell \circ m \circ (m \times \text{Id}_\mathcal{X}) \\
\downarrow \lambda \ast (m \times \text{Id}_\mathcal{X}) \\
\ell \circ \text{pr}_2 \circ (m \times \text{Id}_\mathcal{X}) \\
\end{array}
\text{(2.8.1)}
\quad
\begin{array}{c}
\ell \circ \text{pr}_2 \circ (m \times \text{Id}_\mathcal{X}) \\
\downarrow \lambda \ast \text{pr}_{23} \\
\ell \circ \text{pr}_2 \circ \text{pr}_{23} \\
\end{array}
\]
\[ \lambda \ast (m \times \text{Id}_\mathcal{X}) \]
\[ \lambda \ast (\text{Id}_\mathcal{G} \times m) \]
\[ \ell \circ \text{pr}_2 \circ (\text{Id}_\mathcal{G} \times m) \]
\[ \ell \circ \text{pr}_2 \circ \text{pr}_{23} \]
\[ \ell \circ m \circ (1 \times \text{Id}_\mathcal{X}) \]
\[ \ell \ast e \]
\[ \ell \ast (1 \times \text{Id}_\mathcal{X}) \]
\[ \ell \]
\[ \ell \]
\[ \ell \]
\[ \text{(strictly) commute.} \]
REMARK 2.9. — If $g_1, g_2$ are objects of $G$ and $d$ is an object of $X$, the commutativity of diagram (2.8.1) means that the diagram

$$\begin{array}{ccc}
\mathcal{L}_{g_1(g_2)} & \sim & \mathcal{L}_{g_2(x)} \\
\downarrow & & \downarrow \\
\mathcal{L}_x & \sim & \mathcal{L}_{g_2(x)}
\end{array}$$

is commutative and the commutativity of (2.8.2) that the two isomorphisms $\mathcal{L}_{1,x} \simeq \mathcal{L}_x$ defined by the linearization $\lambda$ and $\epsilon$ respectively are the same.

3. An example.

Let me recall that a closed point $x$ of $X$ has been fixed. Let $S$ be a $k$-scheme. The $S$-points of the jacobian variety of $X$ are by definition isomorphism classes of line bundles on $X_S$ together with a trivialization along $\{x\} \times S$ (such a pair will be called a rigidified line bundle). For the convenience of the reader, let me state this well known lemma which can be found in SGA4, exp. XVIII, (1.5.4).

**Lemma 3.1.** — The Picard stack $\mathcal{J}(X)$ is canonically isomorphic (as a $k$-group stack) to $JX \times BG_m$.

**Proof.** — Let $f : \mathcal{J}(X) \to JX \times BG_m$ be the morphism which associates

- to the line bundle $L$ on $X_S$ the pair $(L \boxtimes L_{\{x\}}^{-1} \times S, L_{\{x\}} \times S)$ where $\boxtimes$ denotes the external tensor product (this pair is thought of as an object of $JX \times BG_m$ over $S$);
- to an isomorphism $L \sim \sim L'$ on $X_S$ its restriction to $\{x\} \times S$.

Let $f' : JX \times BG_m \to \mathcal{J}(X)$ be the morphism which associates

- to the pair $(L, V)$ where $L$ is a rigidified bundle on $X_S$ and $V$ a line bundle on $S$ (thought of as an object of $JX \times BG_m$ over $S$), the line bundle $L \boxtimes X_S \times V$;
- to an isomorphism $(\ell, v) : (L, V) \sim (L', V')$ the tensor product $\ell \boxtimes X_S \times v$.

The morphisms $f$ and $f'$ are (quasi)-inverse to each other and are morphisms of $k$-stacks. \[\square\]

We will identify from now $\mathcal{J}(X)$ and $JX \times BG_m$. Let $\mathcal{L}$ (resp. $\mathcal{P}$ and $\mathcal{T}$) be the universal bundle on $X \times \mathcal{J}(X)$ (resp. on $X \times JX$ and $BG_m$) and let $\Theta = (\det R\Gamma\mathcal{P})^{-1}$ be the theta line bundle on $JX$. The isomorphism $\mathcal{L} \sim \sim \mathcal{P} \boxtimes \mathcal{T}$ yields an isomorphism

$$\det R\Gamma\mathcal{L}(m,x) \sim \sim \Theta^{-n} \boxtimes T^{(m+1-g)}.$$

The object of this section is to prove the following statement.

**Theorem 4.1.** — Let $f: X \rightarrow Y$ be a $G$-torsor as above. Let $\text{Pic}^G(X)$ be the group of isomorphism classes of $G$-linearized line bundles on $X$. Then, the pull-back morphism $f^*: \text{Pic}(Y) \rightarrow \text{Pic}^G(X)$ is an isomorphism.

The descent theory of Grothendieck has been adapted in the case of algebraic 1-stacks in [L-M], essentially in proposition (6.23).

Let $X_\ast \rightarrow Y$ be the (augmented) simplicial complex of stacks coskeleton of $f$ (as defined in [De, (5.1.4)]) for instance. By proposition (6.23) of [L-M], one just has to construct a cartesian $\mathcal{O}_{D_\ast}$-module $L_\ast$ such that $L_0$ is the $\mathcal{O}_X$-module $L$ to prove the theorem. The $n$-th piece $X_n$ is inductively defined by

$$X_0 = X, \quad X_n = X \times_Y X_{n-1} \quad \text{for } n > 0.$$  

Let $p_n: X_n \rightarrow X$ be the projection on the first factor. It is the simplicial morphism associated to the map

$$\tilde{p}_n : \{ \Delta_0 \rightarrow \Delta_n, \quad 0 \rightarrow 0 \}.$$

Let $L_n$ be the line bundle defined by the morphism (see (2.7))

$$\ell_n: X_n \xrightarrow{p_n} X \xrightarrow{\ell} B\mathbb{G}_m.$$  

4.2. — Let $\delta_i$ (resp. $s_j$) be the face (resp. degeneracy) operators (see for instance [De, 5.1.1]). By abuse of notation, we use the same notation for $\delta_j, s_j$ and their image by $X$. The category $(\Delta_\ast)$ is generated by the face and degeneracy operators with the following relations (see for instance the proposition VII.5.2, p. 174 of [McL])

$$\begin{align*}
\delta_i \circ \delta_j &= \delta_{j+1} \circ \delta_i \quad \text{if } i \leq j, \\
\delta_j \circ s_i &= s_i \circ \delta_{j+1} \quad \text{if } i \leq j, \\
s_j \circ \delta_i &= \begin{cases} 
\delta_i \circ \delta_{j-1} & \text{if } i < j, \\
1 & \text{if } i = j, i = j + 1, \\
\delta_{i-1} \circ s_j & \text{if } i > j + 1.
\end{cases}
\end{align*}$$  

Therefore, the datum of a cartesian $\mathcal{O}_X_\ast$-module $L_\ast$ is equivalent to the data of isomorphisms

$$\begin{align*}
\alpha_j: \delta_j^* L_n \xrightarrow{\sim} L_{n+1}, & \quad j = 0, \ldots, n + 1, \\
\beta_j: s_j^* L_{n+1} \xrightarrow{\sim} L_n, & \quad j = 0, \ldots, n,
\end{align*}$$  

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(where \( n \) is a non negative integer) which are compatible with relations (4.2.1), (4.2.2) and (4.2.32).

Let \( n \) be a non negative integer.

4.3. — We have first to define for \( j = 0, \ldots, n+1 \) an isomorphism

\[
\alpha_j : \delta_j^* \mathcal{L}_n \sim \mathcal{L}_{n+1}.
\]

The line bundle \( \delta_j^* \mathcal{L}_n \) is defined by the morphism \( \ell \circ p_n \circ \delta_j : \mathcal{X}_{n+1} \to BG_m \) and \( \tilde{p}_n \circ \delta_j \) is associated to the map

\[
\begin{cases}
\Delta_0 \to \Delta_{n+1}, \\
0 \to \delta_j(0).
\end{cases}
\]

- If \( j \neq 0 \), one has therefore \( \tilde{p}_n \circ \delta_j = \tilde{p}_{n+1} \) and \( \delta_j^* \mathcal{L}_n = L_{n+1} \). We define \( \alpha_j \) by the identity in this case.

- Suppose now that \( j = 0 \). Let \( \pi_n : \mathcal{X}_n \to \mathcal{X}_1 \) be the projection on the two first factors (associated to the canonical inclusion \( \Delta_1 \hookrightarrow \Delta_n \)). The commutativity of the two diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{n+1} & \delta_0 & \mathcal{X}_n \\
\downarrow \pi_{n+1} & \downarrow p_n & \\
\mathcal{X}_1 & \delta_0 & \mathcal{X}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X}_{n+1} & p_{n+1} & \mathcal{X} \\
\downarrow \pi_{n+1} & \uparrow \delta_1 & \\
\mathcal{X}_1 & & \mathcal{X}_1
\end{array}
\]

allows to reduce the problem to the construction of an isomorphism

\[
\delta_0^* \mathcal{L} \sim \delta_1^* \mathcal{L}
\]

where \( \delta_i : \mathcal{X}_1 \to \mathcal{X} \) for \( i = 0,1 \) are the face morphisms or, what amounts to the same, to the construction of a 2-morphism \( \nu : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1 \) (the morphism \( \alpha_j \) will be \( \alpha_j = \nu \ast \pi_{n+1} \)). Now the diagram

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{X} & \overset{pr_2 \times m}{\longrightarrow} & \mathcal{X} \times \mathcal{X} \\
\downarrow \ell \circ pr_2 & & \downarrow \ell \circ \delta_1 \\
BG_m & \overset{\ell \circ \delta_0}{\longrightarrow} & BG_m
\end{array}
\]

is strictly commutative and \( pr_2 \times m \) is an equivalence by definition of a torsor. According to lemma 1.2, the 2-morphism \( \lambda \) induces a canonical 2-morphism

\[
\lambda \ast (pr_2 \times m)^{-1} : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1
\]

which is the required 2-morphism \( \nu \).
4.4. — We then have to define for \( j = 0, \ldots, n \) an isomorphism

\[ \beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n. \]

The line bundle \( s_j^* \mathcal{L} \) is defined by the morphism \( \ell \circ p_{n+1} \circ s_j \) and \( p_{n+1} \circ s_j \) is associated to the canonical inclusion \( \Delta_0 \hookrightarrow \Delta_n \) which means \( p_{n+1} \circ s_j = p_n \). Therefore, one has a canonical isomorphism \( s_j^* \mathcal{L}_{n+1} = \mathcal{L}_n \) and we define \( \beta_j \) to be the identity.

4.5. — We have to show that the data \( \mathcal{L}_\bullet \) and \( \alpha_j, \beta_j \), for \( j \geq 0 \) define a line bundle on the simplicial stack \( \mathcal{X}_\bullet \) as explained in 4.2. Notice that the fact that the definition of the \( \beta_j \) is compatible with relations (4.2.2) is tautological (\( \beta_j \) is the identity on the relevant \( \mathcal{L}_n \)).

4.6. — In terms of \( \ell \), relation (4.2.1) means the following. We have the two strictly 2-commutative diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{n+2} & \xrightarrow{\delta_i} & \mathcal{X}_{n+1} & \xrightarrow{\delta_j} & \mathcal{X}_n \\
p_{n+2} & & \downarrow & & \downarrow p_n \\
\mathcal{X} & \xrightarrow{\ell} & BG_m \\
P_{n+1} & & \downarrow p_n & & \downarrow p_n \\
\mathcal{X} & \xrightarrow{\ell} & BG_m
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{X}_{n+2} & \xrightarrow{\delta_{j+1}} & \mathcal{X}_{n+1} & \xrightarrow{\delta_i} & \mathcal{X}_n \\
p_{n+2} & & \downarrow & & \downarrow p_n \\
\mathcal{X} & \xrightarrow{\ell} & BG_m \\
P_{n+1} & & \downarrow p_n & & \downarrow p_n \\
\mathcal{X} & \xrightarrow{\ell} & BG_m
\end{array}
\]

inducing the two 2-morphisms

\[ \alpha_i \circ (\alpha_j \ast \delta_i) : \ell \circ p_n \circ \delta_j \circ \delta_i = \xrightarrow{\alpha_j \ast \delta_i} \ell \circ p_{n+1} \circ \delta_i = \xrightarrow{\alpha_i} \ell \circ p_{n+2} \]

and

\[ \alpha_{j+1} \circ (\alpha_i \ast \delta_{j+1}) : \ell \circ p_n \circ \delta_i \circ \delta_{j+1} = \xrightarrow{\alpha_i \ast \delta_{j+1}} \ell \circ p_{n+1} \circ \delta_{j+1} = \xrightarrow{\alpha_{j+1}} \ell \circ p_{n+2}. \]

The relation (4.2.1) means exactly the equality

(4.2.1') \[ \alpha_i \circ (\alpha_j \ast \delta_i) = \alpha_{j+1} \circ (\alpha_i \ast \delta_{j+1}), \quad \text{for } i \leq j. \]

- If \( j = 0 \), the relation (4.2.1') is just by definition of \( \alpha_j \) the condition (2.8.1) (see remark 2.9).
- If \( j > 0 \), both isomorphisms \( \alpha_j \) and \( \alpha_{j+1} \) are the relevant identities and the relation (4.2.1') is tautological.
4.7. — The only non tautological relation in (4.2.3) corresponds to the equality $s_0 \circ \delta_0 = 1$ in ($\Delta_*$) which means as before that $\alpha_0 \circ \delta_0$ is the identity functor of $\ell \circ p_n = \ell \circ \phi_n \circ \delta_0 \circ s_0$. But, this is exactly the meaning of the relation (2.8.2) (see remark 2.9).

5. Application to the Picard groups of some moduli spaces.

Let us choose three integers $r, s, d$ such that

$$r \geq 2 \quad \text{and} \quad s \mid r \mid ds.$$

If $G$ is the group $\text{SL}_r/\mu_s$ we denote as in [B-L-S] by $\mathcal{M}_G(d)$ the (connected) moduli stack of $G$-bundles on $X$ of degree $\exp(2i\pi d/r) \in H^2_{\text{et}}(X, \mu_s) = \mu_s$ and by $\mathcal{M}_{\text{SL}_r}(d)$ the moduli stack of rank $r$ vector bundles and determinant $\mathcal{O}(d\cdot x)$. If $r = s$ (i.e. $G = \text{PSL}_r$), the natural morphism of algebraic stacks

$$\pi : \mathcal{M}_{\text{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)$$

is a $\mathcal{J}_r$-torsor (see the corollary of proposition 2 of [Gr] for instance). Let me explain how to deal with the general case.

5.1. — Let $E$ be a rank $r$ vector bundle on $X_s$ endowed with an isomorphism

$$\tau : D^{r/s} \sim \text{det}(E)$$

where $D$ is some line bundle. Let me define the $\text{SL}_r/\mu_s$-bundle $\pi(E)$ associated to $E$ (more precisely to the pair $(E, \tau)$).

**Definition 5.2.**

- An $s$-trivialization of $E$ on an étale neighborhood $T \to X_s$ is a triple $(M, \alpha, \sigma)$ where
  
  $\alpha : D \sim M^s$ is an isomorphism ($M$ is a line bundle on $T$);
  $\sigma : M^{\oplus r} \sim E_T$ is an isomorphism;
  $\det(\sigma) \circ \alpha^{r/s} : D^{r/s} \sim \text{det}(E)$ is equal to $\tau$.

- Two $s$-trivializations $(M, \alpha, \sigma)$ and $(M', \alpha', \sigma')$ of $E$ will be said equivalent if there exists an isomorphism $\iota : M \sim M'$ such that $\iota^s \circ \alpha = \alpha'$.

The principal homogeneous space

$$T \longmapsto \{\text{equivalence classes of } s\text{-trivializations of } E_T\}$$
defines the $\text{SL}_r/\mu_s$-bundle $\pi(E)^\dagger$. Now, the construction is obviously functorial and therefore defines the morphism $\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$ (observe that an object $E$ of $\mathcal{M}_{\text{SL}_r}(d)$ has determinant $O(d s/r \cdot x)^{r/s}$).

Let $L$ be a line bundle and $(M, \alpha, \tau)$ an $s$-trivialization of $E_T$. Then, $(M \otimes L, \alpha \otimes \text{Id}_{L^s}, \sigma \otimes \text{Id}_L)$ is an $s$-trivialization of $E \otimes L$ (which has determinant $(D \otimes L^s)^{r/s}$). This shows that there exists a canonical functorial isomorphism

\[(5.2.1) \quad \pi(E) \sim \pi(E \otimes L)\]

In particular, $\pi$ is $\mathcal{J}_s$-equivariant.

**Lemma 5.3.** — The natural morphism of algebraic stacks

\[\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)\]

is a $\mathcal{J}_s$-torsor.

**Proof.** — Let $E, E'$ be two rank $r$ vector bundles on $X_S$ (with determinant equal to $O(d \cdot x)$) and let $\iota : \pi(E) \sim \pi(E)'$ be an isomorphism. As in the proof of the lemma 13.4 of [B-L-S], we have the exact sequence of sets

\[1 \to \mu_s \to \text{Isom}(E, E') \to \text{Isom}(\pi(E), \pi(E)') \xrightarrow{\pi_{E, E'}} \text{Isom}_{\mu_s}(X_S, \mu_s).\]

Let $L$ be a $\mu_s$-torsor such that $\pi_{E, E'}(\iota) = [L]$. Then, $\pi(E \otimes L)$ is equal to $\pi(E)$, $\pi_{E \otimes L, E'} = 0$ and $\iota$ is induced by an isomorphism $E \otimes L \sim E'$ well defined up to multiplication by $\mu_s$. The lemma follows. \[\square\]

5.4. — Let $\mathcal{U}$ be the universal bundle on $X \times \mathcal{M}_{\text{SL}_r}(d)$. We would like to know which power of the determinant bundle $\mathcal{D} = (\text{det } R\mathcal{U})^{-1}$ on $\mathcal{M}_{\text{SL}_r}(d)$ descends to $\mathcal{M}_G(d)$. As in I.3 of [B-L-S], the rank $r$ bundle

\[\mathcal{F} = \mathcal{L}^{r-1} \oplus \mathcal{L}^{1-r}(d \cdot x)\]

on $X \times \mathcal{J}(X)$ has determinant $O(d \cdot x)$ and therefore defines a morphism

\[f : \mathcal{J}(X) = JX \times \text{BG}_m \to \mathcal{M}_{\text{SL}_r}(d)\]

which is $\mathcal{J}_s$-equivariant.

\[\dagger\] We see here a $G$-bundle as a formal homogeneous space under $G$. BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE
The vector bundle
\[ \mathcal{F'} = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{L}^{-r/s}(d \cdot x) \]
on \(X \times \mathcal{J}(X)\) has determinant \([\mathcal{L}^{-1}(ds/r \cdot x)]^{r/s}\). The \(G\)-bundle \(\pi(\mathcal{F'})\) on \(X \times \mathcal{J}(X)\) defines a morphism \(f' : \mathcal{J} \to \mathcal{M}_G(d)\). The relation
\[ \mathcal{L} \otimes (\text{Id}_X \times s \mathcal{J})^* (\mathcal{F'}) = \mathcal{F} \]
and (5.2.1) gives an isomorphism
\[ \pi(\mathcal{F}) = (\text{Id}_X \times s \mathcal{J})^* \pi(\mathcal{F'}) \]
which means that the diagram
\[ \begin{array}{ccc}
\mathcal{J}(X) & \xrightarrow{f} & \mathcal{M}_{\text{SL}_r}(d) \\
\downarrow s \mathcal{J} & & \downarrow \pi \\
\mathcal{J}(X) & \xrightarrow{f'} & \mathcal{M}_G(d)
\end{array} \]
is 2-commutative. Exactly as in I.3 of [B-L-S], let me prove the

**Lemma 5.5.** — The line bundle \(f^* \mathcal{D}^k\) on \(\mathcal{J}(X)\) descends through \(s \mathcal{J}\) if and only if \(k\) multiples of \(s/(s,r/s)\).

**Proof.** — Let \(\chi = r(g-1)-d\) be the opposite of the Euler characteristic of \((k-)\)points of \(\mathcal{M}_{\text{SL}_r}(d)\). By (3.1.1), one has an isomorphism
\[ f^* \mathcal{D}^k \sim \Theta^{kr(r-1)} \otimes \mathcal{T}^k \chi. \]
The theory of Mumford groups says that \(\Theta^{kr(r-1)}\) descends through \(s \mathcal{J}\) if and only if \(k\) is a multiple of \(s/(s,r/s)\). The line bundle \(\mathcal{T}^k \chi\) on \(BG_m\) descends through \(s_{BG_m}\) if and only if \(k \chi\) is a multiple of \(s\). The lemma follows from the above isomorphism and from the observation that the condition \(s \mid r \mid ds\) forces \(s \chi\) to be a multiple of \(s\). []

**Remark 5.6.** — If \(g = 0\), the jacobian \(J\) is a point and the condition on \(\Theta\) is empty. The only condition in this case is that \(k \chi\) is a multiple of \(s\).

Let me recall that \(\mathcal{D}\) is the determinant bundle on \(\mathcal{M}_{\text{SL}_r}(d)\) and \(G = \text{SL}_r/\mu_s\).

**Theorem 5.7.** — Assume that the characteristic of \(k\) is 0. The integers \(k\) such that \(\mathcal{D}^k\) descends to \(\mathcal{M}_G(d)\) are the multiple of \(s/(s,r/s)\).

By the proposition 1.5 of [BLS], one gets the
Corollary 5.8. — The natural morphism

\[ \text{Pic}(\mathcal{M}_C(d)) \rightarrow \text{Pic}(\mathcal{M}_{\text{SL}_r}(d)) = \mathbb{Z} \cdot \mathcal{D} \]

makes the Picard group of \( \mathcal{M}_C(d) \) an extension of \( \mathbb{Z} = \mathbb{Z} \cdot \mathcal{D}^{s/(s,r/s)} \) by \( H^1_{\text{ét}}(X, \mathbb{Z}/d\mathbb{Z}) \sim (\mathbb{Z}/d\mathbb{Z})^{3g} \).

Proof of the theorem.— By lemma 5.5 and diagram (5.4.1), we just have to prove that \( \mathcal{D}^k \) effectively descends when \( k = s/(s,r/s) \). By theorem 4.1 and lemma 5.3, this means exactly that \( \mathcal{D}^k \) has a \( \mathcal{J}_s \)-linearization. We know by lemma 5.5 that the pull-back \( f^* \mathcal{D}^k \) has such a linearization.

Lemma. — The pull-back morphism

\[ \text{Pic}(\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d)) \rightarrow \text{Pic}(\mathcal{J}_s \times \mathcal{J}(X)) \]

is injective.

Proof. — By lemma 3.1, one is reduced to prove that the natural morphism

\[ \text{Pic}(B\mu_s \times \mathcal{M}_{\text{SL}_r}(d)) \rightarrow \text{Pic}(B\mu_s \times \mathcal{J}(X)) \]

is injective. Let \( \mathcal{X} \) be any stack. The canonical morphism \( \mathcal{X} \rightarrow \mathcal{X} \times B\mu_s \) is a \( \mu_s \)-torsor (with the trivial action of \( \mu_s \) on \( \mathcal{X} \)). By theorem 4.1, one has the equality

\[ \text{Pic}(\mathcal{X} \times B\mu_s) = \text{Pic}^{\mu_s}(\mathcal{X}). \]

Assume further that \( H^0(\mathcal{X}, \mathcal{O}) = k \). The later group is then canonically isomorphic to

\[ \text{Pic}(\mathcal{X}) \times \text{Hom}(\mu_s, \mathbb{G}_m) = \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s). \]

Eventually, we get a functorial isomorphism

\[ (5.9.1) \quad \iota_{\mathcal{X}} : \text{Pic}(\mathcal{X} \times B\mu_s) \sim \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s). \]

By [L-S], the Picard group of \( \mathcal{M}_{\text{SL}_r}(d) \) is the free abelian group \( \mathbb{Z} \cdot \mathcal{D} \) and the formula (3.1.1) proves that

\[ f^* : \text{Pic}(\mathcal{M}_{\text{SL}_r}(d)) \rightarrow \text{Pic}(\mathcal{J}(X)) \]
is an injection. The diagram
\[
\begin{array}{ccc}
\text{Pic}(\mathcal{M}_{\text{SL}_r}(d)) \times \text{Pic}(B\mu_s) & \hookrightarrow & \text{Pic}(\mathcal{J}(X)) \times \text{Pic}(B\mu_s) \\
\varepsilon_{\mathcal{M}} \downarrow & & \varepsilon_{\mathcal{J}} \downarrow \\
\text{Pic}(\mathcal{M}_{\text{SL}_r}(d) \times B\mu_s) & \longrightarrow & \text{Pic}(\mathcal{J}(X) \times B\mu_s)
\end{array}
\]
is commutative and the lemma follows from this commutative diagram. 

Let \( H \) (resp. \( H_J \)) be the line bundle on \( \mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) \) (resp. \( \mathcal{J}_s \times \mathcal{J}(X) \))
\[
H = \text{Hom}(m^*_M D^k, \text{pr}_2^* D^k)
\]
(resp. \( H_J = \text{Hom}(m^*_M f^* D^k, \text{pr}_2^* f^* D^k) \)).

Let us choose a \( \mathcal{J}_s \)-linearization \( \lambda_J \) of \( f^* D^k \). It defines a trivialization of the line bundle \( H_J \). The equivariance of \( f \) implies (cf. 2.3) that there exists a (compatible) 2-morphism
\[
q : m_M \circ (\text{Id} \times f) \Longrightarrow f \circ m_J
\]
making the diagram
\[
\begin{array}{ccc}
\mathcal{J}_s \times \mathcal{J}(X) & \longrightarrow & \mathcal{J}(X) \\
\text{Id} \times f \downarrow & & f \downarrow \\
\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) & \longrightarrow & \mathcal{M}_{\text{SL}_r}(d)
\end{array}
\]
2-commutative. The 2-morphism \( q \) defines an isomorphism from the pull-back \( m^*_M D^k \) on \( \mathcal{J}_s \times \mathcal{J}(X) \) to \( m^*_M (f^* D^k) \). The pull-back of \( \text{pr}_2^* D^k \) on \( \mathcal{J}_s \times \mathcal{J}(X) \) is tautologically isomorphic to \( \text{pr}_2^* (f^* D^k) \). The preceding isomorphisms induce an isomorphism
\[
(\text{Id} \times f)^* H \sim H_J.
\]
The later line bundle being trivial, so is \( (\text{Id} \times f)^* H \). The lemma above proves therefore that \( H \) itself is trivial. Each \((k-)\)point \( j \) of \( \mathcal{J}_s \) defines a morphism
\[
\mathcal{M}_{\text{SL}_r}(d) \rightarrow \mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) \quad \text{(resp.} \mathcal{J}(X) \rightarrow \mathcal{J}_s \times \mathcal{J}(X));
\]
let me denote by \( H_j \) (resp. \( f^* H_j \)) the pull-back of \( H \) (resp. \( (\text{Id} \times f)^* H \)) by this morphism. The pull-back morphism
\[
H^0(\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d), H) \longrightarrow H^0(\mathcal{J}_s \times \mathcal{J}(X), (\text{Id} \times f)^* H)
\]
can be identified to the direct sum
\[ \bigoplus_{j \in J_s(k)} H^0(\mathcal{M}_{\text{SL}_r}(d), \mathcal{H}_j) \rightarrow H^0(\mathcal{J}(X), f^*\mathcal{H}_j). \]

As
\[ (5.9.2) \quad H^0(\mathcal{M}_{\text{SL}_r}(d), \mathcal{O}) = H^0(\mathcal{J}(X), \mathcal{O}) = k, \]
this morphism is a direct sum of non-zero homomorphisms of vector spaces of dimension 1 and therefore an isomorphism. In particular, a linearization \( \lambda_{\mathcal{J}} \) of \( f^*\mathcal{D}^k \) defines canonically an isomorphism
\[ \lambda_M : m_{\mathcal{M}}^*\mathcal{D}^k \sim \text{pr}_2^*\mathcal{D}^k \]
such that \( (\text{Id} \times f)^*\lambda_M = \lambda_{\mathcal{J}}. \)

Explicitly, \( \lambda_M \) is characterized as follows. Let \( x \) be an object of \( \mathcal{M}_{\text{SL}_r}(d) \) over a connected scheme \( S \) and \( g \) an object of \( J_s(S) = J_s(k) \). The preceding discussion means that the functorial isomorphisms
\[ \lambda_M(g, x) : \mathcal{D}_{g,x}^k \rightarrow \mathcal{D}_x^k \]
are determined when \( x \) lies in the essential image of \( f \). In this case, let us choose an isomorphism \( f(x') \sim x \) (inducing an isomorphism \( g \cdot f(x') \sim g \cdot x \). Then, the diagram of isomorphisms of line bundles on \( S \)
\[
\begin{array}{ccc}
L'_{x'} & \cong & L_{f(x')} \\
\downarrow \lambda_{\mathcal{J}}(g, x') & & \downarrow \lambda_M(g, x) \\
L'_{g \cdot x'} & \xrightarrow{q_{g, x'}} & L_{g \cdot f(x')} & \rightarrow & L_{g \cdot x}
\end{array}
\]
is commutative (where \( L = \mathcal{D}^k \) and \( L' = f^*\mathcal{D}^k \)).

Now, the pull-back of \( \lambda_M \) on \( J_s \times \mathcal{J}(X) \) satisfies conditions (2.8.1) and (2.8.2). Using (5.9.2) and the equivariance of \( f \) as above, this shows that \( \lambda_M \) is a linearization. For instance, keeping the notation above, let us verify condition (2.8.2). We have to check that the isomorphism \( \iota \) of \( L \) induced by \( \epsilon \) is the identity. As above, it is enough to check that on \( \mathcal{J}(X) \). With a slight abuse of notations, the two diagrams
\[
\begin{array}{ccc}
L'_{x'} & \xrightarrow{L'_{f(x')}} & L_x \\
\downarrow \lambda_{\mathcal{J}}(1, x') & & \downarrow \lambda_M(1, x) \\
L_{1, x'} & \xrightarrow{q_{1, x'}} & L_{1, f(x')} & \rightarrow & L_{1, x}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L_{x} & \xrightarrow{\iota} & L_x \\
\downarrow \lambda_M(1, x) & & \downarrow \epsilon(x) \\
L_{1, x} & & \text{and} \quad \epsilon(x)
\end{array}
\]

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are commutative (the commutativity of the first diagram follows from
the equivariance of \( f \) and the commutativity of the second diagram
follows exactly from the definition of \( \iota \)). Because \( \lambda_J \) is a linearization,
condition (2.8.2) shows that the diagram

\[
\begin{array}{ccc}
L_x' & \rightarrow & L_x' \\
\lambda_J(x') & \downarrow & \downarrow \epsilon(x') \\
L_1 \cdot x' & \leftarrow &
\end{array}
\]

is commutative. It follows that the equality \( \iota = \text{Id} \) will follow from the
commutativity of the diagram

\[
\begin{array}{ccc}
L_f(x') & \rightarrow & L_f(x') \\
\epsilon' & \downarrow & \downarrow \\
q_1 \cdot x' & \rightarrow & L_1 \cdot f(x')
\end{array}
\]

(5.9.3)

But \( q \) is compatible with \( \epsilon \) as required in 2.3. The diagram

\[
\begin{array}{ccc}
f(1 \cdot x') & \rightarrow & f(x') \\
\epsilon' & \downarrow & \downarrow \\
q_1 \cdot x' & \rightarrow & 1 \cdot f(x')
\end{array}
\]

is therefore commutative from which the commutativity of (5.9.3) follows.
One would check condition (2.8.1) in an analogous way. \( \square \)

**Remark 5.10.** — In the case \( g = 0 \), the condition is an in remark 5.6.

**Remark 5.11.** — This linearization can be certainly also deduced from
a careful analysis of the first section of [Fa], but the method above seems
simpler.

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