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Linearization of group stack actions and the Picard group of the moduli of $SL_r/\mu_s$-bundles on a curve


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LINEARIZATION OF GROUP STACK ACTIONS AND
THE PICARD GROUP OF THE MODULI OF
SL_r/\mu_s-BUNDLES ON A CURVE
PAR YVES LASZLO (*)

ABSTRACT. — In this paper, we consider morphisms of algebraic stacks \( X \to Y \)
which are torsors under a group stack \( \mathcal{G} \). We show that line bundles on \( Y \) correspond
exactly with \( \mathcal{G} \)-linearized line bundles on \( X \) (with a suitable definition of a \( \mathcal{G} \)
linearization). We use this fact to determine the precise structure of the Picard group
of the moduli stack of \( G \)-bundles on an algebraic curve when \( G \) is any group of type \( A_n \).

1. Introduction

Let \( G \) be a complex simple group and \( \tilde{G} \to G \) the universal covering.
For simplicity, let us consider the moduli stack \( \mathcal{M}_G \) (resp. \( \mathcal{M}_{\tilde{G}} \)) of degree
1 \( \in \pi_1(G) \) principal \( G \)-bundles (resp. \( \tilde{G} \)-bundles) over a curve \( X \). In
[B-L-S], we have studied the natural morphism

\[ \iota : \text{Pic}(\mathcal{M}_G) \to \text{Pic}(\mathcal{M}_{\tilde{G}}), \]

the group \( \text{Pic}(\mathcal{M}_{\tilde{G}}) \) being infinite cyclic by [L-S]. It is proved in
[B-L-S] that the kernel of \( \iota \) is naturally identified with the finite group
\( H^1_{\text{et}}(X, \pi_1(G)\vee) \) reducing the study of \( \text{Pic}(\mathcal{M}_G) \) to the computation of
the cardinality of \( \text{Coker}(\iota) \). Among other things, it has been possible to

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perform this computation in the case where $G = \text{PSL}_r$ but not in the case where $G = \text{SL}_r/\mu_s$, where $s \mid r$, although we were able to give partial results. The reason was that the technical background to study the descent of modules through the morphism $p : \mathcal{M}_G \rightarrow \mathcal{M}_G$ wasn’t at our disposal.

The aim of this paper is to compute $\text{card Coker}(\iota)$ when $G = \text{SL}_r/\mu_s$.

It turns out to be that $p$ is a torsor under some group stack, not far from a Galois étale cover in the usual schematic picture. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a torsor under a group scheme $\mathcal{G}$. We know that a line bundle on $\mathcal{X}$ descends if and only if it has a $\mathcal{G}$-linearization (easy consequence of descent theory).

Now, the descent theory of Grothendieck has been adapted to the set-up of fpqc morphisms of stacks in [L-M]. If $\mathcal{G}$ is now only assumed to be a group stack, there is a notion of $\mathcal{G}$-linearization of line bundles on $\mathcal{X}$ (see section 2). One obtains (theorem 4.1) that a line bundle on $\mathcal{X}$ descends if and only if it admits a linearization.

We then use this technical result to compute $\text{card Coker}(\iota)$ when $G = \text{SL}_r/\mu_s$ (theorem 5.7 and its corollary).

I would like to thank L. Breen for having taught me both the notions of torsor and of linearization of a vector bundle in the set-up of group-stack actions and for his comments on a preliminary version of this paper.

**Notations.**

Throughout this paper, all the stacks will be implicitly assumed to be algebraic over an algebraically closed field $k$ and the morphisms locally of finite type. We fix once and for all a projective, smooth, connected genus $g$ curve $X$ and a closed point $x$ of $X$. For simplicity, we assume $g > 0$ (see remarks 5.6 and 5.10 for the case of $\mathbb{P}^1$). The Picard stack parametrizing families of line bundles of degree 0 on $X$ will be denoted by $\mathcal{J}(X)$ and the jacobian variety of $X$ by $J_X$. If $G$ is an algebraic group over $k$, the quotient stack $\text{Spec}(k)/G$ (where $G$ acts trivially on $\text{Spec}(k)$) whose category over a $\mathcal{S}$-scheme $\mathcal{S}$ is the category of $G$-torsors (or $G$-bundles) over $\mathcal{S}$ will be denoted by $BG$. If $n$ is an integer and $A = \mathcal{J}(X), J_X$ or $BG_m$ we denote by $n_A$ the $n^{th}$-power morphism $a \mapsto a^n$. We denote by $J_n$ (resp. $J_n$) the 0-fiber $A \times_A \text{Spec}(k)$ of $n_A$ when $A = \mathcal{J}(X)$ (resp. $A = J_X$).

1. **Generalities.** — Following [Br], for any diagram

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \uparrow_{\lambda} \\
C & \xrightarrow{\ell} & D
\end{array}
$$

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of 2-categories, we will denote by

$$\ell \ast \lambda : \ell \circ f \Rightarrow \ell \circ g \quad \text{(resp. } \lambda \ast h : f \circ h \Rightarrow g \circ h)$$

the 2-morphism deduced from $\lambda$.

1.1. — For the convenience of the reader, let us prove a simple formal lemma which will be useful in section 4. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three 2-categories. Let diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\downarrow{\delta_0} & & \downarrow{d_0} \\
\mathcal{C} & \xleftarrow{\delta_1} & \downarrow{d_1} \\
\end{array}
$$

be a 2-commutative diagram and let $\mu : \delta_0 \Rightarrow \delta_1$ be a 2-morphism.

**Lemma 1.2.** Assume that $f$ is an equivalence. There exists a unique 2-morphism $\mu \ast f^{-1} : d_0 \Rightarrow d_1$ such that $(\nu, \mu \ast f^{-1}) \ast \mu = \delta_0$.

**Proof.** Let $\epsilon_k$, for $k = 0, 1$ be the 2-morphism $d_k \circ f \Rightarrow \delta_k$. Let $b$ be an object of $\mathcal{B}$. Pick an object $a$ of $\mathcal{A}$ and an isomorphism $\alpha : f(a) \cong b$. Let $\varphi_\alpha : d_0(b) \cong d_1(b)$ be the unique isomorphism making the diagram

$$
\begin{array}{ccc}
\delta_0(a) & \xrightarrow{\epsilon_0(a)} & d_0 \circ f(a) \\
\downarrow{\mu_\alpha} & & \downarrow{d_0(\alpha)} \\
\delta_1(a) & \xrightarrow{\epsilon_1(a)} & d_1 \circ f(a) \\
\end{array}
$$

commutative. We have to show that $\varphi_\alpha$ does not depend on $\alpha$ but only on $b$. Let $\alpha' : f(a') \cong b$ be another isomorphism. There exists a unique isomorphism $\iota : a' \cong a$ such that $\alpha \circ f(\iota) = \alpha'$. Then one has the equality

$$\varphi_{\alpha'} = d_1(\alpha) \circ \Phi \circ d_0(\alpha)^{-1}$$

where

$$\Phi = [d_1 \circ f(\iota)] \circ \epsilon_1(a') \circ \mu_{a'} \circ \epsilon_0(a')^{-1} \circ [d_0 \circ f(\iota)]^{-1}.$$
and

\[ \mu_a = \delta_1(i) \circ \mu_{a'} \circ \delta_0(i)^{-1}. \]

This shows the equality

\[ \Phi = \epsilon_1(a) \circ \mu_a \circ \epsilon_0(a)^{-1} \]

which proves the equality \( \varphi_a = \varphi_{a'} \). We can therefore define \( \mu_b \) to be the isomorphism \( \varphi_a \) for one isomorphism \( \alpha : f(a) \sim b \). We check that the construction is functorial in \( b \) and the lemma follows.

2. Linearizations of line bundles on stacks.

Let us first recall the notion of torsor in the stack context according to [Br].

2.1. — Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a faithfully flat morphism of stacks. Let us assume that an algebraic \( \mathcal{G} \)-stack \( \mathcal{G} \) acts on \( f \) (the product of \( \mathcal{G} \) is denoted by \( m_{\mathcal{G}} \) and the unit object by 1). Following [Br], this means that there exists a 1-morphism of \( \mathcal{Y} \)-stacks \( m : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X} \) and a 2-morphism \( \mu : m \circ (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) \Rightarrow m \circ (\text{Id}_{\mathcal{G}} \times m) \) such that the obvious associativity condition (see diagram (6.1.3) in [Br]) is satisfied and such that there exists a 2-morphism \( \epsilon : m \circ (1 \times \text{Id}_{\mathcal{X}}) \Rightarrow \text{Id}_{\mathcal{X}} \) which is compatible to \( \mu \) in the obvious sense (see (6.1.4) of [Br]).

**Remark 2.2.** — To say that \( m \) is a morphism of \( \mathcal{Y} \)-stacks means that the diagram

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{X} & \xrightarrow{m} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \searrow & \mathcal{Y}
\end{array}
\]

is 2-commutative. In other words, if we denote for simplicity the image of a pair of objects \( m(g, x) \) by \( g \cdot x \). This means that there exists a functorial isomorphism \( \iota_{g,x} : f(g \cdot x) \rightarrow f(x) \).

2.3. — Suppose that \( \mathcal{G} \) acts on such another morphism \( f' : \mathcal{X}' \rightarrow \mathcal{Y} \). A morphism \( p : \mathcal{X}' \rightarrow \mathcal{X} \) will be said **equivariant** if there exists a 2-morphism

\[ q : m \circ (\text{Id} \times p) \Rightarrow p \circ m' \]

which is compatible to \( \mu \) (as in [Br, (6.1.6)]) and \( \epsilon \) (which is implicit in [Br]) in the obvious sense.

**Definition 2.4.** — With the above notations, we say that \( f \) (or \( \mathcal{X} \)) is a \( \mathcal{G} \)-torsor over \( \mathcal{Y} \) if the morphism \( \text{pr}_X \times m : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{Y} \mathcal{X} \) is an isomorphism (of stacks) and the geometrical fibers of \( f \) are not empty.
Remark 2.5. — In down to earth terms, this means that if

\[ \iota : f(x) \rightarrow f(x') \]

is an isomorphism in \( \mathcal{Y} \) (\( x, x' \) being objects of \( \mathcal{X} \)), there exists an object \( g \) of \( G \) and a unique isomorphism \( (x, g \cdot x) \sim (x, x') \) which induces \( \iota \) by way of \( \iota_{g,x} \) (cf. 2.2).

Example 2.6. — Let \( \mathcal{M}_X(G_m) \) be the Picard stack of \( X \). Then, the morphism

\[ \mathcal{M}_X(G_m) \rightarrow \mathcal{M}_X(G_m) \]

of multiplication by \( n \in \mathbb{Z} \) is a torsor under \( B\mu_n \times J_n(X) \) (cf. (3.1)).

2.7. — Let \( \mathcal{L} \) be a line bundle on \( \mathcal{X} \). By definition, the datum \( \mathcal{L} \) is equivalent to the datum of a morphism \( \ell : \mathcal{X} \rightarrow BG_m \) (see [L-M, prop. 6.15]). If \( \mathcal{L}, \mathcal{L}' \) are two line bundles on \( \mathcal{X} \) defined by \( \ell, \ell' \), we will view an isomorphism \( \mathcal{L} \sim \mathcal{L}' \) as a 2-morphism \( \ell \Rightarrow \ell' \).

Definition 2.8. — A \( G \)-linearization of \( \mathcal{L} \) is a 2-morphism

\[ \lambda : \ell \circ m \Rightarrow \ell \circ \text{pr}_2 \]

such that the two diagrams of 2-morphisms

\[
\begin{align*}
\ell \circ m \circ (m \times \text{Id}_\mathcal{X}) & \xrightarrow{\ell \times \mu} \ell \circ m \circ (\text{Id}_G \times m) \\
\lambda \times (m \times \text{Id}_\mathcal{X}) & \downarrow \quad \quad \downarrow \lambda \times (\text{Id}_G \times m) \\
\ell \circ \text{pr}_2 \circ (m \times \text{Id}_\mathcal{X}) & \xleftarrow{\lambda \times \text{pr}_{23}} \ell \circ \text{pr}_2 \circ (\text{Id}_G \times m) \\
\ell \circ \text{pr}_2 \circ \text{pr}_{23} & \quad \quad \quad \ell \circ m \circ \text{pr}_{23}
\end{align*}
\]

(2.8.1)

and

\[
\begin{align*}
\ell \circ m \circ (1 \times \text{Id}_\mathcal{X}) & \xrightarrow{\ell \times e} \ell \\
\lambda \times (1 \times \text{Id}_\mathcal{X}) & \downarrow \quad \quad \downarrow \\
\ell & \quad \quad \ell
\end{align*}
\]

(2.8.2)

(strictly) commute.
REMARK 2.9. — If \( g_1, g_2 \) are objects of \( G \) and \( d \) is an object of \( X \), the commutativity of diagram (2.8.1) means that the diagram

\[
\begin{array}{ccc}
\mathcal{L}_{(g_1, g_2)x} & \xrightarrow{\sim} & \mathcal{L}_{g_1(g_2, x)} \\
\downarrow & & \downarrow \\
\mathcal{L}_x & \xleftarrow{\sim} & \mathcal{L}_{g_2, x}
\end{array}
\]

is commutative and the commutativity of (2.8.2) that the two isomorphisms \( \mathcal{L}_{1,x} \simeq \mathcal{L}_x \) defined by the linearization \( \lambda \) and \( \epsilon \) respectively are the same.

3. An example.

Let me recall that a closed point \( x \) of \( X \) has been fixed. Let \( S \) be a \( k \)-scheme. The \( S \)-points of the jacobian variety of \( X \) are by definition isomorphism classes of line bundles on \( X_S \) together with a trivialization along \( \{x\} \times S \) (such a pair will be called a rigidified line bundle). For the convenience of the reader, let me state this well known lemma which can be found in SGA4, exp. XVIII, (1.5.4).

**Lemma 3.1.** — The Picard stack \( \mathcal{J}(X) \) is canonically isomorphic (as a \( k \)-group stack) to \( JX \times BG_m \).

**Proof.** — Let \( f : \mathcal{J}(X) \rightarrow JX \times BG_m \) be the morphism which associates

- to the line bundle \( L \) on \( X_S \) the pair \((L \boxtimes L^{-1}_{\{x\} \times S}, L^{\{x\} \times S})\) where \( \boxtimes \) denotes the external tensor product (this pair is thought of as an object of \( JX \times BG_m \) over \( S \));
- to an isomorphism \( L \overset{\sim}{\rightarrow} L' \) on \( X_S \) its restriction to \( \{x\} \times S \).

Let \( f' : JX \times BG_m \rightarrow \mathcal{J}(X) \) be the morphism which associates

- to the pair \((L, V)\) where \( L \) is a rigidified bundle on \( X_S \) and \( V \) a line bundle on \( S \) (thought of as an object of \( JX \times BG_m \) over \( S \)), the line bundle \( L \boxtimes X_S V \);
- to an isomorphism \((\ell, v) : (L, V) \overset{\sim}{\rightarrow} (L', V')\) the tensor product \( \ell \boxtimes X_S v \).

The morphisms \( f \) and \( f' \) are (quasi)-inverse to each other and are morphisms of \( k \)-stacks. \( \square \)

We will identify from now \( \mathcal{J}(X) \) and \( JX \times BG_m \). Let \( \mathcal{L} \) (resp. \( \mathcal{P} \) and \( T \)) be the universal bundle on \( X \times \mathcal{J}(X) \) (resp. on \( X \times JX \) and \( BG_m \)) and let \( \Theta = (\det R\Gamma \mathcal{P})^{-1} \) be the theta line bundle on \( JX \). The isomorphism \( \mathcal{L} \overset{\sim}{\rightarrow} \mathcal{P} \boxtimes T \) yields an isomorphism

\[
(3.1.1) \quad \det R\Gamma \mathcal{L}^m(m \cdot x) \overset{\sim}{\rightarrow} \Theta^{-n^2} \boxtimes T^{(m+1-g)}.
\]

The object of this section is to prove the following statement.

**Theorem 4.1.** — Let $f : \mathcal{X} \to \mathcal{Y}$ be a $G$-torsor as above. Let $\text{Pic}^G(\mathcal{X})$ be the group of isomorphism classes of $G$-linearized line bundles on $\mathcal{X}$. Then, the pull-back morphism $f^* : \text{Pic}(\mathcal{Y}) \to \text{Pic}(\mathcal{X})$ is an isomorphism.

The descent theory of Grothendieck has been adapted in the case of algebraic 1-stacks in [L-M], essentially in proposition (6.23).

Let $\mathcal{X}, \to \mathcal{Y}$ be the (augmented) simplicial complex of stacks coskeleton of $f$ (as defined in [De, (5.1.4)]) for instance). By proposition (6.23) of [L-M], one just has to construct a cartesian $\mathcal{O}_{D_*}$-module $\mathcal{L}_*$ such that $\mathcal{L}_0$ is the $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ to prove the theorem. The $n$-th piece $\mathcal{X}_n$ is inductively defined by

$$\mathcal{X}_0 = \mathcal{X}, \quad \mathcal{X}_n = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}_{n-1} \quad \text{for} \quad n > 0.$$ 

Let $p_n : \mathcal{X}_n \to \mathcal{X}$ be the projection on the first factor. It is the simplicial morphism associated to the map

$$\tilde{p}_n : \{ \Delta_0 \to \Delta_n, \quad 0 \to 0 \}.$$

Let $\mathcal{L}_n$ be the line bundle defined by the morphism (see (2.7))

$$\ell_n : \mathcal{X}_n \xrightarrow{p_n} \mathcal{X} \xrightarrow{\ell} BG_m.$$ 

4.2. — Let $\delta_i$ (resp. $s_j$) be the face (resp. degeneracy) operators (see for instance [De, 5.1.1]). By abuse of notation, we use the same notation for $\delta_j, s_j$ and their image by $\mathcal{X})$. The category $(\Delta_*)$ is generated by the face and degeneracy operators with the following relations (see for instance the proposition VII.5.2, p. 174 of [McL])

\begin{align*}
(4.2.1) & \quad \delta_i \circ \delta_j = \delta_{j+1} \circ \delta_i \quad \text{if} \quad i \leq j, \\
(4.2.2) & \quad s_j \circ s_i = s_i \circ s_{j+1} \quad \text{if} \quad i \leq j, \\
(4.2.3) & \quad s_j \circ \delta_i = \begin{cases} 
\delta_i \circ s_{j-1} & \text{if} \quad i < j, \\
1 & \text{if} \quad i = j, \ i = j + 1, \\
\delta_{i-1} \circ s_j & \text{if} \quad i > j + 1.
\end{cases}
\end{align*}

Therefore, the datum of a cartesian $\mathcal{O}_{\mathcal{X}_*}$-module $\mathcal{L}_*$ is equivalent to the data of isomorphisms

$$\alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1}, \quad j = 0, \ldots, n + 1,$$

$$\beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n, \quad j = 0, \ldots, n,$$
(where \( n \) is a non negative integer) which are compatible with relations (4.2.1), (4.2.2) and (4.2.32).

Let \( n \) be a non negative integer.

4.3. — We have first to define for \( j = 0, \ldots, n + 1 \) an isomorphism

\[
\alpha_j : \delta_j^* \mathcal{L}_n \sim \mathcal{L}_{n+1}.
\]

The line bundle \( \delta_j^* \mathcal{L}_n \) is defined by the morphism \( \ell \circ p_n \circ \delta_j : \mathcal{X}_{n+1} \to BG_m \)
and \( \tilde{p}_n \circ \delta_j \) is associated to the map

\[
\begin{cases}
\Delta_0 \to \Delta_{n+1}, \\
0 \to \delta_j(0).
\end{cases}
\]

- If \( j \neq 0 \), one has therefore \( \tilde{p}_n \circ \delta_j = \tilde{p}_{n+1} \) and \( \delta_j^* \mathcal{L}_n = \mathcal{L}_{n+1} \). We define \( \alpha_j \) by the identity in this case.
- Suppose now that \( j = 0 \). Let \( \pi_n : \mathcal{X}_n \to \mathcal{X}_1 \) be the projection on the two first factors (associated to the canonical inclusion \( \Delta_1 \hookrightarrow \Delta_n \)). The commutativity of the two diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{n+1} & \overset{\delta_0}{\longrightarrow} & \mathcal{X}_n \\
\downarrow{\pi}_{n+1} & & \downarrow{p}_n \\
\mathcal{X}_1 & \overset{\delta_0}{\longrightarrow} & \mathcal{X}
\end{array}
\hspace{1cm}
\begin{array}{ccc}
\mathcal{X}_{n+1} & \overset{p_{n+1}}{\longrightarrow} & \mathcal{X} \\
\downarrow{\pi}_{n+1} & & \uparrow{\delta_1} \\
\mathcal{X}_1 & = & \mathcal{X}_1
\end{array}
\]

allows to reduce the problem to the construction of an isomorphism

\[
\delta_0^* \mathcal{L} \sim \delta_1^* \mathcal{L}
\]

where \( \delta_i : \mathcal{X}_1 \to \mathcal{X} \) for \( i = 0, 1 \) are the face morphisms or, what amounts to the same, to the construction of a 2-morphism \( \nu : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1 \) (the morphism \( \alpha_j \) will be \( \alpha_j = \nu \circ \pi_{n+1} \)). Now the diagram

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{X} & \overset{pr_2 \times m}{\longrightarrow} & \mathcal{X} \times \mathcal{X} \\
\downarrow{\ell \circ \text{opr}_2} & & \downarrow{\ell \circ \delta_1} \\
BG_m & \overset{\ell \circ \delta_0}{\longrightarrow} & BG_m
\end{array}
\tag{4.3.1}
\]

is strictly commutative and \( pr_2 \times m \) is an equivalence by definition of a torsor. According to lemma 1.2, the 2-morphism \( \lambda \) induces a canonical 2-morphism

\[
\lambda * (pr_2 \times m)^{-1} : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1
\]

which is the required 2-morphism \( \nu \).

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4.4. — We then have to define for \( j = 0, \ldots, n \) an isomorphism

\[ \beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n. \]

The line bundle \( s_j^* \mathcal{L} \) is defined by the morphism \( \ell \circ p_{n+1} \circ s_j \) and \( p_{n+1} \circ s_j \) is associated to the canonical inclusion \( \Delta_0 \hookrightarrow \Delta_n \) which means \( p_{n+1} \circ s_j = p_n \). Therefore, one has a canonical isomorphism \( s_j^* \mathcal{L}_{n+1} = \mathcal{L}_n \) and we define \( \beta_j \) to be the identity.

4.5. — We have to show that the data \( \mathcal{L}_n \) and \( \alpha_j, \beta_j \), for \( j \geq 0 \) define a line bundle on the simplicial stack \( \mathcal{X}_\bullet \) as explained in 4.2. Notice that the fact that the definition of the \( \beta_j \) is compatible with relations (4.2.2) is tautological (\( \beta_j \) is the identity on the relevant \( \mathcal{L}_n \)).

4.6. — In terms of \( \ell \), relation (4.2.1) means the following. We have the two strictly 2-commutative diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{n+2} & \xrightarrow{\delta_i} & \mathcal{X}_{n+1} \\
\downarrow p_{n+2} & & \downarrow p_{n+1} \\
\mathcal{X} & \xrightarrow{\ell} & BG_m \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{X}_{n+2} & \xrightarrow{\delta_{j+1}} & \mathcal{X}_{n+1} \\
\downarrow p_{n+2} & & \downarrow p_{n+1} \\
\mathcal{X} & \xrightarrow{\ell} & BG_m \\
\end{array}
\]

inducing the two 2-morphisms

\[ \alpha_i \circ (\alpha_j \ast \delta_i) : \ell \circ p_n \circ \delta_j \circ \delta_i \xrightarrow{\alpha_j \ast \delta_i} \ell \circ p_{n+1} \circ \delta_i \xrightarrow{\alpha_i} \ell \circ p_{n+2} \]

and

\[ \alpha_{j+1} \circ (\alpha_i \ast \delta_{j+1}) : \ell \circ p_n \circ \delta_i \circ \delta_{j+1} \xrightarrow{\alpha_i \ast \delta_{j+1}} \ell \circ p_{n+1} \circ \delta_{j+1} \xrightarrow{\alpha_{j+1}} \ell \circ p_{n+2}. \]

The relation (4.2.1) means exactly the equality

\[ (4.2.1') \quad \alpha_i \circ (\alpha_j \ast \delta_i) = \alpha_{j+1} \circ (\alpha_i \ast \delta_{j+1}), \quad \text{for } i \leq j. \]

- If \( j = 0 \), the relation (4.2.1') is just by definition of \( \alpha_j \) the condition (2.8.1) (see remark 2.9).
- If \( j > 0 \), both isomorphisms \( \alpha_j \) and \( \alpha_{j+1} \) are the relevant identities and the relation (4.2.1') is tautological.
4.7. — The only non tautological relation in (4.2.3) corresponds to the equality \( s_0 \circ \delta_0 = 1 \) in (\( \Delta_* \)) which means as before that \( \alpha_0 \circ \delta_0 \) is the identity functor of \( \ell \circ p_n = \ell \circ p_n \circ \delta_0 \circ s_0 \). But, this is exactly the meaning of the relation (2.8.2) (see remark 2.9).

5. Application to the Picard groups of some moduli spaces.

Let us choose three integers \( r, s, d \) such that

\[
r \geq 2 \quad \text{and} \quad s \mid r \mid ds.
\]

If \( G \) is the group \( \text{SL}_r/\mu_s \) we denote as in [B-L-S] by \( \mathcal{M}_G(d) \) the (connected) moduli stack of \( G \)-bundles on \( X \) of degree \( \exp(2i\pi d/r) \in H^2_{et}(X, \mu_s) = \mu_s \) and by \( \mathcal{M}_{\text{SL}_r}(d) \) the moduli stack of rank \( r \) vector bundles and determinant \( \mathcal{O}(d \cdot x) \). If \( r = s \) (i.e. \( G = \text{PSL}_r \)), the natural morphism of algebraic stacks

\[
\pi : \mathcal{M}_{\text{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)
\]

is a \( \mathcal{J}_r \)-torsor (see the corollary of proposition 2 of [Gr] for instance). Let me explain how to deal with the general case.

5.1. — Let \( E \) be a rank \( r \) vector bundle on \( X_S \) endowed with an isomorphism

\[
\tau : D^{r/s} \sim \det(E)
\]

where \( D \) is some line bundle. Let me define the \( \text{SL}_r/\mu_s \)-bundle \( \pi(E) \) associated to \( E \) (more precisely to the pair \( (E, \tau) \)).

**Definition 5.2.**

- An \emph{s-trivialization} of \( E \) on an étale neighborhood \( T \to X_S \) is a triple \((M, \alpha, \sigma)\) where
  \[
  \alpha : D \sim M^s \text{ is an isomorphism (} M \text{ is a line bundle on } T; \\
  \sigma : M^{\oplus r} \sim E_T \text{ is an isomorphism; } \\
  \text{det}(\sigma) \circ \alpha^{r/s} : D^{r/s} \sim \det(E) \text{ is equal to } \tau.
  \]
- Two \emph{s-trivializations} \((M, \alpha, \sigma)\) and \((M', \alpha', \sigma')\) of \( E \) will be said \emph{equivalent} if there exists an isomorphism \( \iota : M \sim M' \) such that \( \iota^s \circ \alpha = \alpha' \).

The principal homogeneous space

\[
T \longmapsto \{ \text{equivalence classes of } s\text{-trivializations of } E_T \}
\]
defines the $\text{SL}_r/\mu_s$-bundle $\pi(E)$. Now, the construction is obviously functorial and therefore defines the morphism $\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$ (observe that an object $E$ of $\mathcal{M}_{\text{SL}_r}(d)$ has determinant $O(ds/r \cdot x)^{r/s}$).

Let $L$ be a line bundle and $(M,\alpha,\tau)$ an $s$-trivialization of $E_T$. Then, $(M \otimes L, \alpha \otimes \text{Id}_L, \sigma \otimes \text{Id}_L)$ is an $s$-trivialization of $E \otimes L$ (which has determinant $(D \otimes L^s)^{r/s}$). This shows that there exists a canonical functorial isomorphism

$$(5.2.1) \quad \pi(E) \xrightarrow{\sim} \pi(E \otimes L)$$

In particular, $\pi$ is $\mathcal{J}_s$-equivariant.

**Lemma 5.3.** — The natural morphism of algebraic stacks

$$\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$$

is a $\mathcal{J}_s$-torsor.

**Proof.** — Let $E, E'$ be two rank $r$ vector bundles on $X_s$ (with determinant equal to $O(d \cdot x)$) and let $\iota : \pi(E) \xrightarrow{\sim} \pi(E)'$ be an isomorphism. As in the proof of the lemma 13.4 of [B-L-S], we have the exact sequence of sets

$$1 \to \mu_s \to \text{Isom}(E, E') \to \text{Isom}(\pi(E), \pi(E)') \xrightarrow{\pi_{E,E'}} H^1_{\mathbb{A}}(X_S, \mu_s).$$

Let $L$ be a $\mu_s$-torsor such that $\pi_{E,E'}(\iota) = [L]$. Then, $\pi(E \otimes L)$ is equal to $\pi(E)$, $\pi_{E \otimes L, E'} = 0$ and $\iota$ is induced by an isomorphism $E \otimes L \xrightarrow{\sim} E'$ well defined up to multiplication by $\mu_s$. The lemma follows. ♦

5.4. — Let $\mathcal{U}$ be the universal bundle on $X \times \mathcal{M}_{\text{SL}_r}(d)$. We would like to know which power of the determinant bundle $\mathcal{D} = (\det RTU)^{-1}$ on $\mathcal{M}_{\text{SL}_r}(d)$ descends to $\mathcal{M}_G(d)$. As in I.3 of [B-L-S], the rank $r$ bundle

$$\mathcal{F} = \mathcal{L}^{\oplus(r-1)} \oplus \mathcal{L}^{1-r}(d \cdot x)$$

on $X \times \mathcal{J}(X)$ has determinant $O(d \cdot x)$ and therefore defines a morphism

$$f : \mathcal{J}(X) = JX \times BG_m \to \mathcal{M}_{\text{SL}_r}(d)$$

which is $\mathcal{J}_s$-equivariant.

---

† We see here a $G$-bundle as a formal homogeneous space under $G$. 

---
The vector bundle
\[ \mathcal{F}' = \mathcal{O}^{(r-1)} \oplus \mathcal{L}^{-r/s}(d \cdot x) \]
on $X \times \mathcal{J}(X)$ has determinant $[\mathcal{L}^{-1}(ds/r \cdot x)]^{r/s}$. The $G$-bundle $\pi(\mathcal{F}')$ on $X \times \mathcal{J}(X)$ defines a morphism $f': \mathcal{J} \to \mathcal{M}_G(d)$. The relation
\[ \mathcal{L} \otimes (\text{Id}_X \times s_{\mathcal{J}})^* (\mathcal{F}') = \mathcal{F} \]
and (5.2.1) gives an isomorphism
\[ \pi(\mathcal{F}) = (\text{Id}_X \times s_{\mathcal{J}})^* \pi(\mathcal{F}') \]
which means that the diagram
\[
\begin{array}{ccc}
\mathcal{J}(X) & \xrightarrow{f} & \mathcal{M}_{\text{SL}_r}(d) \\
\downarrow s_{\mathcal{J}} & & \downarrow \pi \\
\mathcal{J}(X) & \xrightarrow{f'} & \mathcal{M}_G(d)
\end{array}
\]
is 2-commutative. Exactly as in I.3 of [B-L-S], let me prove the

**Lemma 5.5.** — The line bundle $f^* \mathcal{D}^k$ on $\mathcal{J}(X)$ descends through $s_{\mathcal{J}}$ if and only if $k$ multiples of $s/(s,r/s)$.

**Proof.** — Let $\chi = r(g-1)-d$ be the opposite of the Euler characteristic of $(k)$-points of $\mathcal{M}_{\text{SL}_r}(d)$. By (3.1.1), one has an isomorphism
\[ f^* \mathcal{D}^k \sim \Theta^{kr(r-1)} \otimes \mathcal{T}^{k\chi}. \]
The theory of Mumford groups says that $\Theta^{kr(r-1)}$ descends through $s_{\mathcal{J}}$ if and only if $k$ is a multiple of $s/(s,r/s)$. The line bundle $\mathcal{T}^{k\chi}$ on $BG_m$ descends through $s_{BG_m}$ if and only if $k\chi$ is a multiple of $s$. The lemma follows from the above isomorphism and from the observation that the condition $s \mid r \mid ds$ forces $s\chi$ to be a multiple of $s$. \[ \]

**Remark 5.6.** — If $g = 0$, the jacobian $J$ is a point and the condition on $\Theta$ is empty. The only condition in this case is that $k\chi$ is a multiple of $s$.

Let me recall that $\mathcal{D}$ is the determinant bundle on $\mathcal{M}_{\text{SL}_r}(d)$ and $G = \text{SL}_r/\mu_s$.

**Theorem 5.7.** — Assume that the characteristic of $k$ is 0. The integers $k$ such that $\mathcal{D}^k$ descends to $\mathcal{M}_G(d)$ are the multiple of $s/(s,r/s)$.

By the proposition 1.5 of [BLS], one gets the
COROLLARY 5.8. — The natural morphism

\[ \text{Pic}(M_G(d)) \rightarrow \text{Pic}(M_{SLr}(d)) = \mathbb{Z} \cdot D \]

makes the Picard group of \( M_G(d) \) an extension of \( \mathbb{Z} = \mathbb{Z} \cdot D^s / (s, r/s) \) by \( H^1_{\text{ét}}(X, \mathbb{Z}/d\mathbb{Z}) \rightarrow (\mathbb{Z}/d\mathbb{Z})^2 \).

Proof of the theorem. — By lemma 5.5 and diagram (5.4.1), we just have to prove that \( D^k \) effectively descends when \( k = s/(s, r/s) \). By theorem 4.1 and lemma 5.3, this means exactly that \( D^k \) has a \( J_s \)-linearization. We know by lemma 5.5 that the pull-back \( f^* D^k \) has such a linearization.

LEMMA. — The pull-back morphism

\[ \text{Pic}(J_s \times M_{SLr}(d)) \rightarrow \text{Pic}(J_s \times J(X)) \]

is injective.

Proof. — By lemma 3.1, one is reduced to prove that the natural morphism

\[ \text{Pic}(B^s \times M_{SLr}(d)) \rightarrow \text{Pic}(B^s \times J(X)) \]

is injective. Let \( \mathcal{X} \) be any stack. The canonical morphism \( \mathcal{X} \rightarrow \mathcal{X} \times B^s \) is a \( \mu_s \)-torsor (with the trivial action of \( \mu_s \) on \( \mathcal{X} \)). By theorem 4.1, one has the equality

\[ \text{Pic}(\mathcal{X} \times B^s) = \text{Pic}^{\mu_s} \mathcal{X}. \]

Assume further that \( H^0(\mathcal{X}, \mathcal{O}) = k \). The latter group is then canonically isomorphic to

\[ \text{Pic}(\mathcal{X}) \times \text{Hom}(\mu_s, G_m) = \text{Pic}(\mathcal{X}) \times \text{Pic}(B^s). \]

Eventually, we get a functorial isomorphism

\[ (5.9.1) \quad \vartheta_{\mathcal{X}}: \text{Pic}(\mathcal{X} \times B^s) \rightarrow \text{Pic}(\mathcal{X}) \times \text{Pic}(B^s). \]

By [L-S], the Picard group of \( M_{SLr}(d) \) is the free abelian group \( \mathbb{Z} \cdot D \) and the formula (3.1.1) proves that

\[ f^*: \text{Pic}(M_{SLr}(d)) \rightarrow \text{Pic}(J(X)) \]
is an injection. The diagram
\[
\begin{array}{ccc}
\text{Pic}(&\mathcal{M}_{\text{SL}_r}(d)) \times \text{Pic}(B\mu_s) & \hookrightarrow & \text{Pic}(\mathcal{J}(X)) \times \text{Pic}(B\mu_s) \\
\downarrow & & & \downarrow \\
\text{Pic}(\mathcal{M}_{\text{SL}_r}(d) \times B\mu_s) & \longrightarrow & \text{Pic}(\mathcal{J}(X) \times B\mu_s)
\end{array}
\]
is commutative and the lemma follows from this commutative diagram.

Let \( H \) (resp. \( \mathcal{H}_\mathcal{J} \)) be the line bundle on \( \mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) \) (resp. \( \mathcal{J}_s \times \mathcal{J}(X) \))
\[
\mathcal{H} = \text{Hom}(m_{\mathcal{M}}^* D^k, \text{pr}_2^* D^k)
\]
(resp. \( \mathcal{H}_\mathcal{J} = \text{Hom}(m_{\mathcal{M}}^* f^* D^k, \text{pr}_2^* f^* D^k) \)).

Let us choose a \( \mathcal{J}_s \)-linearization \( \lambda_{\mathcal{J}} \) of \( f^* D^k \). It defines a trivialization of the line bundle \( \mathcal{H}_\mathcal{J} \). The equivariance of \( f \) implies (cf. 2.3) that there exists a (compatible) 2-morphism
\[
q : m_{\mathcal{M}} \circ (\text{Id} \times f) \Longrightarrow f \circ m_{\mathcal{J}}
\]
making the diagram
\[
\begin{array}{ccc}
\mathcal{J}_s \times \mathcal{J}(X) & \xrightarrow{m_{\mathcal{J}}} & \mathcal{J}(X) \\
\text{Id} \times f & & \downarrow f \\
\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) & \xrightarrow{m_{\mathcal{M}}} & \mathcal{M}_{\text{SL}_r}(d)
\end{array}
\]
2-commutative. The 2-morphism \( q \) defines an isomorphism from the pull-back \( m_{\mathcal{M}}^* D^k \) on \( \mathcal{J}_s \times \mathcal{J}(X) \) to \( m_{\mathcal{J}}^* (f^* D^k) \). The pull-back of \( \text{pr}_2^* D^k \) on \( \mathcal{J}_s \times \mathcal{J}(X) \) is tautologically isomorphic to \( \text{pr}_2^* (f^* D^k) \). The preceding isomorphisms induce an isomorphism
\[
(\text{Id} \times f)^* \mathcal{H} \xrightarrow{\sim} \mathcal{H}_\mathcal{J}.
\]
The later line bundle being trivial, so is \( (\text{Id} \times f)^* \mathcal{H} \). The lemma above proves therefore that \( \mathcal{H} \) itself is trivial. Each \((k-)\)point \( j \) of \( \mathcal{J}_s \) defines a morphism
\[
\mathcal{M}_{\text{SL}_r}(d) \rightarrow \mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d) \quad \text{(resp. } \mathcal{J}(X) \rightarrow \mathcal{J}_s \times \mathcal{J}(X))
\]
let me denote by \( \mathcal{H}_j \) (resp. \( f^* \mathcal{H}_j \)) the pull-back of \( \mathcal{H} \) (resp. \( (\text{Id} \times f)^* \mathcal{H} \)) by this morphism. The pull-back morphism
\[
H^0(\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d), \mathcal{H}) \longrightarrow H^0(\mathcal{J}_s \times \mathcal{J}(X), (\text{Id} \times f)^* \mathcal{H})
\]
can be identified to the direct sum
\[ \bigoplus_{j \in J_\infty(k)} H^0(\mathcal{M}_{\text{SL}_d}(d), \mathcal{H}_j) \longrightarrow H^0(\mathcal{J}(X), f^*\mathcal{H}_j). \]

As
\[ (5.9.2) \quad H^0(\mathcal{M}_{\text{SL}_d}(d), \mathcal{O}) = H^0(\mathcal{J}(X), \mathcal{O}) = k, \]
this morphism is a direct sum of non-zero homomorphisms of vector spaces of dimension 1 and therefore an isomorphism. In particular, a linearization \(\lambda_J\) of \(f^*\mathcal{D}^k\) defines canonically an isomorphism
\[ \lambda_M : m_M^*\mathcal{D}^k \xrightarrow{\sim} \text{pr}_2^*\mathcal{D}^k \]
such that \((\text{Id} \times f)^*\lambda_M = \lambda_J\).

Explicitly, \(\lambda_M\) is characterized as follows. Let \(x\) be an object of \(\mathcal{M}_{\text{SL}_d}(d)\) over a connected scheme \(S\) and \(g\) an object of \(J_\infty(S) = J_\infty(k)\). The preceding discussion means that the functorial isomorphisms
\[ \lambda_M(g, x) : \mathcal{D}^k \xrightarrow{\sim} \mathcal{D}^k \]
are determined when \(x\) lies in the essential image of \(f\). In this case, let us choose an isomorphism \(f(x') \xrightarrow{\sim} x\) (inducing an isomorphism \(g \cdot f(x') \xrightarrow{\sim} g \cdot x\)). Then, the diagram of isomorphisms of line bundles on \(S\)
\[ \begin{array}{ccc}
L_{x'} &=& L_{f(x')} \\
\downarrow \lambda_J(g, x') & & \lambda_M(g, x) \\
L'_{g \cdot x'} &=& L_{g \cdot f(x')}
\end{array} \xrightarrow{q_{g,x'}} L_{g \cdot x} \]
is commutative (where \(L = \mathcal{D}^k\) and \(L' = f^*\mathcal{D}^k\)).

Now, the pull-back of \(\lambda_M\) on \(J_\infty \times \mathcal{J}(X)\) satisfies conditions (2.8.1) and (2.8.2). Using (5.9.2) and the equivariance of \(f\) as above, this shows that \(\lambda_M\) is a linearization. For instance, keeping the notation above, let us verify condition (2.8.2). We have to check that the isomorphism \(\iota\) of \(L\) induced by \(\epsilon\) is the identity. As above, it is enough to check that on \(\mathcal{J}(X)\). With a slight abuse of notations, the two diagrams
\[ \begin{array}{ccc}
L_{x'} &=& L_{f(x')} \\
\downarrow \lambda_J(1, x') & & \lambda_M(1, x) \\
L'_{1 \cdot x'} &=& L_{1 \cdot f(x')}
\end{array} \xrightarrow{q_{1, x'}} L_{1 \cdot x} \]
and
\[ \begin{array}{ccc}
L_x &\xrightarrow{\iota} & L_x \\
\downarrow \lambda_M(1, x) & & \epsilon(x) \\
L_{1 \cdot x} &\xrightarrow{\lambda_M(1, x)} & L_{1 \cdot x}
\end{array} \]
are commutative (the commutativity of the first diagram follows from
the equivariance of \( f \) and the commutativity of the second diagram
follows exactly from the definition of \( \iota \)). Because \( \lambda_J \) is a linearization,
condition (2.8.2) shows that the diagram

\[
\begin{array}{c}
L'_{x'} \\
\downarrow \lambda_J(1,x') \\
\downarrow L_{1,x'} \\
\end{array}
\begin{array}{c}
\lambda_J(1,x') \\
\downarrow \lambda_J(1,x') \\
\downarrow \lambda_J(1,x') \\
\end{array}
\begin{array}{c}
\iota'(x') \\
\iota'(x') \\
\iota'(x') \\
\end{array}
\]

is commutative. It follows that the equality \( \iota = \text{Id} \) will follow from the
commutativity of the diagram

\[
\begin{array}{c}
L_{f(1,x')} \\
\downarrow q_{1,x'} \\
\downarrow L_{1,f(x')} \\
\end{array}
\begin{array}{c}
\iota'(x') \\
\iota'(x') \\
\iota'(x') \\
\end{array}
\begin{array}{c}
L_{f(x')} \\
\iota'(x') \\
\iota'(x') \\
\end{array}
\]

But \( q \) is compatible with \( \iota \) as required in 2.3. The diagram

\[
\begin{array}{c}
f(1,x') \\
\downarrow q_{1,x'} \\
1 \cdot f(x') \\
\end{array}
\begin{array}{c}(x') \\
\f(x') \\
\f(x') \\
\end{array}
\begin{array}{c}
f(x') \\
\f(x') \\
\f(x') \\
\end{array}
\]

is therefore commutative from which the commutativity of (5.9.3) follows.

One would check condition (2.8.1) in an analogous way. \( \Box \)

REMARK 5.10. — In the case \( g = 0 \), the condition is an in remark 5.6.

REMARK 5.11. — This linearization can be certainly also deduced from
a careful analysis of the first section of \([Fa]\), but the method above seems
simpler.

BIBLIOGRAPHIE

[B-L-S] BEAUVILLE (A.), LASZLO (Y.), SORGER (C.). — The Picard group
of the moduli of G-bundles on a curve, preprint alg-geom/9608002,


