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EXISTENCE OF COMPACT QUOTIENTS OF HOMOGENEOUS SPACES, MEASURABLY PROPER ACTIONS, AND DECAY OF MATRIX COEFFICIENTS

BY GREGORY MARGULIS (*)

ABSTRACT. — The main purpose of the present paper is to give a new approach for constructing examples of homogeneous spaces G/H with no compact quotients where G is a Lie group and H is a closed noncompact subgroup. This approach is based on the study of the restriction to H of matrix coefficients of unitary representations of G. A similar method also gives a criterion when the restriction to H of an action of G on a locally compact space X with a G-invariant infinite measure is measurably proper in the sense that, for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper.

RÉSUMÉ. — Le but principal de cet article est de donner une nouvelle méthode pour construire des exemples d'espaces homogènes G/H qui n'admettent pas de quotients compacts où G est un groupe de Lie et H est un sous-groupe fermé non compact. Cette méthode est basée sur l'étude de la restriction à H des coefficients matriciels de représentations unitaires de G. Une méthode similaire donne un critère pour que la restriction à H d'une action de G sur un espace localement compact X qui admet une mesure G-invariante infinie soit mesurablement propre ce qui veut dire que l'application naturelle $H \to Hx$, $h \mapsto hx$, est propre pour presque tout $x \in X$.

Let G be a Lie group, and H a closed subgroup of G. There is a natural question: when does G/H have a compact quotient? More precisely when can one find a discrete subgroup Γ of G such that Γ acts properly on G/H and the quotient space $\Gamma\backslash G/H$ is compact? If G is semisimple and H is compact then according to a theorem of Borel G/H always has a compact form. But if H is not compact the answer to the question is unknown even for semisimple G. For a connected semisimple group G and a connected reductive subgroup H all known examples of homogeneous

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spaces G/H which have compact quotients by discrete subgroups are based on the following construction. Suppose that there exists a connected closed subgroup $F \subset G$ such that $G = F \cdot H$ and $F \cap H$ is compact. Then G/H is naturally isomorphic to $F/F \cap H$. In particular if F is semisimple or, more generally, reductive we can use Borel's theorem to construct a compact quotient of G/H by a discrete subgroup.

On the other hand, there are many examples of homogeneous spaces G/H without compact quotients (see surveys [1], [2] and references therein). To prove that G/H has no compact quotients several criteria are used. These criteria are mostly based on considerations from topology, ergodic theory and the theory of linear groups. In this paper we give a new criterion which is based on the study of the restriction to H of matrix coefficients of unitary representations of G. This criterion gives many new examples of homogeneous spaces G/H without compact quotients.

The study of matrix coefficients also gives a criterion when the restriction to H of an action of G on a locally compact space X with a G-invariant (infinite) measure μ is measurably proper (in the sense that for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper).

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1. (G, K, H)-tempered actions

In this section G is a locally compact group, K is a compact subgroup of G, and H is a closed subgroup of G. Let θ denote a (left invariant) Haar measure on H.

Let G act continuously by measure preserving transformations on a (noncompact) locally compact space X with an infinite regular Borel measure μ. Consider the regular unitary representation ρ of G on $L^2(X, \mu)$:

$$\big(\rho(g)f\big)(x) = f(g^{-1}x); \quad g \in G,\ x \in X,\ f \in L^2(X, \mu).$$

DEFINITION 1. — We say that the action of G on X is (G,K,H)-*tempered* if there exists a (positive) function $q \in L^1(H,\theta)$ such that

$$\big|\langle \rho(h)f_1, f_2\rangle\big| \le q(h)\|f_1\| \cdot \|f_2\| \tag{1}$$

for any $h \in H$ and any $\rho(K)$-invariant functions $f_1, f_2 \in L^2(X,\mu)$.

PROPOSITION 1. — *If the action of G on X is (G,K,H)-tempered then $\mu(X - HM) > 0$ and, consequently, $HM \neq X$ for any compact subset M of X.*

Proof. — Let M be a compact subset of X. Then there exists a non-negative K-invariant continuous function f on X with compact support such that $f(x) > 1$ for any $x \in M$. Consider a function

$$\varphi = \int_H \rho(h) f \, \mathrm{d}\theta(h), \quad \varphi(x) = \int_H f(h^{-1}x) \, \mathrm{d}\theta(h).$$

(The function φ can be infinite, and if HM is not compact then usually φ is not in $L^2(X,\mu)$.) Since f is continuous, M is compact and $f(x) > 1$ for any $x \in M$, there exists a neighborhood W of e in H such that $f(w^{-1}x) > \frac{1}{2}$ for all $x \in M$ and $w \in W$. Now if $x \in HM$ then $\varphi(x) > \frac{1}{2}\theta(W)$ (because if $h^{-1}x \in M$ then $f((hw)^{-1}x) > \frac{1}{2}$ for any $w \in W$). Thus

$$\varphi(x) > \tfrac{1}{2}\theta(W) \quad \text{for any} \quad x \in HM. \tag{2}$$

Take a compact subset L of H such that

$$\int_{H-L} q(h) \, \mathrm{d}\mu(h) < \frac{1}{2\|f\|}\theta(W) \tag{3}$$

where $\|f\| = \sup\{f(x) \mid x \in X\}$. Since the measure μ is Borel and infinite and the support supp$f$ of f is not compact, there exists a K-invariant set $A \subset X$ such that $\mu(A) = 1$ and $(L \cdot \operatorname{supp} f) \cap A = \emptyset$. Let χ_A denote the characteristic function of A. Then using (1) and (3) we get

$$\begin{aligned} \int_A \varphi(x) \, \mathrm{d}\mu(x) &= \int_X (\varphi \cdot \chi_A)(x) \, \mathrm{d}\mu(x) \\ &= \int_H \Big(\int_X ((\rho(h)f)\chi_A)(x) \, \mathrm{d}\mu(x) \Big) \mathrm{d}\theta(h) \\ &= \int_H \langle \rho(h)f, \chi_A \rangle \, \mathrm{d}\theta(h) = \int_{H-L} \langle \rho(h)f, \chi_A \rangle \, \mathrm{d}\theta(h) \\ &\le \int_{H-L} q(h)\|f\| \cdot \|\chi_A\| \, \mathrm{d}\theta(h) = \|f\| \int_{H-L} q(h) \, \mathrm{d}\theta(h) \\ &< \tfrac{1}{2}\theta(W). \end{aligned} \tag{4}$$

The equality $\mu(A) = 1$ and the inequalities (2) and (4) imply that $\mu(A - HM) > 0$ and, consequently $\mu(X - HM) > 0$. ∎

PROPOSITION 2. — *Let A be a bounded Borel subset of X. For any $x \in X$, let $\psi_A(x)$ denote the θ-measure of the set $\{h \in H \mid hx \in A\}$. Suppose that the action of G on X is (G,K,H)-tempered.*

(a) *The function* ψ_A *is locally integrable, that is*

$$\int_B \psi_A(x)\,\mathrm{d}\mu(x) < \infty$$

for any bounded Borel subset B *of* X.

(b) *If* X *is* σ*-compact then* $\psi_A(x) < \infty$ *for almost all* $x \in X$.

Proof. — Clearly (a) implies (b). Let us prove (a). Replacing A by KA and B by KB, we can assume that A and B are K-invariant. Let χ_A and χ_B denote the characteristic functions of A and B. It is easy to see that

$$\psi_A = \int_H \rho(h)\chi_A\,\mathrm{d}\theta(h).$$

Then using (1) we get

$$\int_B \psi_A(x)\,\mathrm{d}\mu(x) = \langle \psi_A, \chi_B\rangle = \int_H \langle \rho(h)\chi_A, \chi_B\rangle\,\mathrm{d}\theta(h)$$
$$\leq \int_H q(h)\langle \chi_A, \chi_B\rangle\,\mathrm{d}\theta(h) < \infty. \quad \square$$

2. (G, K)-tempered subgroups

In this section G, K, H and θ denote the same as in §1.

Definition 2. — We say that H is (G,K)-*tempered* if there exists a function $q \in L^1(H,\theta)$ such that

$$\big|\langle \pi(h)w_1, w_2\rangle\big| \leq q(h)\|w_1\|\cdot\|w_2\| \tag{5}$$

for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any unitary representation π of G without non-trivial $\pi(G)$-invariant vectors.

Remark 1. — As in §1 let us consider a continuous action of G by measure preserving transformations on a locally compact space X with an infinite regular Borel measure μ, and let us denote by ρ the regular representation of G on $L^2(X,\mu)$. If $a > 0, f \in L^2(X,\mu)$ and $\rho(G)f = f$, then the sets

$$\{x \in X \mid f(x) > a\} \quad \text{and} \quad \{x \in X \mid f(x) < -a\}$$

have finite measure and they are G-invariant (modulo sets of measure 0). Hence if X has no G-invariant subsets of finite nonzero measure and the subgroup H is (G,K)-tempered then the action of G on X is (G,K,H)-tempered.

REMARK 2. — Let

$$\pi = \int_y \pi_y \, \mathrm{d}\sigma(y)$$

be a decomposition of π into a continuous sum of irreducible unitary representations, and let

$$W = \int_Y W_y \, \mathrm{d}\sigma(y), \quad w_1 = \int_Y w_{1y} \, \mathrm{d}\sigma(y), \quad w_2 = \int_Y w_{2y} \, \mathrm{d}\sigma(y),$$

w_{1y}, $w_{2y} \in W_y$, be corresponding decompositions of the space W of the representation π and of vectors $w_1, w_2 \in W$. Suppose that for all $y \in Y$

$$|\langle \pi_y(h) w_{1y}, w_{2y} \rangle| \leq q(h) \|w_{1y}\| \cdot \|w_{2y}\|.$$

Then using Cauchy-Schwartz inequality we get

$$\begin{aligned}
\big|\langle \pi(h) w_1, w_2 \rangle\big| &= \Big| \int_Y \langle \pi_y(h) w_{1y}, w_{2y} \rangle \, \mathrm{d}\sigma(y) \Big| \\
&\leq q(h) \int_Y \|w_{1y}\| \cdot \|w_{2y}\| \, \mathrm{d}\sigma(y) \\
&\leq q(h) \sqrt{\int_Y \|w_{1y}\|^2 \, \mathrm{d}\sigma(y) \int_Y \|w_{2y}\|^2 \, \mathrm{d}\sigma(y)} \\
&= q(h) \|w_1\| \cdot \|w_2\|.
\end{aligned}$$

Thus H is (G,K)-tempered if and only if the inequality (5) is true for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any non-trivial irreducible unitary representation π of G.

Let us now give some examples of (G,K)-tempered subgroups. We give only indications of the proofs because more precise and general results are obtained by Hee Oh (see [3]).

EXAMPLES.

(a) Let G be a connected semisimple Lie group having Kazhdan's property (T) and K a maximal compact subgroup of G. Then any commutative diagonalizable subgroup H of G is (G,K)-tempered. To show this it is enough to use Howe-Moore estimates which provide uniform exponential decay for matrix coefficients corresponding to K-invariant vectors and irreducible nontrivial unitary representations of semisimple groups with property (T).

(b) Let $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n)$, and α_n the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Suppose that $n \geq 4$. Then the subgroup $H = \alpha_n(\mathrm{SL}(2,\mathbb{R}))$ is (G,K)-tempered. Let us show this in the case where $n = 4$ and

$$\alpha_4(d_t) = r_t, \quad d_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad r_t = \begin{pmatrix} e^{3t} & & & 0 \\ & e^t & & \\ & & e^{-t} & \\ 0 & & & e^{-3t} \end{pmatrix}.$$

It is well known that the restriction of any nontrivial irreducible unitary representation π of $\mathrm{SL}_4(\mathbb{R})$ to the subgroup

$$F = \left\{ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

does not contain complementary series. But r_t belongs to the subgroup

$$\left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & & & 0 \\ 0 & & A & 0 \\ c & 0 & 0 & d \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

which is the direct product of two conjugates of F.

Using these facts and formulas for matrix coefficients of the principal series of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ we easily get that for some $c > 0$

$$\left|\langle \pi(r_t)w, w\rangle\right| \leq c e^{-4t} t^2 \cdot \left|\langle w, w\rangle\right|, \quad t \geq 0,$$

for any $\pi(K)$-invariant vector w. Now it remains to notice that the function

$$f(k_1 d_t k_2) = e^{-4t} t^2, \quad k_1, k_2 \in \mathrm{SO}(2),\ t \geq 0,$$

is integrable on $\mathrm{SL}_2(\mathbb{R})$ because the Haar measure of the set

$$\left\{ k_1 d_t k_2 \mid k_1, k_2 \in \mathrm{SO}(2),\ 0 \leq t \leq T \right\}$$

is asymptotically $c e^{2T}$ when $T \to +\infty$.

(c) Let L be a connected simple Lie group, $n \geq 3$, $\varphi : L \to \mathrm{SL}_n(\mathbb{R})$ an n-dimensional representation of L, and $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_i$ a decomposition of φ into the sum of irreducible representations of L. Let us denote by β the sum of the positive roots of L with respect to a maximal $\mathbb{R}$-split torus $S \subset L$ and an ordering on the character group $X(S)$ of S, and by χ_j the highest weight of the representation φ_j, $1 \leq j \leq i$. Then using arguments similar to those from the example (b) one can prove that the subgroup $\varphi(L)$ is $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$-tempered whenever

$$(*) \qquad \sum_{j \in \mathcal{J}} \chi_j > \beta(1+\varepsilon) \text{ for some } \varepsilon > 0, \text{ where } \mathcal{J} = \{j \mid \dim \varphi_j \geq 2\}.$$

From this we easily deduce the existence of $N > 0$ such that if

$$\sum_{j \in \mathcal{J}} \dim \varphi_j > N$$

then $\varphi(L)$ is $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$-tempered. (Let us note that $\sum_{j \in \mathcal{J}} \dim \varphi_j$ is the codimension in $\mathbb{R}^n$ of the subspace of $\varphi(L)$-invariant vectors.)

3. Compact quotients of homogeneous spaces

As usual we say that a continuous action of a locally compact group G on a locally compact space X is *proper* if, for every compact subset $L \subset X$, the set $\{g \in G \mid gL \cap L \neq \emptyset\}$ is compact. If G acts properly on X then the quotient space $G \backslash X$ is Hausdorff. We say that the action of G on X is *cocompact* if there exists a compact subset L of X such that $X = GL$. For proper actions this property is equivalent to the compactness of $G \backslash X$.

It is well known and easy to check that, for any locally compact group G and any closed subgroups P and Q of G, the following conditions are equivalent:

(I) the action of P on G/Q by left translations is proper (resp. cocompact);

(II) the action of Q on $P \backslash G$ by right translations is proper (resp. cocompact);

(III) the action $(p,q)g = pgq^{-1}$, $p \in P$, $q \in Q$, $g \in G$, of $P \times Q$ on G is proper (resp. cocompact).

It is natural to call the equivalence (I) $\Leftrightarrow$ (II) the *duality principle.*

THEOREM 1. — *Let G be a unimodular locally compact group, H a closed subgroup of G, and F a closed subgroup of H. Suppose that H is (G,K)-tempered for some compact subgroup K of G.*

(a) *If* Γ *is a discrete subgroup of* G *such that the volume of* $\Gamma\backslash G$ *with respect to Haar measure is infinite then the action of* Γ *on* G/F *by left translations is not cocompact.*

(b) *If* F *is not compact then there are no discrete subgroups* Γ *of* G *such that* Γ *acts properly on* G/F *by left translations and the quotient* $\Gamma\backslash(G/F)$ *is compact.*

Proof.

(a) The group G is unimodular. Therefore the action of G on $\Gamma\backslash G$ by right translations preserves Haar measure μ. Since $\mu(\Gamma\backslash G) = \infty$ there are no G-invariant subsets in $\Gamma\backslash G$ of finite measure. Hence (see Remark 1 after Definition 2) the action of G on $\Gamma\backslash G$ is (G, K, H)-tempered. Now applying Proposition 1 we get that the action of H on $\Gamma\backslash G$ and, consequently, the action of F on $\Gamma\backslash G$ are not cocompact. From this, using the above mentioned duality principle, we deduce that the action of Γ on G/F is not cocompact.

(b) In view of (a) it is enough to consider the case where $\mu(\Gamma\backslash G) < \infty$, but in this case F can not act properly on $\Gamma\backslash G$ because any continuous action of a noncompact group by transformations preserving a finite nonzero regular Borel measure is not proper. □

Combining Theorem 1 with examples (b) and (c) from §2 we get the following two corollaries.

COROLLARY 1. — *Let* α_n *denote the* n*-dimensional irreducible representation of* $\mathrm{SL}_2(\mathbb{R})$. *Let* $G = \mathrm{SL}_n(\mathbb{R})$, $H = \alpha_n(\mathrm{SL}_2(\mathbb{R})) \subset G$, *and* F *a closed subgroup of* H. *Suppose that* $n \geq 4$. *Then for* G,H *and* F *the statements* (a) *and* (b) *in Theorem* 1 *are true. In particular* G/H *has no compact quotients by discrete subgroups.*

COROLLARY 2. — *Let* L *be a connected simple Lie group,* $n \geq 3$, *and let* $\varphi: L \to \mathrm{SL}_n(\mathbb{R})$ *be an* n*-dimensional representation of* L *such that the condition from example* (c) *of* §2 *is satisfied. Then the statements* (a) *and* (b) *of Theorem* 1 *are true for* $G = \mathrm{SL}_n(\mathbb{R})$,$H = \varphi(L)$ *and a closed subgroup* F *of* H.

4. Measurably proper actions

Let H be a locally compact second countable group acting continuously on a locally compact second countable space X with an H-quasi-invariant Borel measure μ. Let θ be a left invariant Haar measure on H. Then the following conditions are equivalent:

(a) for almost all (with respect to μ) points $x \in X$, the orbit Hx is closed in X and the stabilizer $H_x = \{h \in H \mid hx = x\}$ is compact;

(b) for almost all $x \in X$, the stabilizer H_x is compact and the natural map $hH_x \mapsto hx$ of H/H_x onto Hx is a homeomorphism;

(c) for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper or, in other words, the set $\{h \in H \mid Hx \in A\}$ is bounded in H for any bounded subset A of X;

(d) for almost all $x \in X$ and any bounded subset A of X, the θ-measure of the set $\{h \in H \mid hx \in A\}$ is finite.

The equivalences (a)$\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) are standard facts about group actions. The implication (c) $\Rightarrow$ (d) is trivial. To prove (d) $\Rightarrow$ (c) let us consider a bounded neighborhood U of e in H. Then

$$\{h \in H \mid hx \in UA\} = U\{h \in H \mid hx \in A\}.$$

Therefore if $\{h \in H \mid hx \in A\}$ is unbounded then $\{h \in H \mid hx \in UA\}$ has infinite measure. It remains to notice that if A is bounded then UA is also bounded.

If the conditions (a)–(d) are satisfied then we say the action of H on X is *measurable proper.* It is easy to see that if the action of H on X is measurably proper then almost all components in the decomposition of μ into H-ergodic measures are supported on closed H-orbits Hx with compact stabilizers H_x. In particular if the measure μ is H-ergodic then there exists $x \in X$ such that $\mu(X - Hx) = 0$, Hx is closed in X and H_x is compact. Let us also note that if H acts measurably proper on X and F is a closed subgroup of H then the action of F is also measurably proper.

THEOREM 2. — *Let G be a locally compact second countable group acting continuously on a locally compact second countable space X with a G-invariant (regular infinite) Borel measure μ, let H be a closed subgroup of G, and K a compact subgroup of G.*

(a) *If the action of G on X is (G,K,H)-tempered then the restriction of this action to H is measurably proper.*

(b) *If the subgroup H is (G,K)-tempered and X has no G-invariant subsets of finite nonzero measure then the action of H on X is measurably proper.*

Proof.

(a) follows from Proposition 2 (b). In view of Remark 1 from §2, (a) implies (b). □

REMARKS.

(I) In view of examples (a)–(c) from §2, Theorem 2 (b) can be applied in the following cases:

(a) G is a connected semisimple Lie group having Kazhdan's property (T) and H is a commutative diagonalizable subgroup of G;
(b) $G = \mathrm{SL}_n(\mathbb{R})$ and $H = \pi_n(\mathrm{SL}_2(\mathbb{R}))$ where $n \geq 4$ and π_n is the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$.
(c) $G = \mathrm{SL}_n(\mathbb{R})$ and $H = \varphi(L)$ where L is a connected simple Lie group and $\varphi : L \to \mathrm{SL}_n(\mathbb{R})$ is an n-dimensional representation of L such that the condition (*) from example (c) of §2 is satisfied.

(II) Let G be a unimodular locally compact second countable group, H a closed subgroup of G, and Γ a discrete subgroup of G. Suppose that the Haar measure of G/Γ is infinite and that H is (G, K)-tempered for some compact subgroup K of G. Then as a corollary of Theorem 2 we have that the action of H on G/Γ by left translations is measurably proper. In particular if in addition H is not open then the action of H on G/Γ is not ergodic.

BIBLIOGRAPHIE

[1] KOBAYASHI (T.). — *Discontinuous groups and Clifford-Klein forms on pseudoriemannian homogeneous manifolds*, in 'Algebraic and analytic methods in representation theory'. — B. Orsted and H. Schlichtkrull eds, Perspectives in Mathematics, vol. **17**, Academic Press, p. 99–165.
[2] LABOURIE (F.). — *Quelques résultats récents sur les espaces localement homogènes compacts*, in 'Manifolds and Geometry', actes d'un colloque (Pise 1993) en l'honneur d'Eugenio Calabi. — P. de Bartolomeis, F. Tricerri and E. Vesentini eds, Symposia Mathematica, vol. **XXXVI**, Cambridge University Press 1996.
[3] OH (H.). — *Representations with minimal decay of matrix coefficients and tempered subgroups*, Preprint.