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PURELY INFINITE C^*-ALGEBRAS ARISING FROM DYNAMICAL SYSTEMS

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ABSTRACT. — We give a condition sufficient to ensure that the reduced C^*-algebra associated with an r-discrete groupoid is purely infinite. As an application, we get many examples of purely infinite C^*-algebras, from discrete groups of isometries of hyberbolic metric spaces, or of Hadamard manifolds, acting on their limit set. The expanding continuous surjective maps from a compact metric space onto itself provide also interesting examples. Actually, many of the examples considered here give purely infinite, simple, nuclear, separable C^*-algebras, satisfying to the Universal Coefficient Theorem. Therefore, they are completely classified by their K-theory groups, thanks to the recent work of Kirchberg.

RÉSUMÉ. — Nous donnons une condition suffisante pour que la C^*-algèbre réduite associée à un groupoïde r-discret soit purement infinie. Comme application, nous obtenons de nombreux exemples de C^*-algèbres purement infinies, à partir de groupes discrets d'isométries d'espaces métriques hyperboliques, ou de variétés de Hadamard, agissant sur leur ensemble limite. Les surjections continues dilatantes d'un espace métrique compact sur lui-même sont une autre source intéressante d'exemples. Beaucoup d'exemples étudiés ici produisent en fait des C^*-algèbres purement infinies, simples, nucléaires, séparables, satisfaisant au théorème des coefficients universels. Celles-ci sont donc entièrement classifiées par leurs groupes de K-théorie, d'après le travail récent de Kirchberg.

Introduction

The recent remarkable result of E. Kirchberg [24] (see also [29]) concerning the classification of simple purely infinite separable nuclear C^*-algebras has drawn considerable attention to this class of algebras.

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Previously, the work of J. Cuntz and W. Krieger [15] had brought out interesting links between some of these algebras (now called Cuntz-Krieger algebras) and the theory of topological Markov chains. Later, J. Spielberg has shown that certain boundary actions of finitely generated free products of cyclic groups yield a Cuntz-Krieger algebra by the crossed product construction (see [33], [34]).

It is natural to expect that C^*-algebras arising from geometric contexts, or dynamics, which prevent the existence of a tracial state are good candidates to being purely infinite. From this observation, we give in Section 2 a condition sufficient to ensure that the reduced C^*-algebra defined by an r-discrete groupoid is purely infinite. In the following sections, we illustrate this result by a series of examples.

Section 3 is devoted to the case of r-discrete groupoids defined by actions of discrete countable groups on locally compact spaces. We show that a non elementary group of isometries of a hyperbolic geodesic metric space (in Gromov's sense), acting on its limit set, yields a simple purely infinite C^*-algebra as crossed product. This includes the interesting example of non elementary fuchsian groups, as well as the case of word hyperbolic groups acting on the boundary of their Cayley graphs. Moreover in these two last examples we obtain a nuclear C^*-algebra.

We show also that any lattice of a real connected semi-simple Lie group G without compact factors and with trivial centre, acting on its Furstenberg boundary G/P gives rise to a purely infinite simple nuclear C^*-algebra belonging to the bootstrap class $\mathcal{N}$ for which the classification by K-theory is complete. Actually, it appears that there are plenty of examples where the action on its limit set of a discrete group of isometries of a complete simply connected Riemannian manifold with nonpositive curvature, yields a simple nuclear purely infinite C^*-algebra as crossed product.

In Section 4, we study the dynamical system formed by a compact space X and a surjective local homeomorphism σ. As pointed out by J. Renault, there is an r-discrete groupoid which is quite well suited to describe this situation. Its associated C^*-algebra will be denoted by $C^*(X,\sigma)$. When σ is expanding and is such that its set of eventually periodic points is dense with empty interior, we show that $C^*(X,\sigma)$ is purely infinite and nuclear. As particular examples we get the Cuntz-Krieger algebras defined by subshifts of finite type satisfying condition (I) of [15], as well as many C^*-algebras defined by subshifts which are not of finite type. This family of examples includes also the C^*-algebras associated with the differentiable expanding endomorphisms studied by M. Shub in [31].

Our Section 1 is devoted to preliminaries on r-discrete groupoids. We limit ourselves to this class of groupoids, in order to avoid the introduction of Haar systems, though most of the basic facts could be defined in a more general setting. This section is a survey of results essentially due to J. Renault.

Most of the results of this paper had been previously explained in a first draft [4]. M. Laca and J. Spielberg had also found independently similar results in [25]. Here our work is written in the context of r-discrete groupoids, which is a natural framework to the description of many relevant situations, including for instance foliated manifolds, and dynamical systems defined by local homeomorphisms. Our notion of locally contracting groupoid (see Definition 2.1) is inspired by that of locally boundary action in [25]. It appears in fact that the context of groupoids extends the scope of applications of the results of [4] and [25], to the price of simpler proofs.

In view of Kirchberg's classification result, it is interesting to compute the K-theory of the C^*-algebras described in this paper. For instance, in [4], this allowed us to check that the C^*-algebras obtained from non elementary fuchsian groups of the first kind, acting on their limit set, are Cuntz-Krieger algebras. Since $C^*(X,\sigma)$ above is a crossed product $B \rtimes_\rho \mathbb{N}$, where ρ is a proper corner endomorphism of B, and B is an inductive limit of C^*-algebras strongly Morita equivalent to commutative ones, we may compute the K-theory of $C^*(X,\sigma)$ from the Pimsner-Voiculescu exact sequence. Such computations, and applications, will be given in a subsequent paper.

I am grateful to J. Renault for many stimulating discussions on the subject. I wish also to thank G. Skandalis, who drew my attention to the problems studied in this paper and to the reference [10], and F. Ledrappier for useful conversations on manifolds with nonpositive curvature.

1. Locally compact groupoids

1.1 Basic definitions.

We refer to the work of J. Renault [35] for the detailed theory of topological groupoids and of their associated C^*-algebras. In a concise way, a groupoid is a small category with inverse. More explicitely:

DEFINITION 1.1.1. — A *groupoid* consists of a set G, a distinguished subset $G^0 \subset G$, two maps $d,r: G \mapsto G^0$ and a law of composition $(\gamma_1,\gamma_2) \in G^2 \mapsto \gamma_1\gamma_2 \in G$, where

$$G^2 = \{(\gamma_1,\gamma_2) \in G \times G; d(\gamma_1) = r(\gamma_2)\},$$

such that

1) $d(\gamma_1\gamma_2) = d(\gamma_2)$, $r(\gamma_1\gamma_2) = r(\gamma_1)$ for all $(\gamma_1, \gamma_2) \in G^2$;

2) $d(x) = r(x) = x$ for all $x \in G^0$;

3) $\gamma d(\gamma) = \gamma, \quad r(\gamma)\gamma = \gamma$ for all $\gamma \in G$;

4) $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$;

5) each γ has a two-sided inverse γ^{-1}, with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = d(\gamma)$.

For $x \in G^0$, we denote by $G(x)$ the isotropy subgroup $r^{-1}(x) \cap d^{-1}(x)$. A subset V of G^0 which coincides with its saturation $[V] = r(d^{-1}(V))$ is said to be *invariant*.

DEFINITIONS 1.1.2. — A *locally compact groupoid* consists of a groupoid G and a locally compact topology compatible with the groupoid structure:

1) $\gamma \mapsto \gamma^{-1}$ is continuous from G onto G;

2) $(\gamma_1, \gamma_2) \in G^2 \mapsto \gamma_1\gamma_2$ is continuous, with G^2 given the induced topology.

We say that G is *essentially free* (or *principal*) if the set of all $x \in G^0$ whose isotropy group $G(x)$ is reduced to $\{x\}$ is dense in G^0. When the only closed invariant subsets of G^0 are the empty set and G^0 itself, we say that G is *minimal*.

The groupoid G is said to be *r-discrete* if every $\gamma \in G$ has an open neighbourhood V such that $r_{|V}$ is an homeomorphism onto an open subset of G^0.

A subset S of a locally compact groupoid G is called a *bisection* if the restrictions $r_{|S}$ and $d_{|S}$ are one to one.

Note that G is r-discrete if and only if it has a basis of open bisections (see [35, p. 19]), and that in this case the fibers $G^x = r^{-1}(x)$ are discrete for all $x \in G^0$.

Let G be an r-discrete groupoid, and S a bisection. For $x \in r(S)$, we put $\alpha_S(x) = d(xS)$. This map α_S is an homeomorphism from $r(S)$ onto $d(S)$, which is called the *G-map* associated with S.

We say that a measure μ on G^0 is *quasi-invariant* if the induced measure

$$\nu = \mu \circ \lambda : f \longmapsto \int \Big(\sum_{r(\gamma)=x} f(\gamma) \Big) \, \mathrm{d}\mu(x)$$

is equivalent to

$$\nu^{-1} : f \mapsto \int \Big(\sum_{d(\gamma)=x} f(\gamma) \Big) \, \mathrm{d}\mu(x).$$

This means also that for every open bisection S, the measures $A \mapsto \mu(A)$ and $A \mapsto \mu(\alpha_S(A))$ defined on $r(S)$ are equivalent.

In this paper, we mainly consider second countable locally compact r-discrete groupoids. They are very tractable, and include many interesting examples. Among them we give the following ones.

1.2. Examples.

(a) *Group actions.* — To a discrete group Γ acting on the right on a locally compact space X, is associated the groupoid $G = X \times \Gamma$, with the product topology, and the multiplication $(x,t)(xt,s) = (x,ts)$. Here $G^0 = X$, $r(x,t) = x$, and $d(x,t) = xt$. When Γ is countable, this groupoid is essentially free if and only if the set of fixed points of any t, distinct of the identity e, has an empty interior.

(b) *Foliations.* — Let $(V,\mathcal{F})$ be a foliated manifold with a transverse faithful submanifold X. For the construction of the holonomy groupoid G of the foliation, we refer to [13] and [14]. For simplicity we assume that G is Hausdorff. We denote by $G(X)$ the reduced groupoid

$$\big\{\gamma \in G\,;\ r(\gamma) \in X,\ d(\gamma) \in X\big\}.$$

Then $G(X)$ is a locally compact r-discrete groupoid, which is essentially free by [18, Lemme 2.3].

(c) *Local homeomorphisms.* — Let X be a locally compact space, and $\sigma : X \to X$ a continuous surjective map which is a local homeomorphism. J. Renault has constructed in [35] the following locally compact r-discrete groupoid which is a useful tool for the study of the dynamical system (X,σ). One takes

$$G = \big\{(x,n,y) \in X \times \mathbb{Z} \times X\,;\ \exists\, k,\ell \geq 0,\ n = k-\ell,\ \sigma^k(x) = \sigma^\ell(y)\big\}$$

with the range map, the source map, and the multiplication given respectively by

$$r(x,n,y) = x, \quad d(x,n,y) = y, \quad (x,m,y)(y,n,z) = (x,m+n,z).$$

Then $G^0 = X$ embedded in G by $x \mapsto (x,0,x)$. Moreover G is given the topology having as a basis of open sets those of the form

$$\Omega(U,V,k,\ell) = \big\{(x,k-\ell,\sigma_V^{-\ell} \circ \sigma^k(x))\,;\ x \in U\big\},$$

where U and V are open subsets of X, and $k,\ell \geq 0$ are such that ${\sigma^k}_{|U}$ and ${\sigma^\ell}_{|V}$ are homeomorphisms with the same open range (and $\sigma_V^{-\ell}$ is the inverse of ${\sigma^\ell}_{|V}$).

We denote by

- $O^+(x)$ the *positive orbit* $\{\sigma^k(x)\,;\ k \geq 0\}$ of x, and by
- $O(x)$ its *full orbit* $\bigcup_{\ell \geq 0} (\sigma^\ell)^{-1}(O^+(x))$.

Then G is minimal if and only if every orbit $O(x)$ is dense in X. We remark also that the isotropy group $G(x)$ of x is non trivial if and only if its positive orbit $O^+(x)$ is finite.

When σ is an homeomorphism, the map $(x,n,y) \mapsto (x,n)$ is an isomorphism from G onto the groupoid $X \times \mathbb{Z}$ defined as in (*a*) by the obvious $\mathbb{Z}$-action on X.

1.3. The reduced C^*-algebra of a groupoid.

1.3.1. — Let G be a second countable locally compact r-discrete groupoid (for simplicity). The set $C_c(G)$ of all continuous functions on G with compact support has a natural structure of involutive algebra given by:

$$(f \star g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\, g(\gamma_2), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

For simplicity, since there should not be any risk of confusion, we will denote by fg the product of f and g in $C_c(G)$, instead of $f \star g$. Note that for f, $g \in C_c(G^0)$ this is the usual pointwise product.

Let $C_0(G^0)$ be the algebra of continuous functions on G^0 that vanish at infinity. Then $C_c(G)$ is a $C_0(G^0)$-module, endowed with an inner-product with values in $C_0(G^0)$, as follows:

$$(\xi f)(\gamma) = \xi(\gamma) f \circ r(\gamma) \qquad \text{for} \quad \xi \in C_c(G), f \in C_0(G^0),$$

$$\langle \xi, \eta \rangle (x) = \sum_{r(\gamma)=x} \overline{\xi(\gamma)}\, \eta(\gamma) \qquad \text{for} \quad \xi, \eta \in C_c(G), x \in G^0.$$

We denote by $\ell^2(G)$ the completion of the inner-product $C_0(G^0)$-module $C_c(G)$. It is a Hilbert C^*-module over the commutative C^*-algebra $C_0(G^0)$, and $\mathcal{L}(\ell^2(G))$ will be the C^*-algebra of bounded $C_0(G^0)$-linear operators on $\ell^2(G)$ which admit an adjoint (see [26]).

Let π be the $*$-homomorphism of $C_c(G)$ into $\mathcal{L}(\ell^2(G))$ defined by

$$(\pi(f)\xi)(\gamma) = \sum_{r(\gamma')=r(\gamma)} f(\gamma^{-1}\gamma')\, \xi(\gamma') \quad \text{for} \quad f, \xi \in C_c(G).$$

Then the closure of $\pi(C_c(G))$ in $\mathcal{L}(\ell^2(G))$ is called the *reduced* C^*-algebra of the groupoid, and denoted by $C^*_r(G)$. Note that the norm of $f \in C_c(G)$ in $C^*_r(G)$ is

$$\|f\| = \sup_{x\in G^0} \|\pi_x(f)\|$$

where π_x is the representation of $C_c(G)$ in the usual Hilbert space $\ell^2(G^x)$, defined by

$$(\pi_x(f)\xi)(\gamma) = \sum_{r(\gamma')=x} f(\gamma^{-1}\gamma')\xi(\gamma') \quad \text{for} \quad f \in C_c(G),\ \xi \in \ell^2(G^x).$$

The universal C^*-norm on $C_c(G)$ defines the full C^*-algebra of the groupoid, that will not be considered in this paper.

1.3.2. — Let us recall that, in the case of an r-discrete groupoid G, the inclusion $C_c(G) \to C_0(G)$ extends to an injection $C^*_r(G) \to C_0(G)$. Therefore, the elements of $C^*_r(G)$ will be viewed as functions on G (see [35, p. 99]). The subset G^0 is open in G, and $C_0(G^0)$ is canonically identified to an abelian subalgebra of $C^*_r(G)$. Moreover, the restriction map $E: C^*_r(G) \to C_0(G^0)$ is a faithful conditional expectation (see [35, p. 104]). Note also that $C^*_r(G)$ has a tracial state if and only if there exists a finite invariant probability μ on G^0 (that is $\mu \circ \alpha_S = \mu$, for all G-maps S).

If we consider now an essentially free r-discrete groupoid G, we recall also that $C^*_r(G)$ is simple if and only if G is minimal, by [35, Prop. 4.6, p. 103].

1.3.3. — When G is defined by a group action α of Γ on X as in example 1.2 (a), then $C^*_r(G)$ is the usual reduced crossed product $C_0(X) \rtimes_{\alpha r} \Gamma$. In example 1.2 (b), the reduced C^*-algebra $C^*_r(V, \mathcal{F})$ associated by A. Connes to the foliation (see [13]) is isomorphic to the C^*-tensor product of $C^*_r(G(X))$ by the C^*-algebra $\mathcal{K}$ of all compact operators on a separable Hilbert space (see [22]).

1.3.4. — Let $\sigma: X \to X$ be a continuous surjective map which is a local homeomorphism, as in example 1.2 (c), and let G be the corresponding r-discrete groupoid. We will denote by $C^*(X, \sigma)$ the C^*-algebra that it defines. In [35, p. 138], the Cuntz algebras $\mathcal{O}_n$ are exhibited as C^*-algebras of such groupoids associated to subshifts of finite type. More generally, the Cuntz-Krieger algebras can be described in this way, as well as many other interesting examples.

A detailed study of the groupoid G gives a lot of informations on the structure of $C^*(X, \sigma)$, as we will see now. After having been initiated in [35], this approach was considered by V. Azurmanian and A. Vershik in [7], and later on by V. Deaconu. Let us point out that in Deaconu's work [16], it is assumed that σ is a p-fold covering, but in fact this hypothesis is not necessary. For simplicity, we take X compact (otherwise we must suppose that σ is proper).

For $n \geq 0$, we set

$$R_n = \{(x,y) \in X \times X\ ;\ \sigma^n(x) = \sigma^n(y)\}.$$

Notice that R_n is open in R_{n+1}, and its topology is the one induced by R_{n+1}. It follows that $G_0 := \bigcup_n R_n$ endowed with the inductive limit topology is an r-discrete principal groupoid. We denote by $\sim$ the equivalence relation that it defines. Of course, G_0 is the open sub-groupoid of G consisting of its elements of the form $(x,0,y)$ (where we write (x,y) instead of $(x,0,y)$). More generally, for $n \in \mathbb{Z}$, we denote by G_n the subset of G formed by the (x,n,y)'s. Then $G = \bigcup_{n\in\mathbb{Z}} G_n$ is a partition of G by compact open sets.

For $x \in X$, let $p(x)$ be the number of z such that $\sigma(z) = x$. By a result of Eilenberg (see [2, Th. 2.1.1]), we know that the map $x \mapsto p(x)$ is continuous. The local homeomorphism σ induces a $*$-endomorphism ρ of $B = C_r^*(G_0)$ by the formula

$$\rho(f)(x,y) = \frac{1}{(p\circ\sigma(x)\,p\circ\sigma(y))^{1/2}} f\big(\sigma(x),\sigma(y)\big), \quad \forall f \in C_c(G_0).$$

Indeed we have

$$\begin{aligned}
&(\rho(f)\rho(g))(x,y)\\
&\quad= \frac{1}{(p\circ\sigma(x)\,p\circ\sigma(y))^{1/2}} \sum_{z\sim x} \frac{1}{p\circ\sigma(z)} f\big(\sigma(x),\sigma(z)\big)\, g\big(\sigma(z),\sigma(y)\big)\\
&\quad= \frac{1}{(p\circ\sigma(x)\,p\circ\sigma(y))^{1/2}} \sum_{v\sim\sigma(x)} f\big(\sigma(x),v\big)\, g\big(v,\sigma(y)\big)\\
&\quad= \rho(fg)(x,y).
\end{aligned}$$

Moreover, consider the element v of $C_cG)$ defined by

$$\begin{aligned}
v\big(x,1,\sigma(x)\big) &= \big(p\circ\sigma(x)\big)^{-1/2} &&\text{for}\quad x \in X,\\
v(x,n,y) &= 0 &&\text{otherwise.}
\end{aligned}$$

Straightforward computations give

$$v^*v = 1, \quad v^*Bv \subset B, \quad vfv^* = \rho(f) \ \text{ for } f \in C_c(G_0).$$

It follows that ρ can be extended to an injective $*$-endomorphism of B. We have

$$vv^*(x,0,y) = \frac{1}{p \circ \sigma(x)} \qquad \text{if} \quad \sigma(x) = \sigma(y),$$
$$vv^*(x,n,y) = 0 \qquad \text{otherwise.}$$

Clearly, if σ is non invertible, which will be our assumption from now on, v is a non unitary isometry. Thus ρ is a proper corner endomorphism of B.

Now, for $f \in C_c(G)$ and $k \in \mathbb{Z}$, we denote by f_k the pointwise product of f by the characteristic function of G_k. We put

$$F_0 = f_0, \quad F_k = f_k(v^*)^k, \ F_{-k} = v^k f_k \quad \text{for} \quad k \geq 1.$$

Then we have $F_k \in C_c(G_0)$ for all k, and

$$F_k p_k = F_k, \ p_k F_{-k} = F_{-k} \quad \text{for} \quad k \geq 0,$$

where $p_k = v^k(v^*)^k$. Moreover

$$f = \sum_{k\geq 1} (v^*)^k F_{-k} + F_0 + \sum_{k \geq 1} F_k v^k$$

is the unique decomposition of f as a finite sum with $F_k \in C_c(G_0)$ for all $k \in \mathbb{Z}$ and

$$F_k p_k = F_k, \quad p_k F_{-k} = F_{-k} \ \text{for} \ \ k \geq 1.$$

From this observation, it is not difficult to prove that $C^*(X,\sigma)$ is canonically isomorphic to the crossed product $B \rtimes_\rho \mathbb{N}$, which is the universal C^*-algebra generated by a copy of B and an isometry implementing ρ. In particular, since $B = \varinjlim C_r^*(R_n)$, where each $C_r^*(R_n)$ is strongly Morita equivalent to a commutative C^*-algebra, we see that $C^*(X,\sigma)$ is nuclear.

1.4. Amenability.

The nuclearity of the C^*-algebras defined by a groupoid is related to amenability properties that will be introduced now.

DEFINITION 1.4.1. — We say that a r-discrete locally compact groupoid G is *amenable* if there exists a net (f_i) in $C_c(G)$ such that

1) $\sum_{r(\gamma)=x} \left|f_i(\gamma)\right|^2 \leq 1$ for all $x \in G^0$ and i;

2) we have

$$\lim_i \sum_{r(\gamma')=r(\gamma)} f_i(\gamma')\,\overline{f_i(\gamma^{-1}\gamma')} = 1$$

uniformly on any compact subset of G.

This notion is introduced in [35, p. 92] (see also [3] for the case of a discrete group action). Let us recall now the notion of amenable measured groupoid, defined as in [36], by means of a fixed point property.

Let μ be a quasi-invariant measure on G^0, and E a separable Banach space, and consider a Borel cocycle α from G into the isometry group of E.

An *affine G-space* (with respect to α) is a Borel field $A = (A_x)_{x\in G^0}$ of compact convex subsets of the unit ball of E^*, such that

$$\alpha(\gamma^{-1})^* A_{d(\gamma)} = A_{r(\gamma)}$$

almost everywhere (see [36] for details).

DEFINITION 1.4.2. — The measured groupoid (G,μ) is said to be *amenable* if every affine G-space $A = (A_x)_{x\in G^0}$ has a fixed point, i.e. there is a Borel section $x \mapsto \varphi(x)$, with $\varphi(x) \in A_x$ a.e., such that

$$\alpha(\gamma^{-1})^*\varphi\big(d(\gamma)\big) = \varphi\big(r(\gamma)\big) \quad \text{a.e.}$$

DEFINITION 1.4.3. — A r-discrete locally compact groupoid G is said to be *measurewise amenable* if (G,μ) is amenable for every quasi-invariant measure μ on G^0.

The following proposition is true for any r-discrete groupoid (see [6]). For simplicity, we give here the proof in the case of a discrete group action, which is the only case considered in Section 3.

PROPOSITION 1.4.4. — *Let Γ be a countable discrete group acting by homeomorphisms on a second countable locally compact space X. The following conditions are equivalent:*

1) *the action α of Γ on X is amenable;*

2) *for each quasi-invariant measure μ on X, the crossed product von Neumann algebra $L^\infty(X,\mu) \rtimes \Gamma$ is injective;*

3) *the action α is measurewise amenable;*

4) *the full crossed product $C_0(X) \rtimes_\alpha \Gamma$ is a nuclear C^*-algebra;*

5) *the reduced crossed product $C_0(X) \rtimes_{\alpha,r} \Gamma$ is a nuclear C^*-algebra.*

Proof.

(1) $\Rightarrow$ (2). — Let μ be a quasi-invariant measure on X. We put

$$M = L^\infty(X,\mu) \rtimes \Gamma,$$

and $t \mapsto u(t) \in M$ will be the unitary representation of Γ implementing its action (still denoted by α) on $L^\infty(X,\mu)$. We denote by $\ell^2(\Gamma) \otimes_w M$ the self-dual right Hilbert M-module consisting of the families $(f(t))_{t\in\Gamma}$ of elements of M such that $\sum_{t\in\Gamma} f(t)^* f(t)$ is σ-weakly convergent, with inner product

$$\langle f, g\rangle = \sum_{t\in\Gamma} f(t)^* g(t) \in M$$

and obvious right M-action. Then the von Neumann algebra of all bounded M-linear maps of the Hilbert M-module $\ell^2 \otimes_w M$ is $\mathcal{L}(\ell^2(\Gamma)) \otimes M$.

Since the action of Γ on X is amenable, there is a net (f_i) of functions $f_i \in C_c(X\times\Gamma)$ satisfying the conditions (1) and (2) of Definition 1.4.1. We will view f_i as a map, with finite support, from Γ into $L^\infty(X,\mu) \subset M$, in an obvious way. Then conditions (1) and (2) give immediately

$$\sum_t f_i(t)^* f_i(t) \le 1 \quad \text{for all} \quad i,$$

$$\lim_i \langle f_i, \widetilde{\alpha}_s f_i\rangle = 1 \quad \sigma\text{-weakly, for all } s\in\Gamma,$$

where $(\widetilde{\alpha}_s f)(t) = \alpha_s(f(s^{-1}t))$.

Let us denote by ρ the normal faithful representation of M into $\ell^2(\Gamma) \otimes_w M$ defined by

$$(\rho(h)f)(t) = \alpha_{t^{-1}}(h) f(t), \quad \text{for all} \quad h\in L^\infty(X,\mu),\ f\in \ell^2(\Gamma)\otimes_w M,$$

$$(\rho(u(s))f)(t) = f(s^{-1}t), \quad \text{for all} \quad s\in\Gamma,\ f\in \ell^2(\Gamma)\otimes_w M,$$

and let w be the unitary in $\mathcal{L}(\ell^2(\Gamma)) \otimes M$ defined by

$$(wf)(t) = u(t)^* f(t), \quad \text{for all } f\in \ell^2(\Gamma)\otimes_w M.$$

Note that

$$\rho(M) \subset \mathcal{L}(\ell^2(\Gamma)) \otimes L^\infty(X,\mu).$$

Let ϕ_i be the completely positive map $y \mapsto \langle wf_i, ywf_i\rangle$ from $\mathcal{L}(\ell^2(\Gamma)) \otimes L^\infty(X,\mu)$ into M, where f_i is viewed as an element of $\ell^2(\Gamma)\otimes_w M$. We have $\|\phi_i\| \le 1$ for all i. Using a standard compacity argument, we may suppose that the net (ϕ_i) converges to ϕ in the topology of σ-weak pointwise convergence. For $s \in \Gamma$ and $h \in L^\infty(X, \mu)$, we have

$$\begin{aligned}\phi_i\big(\rho(u(s)h)\big) &= \sum_{t\in\Gamma} f_i(t)^*\, u(t)\, \alpha_{t^{-1}s}(h)\, u(s^{-1}t)\, f_i(s^{-1}t)\\ &= \sum_{t\in\Gamma} f_i(t)^*\, u(s)\, h\, f_i(s^{-1}t)\\ &= \langle f_i, \widetilde{\alpha}_s f_i\rangle\, u(s)\, h,\end{aligned}$$

so that $\phi \circ \rho(m) = m$ for all $m \in M$.

Since ϕ is a completely positive map from the injective von Neumann algebra $\mathcal{L}(\ell^2(\Gamma)) \otimes L^\infty(X, \mu)$ onto M, we see that M is injective.

(2) $\Rightarrow$ (3) follows from [38].

(3) $\Rightarrow$ (4). — Let (π, U) be a covariant representation of the dynamical system (X, Γ, α) into a separable Hilbert space H. Then there exist a quasi-invariant measure μ on X, a disintegration

$$H = \int_X^\oplus H(x)\, \mathrm{d}\mu(x),$$

and for every $(x, s) \in X \times \Gamma$ a unitary $u(x, s) : H(xs) \to H(x)$ such that $(x, s) \mapsto u(x, s)$ is a representation of the groupoid $X \times \Gamma$ and

$$\big(\pi(f)\xi\big)(x) = f(x)\, \xi(x) \qquad \text{for } f \in C_0(X),\ \xi \in \int_X^\oplus H(x)\, \mathrm{d}\mu(x),$$

$$(U_s\xi)(x) = r(x, s)^{1/2}\, u(x, s)\, \xi(xs) \quad \text{a.e. for } \xi \in \int_X^\oplus H(x)\, \mathrm{d}\mu(x),$$

where $r(x, s) = \mathrm{d}\mu(xs)/\mathrm{d}\mu(x)$.

Since for $n = 0, 1, \ldots, +\infty$, the measurable set X_n formed by the $x \in X$ with $\dim H(x) = n$ is Γ-invariant, (π, U) is a direct sum of covariant representations in Hilbert spaces whose disintegration gives fibers of constant dimension (a.e.). Therefore, we may suppose that

$$H = L^2(X, \mu) \otimes K$$

where μ is a quasi-invariant measure, K is a Hilbert space, and $(x, s) \mapsto u(x, s)$ is a Borel cocycle with values in the unitary group of K. By a close inspection of the proof of Theorem 2.1 in [37], we see that it can be easily adapted, in order to show that the amenability of the measured groupoid

$(X \times \Gamma, \mu)$ implies the injectivity of the von Neumann algebra generated by $\pi(C_0(X)) \cup \{U_s\,;\ s \in \Gamma\}$.

(4) $\Rightarrow$ (5) is obvious, and

(5) $\Rightarrow$ (1) has been proved in [3]. $\square$

2. Locally contracting r-discrete groupoids

In this section, G is a second countable locally compact r-discrete groupoid and we put $A = C_r^*(G)$.

DEFINITION 2.1. — We say that G is *locally contracting* if for every non empty open subset U of G^0, there exist an open subset V in U and an open bisection S with $\overline{V} \subset d(S)$ and $\alpha_{S^{-1}}(\overline{V}) \subsetneqq V$.

We will give later many examples of locally contracting groupoids.

REMARK. — Clearly, S above is not contained in G^0, since α_S is not the identity map, and replacing S by $S \setminus G^0$ we will always assume that $S \cap G^0 = \emptyset$.

For any open bisection S, we denote by χ_S its characteristic function. Although it does not belong to A in general, for every continuous function h with compact support in $d(S)$, note that

$$\gamma \mapsto (\chi_S h)(\gamma) = \chi_S(\gamma) h\big(d(\gamma)\big)$$

belongs to $C_c(G)$. Moreover, we have

$$(\chi_S h \chi_{S^{-1}})(\gamma) = \begin{cases} h \circ \alpha_S(\gamma) & \text{if } \gamma \in r(S), \\ 0 & \text{otherwise.} \end{cases}$$

By a slight abuse of notation, we set

$$\chi_S h \chi_{S^{-1}} = h \circ \alpha_S$$

in this case. We have $h \circ \alpha_S \in C_c(G^0)$, and the support Supp $h \circ \alpha_S$ of $h \circ \alpha_S$ is $\alpha_{S^{-1}}(\text{Supp } h)$.

PROPOSITION 2.2. — *Let G be a locally contracting r-discrete groupoid, and let f be a non zero positive element of $C_0(G^0)$. Then there exist an infinite projection p in $A = C_r^*(G)$ and $\lambda > 0$ in $\mathbb{R}$ such that $\lambda p \le f$.*

Proof. — We note first that, with our assumption, it is very easy to get projections in A. Following an idea of [12], we begin by constructing scaling elements, that is elements $x \in A$ with $x^* x x = x$.

Let V be an open subset of G^0 and S an open bisection in $G \setminus G^0$ with $\overline{V} \subset d(S)$ and $\alpha_{S^{-1}}(\overline{V}) \subsetneq V$. Choose $h: G^0 \to [0,1]$, continuous with compact support in V, and set $x = \chi_S h$. Then

$$x^* x = h^2 \quad \text{and} \quad xx^* = \chi_S h^2 \chi_{S^{-1}} = h^2 \circ \alpha_S.$$

Therefore, taking h equal to 1 on a neighbourhood of $\alpha_{S^{-1}}(\overline{V})$, we get $h h \circ \alpha_S = h \circ \alpha_S$, since $\operatorname{Supp} h \circ \alpha_S = \alpha_{S^{-1}}(\operatorname{Supp} h)$. It follows that $x^* x x x^* = xx^*$, and thus $x^* x x = x$.

In $\widetilde{A}$, we put

$$v = x + (1 - x^* x)^{1/2}.$$

Then $v^* v = 1$, and

$$vv^* = 1 - \big(x^* x - xx^* - x(1 - x^* x)^{1/2} - (1 - x^* x)^{1/2} x^*\big).$$

Hence

$$p = x^* x - xx^* - x(1 - x^* x)^{1/2} - (1 - x^* x)^{1/2} x^*$$

is a projection in A. Moreover, we have $p \neq 0$ since $x^* x - xx^*$ is non zero with support in G^0 and $x(1 - x^* x)^{1/2} + (1 - x^* x)^{1/2} x^*$ is a function with support in $G \setminus G^0$.

Now, given a positive function g in $C_0(G^0)$, which is equal to 1 on a non empty open subset U of G^0, we can easily construct x such that $gx = xg = x$ by taking the above V contained in U. Then we have

$$gp = pg = p.$$

To achieve the proof of proposition 2.2, we need to show that p is infinite. Using an idea of M. Laca and J. Spielberg [25], we construct, in the same way as x above, a new scaling element y with $py = yp = y$. Then

$$w = p(y + (1 - y^* y)^{1/2})$$

will be a partial isometry in A with $w^* w = p$ and

$$ww^* = p - y^* y + yy^* + y(1 - y^* y)^{1/2} + (1 - y^* y)^{1/2} y^*$$

strictly less than p since $y^* y - yy^* - y(1 - y^* y)^{1/2} - (1 - y^* y)^{1/2} y^* \neq 0$.

For the construction of y, we choose a positive function $g' \in C_0(G^0)$, equal to 1 on a non empty open set, such that $g' x^* x = g'$ and $g' xx^* = 0$. Then we take the scaling element y such that $g' y = y g' = y$. It follows that $x^* x y = y x^* x = y$ and $yx = x^* y = 0$, and hence $py = yp = y$. □

LEMMA 2.3. — *Let G be an essentially free r-discrete groupoid. Given a compact subset K of $G \setminus G^0$ and a non empty open subset U of G^0, there exists a non empty open subset V of U such that $r^{-1}(V) \cap d^{-1}(V) \cap K = \emptyset$.*

Proof. — There exist open bisections $S'_1, \dots, S'_n$ in $G \setminus G^0$ and, for $i = 1, \dots, n$, an open set S_i with $S_i \subset \overline{S}_i \subset S'_i$, such that $K \subset \bigcup_{i=1}^{n} S_i$. We set $\alpha_i = \alpha_{S_i}$. Let $u \in U$ with trivial isotropy. For each i with $u \in r(S'_i)$, we denote by γ_i the element of S'_i such that $r(\gamma_i) = u$. Since $d(\gamma_i) \neq u$, there exists a neighbourhood V_i of u with $\alpha_i(V_i) \cap V_i = \emptyset$. Put

$$V = \Big(U \setminus \bigcup_{u \notin r(S'_i)} r(\overline{S}_i)\Big) \cap \Big(\bigcap_{u \in r(S'_i)} V_i\Big),$$

and consider $\gamma \in r^{-1}(V) \cap K$. Let i such $\gamma \in S_i$. Since $r(\gamma) \in r(S_i) \cap V$, we have $u \in r(S'_i)$ and $d(\gamma) \notin V$. Therefore $r^{-1}(V) \cap d^{-1}(V) \cap K = \emptyset$. □

PROPOSITION 2.4. — *Let G be a r-discrete groupoid, essentially free and locally contracting. Then every non zero hereditary sub-C^*-algebra of $A = C_r^*(G)$ contains an infinite projection.*

Proof. — Let $a \in A_+$ and $a \neq 0$. We will construct an infinite projection p, and a real $\lambda > 0$ such that $\lambda p \leq a$. We may suppose that $\|E(a)\| = 1$. We first choose $b \in A_+ \cap C_c(G)$ with

$$\|a - b\| < \tfrac{1}{4}\cdot$$

Then $b_0 = E(b)$ satisfies $\|b_0\| > \frac{3}{4}$, and $b_1 = b - b_0$ has its compact support K contained in $G \setminus G^0$. Let

$$U = \big\{\gamma \in G^0\,;\ b_0(\gamma) > \tfrac{3}{4}\big\}.$$

Using Lemma 2.3, we consider an open subset V of U with

$$r^{-1}(V) \cap d^{-1}(V) \cap K = \emptyset.$$

Let $f : G^0 \to [0,1]$ be a continuous function with compact support in V, such that $\{\gamma, f(\gamma) = 1\}$ has a non empty interior. Since

$$(fb_1f)(\gamma) = f \circ r(\gamma) f \circ d(\gamma)\, b_1(\gamma),$$

we see that $fbf = fb_0f$. By the previous proposition (and its proof), we may find a projection p and a partial isometry w with $w^*w = p$, $ww^* < p$, and $pf = p$. Then we get

$$w^*bw = w^*fb_0fw \geq \tfrac{3}{4} w^*f^2w = \tfrac{3}{4}p,$$

$$w^*aw \geq w^*bw - \tfrac{1}{4}p \geq \tfrac{1}{2}p.$$

It follows that w^*aw is invertible in pAp. We denote by c its inverse and we put $u = c^{1/2}w^*a^{1/2}$. We have $uu^* = p$ and $u^*u \leq \|c\|a$. Therefore u^*u is an infinite projection in the hereditary sub-C^*-algebra of A generated by a. ∎

REMARK 2.5. — Note that the assumption of Proposition 2.4 is fulfilled when G is a r-discrete, essentially free, minimal groupoid with a G-map α_S having a non isolated fixed point ℓ_0 which is an attractor in the following sense: one may find a neighbourhood W of ℓ_0 such that for every neighbourhood U of ℓ_0 there exists $n \geq 0$ with $(\alpha_S)^n(W) \subset U$. We will give in the two next sections several examples where this situation occurs.

3. Purely infinite transformation group C^*-algebras

Let us first recall that a C^*-algebra is said to be *purely infinite* if each of its non zero hereditary sub-C^*-algebras contains an infinite projection.

The bootstrap class $\mathcal{N}$ is defined as the smallest class of C^*-algebras containing the separable commutative ones, which is closed under stable isomorphisms, inductive limits, extensions and crossed products by $\mathbb{Z}$ or $\mathbb{R}$. These C^*-algebras satisfy to the Universal Coefficient Theorem. Therefore, the purely infinite, simple, nuclear C^*-algebras belonging to the class $\mathcal{N}$ are completely classified by their K-theory (see [24] and [29]).

For all the basic facts on hyperbolic metric spaces, we refer to [20] and [19]. Let X be an hyperbolic space in Gromov's sense. We suppose that X is geodesic (i.e. every pair of points in X can be joined by a geodesic segment) and proper, which means that the closed balls are compacts. Let Γ be a countable group of isometries of X. The set of points in the hyperbolic boundary ∂X of X which are limit points of the orbit Γx does not depend on the choice of $x \in X$. This set, called the limit set of Γ, will be denoted $L(\Gamma)$. One says that Γ is non elementary if $L(\Gamma)$ has at least three points, and in this case $L(\Gamma)$ is a compact perfect metric space.

PROPOSITION 3.1. — *Let X be an hyperbolic, geodesic, proper metric space, and let Γ be a non elementary group of isometries of X. We suppose that Γ has no fixed point in ∂X and we denote by α the natural action of Γ on $L(\Gamma)$. Then $C(L(\Gamma)) \rtimes_{\alpha r} \Gamma$ is a simple purely infinite C^*-algebra.*

Proof. — Since every $t \in \Gamma$ has at most 2 fixed points in ∂X, the action α is essentially free. Moreover it is minimal (see [19, p. 153]), and therefore $C(L(\Gamma)) \rtimes_{\alpha r} \Gamma$ is a simple C^*-algebra. It follows from [20, § 8.2.F], that for every non empty open subset U of $L(\Gamma)$ there exits an hyperbolic element $t \in \Gamma$ such that U contains its attractive fixed point $t^{+\infty}$. Let V_+

be an open neighbourhood of $t^{+\infty}$ and V_- an open neighbourhood of the repulsive fixed point $t^{-\infty}$ of t, with $V_+ \cap V_- = \emptyset$. By [19, th. 16, p. 147], there exists $n_0 \geq 1$ such that $t^n(\partial X \setminus V_-) \subset V_+$ for all $n \geq n_0$. Hence, the action of Γ on $L(\Gamma)$ is locally contracting. □

In particular the above proposition can be applied to any complete simply connected Riemannian manifold X with sectional curvature less than a real $\kappa < 0$. Let us mention that if moreover the curvature is bounded below, then the action of Γ on ∂X is amenable (see [32]).

Consider now a group Γ with a finite system S of generators (where we suppose S symmetric and not containing the neutral element e of Γ). Recall that Γ is said to be *hyperbolic* if its Cayley graph $\mathcal{G}(\Gamma, S)$, endowed with the word metric, is hyperbolic (this property does not depend on the choice of S). We denote by $\partial\Gamma$ the boundary of $\mathcal{G}(\Gamma, S)$ (which is also independent of S, up to quasi-conform equivalence), and remark that $\partial\Gamma$ is the limit set of Γ acting by translations on $\mathcal{G}(\Gamma, S)$. Let us point out that here Γ is non elementary if and only if it does not contain a cyclic subgroup of finite index (see [19, p. 129]) and that, in this case, $\partial\Gamma$ is a perfect compact space of finite dimension without any point fixed by Γ [19, p. 154]). This class of groups includes the free groups and the fundamental groups of the Riemannian compact manifolds with strictly negative curvature.

PROPOSITION 3.2. — *Let Γ be a non elementary hyperbolic group and denote by α its action on its boundary $\partial\Gamma$. Then $C(\partial\Gamma) \rtimes_{\alpha r} \Gamma$ is a simple purely infinite nuclear C^*-algebra.*

Proof. — We only have to remark that $C(\partial\Gamma)) \rtimes_{\alpha r} \Gamma$ is nuclear, and this follows from the result of S. Adams [1] showing that the action of Γ on $\partial\Gamma$ is measurewise amenable. □

Let us now consider the particular case of proposition 3.1 where X is the hyperbolic Poincaré upper half-space H^2. Here the group of orientation preserving isometries of X is $\mathrm{PSL}(2, \mathbb{R})$, and ∂X is the real projective line $P^1(\mathbb{R})$, canonically isomorphic to $\mathrm{PSL}(2, \mathbb{R})/P$, where P is the parabolic subgroup of all upper triangular matrices.

A fuchsian group Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. When Γ is non elementary, then either $L(\Gamma) = P^1(\mathbb{R})$ or $L(\Gamma)$ is the Cantor discontinuum. The group Γ is said to be of the *first kind* in the first case. Note also that a fuchsian group Γ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ (i.e. a discrete subgroup of finite covolume) if and only if it is finitely generated and of the first kind.

PROPOSITION 3.3. — *Let Γ be a fuchsian group of the first kind, and let*

α be its natural action on the boundary $P^1(\mathbb{R})$ of the Poincaré upper half-space. Then $C(P^1(\mathbb{R})) \rtimes_{\alpha r} \Gamma$ is a simple purely infinite nuclear C^*-algebra in the bootstrap class $\mathcal{N}$.

Proof. — That $C(P^1(\mathbb{R})) \rtimes_{\alpha r} \Gamma$ is simple purely infinite is a consequence of Proposition 3.1, since $P^1(\mathbb{R}) = L(\Gamma)$. Moreover, by [30], we know that $C(P^1(\mathbb{R})) \rtimes_{\alpha r} \Gamma$ is Morita equivalent to the crossed product

$$C\big(\Gamma \setminus \mathrm{PSL}(2,\mathbb{R})\big) \rtimes P,$$

which is nuclear and belongs to the class $\mathcal{N}$ because P is a solvable group. ▯

We will now show that this result can be extended to any lattice of any real connected, semi-simple Lie group G without compact factors, and with trivial centre.

PROPOSITION 3.4. — *Let Γ be a lattice of a real connected, semi-simple Lie group G without compact factors, and with trivial centre. Let P be a minimal parabolic subgroup of G. Let us denote by α the natural action of Γ on the Furstenberg boundary G/P. Then $C(G/P) \rtimes_{\alpha r} \Gamma$ is a simple purely infinite nuclear C^*-algebra in the bootstrap class $\mathcal{N}$.*

Proof. — As in Proposition 3.3, $C(G/P) \rtimes_{\alpha r} \Gamma$ is Morita equivalent to $C_0(\Gamma \backslash G) \rtimes P$ which is nuclear in the class $\mathcal{N}$. The lattice Γ acts minimally on the compact space G/P by [27, Lemma 8.5]. Moreover, the action is essentially free since any $\gamma \neq e$ in Γ has a finite number of fixed points. It follows that $C(G/P) \rtimes_{\alpha r} \Gamma$ is simple. It remains to show that the action is locally contracting. For the proof, the paper [10] has been useful to us.

Consider a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Let us denote by Σ the set of roots of $(\mathfrak{g}, \mathfrak{a})$, and for $\lambda \in \Sigma$, let $\mathfrak{g}_\lambda$ be the corresponding root space. Choose a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$, and denote by Σ_+ the set of positive roots. We put

$$\mathfrak{n} = \sum_{\lambda \in \Sigma_+} \mathfrak{g}_\lambda,$$

and we denote by A and N the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively. As usual, KAN is the Iwasawa decomposition of G. Finally, M and M' are the centralizer and the normalizer of $\mathfrak{a}$ in K, so that M'/M is the Weyl group of G. We denote by P the minimal parabolic subgroup MAN of G, and the Furstenberg boundary G/P by B.

Next, we need to recall some facts on the Bruhat decomposition of G. For each $w \in W$, we choose a representative $s_w \in M'$. Then G is the disjoint union

$$\bigcup_{w \in W} P s_w P$$

If $\bar{w}$ is the longest element in W, we know that $P s_{\bar{w}} P$ is an open subset of G, and that the other terms are submanifolds of lower dimension (see [21, p. 407]). It follows in particular that

$$G/P = \bigcup_{w \in W} N b_w,$$

where $b_w = s_w P$, and $N b_{\bar{w}}$ is open. Denoting by $\overline{N}$ the opposite subgroup $s_{\bar{w}} N s_{\bar{w}}^{-1}$ of N, we have also

$$B = \bigcup_{w \in W} s_{\bar{w}} N b_w = \bigcup_{w \in W} \overline{N} b_w$$

where $\overline{N} b_e$ is open in B.

Let β and θ be the Killing form and the Cartan involution on $\mathfrak{g}$, and denote by $\| \,.\, \|$ the norm defined by the inner product

$$(X, Y) \longmapsto -\beta(X, \theta Y) \quad \text{on } \mathfrak{g}.$$

Then M acts isometrically on $\mathfrak{g}$ by Ad. The elements of $A_+ = \exp \mathfrak{a}_+$ centralize $\mathfrak{m} \oplus \mathfrak{a}$, expand $\mathfrak{n}$ and shrink $\bar{\mathfrak{n}}$, in the decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}.$$

Therefore, given $t_0 \in MA_+$, there exists $\lambda \in]0,1[$ such that

$$\| \operatorname{Ad} t_0 \, X \| \le \lambda \|X\|$$

for $X \in \bar{\mathfrak{n}}$. Note that the diffeomorphism $\phi : X \mapsto (\exp X) b_e$ from $\bar{\mathfrak{n}}$ onto $\overline{N} b_e$ is equivariant:

$$\big(\exp(\operatorname{Ad} t_0 \, X)\big) b_e = t_0 (\exp X) b_e$$

for all $X \in \bar{\mathfrak{n}}$. It follows that, given a neighbourhood U of b_e in B, there exist an open neighbourhood V of b_e contained in U, and $n_0 \ge 0$ such that $t_0^n(\overline{V}) \subsetneq V$ for all $n \ge n_0$.

To conclude, we need the following crucial fact (see [11, Appendice]): any lattice in G contains an element t_1 of the form gt_0g^{-1} where $g \in G$ and $t_0 \in MA_+$. Then, let U be a non empty open subset of B. Since the Γ-action on B is minimal, there is $t_2 \in \Gamma$ with $t_2gb_e \in U$. By the above observations, we may find an open subset V' of $\overline{N}b_e \cap g^{-1}t_2^{-1}U$ and $n > 0$ with $t_0^n(\overline{V'}) \subsetneqq V'$. Putting $V = t_2gV'$, we get $t_2t_1^nt_2^{-1}(\overline{V}) \subsetneqq V$. □

Actually, other discrete isometry groups of manifolds with nonpositive curvature yields purely infinite C^*-algebras. For more details on this paragraph, we refer the reader to [17], [8] and [9].

Let us recall that a *Hadamard manifold* X is a complete, simply connected Riemannian manifold of nonpositive curvature. For such a manifold, there is a notion of point at infinity, and the set ∂X of these point has a natural topology which turns it into a compact space. Let Γ be a discrete group of isometries of X. The action of Γ extends to an action on the boundary ∂X. As for hyperbolic metric spaces, the limit set $L(\Gamma)$ of Γ is the closed subset of ∂X defined as the set of limit points of an orbit of Γ in X. When Γ is such that $\mathrm{vol}(\Gamma \setminus X) < +\infty$, we have $L(\Gamma) = \partial X$.

A Hadamard manifold is called a *visibility manifold* if for every two different points $x, y \in \partial X$, there is a geodesic linking x to y. There are visibility manifolds whose curvature is not strictly negative (see [17]). However, many of the properties of a complete simply connected Riemannian manifold with curvature strictly less than $\kappa < 0$ remain true for visibility manifolds. In particular, if Γ is a discrete group of isometries of a visibility manifold X, the number of fixed points in ∂X of any $t \in \Gamma \setminus \{e\}$ is ≤ 2 (see [9, p. 85]). If moreover $\mathrm{vol}(\Gamma \setminus X) < +\infty$, the action of Γ on ∂X is minimal and locally contracting by [8, th. 2.8 and th. 2.2]. Therefore we have

PROPOSITION 3.5. — *Let Γ be a discrete group of isometries of a visibility manifold X, such that $\mathrm{vol}(\Gamma \setminus X) < +\infty$, and denote by α the natural action of Γ on ∂X. Then $C(\partial X) \rtimes_{\alpha r} \Gamma$ is a simple purely infinite C^*-algebra.*

PROBLEM 3.6. — Let X be a Hadamard manifold without flat de Rham component, and let Γ be a discrete group of isometries of X, such that $\mathrm{vol}(\Gamma \setminus X) < +\infty$.

- Is the reduced crossed product $C(\partial X) \rtimes_{\alpha r} \Gamma$ a simple purely infinite C^*-algebra?
- Is it a nuclear C^*-algebra?

4. Purely infinite C^*-algebras associated with expanding maps

DEFINITIONS 4.1. — Let X be a compact metric space and $\sigma: X \to X$ a continuous surjection. We say that σ is *expansive* if there is a constant $c > 0$ such that $x \neq y$ implies $d(\sigma^n(x), \sigma^n(y)) \geq c$ for some integer $n \geq 0$. If moreover σ is open, we say that σ is an *expanding* map.

By a result of W. Reddy, it is known (see [2, Th. 2.2.10]) that if $\sigma: X \to X$ is a continuous expanding surjection, then there exist a compatible metric d' and constants $\delta_0 > 0$, $\lambda > 1$, such that

$$d'\big(\sigma(x), \sigma(y)\big) \geq \lambda d'(x,y)$$

whenever $d'(x,y) < \delta_0$. In particular, an expanding map is a local homeomorphism.

An element $x \in X$ is said to be *eventually periodic* if its positive orbit is finite, that is, if there are two integers $p \neq q$ with $\sigma^p(x) = \sigma^q(x)$.

PROPOSITION 4.2. — *Let X be a compact metric space and $\sigma: X \to X$ an expanding continuous surjection. We suppose that the eventually periodic points of σ form a dense set with empty interior. Then $C^*(X, \sigma)$ is nuclear, purely infinite and belongs to the bootstrap class $\mathcal{N}$.*

Proof. — Since $x \in X$ has a non trivial isotropy if and only if it is eventually periodic, we see that the groupoid G defined by the dynamical system (X, σ) is essentially free. To prove that it is locally contracting, we consider a non empty open subset U of X. Let $y \in U$ and $k \geq 0$ such that $\sigma^k(y) = x$ is periodic, and denote by p its period, that is the smallest integer $p \geq 1$ with $\sigma^p(x) = x$. Replacing U by a smaller open set if necessary, we may suppose that $U = r(S)$, where S is an open bisection of the form $\Omega(U, U', k - 0)$: this means that $\sigma^k{}_{|U}$ is an homeomorphism onto U' and $\alpha_S = \sigma^k{}_{|U}$.

Since σ is expanding, there exist a compatible metric d and $\delta_0 > 0$, $\lambda > 1$, such that $d(x,y) < \delta_0$ implies $d(\sigma(x), \sigma(y)) \geq \lambda d(x,y)$. Let W be an open neighbourhood of x contained in U' with the following properties:

(a) $\sigma^p{}_{|W}$ is an homeomorphism onto an open set W';

(b) for $\ell = 0, \dots, p$, and $z_1, z_2 \in \sigma^\ell(W)$, we have $d(z_1, z_2) < \delta_0$.

It follows that

$$d(z_1, z_2) \leq \frac{1}{\lambda^\ell} d\big(\sigma^\ell(z_1), \sigma^\ell(z_2)\big) \quad \text{for } z_1, z_2 \in W.$$

In particular, we get

$$d(x, z) \leq \frac{1}{\lambda^p} d\big(x, \sigma^p(z)\big) \quad \text{for all } z \in W.$$

Denote by T the bisection

$$\Omega(W', W, 0-p) = \{(\sigma^p(z), 0-p, z)\,;\ z \in W\}.$$

We have $d(x, \alpha_T(z')) \le \lambda^{-p} d(x, z')$ for all $z' \in W'$. Therefore, choosing r with

$$B(x,r) = \{z'\,;\ d(x,z') < r\} \subset W' \cap U',$$

we see that $B(x,r)$ is contained, for all $k > 0$, in the domain of definition of a G-map, which will be denoted α_{T^k}, and satisfies $\alpha_{T^k}(z') = (\alpha_T)^k(z')$ for $z' \in B(x,r)$. Since x is non isolated in X, we may find an open neighbourhood V' of x contained in $B(x,r)$ and $k \ge 1$, with $\alpha_{T^k}(\overline{V'}) \subsetneqq V'$. Then we set

$$V = \alpha_S^{-1}(V') \subset U,$$

and since

$$\alpha_{T^k} \circ \alpha_S(\overline{V}) \subsetneqq \alpha_S(V),$$

we conclude that there exists a well defined G-map α_R with $\overline{V} \subset r(R)$ and $\alpha_R(\overline{V}) \subsetneqq V$.

That $C^*(X,\sigma)$ is nuclear in the class $\mathcal{N}$ follows from 1.3.4. □

Let us consider the finite set of symbols (or states) $S = \{1, \dots, N\}$ with $N \ge 2$. Let $\Sigma^+(N)$ be the set of sequences of the form $x = (x_i)_{i\ge 0}$ where $x_i \in S$. The set $\{1, \dots, N\}$ is equipped with the discrete topology, and $\Sigma^+(N)$ with the product topology, which makes it a Cantor set. For $x, y \in \Sigma^+(N)$, we put

$$v(x,y) = \begin{cases} +\infty & \text{if } x = y, \\ \min\{k \ge 0, x_k \ne y_k\} & \text{otherwise.} \end{cases}$$

Then

$$d\colon (x,y) \longmapsto \mathrm{e}^{-v(x,y)}$$

is a distance on $\Sigma^+(N)$, compatible with the topology. The full one-sided N-shift $(\Sigma^+(N), \sigma)$ consists of the compact space $\Sigma^+(N)$ upon which the shift transformation acts by the formula

$$\big(\sigma(x)\big)_n = x_{n+1} \quad \text{for} \quad n \in \mathbb{N}.$$

Clearly, whenever $d(x,y) < 1$, we have $d(\sigma(x), \sigma(y)) \ge \mathrm{e}\, d(x,y)$, and therefore $(\Sigma^+(N), \sigma)$ is expanding.

A *subshift* (X, σ) is defined as a closed shift-invariant subspace X of $\Sigma^+(N)$, together with the restriction of σ to X. Recall that a word is a finite sequence $w = (w_1, \dots, w_k)$ of elements of the alphabet S. There is a list F of forbidden words such that X is defined as the subset of elements of $\Sigma^+(N)$ that do not contain a word of F as a string of consecutive symbols. A word which appears as a string of consecutive symbols in a sequence $x \in X$ is called *admissible.* We denote by W the set of all admissible words for X. The concatenation of two words w and v is denoted wv. We say that two admissible words w_1 and w_2 are *equivalent,* and we write $w_1 \sim w_2$ if

$$\{v \in W\,;\ w_1 v \in W\} = \{v \in W\,;\ w_2 v \in W\}.$$

PROPOSITION 4.3. — *Let (X,σ) be a subshift and W its list of admissible words. Let us suppose that*

1) *for all $i \in S$, the word reduced to the letter i is admissible, and there exists j with $(ji) \sim (i)$ in W;*

2) *(X,σ) has a point which is not eventually periodic;*

3) *given $w_1, w_2 \in W$, there is a word v such that $w_1 v \sim w_2$ in W.*

Then $C^*(X,\sigma)$ is a simple, nuclear, purely infinite C^*-algebra in the class $\mathcal{N}$.

Proof. — First, $\sigma : X \to X$ is surjective, thanks to the condition 1). The assumption 3) implies obviously that every full orbit $O(x)$ is dense in X. Using the minimality of the system, and assumption 2), we see that there is a dense set of points which are not eventually periodic. The groupoid defined by (X,σ) is therefore essentially free and minimal, and $C^*(X,\sigma)$ is simple. Note that (X,σ) is expanding. To conclude, it is enough to prove that there is an eventually periodic point, but this is obvious since there are words w and v in W with $wv \sim w$. In fact, wvv is also admissible, and by repeating v infinitely many times, we construct an eventually periodic element in X. ∎

It is very easy to construct examples which fulfill the condition of Proposition 4.2. The most famous ones are taken among subshifts of finite type (Σ_A^+, σ), where $A = (a_{i,j})$ is a $N \times N$ matrix whose entries are either zeros or ones and

$$\Sigma_A^+ = \{x \in \Sigma^+(N)\,;\ a_{x_i, x_{i+1}} = 1 \text{ for } i \in \mathbb{N}\}.$$

The matrix A is called *irreducible* if for all i, j there is an $m > 0$ such that $(A^m)_{i,j} > 0$. Clearly, when A is an irreducible matrix which is not a permutation matrix, the conditions 1) to 3) of Proposition 4.3 are satisfied for (Σ_A^+, σ). In fact, when the less stringent property (I) of [15] is satisfied, we see also that $C^*(\Sigma_A^+, \sigma)$ is purely infinite, as a consequence of Proposition 4.2. As observed by J. Renault, $C^*(\Sigma_A^+, \sigma)$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$ defined by A.

Let us give now an example of subshift, not of finite type, which also yields a purely infinite simple C^*-algebra. We take $N = 2$ and the forbidden words are those beginning and ending by 2, with only the symbol 1 inside, repeated an odd number of times. Here also, it is easily checked that the assumptions of Proposition 4.3 are fulfilled.

EXAMPLE 4.4. — Let M be a compact Riemannian manifold. Then a differentiable map $\sigma : M \to M$ is expanding if and only if there exist $\lambda > 1$ and $c > 0$ such that

$$\|D\sigma^n v\| \geq c\lambda^n \|v\|$$

for any tangent vector v and $n \geq 1$. The differentiable expanding maps on a compact manifold M without boundary have been studied by M. Shub in [31]. We note that the existence of such an expanding map imposes serious restrictions on the manifold: its universal covering is diffeomorphic to a space $\mathbb{R}^n$, and its Euler characteristic is zero. M. Shub has also proved that any differentiable expanding map σ has a dense positive orbit, and a dense set of periodic points. It follows that (M, σ) satisfies the properties of Proposition 4.2.

Well known examples of such differentiable expanding maps are constructed as follows. We take $M = \mathbb{R}^n/\mathbb{Z}^n$ and we consider a matrix $A \in M_n(\mathbb{Z})$ having all its eigenvalues of absolute value strictly greater than 1. Obviously, this linear map on $\mathbb{R}^n$ induces an expanding map on M. Morever, every full orbit $O(x)$ is dense, that is, (M, f) is a minimal dynamical system.

REMARK 4.5. — Although we have not considered in details here the case of the reduced C^*-algebra defined by a foliation, let us mention that, as a consequence of Proposition 2.4, we get:

PROPOSITION. — *Let $(V, \mathcal{F})$ be a minimal foliation (i.e. such that every leaf is dense in V). We suppose that there exist a leaf F and a loop γ in F, whose base point is an attractor (in the sense of Remark 2.5) for the element of the pseudogroup of holonomy of $\mathcal{F}$ associated to γ. Then $C_r^*(V, \mathcal{F})$ is simple and purely infinite.*

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