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# LIOUVILLE THEOREMS BASED ON SYMMETRIC DIFFUSIONS

PAR

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*Dedicated to Professor M. Fukushima on his 60th birthday*

RÉSUMÉ. — On étudie dans cet article le théorème de Liouville pour les fonctions sousharmoniques en se basant sur l'espace de Dirichlet. L'intégrale de Hellinger permet d'écrire les théorèmes de Liouville pour les fonctions sousharmoniques sans hypothèse de régularité sur une fonction exhaustive donnée et d'examiner avec précision la croissance des fonctions sousharmoniques non constantes.

ABSTRACT. — In this article, we study Liouville properties for subharmonic functions based on the Dirichlet space theory. In order to describe Liouville theorems for subharmonic functions, we will use the Hellinger integral which eliminates the smoothness requirement on the given exhaustion function and enables us to examine the increasing order of non-constant subharmonic functions.

## 1. Introduction

The study on Liouville theorems has been developed in geometry as well as in complex analysis. Geometrical Liouville theorems ordinarily state that if a Riemannian manifold enjoys certain condition on the curvature, then it does not admit non-constant bounded subharmonic function (*e.g.* [C-Y]). In complex analysis, Liouville theorems assert that any bounded plurisubharmonic function or occasionally pluriharmonic function does not exist except constant, whenever the underlying complex manifold is parabolic in a certain sense (*e.g.* [K], [T1] and [T2]). N. Sibony and P.M. Wong [S-W] suggested that this sort of assertion is usually related to the vanishing of the capacity of the infinity. The notion of capacity for the value distribution theory was originally dealt with by W. Stoll (*e.g.* [Sto1]).

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In the mean time, the capacity is expressed by the Dirichlet integral of 0-equilibrium potential with respect to the corresponding self-adjoint differential operator. M. Fukushima pointed out in [F2] that the vanishing of the Dirichlet integral concerns the non-transience of the corresponding Hunt process. In the recent development of Dirichlet space theory, H. Ôkura [Ok3] found a recurrence criterion for Hunt process in terms of the Hellinger integral which is attributed to [H]. The method was utilized to reproduce a sharp capacitary estimate by K.T. Sturm [Stu2]. Their arguments are based upon the stochastic metric introduced by M. Biroli and U. Mosco in [B-M2].

Throughout this paper, we focus our attention on the case that the Dirichlet space has the strong local property, that is, the case that the corresponding Hunt process is a diffusion with no killing inside. We start with an inequality which will be the primary tool for our Liouville theorems formulated in Section 3. Our theorems not only handle the family of subharmonic functions locally in the domain of the Dirichlet space but imply some of the known results on the absence of bounded functions with subharmonicity such as [C-Y], [T1] and [Stu1]. Our method also provides a Liouville theorem without exhaustion function, which is directly applicable to the plurisubharmonic functions without the approximating procedures as in [T1].

As for notions and notations, the author recommends the reader to consult the book [F-O-T]. The author expresses his thanks to Professor M. Fukushima and Professor M. Takeda for their heartfelt encouragement and the proof of Lemma 1 which is shorter than the author's original one.

## 2. Green's formula

We denote a regular Dirichlet space on $L^2(X,m)$ with the strong local property by $(\mathcal{E},\mathcal{F})$, where $X$ is a locally compact Hausdorff space and $m$ is a Radon measure with $\mathrm{supp}[m]=X$. In this section, we deal with the case that the underlying space $X$ has a continuous exhaustion function $\rho\in\mathcal{F}_{\mathrm{loc}}$. Throughout this paper, all elements in $\mathcal{F}_{\mathrm{loc}}$ are assumed to be quasi-continuous already.

We use the following notations:

$$\begin{aligned} B(s) &= \big\{x\in X\,;\ \rho(x)<s\big\}, && \text{for}\quad s>0\,,\\ B(r,s) &= \big\{x\in X\,;\ r<\rho(x)<s\big\}, && \text{for}\quad s>r>0\,,\\ e_{\langle u,v\rangle}(s) &= \mu_{\langle u,v\rangle}\big(B(s)\big), && \text{for}\quad u,v\in\mathcal{F}_{\mathrm{loc}}\,, \end{aligned}$$

where $\mu_{\langle u,v\rangle}$ is the co-energy of $u,v\in\mathcal{F}_{\mathrm{loc}}$.

The measure $\mu_{\langle u,u\rangle}$ (resp. $e_{\langle u,u\rangle}$) is denoted by $\mu_{\langle u\rangle}$ (resp. $e_{\langle u\rangle}$).

MAIN LEMMA (Green's formula). — *If $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$, then*

$$\int_r^R e_{\langle u,v\rangle}(s)\,\mathrm{d}s - \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} = \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle}$$

*holds for $w_{r,R} = R - (\rho \wedge R) \vee r$, where*

$$x \vee a = \max\{x,a\} \quad \textit{and} \quad x \wedge b = \min\{x,b\}.$$

*Especially, if $u$ is $\mathcal{E}$-subharmonic and $v$ is non-negative, then*

$$\int_r^R e_{\langle u,v\rangle}(s)\,\mathrm{d}s \le \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle}\,.$$

*Proof.* — Since the assertion is local, it suffices to consider the case that $u \in \mathcal{F}$ and $v \in \mathcal{F}_b$. By using Theorem 3.2.2 in [F-O-T], we can derive the desired identity as follows:

$$\begin{aligned}\int_r^R \mathrm{e}_{\langle u,v\rangle}(s)\,\mathrm{d}s &= \int_r^R\int_{B(s)} \mathrm{d}\mu_{\langle u,v\rangle}\,\mathrm{d}s = \int_X w_{r,R}\,\mathrm{d}\mu_{\langle u,v\rangle}\\ &= \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} - \int_X v\,\mathrm{d}\mu_{\langle u,w_{r,R}\rangle}\\ &= \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} + \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle},\end{aligned}$$

where the last equality follows from the identity in [M] (see also [Ok3, Lemma 2.1 (i)]). The first term in the last expression equals $2\mathcal{E}(w_{r,R}v,u)$, which is non-positive when $u$ is subharmonic and $v$ is non-negative. $\square$

REMARK 1. — We consider the case that

$$\mathcal{F} = H^1(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N)\,;\ \frac{\partial u}{\partial x^i} \in L^2(\mathbb{R}^N),\ i = 1,\dots,N\Big\},$$

$$\mathcal{E}(u,v) = \tfrac{1}{2}\int_{\mathbb{R}^N} (\nabla u, \nabla v)\,\mathrm{d}V \quad \text{for } u,v \in \mathcal{F},$$

where $V$ is the Lebesgue measure. If a bounded domain $D$ is exhausted by a smooth function $\rho$ as $D = \{\rho < r\} \subset \mathbb{R}^N$ and $\mathrm{d}\rho \neq 0$ is satisfied on $\partial D$, we can verify the following convergences for functions $u,v$ in $C^2$:

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{r-\varepsilon}^r e_{\langle u,v\rangle}(s)\,\mathrm{d}s &= \int_D (\nabla u,\nabla v)\,\mathrm{d}V,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_X \mathrm{d}\mu_{\langle u,w_{r-\varepsilon,r}v\rangle} &= -\int_D v\,\triangle u\,\mathrm{d}V,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B(r-\varepsilon,r)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle} &= \int_{\partial D} v\frac{\partial u}{\partial \nu}\,\mathrm{d}S,\end{aligned}$$

where $\partial u/\partial \nu$ denotes the outer normal derivative of $u$ and $S$ stands for the volume element on $\partial D$. Therefore, our formula implies the classical Green's formula.

In the sequel, we also use the following lemma.

LEMMA 1. — *A function $u$ in $\mathcal{F}_{\mathrm{loc}}$ is constant quasi-everywhere on $X$ if $\mu_{\langle u\rangle}$ vanishes.*

*Proof.* — By the strong local property, $\mu_{\langle (u-\alpha)\vee 0\rangle} = I_{\{u>\alpha\}}\mu_{\langle u\rangle}$ for any $\alpha \in \mathbb{R}$ and accordingly $\mu_{\langle (u-\alpha)\vee 0\rangle}$ vanishes under the stated condition. This implies the $\mathcal{E}$-harmonicity of $(u-\alpha)\vee 0$ and consequently

$$p_t\big[(u-\alpha)\vee 0\big] = (u-\alpha)\vee 0 \qquad (\forall\,\alpha\in\mathbb{R},\ \forall\, t>0)$$

for the semi-group $\{p_t\}$ generated by $(\mathcal{E},\mathcal{F})$. The $p_t$-invariance of $\{u\le\alpha\}$ follows from this identity. From the irreducibility of the diffusion it turns out that $u$ must equal $\inf\{\alpha\,;\ \mathrm{Cap}(X\setminus\{u\le\alpha\})=0\}$ quasi-everywhere on $X$. $\square$

## 3. The Liouville Theorems

We start with the case that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet space on $L^2(X,m)$ with the strong local property which generates an irreducible diffusion and that $X$ has a continuous exhaustion function $\rho\in\mathcal{F}_{\mathrm{loc}}$. For $u\in\mathcal{F}_{\mathrm{loc}}$, we set

$$m(u,s) = \operatorname{ess\,sup}\big\{u(x)\,;\ \rho(x)<s\big\},$$

which is equal to $\inf\{\alpha\,;\ \mathrm{Cap}(B(s)\setminus\{u\le\alpha\})=0\}$ by the quasi-continuity of $u$. We further set

$$h^*(s) = h(s) - \sum_{\xi\le s}\big(h(\xi+)-h(\xi)\big)$$

for any left-continuous increasing function $h$.

We shall particularly need $e^*_{\langle u\rangle}(\cdot)$.

LEMMA 2. — *If $u\in\mathcal{F}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}$ is non-negative and $\mathcal{E}$-subharmonic, then*

$$\text{(i)}\quad \int_r^R e_{\langle u\rangle}(s)\,\mathrm{d}s \le m(u,R)\sqrt{e_{\langle\rho\rangle}(R)-e_{\langle\rho\rangle}(r)} \times\sqrt{e^*_{\langle u\rangle}(R)-e^*_{\langle u\rangle}(r)}\,,$$

$$\text{(ii)}\quad \int_r^R e^*_{\langle u\rangle}(s)\,\mathrm{d}s \le m(u,R)\sqrt{e_{\langle\rho\rangle}(R)-e_{\langle\rho\rangle}(r)} \times\sqrt{e^*_{\langle u\rangle}(R)-e^*_{\langle u\rangle}(r)}\cdot$$

*Proof.* — Due to the inequality in [B-H, Thm 5.2.3], we have that $\mu_{\langle\rho\rangle}$ never charges any level set $\{\rho = \xi\}$ (where $\xi \geq \inf \rho$). Therefore, we can derive

$$\int_r^R e_{\langle u\rangle}(s)\,\mathrm{d}s \leq \int_{B(r,R)\cap\{\rho\notin E\}} u\,\mathrm{d}\mu_{\langle u,\rho\rangle}$$

for $E = \{\xi\,;\ e_{\langle u\rangle}(\xi+) > e_{\langle u\rangle}(\xi)\}$ from Main Lemma. Assertion (i) follows from this inequality and Lemma 5.6.1 in [F-O-T]. Assertion (ii) is clear from (i). □

For non-negative increasing continuous function $g$ and positive non-decreasing left continuous function $h$ on the interval $[a,b]$, we will need the integral

$$\int_a^b \frac{(\mathrm{d}s)^2}{h(s)\,\mathrm{d}g(s)}$$

to describe our Liouville theorems. For any divisions $\Delta$ and $\Delta'$ of $[a,b]$, we can show that

$$\Delta = \big\{a = \xi_0 < \xi_1 < \cdots < \xi_k = b\big\} \subset \Delta' = \big\{a = \eta_0 < \eta_1 < \cdots < \eta_\ell = b\big\}$$

implies

$$\sum_{i=1}^{k-1} \frac{(\xi_{i+1} - \xi_i)^2}{h(\xi_{i+1})(g(\xi_{i+1}) - g(\xi_i))} \leq \sum_{i=1}^{\ell-1} \frac{(\eta_{i+1} - \eta_i)^2}{h(\eta_{i+1})(g(\eta_{i+1}) - g(\eta_i))}.$$

Therefore, by running $\Delta = \{a = \xi_0 < \xi_1 < \cdots < \xi_k = b\}$ in all divisions of $[a,b]$, we can define the integral

$$\int_a^b \frac{(\mathrm{d}s)^2}{h(s)\,\mathrm{d}g(s)} = \sup_\Delta \sum_{i=1}^{k-1} \frac{(\xi_{i+1} - \xi_i)^2}{h(\xi_{i+1})(g(\xi_{i+1}) - g(\xi_i))}$$

in the same manner as the standard Hellinger integral

$$\int_a^b \frac{(\mathrm{d}s)^2}{\mathrm{d}g(s)} = \sup_\Delta \sum_{i=1}^{k-1} \frac{(\xi_{i+1} - \xi_i)^2}{g(\xi_{i+1}) - g(\xi_i)}.$$

THEOREM 1. — *If $(\mathcal{E},\mathcal{F})$ is irreducible and for large enough $r$*

$$\lim_{R\to\infty} \int_r^R \frac{(\mathrm{d}s)^2}{\mathrm{d}e_{\langle\rho\rangle}(s)} = \infty,$$

*then any non-negative $\mathcal{E}$-subharmonic function $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ satisfying*

$$\int_r^\infty \frac{(\mathrm{d}s)^2}{m(u,s)^2 \,\mathrm{d}e_{\langle\rho\rangle}(s)} = \infty$$

*for any $r$ is constant quasi-everywhere on $X$.*

*Proof.* — We shall first verify that $e^*_{\langle u\rangle}(\cdot) \equiv 0$. Otherwise, $e^*_{\langle u\rangle}(r) > 0$ for some $r > 0$ and, for an arbitrary division $\Delta : r = \xi_0 < \xi_1 < \cdots < \xi_k = R$ of the interval $[r, R]$, we can derive the following estimate from Lemma 2 (ii):

$$\begin{aligned}
\sum_{i=0}^{k-1} & \frac{(\xi_{i+1} - \xi_i)^2}{m(u, \xi_{i+1})^2 (e_{\langle\rho\rangle}(\xi_{i+1}) - e_{\langle\rho\rangle}(\xi_i))} \\
&\le \sum_{i=0}^{k-1} \frac{(e^*_{\langle u\rangle}(\xi_{i+1}) - e^*_{\langle u\rangle}(\xi_i))}{\frac{1}{(\xi_{i+1}-\xi_i)^2}\big(\int_{\xi_i}^{\xi_{i+1}} e^*_{\langle u\rangle}(s)\,\mathrm{d}s\big)^2} \\
&\le \sum_{i=0}^{k-1} \frac{(e^*_{\langle u\rangle}(\xi_{i+1}) - e^*_{\langle u\rangle}(\xi_i))}{e^*_{\langle u\rangle}(\xi_i)^2}.
\end{aligned}$$

The assumption on the Hellinger integral implies that the left-hand side can exceed arbitrarily large number by taking sufficiently large $R$ and sufficiently small $\|\Delta\| = \max_{0\le i\le k-1}(\xi_{i+1} - \xi_i)$. However, the right hand side does not exceed $\dfrac{1}{e^*_{\langle u\rangle}(r)} + 1$, for $\|\Delta\|$ small enough, arriving at a contradiction. The identity $e^*_{\langle u\rangle}(\cdot) \equiv 0$ and Lemma 2 (i) imply $e_{\langle u\rangle}(\cdot) \equiv 0$. The function $u$ is constant quasi-everywhere on $X$ by Lemma 1. $\square$

REMARK 2. — This theorem covers a wider family of subharmonic functions than [C-Y] and [T1] and describes the growth order of non-constant subharmonic function more precisely than the Liouville theorem in [T1].

Here, we give the definition of $e_{\langle u,u^{p-1}\rangle}(\cdot)$ by assuming $p > 1$. For an $\mathcal{E}$-subharmonic function $u \in \mathcal{F}_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$, the method in [Stu1] enables us to define the *co-energy measure* $e_{\langle u,u^{p-1}\rangle}(\cdot)$ by

$$e_{\langle u,u^{p-1}\rangle}(r) = (p-1) \lim_{n\to\infty} \int_{B(r)} \big[u_b^{(n)}\big]^{(p-2)} \,\mathrm{d}\mu_{\langle u_b^{(n)}, u_b^{(n)}\rangle}$$

($r \in [\inf\rho, \sup\rho)$), where $\{u_b^{(n)}\}_{n=1}^\infty \subset \mathcal{F}_b$ is defined by

$$u_b^{(n)} = \psi_n(u) - \psi_n(u-n) \quad (n = 1, 2, \ldots)$$

with $\{\psi_n\}_{n=1}^{\infty} \subset C^2(\mathbb{R})$ satisfying $0 \leq \psi_n' \leq 1$, $\psi_n'' \geq 0$ and

$$\psi_n(x) = \begin{cases} 1/n & \text{if } x \leq 0 \\ x & \text{if } x \geq 2/n. \end{cases}$$

We note the following equivalent expressions of $e_{\langle u,u^{p-1}\rangle}(\cdot)$ involving $\mathcal{E}$-subharmonic functions $u^{(n)} = \psi_n(u)$:

$$\begin{aligned} e_{\langle u,u^{p-1}\rangle}(r) &= (p-1)\lim_{n\to\infty}\int_{B(r)} \big[u_b^{(n)}\big]^{(p-2)}\,\mathrm{d}\mu_{\langle u_b^{(n)},u^{(n)}\rangle} \\ &= (p-1)\lim_{n\to\infty}\int_{B(r)} \big[u_b^{(n)}\big]^{(p-2)}\,\mathrm{d}\mu_{\langle u^{(n)},u^{(n)}\rangle}\,. \end{aligned}$$

LEMMA 3. — *If $\mu_{\langle\rho\rangle} \leq m$ and $p > 1$, then any non-negative $\mathcal{E}$-subharmonic function $u \in \mathcal{F}_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ enjoys the following properties*:

$$\text{(i)}\quad \int_r^R e_{\langle u,u^{p-1}\rangle}(s)\,\mathrm{d}s \leq \sqrt{\ell(u,R)-\ell(u,r)} \times \sqrt{\tfrac{1}{p-1}\big[e^*_{\langle u,u^{p-1}\rangle}(R) - e^*_{\langle u,u^{p-1}\rangle}(r)\big]}\,,$$

$$\text{(ii)}\quad \int_r^R e^*_{\langle u,u^{p-1}\rangle}(s)\,\mathrm{d}s \leq \sqrt{\ell(u,R)-\ell(u,r)} \times \sqrt{\tfrac{1}{p-1}\big[e^*_{\langle u,u^{p-1}\rangle}(R) - e^*_{\langle u,u^{p-1}\rangle}(r)\big]}\,,$$

*where $\ell(u,s) = \int_{B(s)} u^p\,\mathrm{d}m$.*

*Proof.* — It suffices to show (i). Thanks to the inequality $\mu_{\langle\rho\rangle} \leq m$, the same discussion as in the proof of Lemma 2 (i) shows that

$$\begin{aligned} \int_r^R e_{\langle u^{(n)},[u_b^{(n)}]^{(p-1)}\rangle}(s)\,\mathrm{d}s &\leq \int_{B(r,R)\cap\{\rho\notin E\}} \big[u_b^{(n)}\big]^{(p-1)}\,\mathrm{d}\mu_{\langle\rho,u^{(n)}\rangle} \\ &\leq \sqrt{\int_{B(r,R)} \big[u_b^{(n)}\big]^{p}\,\mathrm{d}m} \\ &\qquad \times \sqrt{\int_{B(r,R)\cap\{\rho\notin E\}} \big[u_b^{(n)}\big]^{(p-2)}\,\mathrm{d}\mu_{\langle u^{(n)},u^{(n)}\rangle}} \end{aligned}$$

for $E = \{\xi\,;\ e_{\langle u\rangle}(\xi+) > e_{\langle u\rangle}(\xi)\}$. Letting $n \to \infty$, we have (i). $\square$

Here, we have a generalization of [Stu1, Thm 1].

THEOREM 2. — *If $\mu_{\langle\rho\rangle} \leq m$ and $p > 1$ and $(\mathcal{E},\mathcal{F})$ is irreducible, then there exists no non-negative non-constant $\mathcal{E}$-subharmonic function $u \in \mathcal{F}_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ satisfying, for large enough $r$,*

$$\int_r^\infty \frac{(\mathrm{d}s)^2}{\mathrm{d}\ell(u,s)} = \infty.$$

*Proof.* — We can show that $e^*_{\langle u,u^{(p-1)}\rangle}(\cdot) \equiv 0$ through the same discussion as in the proof of Theorem 1 by using Lemma 3 (ii). From this identity and Lemma 3 (i), we obtain that $e_{\langle (u\wedge b)\vee a\rangle}(\cdot) \equiv 0$ for any positive numbers $b, a$ (where $b > a$), which shows $u$ is constant on $\{u > 0\}$ by Lemma 1. This together with the $\mathcal{E}$-subharmonicity of $u$ implies $\{u > 0\}$ is $p_t$-invariant. Since $(\mathcal{E},\mathcal{F})$ is irreducible, $u$ is constant. □

Henceforth, we fix a non-polar compact set $K$ in $X$ and an increasing sequence of relatively compact open sets $\{G_n\}_{n=1}^\infty$ enjoying

$$\bigcup_{n=1}^\infty G_n = X.$$

$\mathrm{Cap}(K;G_n)$ is defined as the Dirichlet integral $\mathcal{E}(e_K^{G_n}, e_K^{G_n})$ of 0-equilibrium potential $e_K^{G_n}$. We will describe a Liouville theorem without exhaustion function. For that purpose, we set

$$\rho_n = 1 - e_K^{G_n} \quad (n = 1, 2, \ldots),$$

which will be utilized as substitutes for exhaustion function. We introduce the notation

$$e_{\langle u\rangle}(s \mid w) = \mu_{\langle u,u\rangle}\big(\{w < s\}\big)$$

for $u, w \in \mathcal{F}_{\mathrm{loc}}$, especially we denote $e_{\langle w\rangle}(s \mid w)$ by $e^\star_{\langle w\rangle}(s)$. The continuity of $e^\star_{\langle w\rangle}(\cdot)$ is another consequence of Theorem 5.2.3 in [B-H].

THEOREM 3. — *If $(\mathcal{E},\mathcal{F})$ is irreducible and recurrent, then there exist no non-negative non-constant $\mathcal{E}$-subharmonic function $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ satisfying*

$$\varliminf_{n\to\infty} m(u,G_n)^2 \,\mathrm{Cap}(K;G_n) = 0,$$

*where $m(u,G_n) = \operatorname{ess\,sup}\big\{u(x); x \in G_n\big\}$.*

*Proof.* — For any $0 < r < 1$, the definition of the Hellinger integral assures that

$$\int_r^1 \frac{(\mathrm{d}s)^2}{m^{(n)}(u,s)^2\,\mathrm{d}e^\star_{\langle\rho_n\rangle}(s)}$$

dominates

$$\frac{(1-r)^2}{m(u,G_n)^2\,\mathrm{Cap}(K;G_n)},$$

where

$$m^{(n)}(u,s)=\inf\{\alpha\,;\ \mathrm{Cap}(\{\rho_n<s\}\setminus\{u\le\alpha\})=0\}.$$

Thanks to the quasi-continuity of $\rho_n$, the argument up to the proof of Theorem 1 works in deriving the following inequalities:

$$\begin{aligned}\frac{(1-r)^2}{m(u,G_n)^2\,\mathrm{Cap}(K;G_n)} &\le \int_r^1 \frac{(\mathrm{d}s)^2}{m^{(n)}(u,s)^2\,\mathrm{d}e^*_{\langle\rho_n\rangle}(s)}\\ &\le \int_r^1 \frac{\mathrm{d}e^*_{\langle u\rangle}(s\mid\rho_n)}{e^*_{\langle u\rangle}(s\mid\rho_n)^2}\\ &\le \frac{1}{e^*_{\langle u\rangle}(r\mid\rho_n)}.\end{aligned}$$

Therefore $\lim_{n\to\infty} e^*_{\langle u\rangle}(r\mid\rho_n)=0$ for $0<r<1$. Since Lemma 2 (i) holds for $\rho=\rho_n$,

$$\int_r^R e_{\langle u\rangle}(s\mid\rho_n)\,\mathrm{d}s\le m(u,G_n)\sqrt{\mathrm{Cap}(K;G_n)}\,\sqrt{e^*_{\langle u\rangle}(R\mid\rho_n)-e^*_{\langle u\rangle}(r\mid\rho_n)}\,,$$

where $0<r<R<1$. Letting $n\to\infty$, we can conclude that

$$\mu_{\langle u,u\rangle}\big(p_K>1-r\big)=\lim_{n\to\infty} e_{\langle u\rangle}(r\mid\rho_n)=0,$$

where $p_K$ is the 0-order hitting probability of $K$ with respect to the associated diffusion. On account of the irreducibility and [F-O-T, Thm 4.6.6], we see by letting $r\uparrow 1$ that $\mu_{\langle u,u\rangle}(X)=0$ and hence $u$ is constant quasi-everywhere on $X$ by virtue of Lemma 1. □

As an application of Theorem 3, we can now formulate a Liouville theorem for plurisubharmonic functions.

THEOREM 4. — *If $A$ is an irreducible closed complex analytic set in $\mathbb{C}^N$ with pure dimension $k\ge 2$ and if*

$$\int_c^\infty \frac{\mathrm{d}s}{s\,\mathrm{n}(A,s)}=\infty,$$

*then $A$ does not admit any non-constant plurisubharmonic function $u$ satisfying*

$$\varliminf_{R\to\infty} \frac{m(u,R)^2}{\displaystyle\int_c^R \frac{\mathrm{d}s}{s\,\mathrm{n}(A,s)}} = 0,$$

*where*

- $\mathrm{n}(A,s) = \displaystyle\int_{r(A)\cap B_{\mathrm{euc}}(s)} \big[\mathrm{dd}^c \log|z|^2\big]^k$;
- $r(A)$ *denotes the set of regular points in* $A$;
- $|z|^2 = \sum\limits_{i=1}^{N} |z_i|^2$ *and*
- $B_{\mathrm{euc}}(s) = \{z \in \mathbb{C}^N ; |z| < s\}$.

*Proof.* — We may assume that $0 < \operatorname{dist}(0, r(A)) < c$. Setting

$$\theta = \big[\mathrm{dd}^c \log(|z|^2+1)\big]^{k-1}, \quad m = \mathrm{dd}^c|z|^2 \wedge \theta,$$

we obtain a symmetric closable bilinear form on $L^2(r(A), m)$

$$\mathcal{E}_0^\theta(u,v) = \int_{r(A)} \mathrm{d}u \wedge \mathrm{d}^c v \wedge \theta, \qquad u, v \in C_0^\infty(A).$$

Since $\rho_0 = \log(|z|^2+1)$ is strongly plurisubharmonic on $A$, the smallest closed extension $(\mathcal{E}^\theta, \mathcal{F}^\theta)$ of $\mathcal{E}_0^\theta$ generates an irreducible holomorphic diffusion on $r(A)$. From [F-O], we know that $\mathcal{F}_{\mathrm{loc}}^\theta$ contains the family of locally bounded plurisubharmonic functions on $r(A)$ as a subfamily of $\mathcal{E}^\theta$-subharmonic functions. We set

$$\begin{aligned} a(r) &= \int_c^r \frac{t\,\mathrm{d}t}{(t^2+1)\mathrm{n}(A,t)}, \\ w_R(z) &= \Big[\Big(1 - \frac{a(|z|)}{a(R)}\Big) \wedge 1\Big] \vee 0 \in \mathcal{F}^\theta \qquad \big(R > \inf_{z\in A} \rho_0(z)\big). \end{aligned}$$

We note that $w_R$ is equal to 1 on $B_{\mathrm{euc}}(c)$ and 0 on $B_{\mathrm{euc}}(R)^c$ and that

$$\begin{aligned} \mathrm{n}(A,r) &= \int_{r(A)\cap\partial B_{\mathrm{euc}}(r)} \mathrm{d}^c \log|z|^2 \wedge \big[\mathrm{dd}^c \log|z|^2\big]^{k-1} \\ &= \frac{\operatorname{vol}(A \cap \partial B_{\mathrm{euc}}(r))}{r^{2k}}. \end{aligned}$$

We obtain a similar identity to the last one by replacing $|z|^2$ and $r$ with $|z|^2+1$ and with $r+1$ respectively. Thus

$$\lim_{r\to\infty} \frac{\int_{r(A)\cap\partial B_{\mathrm{euc}}(r)} \mathrm{d}^c \log(|z|^2+1)\wedge\theta}{\mathrm{n}(A,r)} = 1.$$

As a result, we obtain an estimate

$$\begin{aligned}
&\mathcal{E}^\theta(w_R, w_R)\\
&= \frac{1}{a(R)^2}\int_{r(A)\cap B_{\mathrm{euc}}(c,R)} \mathrm{d}a\big(|z|\big)\wedge \mathrm{d}^c a\big(|z|\big)\wedge\theta\\
&= \frac{1}{4a(R)^2}\int_{r(A)\cap B_{\mathrm{euc}}(c,R)} \frac{1}{\mathrm{n}(A,|z|)^2}\,\mathrm{d}\log\big(|z|^2+1\big)\wedge \mathrm{d}^c\log(|z|^2+1)\wedge\theta\\
&= \frac{1}{2a(R)^2}\int_c^R \frac{t\,\mathrm{d}t}{(t^2+1)\,\mathrm{n}(A,t)^2}\int_{r(A)\cap\partial B_{\mathrm{euc}}(t)} \mathrm{d}^c\log\big(|z|^2+1\big)\wedge\theta\\
&\le \frac{1}{a(R)^2}\Big(C_1 + C_2\int_c^R \frac{t\,\mathrm{d}t}{(t^2+1)\mathrm{n}(A,t)}\Big) \le \frac{C_3}{a(R)}
\end{aligned}$$

for some constants $C_1$, $C_2$ and $C_3$.

Theorem 4 now follows from Theorem 3, because

$$\mathrm{Cap}\big(\overline{B(c)}, B(R)\big) \le \mathcal{E}^\theta(w_R, w_R) \le \frac{C_3}{a(R)}$$

and $\displaystyle\lim_{R\to\infty} a(R)\Big/\int_c^R \frac{\mathrm{d}s}{s\,\mathrm{n}(A,s)} = 1.$ □

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