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MICROLOCAL BOUNDARY VALUE PROBLEM IN HIGHER CODIMENSIONS

BY

KIYOSHI TAKEUCHI (*)

ABSTRACT. — The aim of this paper is to set up the microlocal study of higher codimensional boundary value problems by solving a part of Schapira's conjecture on the concentration of the complex $\mathcal{C}_{\Omega|X}$ of sheaves. We prove the microlocal injectivity of the higher codimensional boundary value morphism as an application of the new correspondence between the complex $\mathcal{C}_{\Omega|X}$ and the second microfunction $\mathcal{C}_{ML}$ of Kataoka–Tose–Okada and Schapira–Takeuchi. The Kashiwara–Kawaï's extension theorem will be generalized to non elliptic equations.

RÉSUMÉ. — Le but de cet article est de commencer l'étude microlocale des problèmes aux limites en codimension supérieure en résolvant une partie de conjecture de Schapira sur la concentration du complexe $\mathcal{C}_{\Omega|X}$ des faisceaux. On démontre l'injectivité microlocale du morphisme de valeur au bord en codimension supérieure comme une application de la nouvelle correspondence entre le complexe $\mathcal{C}_{\Omega|X}$ et la deuxième microfonction $\mathcal{C}_{ML}$ de Kataoka–Tose–Okada et Schapira–Takeuchi. Le théorème d'extension due à Kashiwara–Kawaï sera généralisé aux équations non elliptiques.

1. Introduction

The aim of the present paper is to set up the microlocal study of higher codimensional boundary value problems by establishing basic theorems. After the definition of the boundary value in the framework of the hyperfunction theory due to Komatsu–Kawaï [6] and Schapira [19], many studies have been done in the case when the codimension of the boundary is equal to one. Around 1986, P. Schapira [21] introduced the complex $\mathcal{C}_{\Omega|X}$ of sheaves for an open subsets Ω of $M := \mathbb{R}^n$ and

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constructed the microlocal boundary value morphism for a submanifold $N \subset M$ of codimension d such that $N \subset \overline{\Omega}$:

$$(1.1) \qquad R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{C}_{N|X}[d]\big),$$

where $X := \mathbb{C}^n$ is a complexification of M and $\mathcal{C}_{N|X}$ is a sheaf introduced by Sato–Kawaï–Kashiwara [18].

This construction enables us to study the boundary value problems microlocally and algebraically. The reader is suggested to refer to the important works before Schapira's definitions in [2], [6], [11], [16] and [20], etc. Many classical results on the estimation of the singularities of the boundary value or on the propagation of regularities up to the boundary and so on were recovered and extended to the systems of differential equations systematically when the codimension of the boundary equals to one. However it seems that the higher codimensional boundary value problems are not yet studied successfully except in the papers [6], [26] and [30] for elliptic equations and in [17] in non microlocal situations. The difficulty lies essentially in the fact that the structure of the complex $\mathcal{C}_{\Omega|X}$ for general open subsets Ω was not completely determined yet. For this purpose, we will clarify the structure of the complex $\mathcal{C}_{\Omega|X}$ when Ω is an open convex cone of $M = \mathbb{R}^n$ with vertex at the origin. In the sequel, we assume $\Omega := \Omega_1 \times \mathbb{R}^{n-d}$ be an open convex cone with the edge $N := \{0\} \times \mathbb{R}^{n-d}$ for an open convex proper cone $\Omega_1 \subset \mathbb{R}^d$. First of all, we show in THEOREM 3.1 (i):

The complex $\mathcal{C}_{\Omega|X}$ is concentrated in degree 0 *on $N \times_M T^*_M X$,*

where $X := \mathbb{C}^n$ is a complexification of M. It gives an affirmative answer to Schapira's conjecture in 1986 [22] in particular cases. (As for the studies of the complex $\mathcal{C}_{\Omega|X}$ for open subsets Ω whose complements $K = M \backslash \Omega$ are closed convex subsets of M, refer to [22], [23], and [29].) This concentration is also important to investigate the propagation of regularities up to the boundary, so-called «Ω–regularity» of [22]. In Section 4, we will completely determine the structure of the complex $\mathcal{C}_{\Omega|X}$ for open quadrants Ω, and it will be effectively used to get several theorems on the extension of hyperfunction solutions in Section 8. In 1988, Oaku [17] proved the local unicity of the boundary value problems in higher codimensional cases, that is, the injectivity of the boundary value morphism:

$$(1.2) \qquad \varinjlim_{\Omega} \Gamma_\Omega \, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_{|N} \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

for coherent $\mathcal{D}_X$-modules $\mathcal{M}$ for which N is non-characteristic. Here the inductive limit is taken by shrinking the open convex proper cone Ω_1

including a fixed vector of $\mathbb{R}^d$. His proof of it relies upon the analytic methods which uses F-mild hyperfunctions. Our main theorem, which is proved purely algebraically, microlocalizes this result as follows.

THEOREM 6.1. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X for which N is non-characteristic. Then the canonical morphism of the boundary value:*

$$\varinjlim_{\Omega} H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Omega|X}) \longrightarrow H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})[d] \simeq \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})$$

is injective on $\pi_X^{-1}(N) = N \times_X T^*X$, where $\pi_X : T^*X \longrightarrow X$.

This theorem can also be considered as an extension of Schapira's result in [23] to arbitrary codimensions and explicitly shows how the singularities of the hyperfunction solutions defined in the interior domain propagate to the boundary as in COLLORARY 8.8. Several theorems on the extension of hyperfunction solutions in Section 8 are essentially deduced from this theorem. In THEOREM 8.6, we give a natural generalization of a theorem of Kashiwara–Kawaï [6] to non elliptic equations. In the course of the proof of these theorems, we find in THEOREM 7.2 a new correspondence between « the sheaf » $\mathcal{C}_{\Omega|X\,|N\times_M T^*_M X}$ and the sheaf $\mathcal{C}_{ML}$ of second microfunction via the Legendre transformation. Here the sheaf $\mathcal{C}_{ML}$ was introduced first by Kataoka–Tose [12] and recently reformulated by Schapira–Takeuchi [24].

As an application of this correspondence, we give an interesting example of the vanishing of the $\mathcal{C}_{ML}$ solution complex to the H. Lewy equation. Section 5 is devoted to the proof of the results announced in Schapira–Takeuchi [24]. Finally Example 8.9 explains how very basic facts in the hyperfunction theory can be considered as a particular case of our results, because the hyperfunction theory as a boundary value of the holomorphic functions is itself an example of the boundary value problems treated in this paper.

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2. Simple sheaves and canonical transformations

Let X be a C^∞ manifold and f, g be C^∞ functions on X such that $\mathrm{d}f \wedge \mathrm{d}g$ never vanishes on X. The notion of simple sheaves was introduced by Kashiwara–Schapira [8]. In this article, we will frequently use the terminologies in [8] and [9].

LEMMA 2.1. — *Set $H_- := \{x \in X \mid f(x) = 0,\ g(x) \le 0\}$. Then $\mathbb{C}_{H_-}$ is a simple sheaf with shift* 1 *on X (along $\Lambda := \{(x; a \cdot \mathrm{d}f + b \cdot \mathrm{d}g) \in T^*X; f(x) = g(x) = 0,\ a \in \mathbb{R},\ b < 0\} \subset T^*X$).*

Proof. — The problem being local, we may assume X is locally a product of two manifolds: $X = X_1 \times X_2$ and there exists C^∞ function f_1 (resp. f_2) on X_1 (resp. X_2) such that $\mathbb{C}_{H_-} = \mathbb{C}_{\{f_1=0\}} \boxtimes \mathbb{C}_{\{f_2\le 0\}}$. Therefore LEMMA 2.1 is a direct consequence of Proposition 7.5.10 (i) and Example 7.5.5 (i), (iii) of [9]. □

Now, suppose X is a complex manifold and f a holomorphic function on X such that $\mathrm{d}f$ never vanishes on X. It follows immediately that $\mathrm{d}\,\mathrm{Re}\, f \wedge \mathrm{d}\,\mathrm{Im}\, f$ never vanishes on X. By virtue of LEMMA 2.1, if we set $H_- := \{z \in X \mid \mathrm{Re}\, f(z) = 0,\ \mathrm{Im}\, f(z) \le 0\}$, $\mathbb{C}_{H_-}[-1]$ is a simple sheaf with shift 0 on X.

Let X and Y be complex manifolds of the same dimension, and $z = x + iy$ (resp. $w = u + iv$) be a local coordinate system of X (resp. Y). Then there are canonical identifications:

$$
(2.1)\quad \begin{cases} \phi_X : T^*X^{\mathbb{R}} \simeq T^*X, & (x, y; \xi\,\mathrm{d}x + \eta\,\mathrm{d}y) \longmapsto (x + iy; (\xi - i\eta)\,\mathrm{d}z), \\ \phi_Y : T^*Y^{\mathbb{R}} \simeq T^*Y, & (u, v; p\,\mathrm{d}x + q\,\mathrm{d}y) \longmapsto (u + iv; (p - iq)\,\mathrm{d}z), \end{cases}
$$

and $\phi_{X\times Y} : T^*(X^{\mathbb{R}} \times Y^{\mathbb{R}}) \simeq T^*(X \times Y)$, where $X^{\mathbb{R}}$ and $Y^{\mathbb{R}}$ denote the underlying real analytic manifold of X and Y respectively. Let Λ be an $\mathbb{R}^+$-conic complex Lagrangian submanifold of $T^*(X \times Y)$ and Ω_X (resp. Ω_Y) be an open subset of T^*X (resp. T^*Y), and assume: $p^a_{1|\Lambda} : \Lambda \simeq \Omega_X$ and $p_{2|\Lambda} : \Lambda \simeq \Omega_Y$ are complex analytic diffeomorphisms, where p_1 and p_2 are the first and second projection from $T^*X \times T^*Y$ to T^*X and T^*Y respectively and p^a_1 is the composition of p_1 and the antipodal mapping of T^*X. Set $\Phi_\Lambda := (p_{2|\Lambda})(p^a_{1|\Lambda})^{-1}$. Then it induces a complex canonical transformation between Ω_X and Ω_Y. We denote by $\Lambda^{\mathbb{R}}$ the real submanifold of $T^*(X^{\mathbb{R}} \times Y^{\mathbb{R}})$ which corresponds to Λ, that is, $\Lambda^{\mathbb{R}} := \phi^{-1}_{X\times Y}(\Lambda)$. If we define an $\mathbb{R}^+$ homogeneous mapping $\Phi_{\Lambda^{\mathbb{R}}}$ between $\Omega^{\mathbb{R}}_X := \phi^{-1}_X(\Omega_X)$ and $\Omega^{\mathbb{R}}_Y := \phi^{-1}_Y(\Omega_Y)$ by $\Phi_{\Lambda^{\mathbb{R}}} := (p_{2|\Lambda^{\mathbb{R}}})(p^a_{1|\Lambda^{\mathbb{R}}})^{-1}$, we have a commutative diagram:

$$
\begin{array}{ccc}
\Omega_X^{\mathbb{R}} & \xrightarrow{\Phi_{\Lambda^{\mathbb{R}}}} & \Omega_Y^{\mathbb{R}} \\
\phi_Y \downarrow \wr & & \phi_Y \downarrow \wr \\
\Omega_X & \xrightarrow[\Phi_\Lambda]{} & \Omega_Y .
\end{array}
\tag{2.2}
$$

Hence $\Phi_{\Lambda^{\mathbb{R}}}$ is a real canonical transformation, and $\Lambda^{\mathbb{R}}$ is a real Lagrangian submanifold of the real symplectic manifold $T^*(X^{\mathbb{R}} \times Y^{\mathbb{R}})$.

3. Vanishing theorem for the complex $\mathcal{C}_{\Omega|X}$

Let M be $\mathbb{R}^n$, $X = \mathbb{C}^n$ its complexification, and $\Omega \subset M$ an open convex cone with a vertex at the origin. The hypotheses on Ω allow us to assume from the beginning: $M = \mathbb{R}^d \times \mathbb{R}^{n-d}$ and $\Omega = \Omega_1 \times \mathbb{R}^{n-d}$ for some $d \geq 0$, where $\Omega_1 \subset \mathbb{R}^d$ is an open convex proper cone. Define the maximal linear subspace N of $\mathbb{R}^n$ passing through the origin such that $N \subset \overline{\Omega}$ by $N := \{0\} \times \mathbb{R}^{n-d}$, and we say Ω is an *open convex cone with the edge* N. In this section, we will show the complex

$$\mathcal{C}_{\Omega|X} := \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)[n]$$

of the sheaves of microfunctions at the boundary due to Schapira [22] is concentrated in degree 0 on $N \times_M T^*_M X$, where

$$\mu hom(\cdot\,,\cdot) : \mathbf{D}^{\mathrm{b}}(X)^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(T^*X)$$

is the bifunctor introduced in [9]. We can also define the complex $\mathcal{C}_{A|X}$ for any locally closed subset A of M as in [22]. Consider the morphisms associated to the inclusion $M \hookrightarrow X$:

$$T^*M \xleftarrow{\ \rho\ } M \times_X T^*X \xrightarrow{\ \varpi\ } T^*X.$$

Since Ω is an open subset of M, we have an estimate of the micro-support $\mathrm{SS}\,\mathbb{C}_\Omega$ of the sheaf $\mathbb{C}_\Omega$:

$$\operatorname{supp} \mathcal{C}_{\Omega|X} \subset \mathrm{SS}\,\mathbb{C}_\Omega \subset \varpi\rho^{-1}\big(N^*(\Omega)^a\big). \tag{3.1}$$

As for the definition of $N^*(\Omega) \subset T^*M$, see Definition 5.3.6 of [9]. Let us take coordinate systems $z = x + iy$ of X and $(z\,;\,\zeta\,\mathrm{d}z)$, $\zeta = \xi + i\eta$ of T^*X associated to the decomposition $M = \mathbb{R}^d \times \mathbb{R}^{n-d}$. Then we have:

$$
\begin{aligned}
(3.2)\quad & \operatorname{supp} \mathcal{C}_{\Omega|X} \cap (N \times_X T^*X) \\
& \subset \Big\{(x + i0\,;\ \xi + i\eta)\,;\ x \in N, (\xi_1, \cdots \xi_d) \in (\Omega_1^0)^a, \\
& \qquad\qquad \xi_{d+1} = \cdots = \xi_n = 0,\ \eta \in \mathbb{R}^n\Big\},
\end{aligned}
$$

where $\Omega_1^0 \subset \mathbb{R}^d$ is the polar set of Ω_1. Finally, define the subset I of $\mathrm{SS}\,\mathbb{C}_\Omega$ by:

$$(3.3)\quad I := \Big\{(x+i0\,;\ \xi+i\eta)\,;\ x\in N,\ (\xi_1,\cdots\xi_d)\in \operatorname{int}(\Omega_1^0)^a;\ \xi_{d+1}=\cdots=\xi_n=0,\ \eta\in\mathbb{R}^n\Big\}\subset N\times_X T^*X.$$

Notice that I is the interior of $\mathrm{SS}\,\mathbb{C}_\Omega \cap T_N^*X$.

THEOREM 3.1.

(i) *The complex* $\mathcal{C}_{\Omega|X}$ *is concentrated in degree* 0 *on* $N\times_M T_M^*X$.

(ii) *The boundary value morphism* $\mathcal{C}_{\Omega|X}\to\mathcal{C}_{N|X}[d]$ *is an isomorphism on* I.

Proof. — The proof essentially goes along the same line as Schapira [22]. Let Y be a copy of X and $w=u+iv$ be the coordinate system of Y. Set $z=(z',z_n)$, $w=(w',w_n)$ and $f(z,w):=z'\cdot w'+z_n-w_n$. Since f is a holomorphic function on $X\times Y$ and $\mathrm{d}f$ never vanishes, it follows from the results in Section 2 that for the set:

$$H_- := \big\{(z,w)\in X\times Y\,;\ \operatorname{Re} f(z,w)=0,\ \operatorname{Im} f(z,w)\le 0\big\},$$

$\mathbb{C}_{H_-}[-1]$ is a simple sheaf with shift 0 on $X\times Y$. Now define an open subset of $\mathrm{SS}\,\mathbb{C}_{H_-}$ by:

$$\Lambda^{\mathbb{R}} := \Big\{(z,w\,;\ a\cdot \mathrm{d}\operatorname{Re} f + b\cdot \mathrm{d}\operatorname{Im} f)\,;\ f(z,w)=0,\ a\in\mathbb{R},\ b<0\Big\}\subset T^*(X^{\mathbb{R}}\times Y^{\mathbb{R}}),$$

and set

$$\Lambda := \phi_{X\times Y}(\Lambda^{\mathbb{R}})\subset T^*(X\times Y),$$

where $\phi_{X\times Y}$ is the canonical identification

$$\phi_{X\times Y}: T^*(X^{\mathbb{R}}\times Y^{\mathbb{R}})\simeq T^*(X\times Y).$$

The set Λ is a complex Lagrangian submanifold of $T^*(X\times Y)$, because it is an open subset of the conormal bundle $T_H^*(X\times Y)$ to the complex submanifold H defined by $H:=\{(z,w)\mid f(z,w)=0\}$. Taking coordinate systems, $(z\,;\ \zeta\,\mathrm{d}z)$, $(w\,;\ \theta\,\mathrm{d}w)$, $(x,y\,;\ \xi\,\mathrm{d}x+\eta\,\mathrm{d}y)$ and $(u,v\,;\ p\,\mathrm{d}u+q\,\mathrm{d}v)$ for T^*X, T^*Y, $T^*X^{\mathbb{R}}$ and $T^*Y^{\mathbb{R}}$ respectively, let us set:

$$(3.4)\quad \begin{cases}\Omega_X := \{\operatorname{Im}\zeta_n<0\}\subset T^*X, & \Omega_Y := \{\operatorname{Im}\theta_n<0\}\subset T^*Y,\\ \Omega_X^{\mathbb{R}} := \{\eta_n>0\}\subset T^*X^{\mathbb{R}}, & \Omega_Y^{\mathbb{R}} := \{q_n>0\}\subset T^*Y^{\mathbb{R}},\end{cases}$$

and define the real (resp. complex) contact transformation $\Phi_{\Lambda^{\mathbb{R}}}$ (resp. Φ_Λ) between $\Omega_X^{\mathbb{R}}$ (resp. Ω_X) and $\Omega_Y^{\mathbb{R}}$ (resp. Ω_Y) as in Section 2. The complex contact transformation Φ_Λ is nothing but the Legendre transformation:

$$w' = \frac{\zeta'}{\zeta_n}, \quad w_n = \left\langle z, \frac{\zeta}{\zeta_n} \right\rangle, \quad \theta' = -\zeta_n \cdot z', \quad \theta_n = \zeta_n. \tag{3.5}$$

Set $K := \mathbb{C}_{H_-}[-1]$ in the derived category $\mathbf{D}^{\mathrm{b}}(X \times Y)$ of sheaves of $\mathbb{C}$−vector space on $X \times Y$. Then K satisfies all conditions of Theorem 7.2.1 of [9] and the functor Φ_K:

$$\begin{array}{rrcl} \Phi_K: & \mathbf{D}^{\mathrm{b}}(X) & \longrightarrow & \mathbf{D}^{\mathrm{b}}(Y) \\ & F & \longmapsto & Rq_{2!}(K \otimes^L q_1^{-1} F) \end{array} \tag{3.6}$$

induces an equivalence of categories between $\mathbf{D}^{\mathrm{b}}(X, \Omega_X^{\mathbb{R}})$ and $\mathbf{D}^{\mathrm{b}}(Y, \Omega_Y^{\mathbb{R}})$, where q_1 and q_2 denote the first and second projections on $X \times Y$ respectively. According to Corollary 11.4.11 of [9], we have the quantized contact transformation induced by the kernel K:

$$\begin{aligned} (\Phi_{\Lambda^{\mathbb{R}}})_* \mathcal{C}_{\Omega|X} &= (\Phi_{\Lambda^{\mathbb{R}}})_* \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)[n] \\ &\simeq \mu hom\big(\Phi_K(\mathbb{C}_\Omega), \mathcal{O}_Y\big), \end{aligned} \tag{3.7}$$

and

$$(\Phi_{\Lambda^{\mathbb{R}}})_* \mathcal{C}_{N|X} \simeq \mu hom\big(\Phi_K(\mathbb{C}_N), \mathcal{O}_Y\big). \tag{3.8}$$

LEMMA 3.2. — *Assume* $\dim^{\mathbb{R}} N = n - d \geq 1$. *Then there are isomorphisms*:

$$\Phi_K(\mathbb{C}_\Omega) \simeq \mathbb{C}_{G_\Omega}[-n] \quad \textit{and} \quad \Phi_K(\mathbb{C}_N) \simeq \mathbb{C}_{G_N}[-(n-d)] \tag{3.9}$$

by setting:

$$\begin{cases} G_\Omega := \big\{u + iv; u \in \mathbb{R}^n,\ (v_1, \dots, v_d) \in (\Omega_1^0)^a, \\ \qquad\qquad v_{d+1} = \cdots = v_{n-1} = 0,\ v_n \geq 0\big\}, \\ G_N := \big\{u + iv; u \in \mathbb{R}^n,\ (v_1, \cdots, v_d) \in \mathbb{R}^d, \\ \qquad\qquad v_{d+1} = \cdots = v_{n-1} = 0,\ v_n \geq 0\big\}. \end{cases} \tag{3.10}$$

Proof. — For every $w = u + iv \in Y$, set:

$$F_{u+iv} := \big\{(z = x + iy, w = u + iv) \in X \times Y; \\ \operatorname{Re} f(z, w) = 0,\ \operatorname{Im} f(z, w) \leq 0,\ x + iy \in \Omega\big\}.$$

Then:

$$\begin{aligned}(3.11)\qquad \Phi_K(\mathbb{C}_\Omega)_{|u+iv} &= Rq_{2!}(\mathbb{C}_{H_-}\otimes^L q_1^{-1}\mathbb{C}_\Omega)_{|u+iv}[-1]\\ &\simeq R\Gamma_c(F_{u+iv}\,;\ \mathbb{C}_{F_{u+iv}})[-1],\end{aligned}$$

and by setting $x=(x',x_n)$, $u=(u',u_n)$ and $v=(v',v_n)$ we have:

$$\begin{aligned}(3.12)\qquad F_{u+iv} &= \{x\in\Omega\,;\ x_n=u_n-x'\cdot u', v_n\geq x'\cdot v'\}\\ &\simeq \{x'\in\Omega_1\times\mathbb{R}^{n-d-1}\,;\ v_n\geq x'\cdot v'\}.\end{aligned}$$

If $(v_1,\cdots,v_d)\in(\Omega_1^0)^a$, $v_{d+1}=\cdots=v_{n-1}=0$ and $v_n\geq 0$, F_{u+iv} is homeomorphic to $\mathbb{R}^{n-1}$, and otherwise, it is homeomorphic to a closed half space of $\mathbb{R}^{n-1}$ or an empty set. It implies:

$$\Phi_K(\mathbb{C}_\Omega)\simeq\mathbb{C}_{G_\Omega}[-n].$$

The proof of the remaining part is similar. □

Let us continue the proof of THEOREM 3.1. By virtue of « the trick of the dummy variables » due to Kashiwara, we may assume $\dim^{\mathbb{R}}N=n-d\geq 1$, and it is enough to consider the problem in $\Omega_X^{\mathbb{R}}:=\{\eta_n>0\}$. Note that for every $(x,0\,;\ \xi\mathrm{d}x,\eta\mathrm{d}y)\in\phi_X^{-1}(I)\cap\Omega_X^{\mathbb{R}}$, $(u,v\,;\ p,q):=\Phi_{\Lambda^{\mathbb{R}}}(x,0\,;\ \xi,\eta)$ satisfies $(v_1,\cdots,v_d)\in\mathrm{int}(\Omega_1^0)^a$, $v_{d+1}=\cdots=v_{n-1}=0$ and $v_n=0$. By LEMMA 3.2, there exists a canonical isomorphism:

$$(3.13)\qquad \Phi_K(\mathbb{C}_N)[-d]=\mathbb{C}_{G_N}[-n]\xrightarrow{\ \sim\ }\Phi_K(\mathbb{C}_\Omega)=\mathbb{C}_{G_\Omega}[-n]$$

in a neighborhood of $u+iv\in Y$, and we have by (3.7) and (3.8):

$$\begin{aligned}(3.14)\qquad \mathcal{C}_{\Omega|X\,|\,(x,0\,;\ \xi,\eta)} &\simeq \mu hom(\mathbb{C}_{G_\Omega},\mathcal{O}_Y)_{|(u,v;p,q)}[n]\\ &\simeq \mu hom(\mathbb{C}_{G_N},\mathcal{O}_Y)_{|(u,v;p,q)}[n]\\ &\simeq \mathcal{C}_{N|X\,|\,(x,0;\xi,\eta)}[d].\end{aligned}$$

This achieves the proof of part (ii).

Finally, we will show part (i). Without loss of generality, we only have to show the problem for $(0,0\,;\ 0,\eta\mathrm{d}y)\in\phi_X^{-1}(N\times_M T_M^*X)\cap\Omega_X^{\mathbb{R}}$. Now $(u,v\,;\ p,q):=\Phi_{\Lambda^{\mathbb{R}}}(0,0\,;\ 0,\eta)$ satisfies: $v_1=v_2=\cdots=v_n=0$, $p=0$, $q_1=\cdots=q_{n-1}=0$ and $q_n=\eta_n>0$. It follows from (3.7) that

$$\mathcal{C}_{\Omega|X\,|\,(0,0;0,\eta)}\simeq\mu hom(\mathbb{C}_{G_\Omega},\mathcal{O}_Y)_{|(u,0;q_n\mathrm{d}v_n)}[n]\quad(q_n>0),$$

and applying Proposition 4.4.4 of [9] we have:

$$H^j \mu hom(\mathbb{C}_{G_\Omega}, \mathcal{O}_Y)|_{(u,0\,;\,dv_n)}[n] = \varinjlim_{U,\gamma} H^{n+j}_{(G_\Omega+\gamma)\cap U}(U\,;\,\mathcal{O}_Y), \tag{3.15}$$

for every $j \in \mathbb{Z}$, where U ranges over the family of open neighborhoods of $u + i0$ in Y and γ ranges the family of closed convex proper cone of $Y = \mathbb{R}^{2n}_{u,v}$ such that $(\gamma \setminus \{0\}) \subset \{v_n > 0\}$. On account of «the abstract edge of the wedge theorem» of Kashiwara [7], the right hand side is concentrated in degree 0, because the closed subset $G_\Omega + \gamma$ contains no complex line passing through the point $u + i0$. Hence we have finished the proof of part (i). □

REMARK 3.3. — When Ω is an open quadrant $\{x_1 > 0, \ldots, x_d > 0\}$ of $M = \mathbb{R}^n$, THEOREM 3.1 (ii) is almost trivial as we shall see it in the next section. Prof. Uchida suggested the author that the proof for general open convex cones with the edge N could also be simplified in the following way. First, notice that the morphism $\mathbb{C}_{\overline{\Omega}} \longrightarrow \mathbb{C}_N$ is an isomorphism in the category $\mathbf{D}^{\mathrm{b}}(X\,;\,I^a)$. Therefore the morphism $D'(\mathbb{C}_N) \longrightarrow D'(\mathbb{C}_{\overline{\Omega}})$ is also an isomorphism in $\mathbf{D}^{\mathrm{b}}(X\,;\,I)$ by Proposition 5.4.14 (ii) of [9]. Applying the functor $\mu hom(\cdot, \mathcal{O}_X)$ to it, we obtain the desired isomorphism $\mathcal{C}_{\Omega|X} \simeq \mathcal{C}_{N|X}[d]$ in an open neighborhood of I. The author thanks him for letting him announce such a sophisticated proof.

REMARK 3.4. — Let Ω_1 and Ω_2 be two open convex cones in $M = \mathbb{R}^n$. Then by using the distinguished triangle:

$$\mathcal{C}_{\Omega_1\cup\Omega_2|X} \longrightarrow \mathcal{C}_{\Omega_1|X} \oplus \mathcal{C}_{\Omega_2|X} \longrightarrow \mathcal{C}_{\Omega_1\cap\Omega_2|X} \xrightarrow[+1]{}$$

and THEOREM 3.1 (i), we can show that $\mathcal{C}_{\Omega_1\cup\Omega_2|X}$ is also concentrated in degree 0 on $\{0\} \times_M T^*_M X$ and the morphism $\mathcal{C}_{\Omega_1\cup\Omega_2|X} \to \mathcal{C}_{\Omega_1|X} \oplus \mathcal{C}_{\Omega_2|X}$ is injective there, which is a generalization of Remarque 3.2 of Schapira [22] (*cf.* Proposition 5.1 of Uchida [29]). By repeating this operation, $\mathcal{C}_{\Omega|X}$ is concentrated in degree 0 on $\{0\} \times_M T^*_M X$ for any union Ω of finite open convex cones. Moreover if Ω_A and Ω_B are open cones of this type such that $\Omega_A \supset \Omega_B$, the «difference» $\mathcal{C}_{(\Omega_A\setminus\Omega_B)|X}$ has the same property. Since any open cone in M is approximated by a sequence of the unions of finite open convex cones, it seems for us that the complex $\mathcal{C}_{\Omega|X}$ is concentrated in degree 0 on $\{0\} \times_M T^*_M X$ for every locally closed cone Ω in M (*cf.* Schapira's conjecture, Conjecture 2.3 of [23]).

4. Structure of the complex $\mathcal{C}_{\Omega|X}$ for open quadrants

In this section, we will completely investigate the structure of the complex $\mathcal{C}_{\Omega|X}$ for open quadrants Ω. Here we say an open convex cone Ω in $M = \mathbb{R}^n$ is an open quadrant if we have the expression: $\Omega = \{x_1 > 0, \cdots, x_d > 0\}$ by a suitable real analytic change of coordinates. In this case, THEOREM 3.1 implies $\mathcal{C}_{\Omega|X\,|N\times_M T^*_M X}$ is concentrated in degree 0 by setting $N := \{x_1 = \cdots = x_d = 0\}$, and

$$\text{(4.1)}\qquad \operatorname{supp}\mathcal{C}_{\Omega|X} \cap (N \times_X T^*X) \subset \big\{\xi_1 \leq 0, \cdots, \xi_d \leq 0,\ \xi_{d+1} = \cdots = \xi_n = 0\big\}$$

in $N \times_X T^*X$ if we take a coordinate system $(z, \zeta \mathrm{d}z)$ $(z = x + iy, \zeta = \xi + i\eta)$ of T^*X. Now the subset I of $N \times_X T^*X$ introduced in Section 3 is expressed in $N \times_X T^*X$ as follows:

$$\text{(4.2)}\qquad I = \big\{\xi_1 < 0, \ldots,\ \xi_d < 0,\ \xi_{d+1} = \cdots = \xi_n = 0\big\}.$$

We have already shown the isomorphism

$$\mathcal{C}_{\Omega|X\,|I} \xrightarrow{\sim} \mathcal{C}_{N|X\,|I}[d]$$

in THEOREM 3.1 (ii), but here we will give a more elementary proof for open quadrants. For the sake of simplicity, we shall restrict ourselves to the case when $d = 3$. The generalization to arbitrary codimensional cases is easy, and we will not develop it here. Therefore, set

$$\begin{aligned} \Omega &:= \{x_1 > 0,\ x_2 > 0,\ x_3 > 0\}, \\ \Omega_1 &:= \{x_1 = 0,\ x_2 > 0,\ x_3 > 0\}, \\ \Omega_2 &:= \{x_1 = 0,\ x_2 = 0,\ x_3 > 0\}, \\ N &= \{x_1 = x_2 = x_3 = 0\} \end{aligned}$$

in M. Let us define the following subsets in $N \times_X T^*X$:

$$\text{(4.3)}\qquad \begin{cases} I_0 := \{\xi_1 = \cdots = \xi_n = 0\}, \\ I_1 := \{\xi_1 < 0,\ \xi_2 = \cdots = \xi_n = 0\}, \\ I_2 := \{\xi_1 < 0,\ \xi_2 < 0,\ \xi_3 = \cdots = \xi_n = 0\}, \\ I_3 := \{\xi_1 < 0,\ \xi_2 < 0,\ \xi_3 < 0,\ \xi_4 = \cdots = \xi_n = 0\} = I. \end{cases}$$

Then we have:

THEOREM 4.1.

(i) *There exist canonical isomorphisms*:

$$\begin{cases} \mathcal{C}_{\Omega|X\,|\,I_1} \xrightarrow{\sim} \mathcal{C}_{\Omega_1|X\,|\,I_1}[1], \\ \mathcal{C}_{\Omega|X\,|\,I_2} \xrightarrow{\sim} \mathcal{C}_{\Omega_2|X\,|\,I_2}[2], \\ \mathcal{C}_{\Omega|X\,|\,I_3} \xrightarrow{\sim} \mathcal{C}_{N|X\,|\,I_3}[3]. \end{cases} \tag{4.4}$$

(ii) $\mathcal{C}_{\Omega|X\,|\,I_k}$ *is concentrated in degree* $-k$ *for* $k = 0,1,2,3$.

Proof. — Since $M = \mathbb{R}^n$, we may forget orientation sheaves. First recall that the boundary value morphisms

$$\mathcal{C}_{\Omega|X} \longrightarrow \mathcal{C}_{\Omega_1|X}[1] \longrightarrow \mathcal{C}_{\Omega_2|X}[2] \longrightarrow \mathcal{C}_{N|X}[3]$$

are constructed by the cut-off morphisms:

$$\mathbb{C}_{\bar{\Omega}} \longrightarrow \mathbb{C}_{\bar{\Omega}_1} \longrightarrow \mathbb{C}_{\bar{\Omega}_2} \longrightarrow \mathbb{C}_N.$$

Taking the stupid dual $D'(\cdot) = R\mathcal{H}om(\cdot, \mathbb{C}_X)$ of the exact sequence:

$$0 \to \mathbb{C}_{\bar{\Omega}\setminus\bar{\Omega}_1} \longrightarrow \mathbb{C}_{\bar{\Omega}} \longrightarrow \mathbb{C}_{\bar{\Omega}_1} \to 0,$$

one has a distinguished triangle:

$$\mathbb{C}_{\Omega_1}[-(n+1)] \longrightarrow \mathbb{C}_{\Omega}[-n] \longrightarrow \mathbb{C}_{\Omega\sqcup\Omega_1}[-n] \xrightarrow[+1]{} .$$

Apply the functor $\mu hom(\cdot, \mathcal{O}_X)$ to this triangle. Then we have

$$\mu hom(\mathbb{C}_{\Omega\sqcup\Omega_1}, \mathcal{O}_X)[n] \longrightarrow \mathcal{C}_{\Omega|X} \longrightarrow \mathcal{C}_{\Omega_1|X}[1] \xrightarrow[+1]{},$$

and it follows from an estimation of the micro-support of $\mathbb{C}_{\Omega\sqcup\Omega_1}$:

$$\operatorname{supp} \mu hom(\mathbb{C}_{\Omega\sqcup\Omega_1}, \mathcal{O}_X) \cap (N \times_X T^*X) \cap \{\xi_1 < 0\} = \emptyset$$

and

$$\mathcal{C}_{\Omega|X\,|\,(N\times_X T^*X)\cap\{\xi_1<0\}} \xrightarrow{\sim} \mathcal{C}_{\Omega_1|X\,|\,(N\times_X T^*X)\cap\{\xi_1<0\}}[1]. \tag{4.5}$$

Similarly, starting from the exact sequence:

$$0 \to \mathbb{C}_{\bar{\Omega}_1\setminus\bar{\Omega}_2} \to \mathbb{C}_{\bar{\Omega}_1} \to \mathbb{C}_{\bar{\Omega}_2} \to 0,$$

we also obtain the isomorphism:

$$\mathcal{C}_{\Omega_1|X\,|\,(N\times_X T^*X)\cap\{\xi_2<0\}}[1] \xrightarrow{\sim} \mathcal{C}_{\Omega_2|X\,|\,(N\times_X T^*X)\cap\{\xi_2<0\}}[2]. \tag{4.6}$$

Now by (4.5) and (4.6), the first and second isomorphisms of (i) are clear. The last isomorphism of (i) can be shown by repeating the above procedure. For part (ii), it is enough to show the following lemma. ∎

LEMMA 4.2. — *Set*

$$N := \{x_1 = \cdots = x_d = 0\} \subset M,$$
$$L := \{x_1 = \cdots = x_p = 0\} \supset N$$

for an integer p *such that* $0 \leq p \leq d$, *and*

$$\Omega_L := \{x_1 = \cdots = x_p = 0,\ x_{p+1} > 0, \ldots, x_d > 0\}.$$

Then, the complex $\mathcal{C}_{\Omega_L|X}$ *is concentrated in degree* 0 *on* $N \times_L T_L^*X$.

Proof. — The proof is almost similar to that of THEOREM 3.1, and it requires the same contact transformation $\Phi_{\Lambda^{\mathbb{R}}}$ and the same kernel K. Therefore we adopt the notations in the proof of THEOREM 3.1. Assume $n - d \geq 1$. Then we have:

$$(\Phi_{\Lambda^{\mathbb{R}}})_* \mathcal{C}_{\Omega_L|X} \simeq \mu hom\big(\Phi_K(\mathbb{C}_{\Omega_L}), \mathcal{O}_Y\big),$$
$$\Phi_K(\mathbb{C}_{\Omega_L}) \simeq \mathbb{C}_{G_{\Omega_L}}\big[-(n-p)\big],$$

where

$$G_{\Omega_L} := \Big\{u + iv\,;\ u \in \mathbb{R}^n,\ (v_1, \ldots, v_p) \in \mathbb{R}^p,\ v_{p+1} \leq 0, \cdots, v_d \leq 0,\ v_{d+1} = \cdots = v_{n-1} = 0,\ v_n \geq 0\Big\}.$$

It follows from «the abstract edge of the wedge theorem» that the complex $\mathcal{C}_{\Omega_L|X}$ is concentrated in degree 0 on $\phi_X^{-1}(N \times_L T_L^*X) \cap \Omega_X^{\mathbb{R}}$. The general cases can be treated by «the trick of the dummy variables». □

REMARK 4.3. — The method of reduction in THEOREM 4.1 (i) by the estimation of micro-supports was used in the proof of Proposition 5.4 of [29].

5. Explicit formula of the stalks of $\mathcal{C}_{\Omega|X}$

Let $\Omega \subset M = \mathbb{R}_x^n$ be an open convex cone with the edge

$$N = \{x_1 = \cdots = x_d = 0\}$$

as in Section 3, and set $K := M \setminus \Omega$. In this section, we will calculate an explicit formula of the stalks of the complex $\mathcal{C}_{\Omega|X}$ and compare it with the complex $\nu\mu_{NM}(\mathcal{O}_X)[n]$ introduced in [24]. First of all, assume the vector

$(1,0,\cdots,0) \in M = \mathbb{R}^n$ is contained in Ω and $\Omega \subset \{x_1 > 0\}$, and take an arbitrary closed convex proper cone γ_M in M such that

$$\bigl(\gamma_M \setminus \{0\}\bigr) \subset \Omega, \quad \text{and} \quad (1,0,\cdots,0) \in \operatorname{int}\gamma_M.$$

Let us define the closed convex subset A_1 of M by:

$$A_1 := \bigl(-\tfrac{1}{2}C, 0, \cdots, 0\bigr) + \bigl(\gamma_M \cap \{x_1 \le C\}\bigr)$$

for some $C > 0$. Note that the origin of M is contained in the interior of A_1, and it is easy to show:

LEMMA 5.1. — *For every $x \in M = \mathbb{R}^n_x$, $(x + A_1) \cap K$ is a star-shaped compact subset of M centered at $x + (-\frac{1}{2}C, 0, \ldots, 0)$, and in particular it is contractible.*

Next, take an arbitrary convex compact subset A_2 in $\mathbb{R}^{n-1}_{y'}$, $y' = (y_1, \cdots, y_{n-1})$ such that $\{0\} \in \operatorname{int} A_2$, and define the closed convex proper cone γ_ε in $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$ for $\varepsilon > 0$ by:

$$\gamma_\varepsilon := \bigl\{(cx, c(y', \varepsilon))\,;\ c \ge 0,\ x \in A_1,\ y' \in A_2\bigr\}. \tag{5.1}$$

The family $\{\gamma_\varepsilon\}_{\varepsilon > 0}$ is cofinal in the family of all closed convex proper cones γ of $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$ such that $(\gamma \setminus \{0\}) \subset \{y_n > 0\}$, and we may use it to calculate the stalk of $\mathcal{C}_{\Omega|X} := \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)[n]$ at $(0; \mathrm{d}y_n) \in T^*X^{\mathbb{R}}$. Recall the calculation in the proof of Proposition 4.4.4 of [9]. Then we have:

$$\begin{aligned} H^j\, \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)_{|(o;\,\mathrm{d}y_n)} &\simeq \varinjlim_{U \ni \{0\},\, \varepsilon > 0} H^j\, \mathrm{RHom}\bigl(\mathbb{C}_{U \cap (M+\gamma_\varepsilon)\setminus(K+\gamma_\varepsilon)}, \mathcal{O}_X\bigr) \\ &\simeq \varinjlim_{\gamma} \mathcal{H}^j_{(M+\gamma)\setminus(K+\gamma)}(\mathcal{O}_X)_0, \end{aligned}$$

for every $j \in \mathbb{Z}$ by LEMMA 5.1, where γ ranges over the closed convex proper cones of $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$ such that $(\gamma \setminus \{0\}) \subset \{y_n > 0\}$. Hence we have shown the proposition below:

PROPOSITION 5.2. — *There exists an isomorphism:*

$$H^j \mathcal{C}_{\Omega|X\,|(o;\,\mathrm{d}y_n)} \simeq \varinjlim_{\gamma} \mathcal{H}^{n+j}_{(M+\gamma)\setminus(K+\gamma)}(\mathcal{O}_X)_0 \tag{5.2}$$

for every $j \in \mathbb{Z}$, where γ ranges through the closed convex proper cones of $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$ such that $(\gamma \setminus \{0\}) \subset \{y_n > 0\}$.

REMARK 5.3. — The right hand side of (5.2) vanishes if $j \neq 0$ by THEOREM 3.1 (i), which can be considered as a variation of the abstract edge of the wedge theorem.

Now set in $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$:

$$\gamma_{\varepsilon_1,\varepsilon_2} := \{y_n \geq \varepsilon_1 |y'|,\ |y| \geq \varepsilon_2 |x|\}$$

for every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. These cones are not convex, but the family $\{\gamma_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1,\varepsilon_2>0}$ is cofinal in the family of the closed cones γ of $\mathbb{R}^n_x \times \mathbb{R}^n_y = X$ which satisfy $(\gamma \setminus \{0\}) \subset \{y_n > 0\}$. Consequently we obtain:

$$\text{(5.3)} \qquad H^j \mathcal{C}_{\Omega|X}|_{(o;\, \mathrm{d}y_n)} \simeq \varinjlim_{\varepsilon_1,\varepsilon_2>0} \mathcal{H}^{n+j}_{(M+\gamma_{\varepsilon_1,\varepsilon_2})\setminus(K+\gamma_{\varepsilon_1,\varepsilon_2})}(\mathcal{O}_X)_0$$

and

$$(M+\gamma_{\varepsilon_1,\varepsilon_2}) \setminus (K+\gamma_{\varepsilon_1,\varepsilon_2}) = \{y_n \geq \varepsilon_1|y'|\} \cap \{x \in \Omega,\ |y| < \varepsilon_2 \operatorname{dist}(x,K)\}.$$

If we take an inductive limit by shrinking open cones Ω with the edge N such that $(1,0,\cdots,0) \in \Omega$, we get an isomorphism by (5.3) and Theorem 3.1 (ii) of [24]:

$$\text{(5.4)} \qquad \varinjlim_{\Omega} H^j \mathcal{C}_{\Omega|X}|_{(0;\, \mathrm{d}y_n)} \simeq H^j\big[\nu\mu_{NM}(\mathcal{O}_X)|_{(\mathrm{d}y_n, \partial/\partial x_1, 0)}[n]\big],$$

in which the right hand side is an object on $T_N M \times_M T^*_M X$ introduced in [24]. For more details about the functor $\nu\mu_{NM}$, see [24]. Hence the complex $\nu\mu_{NM}(\mathcal{O}_X)[n]$ can be considered as a microlocalization of $\nu_N \mathcal{B}_M$, *i.e.* the specialization of the sheaf $\mathcal{B}_M$ of hyperfunctions along N. The following theorem is a consequence of THEOREM 3.1 (i) and the isomorphism (5.4), which is already announced in [24].

THEOREM 5.4. — *The complex $\nu\mu_{NM}(\mathcal{O}_X)[n]$ is concentrated in degree* 0.

6. Microlocal injectivity of the boundary value morphism

In this section, we shall prove the injectivity of the higher codimensional boundary value morphism, which is a generalization of Proposition 3.3 of Schapira [23] to arbitrary codimensional cases. It is also considered as a microlocal extension of the uniqueness theorem due to Oaku [17]. Let

$$\begin{aligned} M &= \mathbb{R}^n_x, \\ N &= \{0\} \times \mathbb{R}^{n-d} = \{x_1 = \cdots = x_d = 0\}, \\ A_0 &:= \{x_1 \geq 0,\ x_2 = \cdots = x_d = 0\} \subset \mathbb{R}^d, \end{aligned}$$

then the precise statement is as follows.

THEOREM 6.1 (Microlocal injectivity of the boundary value morphism). — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X for which N is non-characteristic. Then the canonical morphism of the boundary value:*

$$\begin{aligned}\varinjlim_{\Omega} H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Omega|X}) \\ \longrightarrow H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})[d] \\ \xrightarrow{\sim} \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})\end{aligned}$$

is injective on $\pi_X^{-1}(N) = N \times_X T^*X$, where Ω ranges through open convex cones with the edge N such that $\Omega \supset A_0 \times \mathbb{R}^{n-d} \setminus N$.

Proof. — We will make use of the contact transformation Φ_Λ in the proof of THEOREM 3.1, and we shall mainly follow the notations in it. By the use of «the trick of the dummy variables», we may assume $\dim^{\mathbb{R}} N = n - d \geq 1$ and we only have to show the injectivity on $\Omega_X^{\mathbb{R}} = \{\eta_n > 0\}$. By virtue of THEOREM 3.1 (ii), if we shrink Ω as in the statement of the theorem, the injectivity is clear outside $N \times_M T^*_M X$. Let us take a point $p = (0;\eta \mathrm{d}y)$ from $(N \times_M T^*_M X) \cap \Omega_X^{\mathbb{R}}$. Since the Legendre transformation of the complex $\mathcal{C}_{\Omega|X}$ is supported by $\pi_Y^{-1}(G_\Omega)$ in a neighborhood of $\Phi_{\Lambda^{\mathbb{R}}}(p) \in \Omega_Y^{\mathbb{R}}$, by setting $Z_\Omega := \Phi_{\Lambda^{\mathbb{R}}}^{-1}(\pi_Y^{-1}(G_\Omega))$, there exists a canonical morphism:

$$\mathcal{C}_{\Omega|X} \simeq R\Gamma_{Z_\Omega}\mathcal{C}_{\Omega|X} \longrightarrow R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d] \tag{6.1}$$

in a neighborhood of p. By the same reason, we have the following commutative diagram:

$$\begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Omega|X}) & \longrightarrow & R\Gamma_{Z_\Omega}\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})[d] \\ & \searrow & \downarrow \\ & & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})[d]. \end{array} \tag{6.2}$$

The complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X})[d]$ is concentrated in degree ≥ 0 by Kashiwara–Kawaï's division theorem in [6], and we get a commutative diagram of morphisms between the sheaves by taking the 0-th cohomology group of the above one (6.2):

$$\begin{array}{ccc} H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Omega|X}) & \longrightarrow & \Gamma_{Z_\Omega}\, \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X}) \\ & \searrow & \downarrow \\ & & \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|X}). \end{array} \tag{6.3}$$

Therefore to show the injectivity of the morphism:

$$\varinjlim_{\Omega} H^0 R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X})_p \longrightarrow \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})_p,$$

it is enough to show the injectivity of the canonical morphism:

$$(6.4) \qquad \varinjlim_{\Omega} H^0 R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X})_p \longrightarrow H^0 R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d]\big)_p$$

induced by the morphism in (6.1), because we have a commutative diagram:

$$(6.5) \qquad \begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d]\big) \\ & \searrow & \downarrow \wr \\ & & R\Gamma_{Z_\Omega} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})[d]. \end{array}$$

PROPOSITION 6.2. — *The complex $R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d]|_p$ is concentrated in degree* 0.

Proof. — Now set in $Y = \mathbb{C}^n_w$, $w = u + iv$:

$$(6.6) \qquad \begin{cases} L := \{u + iv\,;\ v_{d+1} = \cdots = v_n = 0\}, \\ A_\Omega := \{u + iv\,;\ (v_1, \cdots, v_d) \in (\Omega_1^0)^a, \\ \qquad\qquad v_{d+1} = \cdots = v_n = 0\} \subset L, \end{cases}$$

and $\pi_L : T^*_L Y \longrightarrow L$. Then in a neighborhood of $\Phi_{\Lambda^{\mathbb{R}}}(p)$ we have by (3.8) and (3.9):

$$\begin{aligned} (\Phi_{\Lambda^{\mathbb{R}}})_*\big(R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d]\big) &\simeq R\Gamma_{\pi_Y^{-1}(G_\Omega)}\mu hom(\mathbb{C}_{G_N}, \mathcal{O}_Y)[n] \\ &\simeq R\Gamma_{\pi_Y^{-1}(G_\Omega)}\mu hom(\mathbb{C}_L, \mathcal{O}_Y)[n] \\ &\simeq R\Gamma_{\pi_L^{-1}(A_\Omega)}\mu_L(\mathcal{O}_Y)[n], \end{aligned}$$

where the second isomorphism is a consequence of the microlocal isomorphism $\mathbb{C}_{G_N} \simeq \mathbb{C}_L$ in $\mathbf{D}^{\mathrm{b}}(Y; \Phi_{\Lambda^{\mathbb{R}}}(p))$ and the third isomorphism is obtained by $\pi_Y^{-1}(G_\Omega) \cap T^*_L Y = \pi_L^{-1}(A_\Omega)$. Hence the assertion of the proposition is shown by using the lemma below and Kashiwara's abstract edge of the wedge theorem because $\Phi_{\Lambda^{\mathbb{R}}}(p) = (u + i0; q_n \mathrm{d}v_n)$ for some $u \in \mathbb{R}^n$ and $q_n > 0$. ∎

LEMMA 6.3 (Lemma 2.4.4 of Kashiwara–Laurent [7]). — *Let L be a submanifold of Y defined by $L = \{v_{d+1} = \cdots = v_n = 0\}$ and A a closed subset of L. Then for every sheaf F on Y and every $j \in \mathbb{Z}$:*

$$H^j R\Gamma_{\pi_L^{-1}(A)}\mu_L(F)|_{(0;\mathrm{d}v_n)} = \varinjlim_{Z} \mathcal{H}^j_Z(F)_0$$

holds, where Z ranges over closed subsets of Y such that

$$Z \subset \big\{v_n \geq \varepsilon|(v_{d+1}, \cdots, v_{n-1})|\big\} \subset Y$$

for some $\varepsilon > 0$ and $Z \cap L \subset A$.

Since we know by THEOREM 3.1 (i) that $\mathcal{C}_{\Omega|X|p}$ is concentrated in degree 0,

$$H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X})_p \simeq \mathcal{H}om_{\mathcal{D}_{X,p}}\big(\mathcal{M}_p, \mathcal{C}_{\Omega|X|p}\big).$$

Similarly by PROPOSITION 5.2, we have:

$$H^0\, R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, R\Gamma_{Z_\Omega}\mathcal{C}_{N|X}[d]\big)_p \simeq \mathcal{H}om_{\mathcal{D}_{X,p}}\big(\mathcal{M}_p, \mathcal{H}^d_{Z_\Omega}\mathcal{C}_{N|X|p}\big).$$

Therefore to show the injectivity of the morphism (6.4), it suffices to show the next proposition, because there exists a commutative diagram for some $N_0 > 0$:

$$\begin{array}{ccc} \varinjlim_{\Omega} \mathcal{H}om_{\mathcal{D}_{X,p}}(\mathcal{M}_p, \mathcal{C}_{\Omega|X|p}) & \longrightarrow & \varinjlim_{\Omega} \big(\mathcal{C}_{\Omega|X|p}\big)^{N_0} \\ \downarrow & & \downarrow \\ \varinjlim_{\Omega} \mathcal{H}om_{\mathcal{D}_{X,p}}\big(\mathcal{M}_p, \mathcal{H}^d_{Z_\Omega}\mathcal{C}_{N|X|p}\big) & \longrightarrow & \varinjlim_{\Omega} \big(\mathcal{H}^d_{Z_\Omega}\mathcal{C}_{N|X|p}\big)^{N_0}, \end{array} \tag{6.7}$$

in which all horizontal arrows are injective.

PROPOSITION 6.4. — *The canonical morphism:*

$$\varinjlim_{\Omega} \mathcal{C}_{\Omega|X}\,|_p \longrightarrow \varinjlim_{\Omega} \mathcal{H}^d_{Z_\Omega}\mathcal{C}_{N|X|p}$$

induced by the morphism (6.1) *is injective.*

Proof. — By performing the quantized contact transformation associated to the Legendre transformation, we get the isomorphisms below:

$$\begin{cases} \mathcal{C}_{\Omega|X|p} \simeq \mu hom(\mathbb{C}_{G_\Omega}, \mathcal{O}_Y)|_{\Phi_{\Lambda^{\mathbb{R}}}(p)}[n] \\ R\Gamma_{Z_\Omega}\mathcal{C}_{N|X|p}[d] \simeq R\Gamma_{\pi_L^{-1}(A_\Omega)}\mu_L(\mathcal{O}_Y)|_{\Phi_{\Lambda^{\mathbb{R}}}(p)}[n]. \end{cases} \tag{6.8}$$

Thus in virtue of the calculations of stalks in the right hand sides of the above isomorphisms done in the proofs of THEOREM 3.1 (i) and PROPOSITION 6.2, the proof of this proposition reduces to the next lemma.

LEMMA 6.5. — *Let $X = \mathbb{C}^n_z$ and $z = x + iy$. Then the natural morphism induced by the inclusion $Z \subset Z'$:*

$$\phi\colon \varinjlim_{Z \in \mathcal{Z}} \mathcal{H}^n_Z(\mathcal{O}_X)_0 \longrightarrow \varinjlim_{Z' \in \mathcal{Z}'} \mathcal{H}^n_{Z'}(\mathcal{O}_X)_0$$

is injective, where $\mathcal{Z}$ and $\mathcal{Z}'$ denote the families of closed subsets of X which satisfy:

$$\begin{aligned} Z \in \mathcal{Z} \iff & \text{In a neighborhood of the origin,} \\ & Z \subset \mathbb{R}^n_x \times \sqrt{-1}\left[(\gamma'_\varepsilon \times \{0\}) + \gamma_\varepsilon\right], \\ & \gamma'_\varepsilon := \left\{y_1 \geq \varepsilon|(y_2, \ldots, y_d)|\right\} \subset \mathbb{R}^d, \\ & \gamma_\varepsilon := \left\{y_n \geq \varepsilon|(y_1, \ldots, y_{n-1})|\right\} \subset \mathbb{R}^n_y, \end{aligned}$$

for some $\varepsilon > 0$, and

$$\begin{aligned} Z' \in \mathcal{Z}' \iff & \text{In a neighborhood of the origin,} \\ & Z' \subset \mathbb{R}^n_x \times \sqrt{-1}\left\{y_n \geq \varepsilon|(y_{d+1}, \cdots, y_{n-1})|\right\}, \end{aligned}$$

and

$$Z' \cap \left\{y_{d+1} = \cdots = y_n = 0\right\} \subset \mathbb{R}^n_x \times \sqrt{-1}\left(\gamma'_\varepsilon \times \{0\}\right)$$

for some $\varepsilon > 0$ respectively.

Proof. — The idea of the proof is similar to that of the injectivity of the morphism from the sheaf $\mathcal{C}_M$ of microfunctions to the sheaf $\mathcal{B}^2_\Lambda$ of second hyperfunctions in the note of Kashiwara [3]. To begin with, let us recall the definition of q-propreness.

DEFINITION 6.6 (Kashiwara [7]). — Let G be a locally closed subset of a complex manifold X. We say G is *q-propre* ($q \in \mathbb{N}$) if for every complex manifold Y and every $j < q$ we have:

$$H^j_{G \times Y}(X \times Y; \mathcal{O}_{X \times Y}) = 0.$$

Now we shall explain the reason why the proof of LEMMA 6.5 can be reduced to the following lemma:

LEMMA 6.7. — *For every $Z \in \mathcal{Z}$, $Z' \in \mathcal{Z}'$ such that $Z \subset Z'$ and every open neighborhood V of the origin of $X = \mathbb{C}^n_z$, we can find closed subsets G and G' of X such that $G \subset G'$ and an open neighborhood U of the origin of X which satisfy:*

$$(6.9) \qquad \begin{cases} Z \cap U \subset G, \quad Z' \cap U \subset G', \quad U \subset V, \quad G \in \mathcal{Z}, \\ G' \setminus G \subset\subset U \quad \text{and} \quad G' \setminus G \text{ is } n\text{-propre.} \end{cases}$$

In fact, if we take an element u of $\varinjlim_{Z\in\mathcal{Z}} \mathcal{H}^n_Z(\mathcal{O}_X)_0$, u can be represented by an element, which we will call again u, of the cohomology group $H^n_{Z\cap V_0}(V_0;\mathcal{O}_X)$ for an open neighborhood V_0 of the origin and some $Z \in \mathcal{Z}$. Assume $\phi(u) \in \varinjlim_{Z'\in\mathcal{Z}'} \mathcal{H}^n_{Z'}(\mathcal{O}_X)_0$ is zero. Then there exist an open neighborhood V of the origin such that $V \subset V_0$ and $Z' \in \mathcal{Z}'$ such that $Z \subset Z'$ and the image of u by the morphism:

$$H^n_{Z\cap V_0}(V_0;\mathcal{O}_X) \longrightarrow H^n_{Z'\cap V}(V;\mathcal{O}_X)$$

is zero. Now take U, G and G' as in LEMMA 6.7, then we have a commutative diagram:

$$\begin{array}{ccc} H^n_{Z\cap V}(V;\mathcal{O}_X) & \longrightarrow & H^n_{Z'\cap V}(V;\mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^n_{G\cap U}(U;\mathcal{O}_X) & \longrightarrow & H^n_{G'\cap U}(U;\mathcal{O}_X), \end{array} \tag{6.10}$$

and the n-propreness of $G' \setminus G$ implies the injectivity of the second horizontal arrow. Therefore the image of u in $H^n_{G\cap U}(U;\mathcal{O}_X)$ is also zero. Because G is an element of $\mathcal{Z}$, it follows that:

$$u = 0 \text{ in } \varinjlim_{Z\in\mathcal{Z}} \mathcal{H}^n_Z(\mathcal{O}_X)_0.$$

Hence the proofs of LEMMA 6.5, PROPOSITION 6.4 and THEOREM 6.1 are obtained at the same time. □

To complete the proof of THEOREM 6.1, it remains to prove LEMMA 6.7.

Proof. — First, we consider the case when $n - d = 1$. Let

$$Y = \mathbb{C}^n_w, \quad w = (w', w_n) = (w_1, w'', w_n) = u + iv$$

be an another copy of

$$X = \mathbb{C}^n_z, \quad z = (z', z_n) = (z_1, z'', z_n) = x + iy,$$

and $R_\theta : \mathbb{R}^n_y \xrightarrow{\sim} \mathbb{R}^n_v$ be the rotation which fixes the linear subspace:

$$\{0\} \times \mathbb{R}^{n-2}_{y''} \times \{0\} \simeq \{0\} \times \mathbb{R}^{n-2}_{v''} \times \{0\}$$

and sends the point $(1, 0, \cdots, 0)$ to the point $(\cos\theta, 0, \ldots, 0, \sin\theta)$ with some $0 < \theta \ll \frac{1}{2}\pi$. We can naturally extend this linear transformation to $X = \mathbb{C}^n_z = \mathbb{R}^n_x \times \sqrt{-1}\,\mathbb{R}^n_y$ and denote it also by $R_\theta : \mathbb{C}^n_z \xrightarrow{\sim} \mathbb{C}^n_w$. Here R_θ sends Z and Z' to closed subsets of $\{v_n \geq (\tan\theta)v_1\} \subset Y$ and we denote them by Z_θ and Z'_θ respectively. The next lemma is a simple observation on the geometric situation.

LEMMA 6.8. — *Fix $\delta > 1$, $\rho > 0$ and $t > 0$, and set in $\mathbb{R}^n_v$:*

$$\tag{6.11} \begin{cases} H := \{v_n = \delta\rho v_1\} \\ C := \{v_n = \tfrac{1}{2}\rho\,|v'|\} \\ C' := \{v_n = \rho\,|v'| - \tfrac{1}{5}t\rho\} \subset \mathbb{R}^n_v. \end{cases}$$

Let $\alpha: H \xrightarrow{\sim} \mathbb{R}^{n-1}_{v'}$ be the isomorphism induced by the orthogonal projection from $\mathbb{R}^n_v$ to $\mathbb{R}^{n-1}_{v'}$. Then:

$$\tag{6.12} \alpha(H \cap C) = \Big\{v_1^2 = \frac{1}{4\delta^2 - 1}v''^2\Big\}$$

and

$$\tag{6.13} \alpha(H \cap C') = \Big\{\Big(\frac{v_1 + \delta t/(5(\delta^2-1))}{t/(5(\delta^2-1))}\Big)^2 - \Big(\frac{|v''|}{t/(5\sqrt{\delta^2-1})}\Big)^2 = 1\Big\}$$

in $\mathbb{R}^{n-1}_{v'}$ and they do not depend on the parameter $\rho > 0$.

Now fix a sufficiently large number $\delta \gg 0$ and set

$$0 < \theta = \arctan(\delta\rho) < \tfrac{1}{2}\pi \quad \text{for } 0 < \rho \ll 1.$$

It follows from the above lemma that for every $t > 0$ there exists $\rho_t > 0$ such that:

$$\tag{6.14} Z_\theta \subset D_0 := \{v_n \geq \tfrac{1}{2}\rho\,|v'|\} \cap \{v_n \geq \rho\,|v'| - \tfrac{1}{5}t\rho\} \cap \{v_n \geq \delta\rho v_1\}$$

and

$$\tag{6.15} Z'_\theta \subset G'_0 := \{v_n \geq \rho\,|v'| - \tfrac{1}{5}t\rho\} \cap \{v_n \geq \delta\rho v_1\}$$

hold in $\{\,|u| < 4\,\delta t\} \times \{\,|v| < t\}$ for every ρ with $0 < \rho < \rho_t$. Let us take $0 < t \ll 1$ which satisfies:

$$\tag{6.16} U_\theta := \{\,|u| < 4\delta t\} \times \{\,|v| < t\} \subset R_\theta(V),$$

and set in $Y = \mathbb{C}^n_w$ for some $\rho > 0$ such that $0 < \rho < \rho_t$:

$$\tag{6.17} \begin{cases} K := \overline{U} = \{\,|u| \leq 4\,\delta t\} \times \{\,|v| \leq t\}, \\ D := D_0 \cap K, \\ E := G'_0 \cap \big(\{2\delta t \leq |u| \leq 4\delta t\} \times \{\,|v| \leq \tfrac{1}{2}t\}\big), \\ G'_\theta := G'_0 \cap K, \end{cases}$$

and finally

$$G_\theta := \text{holomorphically convex hull of } D \cup E.$$

Since $D \cup E \subset G'_\theta$ and G'_θ is a convex subset of Y, $G_\theta \subset G'_\theta$ holds. Notice:

$$\begin{aligned}(6.18)\qquad &\big(\{v_n \geq \rho\,|v'| - \tfrac{1}{5}t\rho\} \setminus \{v_n \geq \tfrac{1}{2}\rho\,|v'|\}\big)\\ &\qquad \subset \big(\{\,|v'| < \tfrac{2}{5}t\} \cap \{\,|v_n| \leq \tfrac{1}{5}t\rho\}\big)\\ &\qquad \subset \big\{\,|v| < \tfrac{1}{2}t\big\} \quad \text{if} \quad \rho < \tfrac{1}{2},\end{aligned}$$

and it implies:

$$(6.19)\qquad G'_\theta \setminus G_\theta \subset\subset U_\theta.$$

Furthermore, the previous formulas (6.14) and (6.15) entail:

$$(6.20)\qquad Z_\theta \cap U_\theta \subset G_\theta \quad \text{and} \quad Z'_\theta \cap U_\theta \subset G'_\theta.$$

Since G_θ and G'_θ are compact holomorphically convex subsets in $Y = \mathbb{C}^n_w$, we have by Martineau's theorem (Theorem 1.2.3 of [7]):

$$(6.21)\qquad G'_\theta \setminus G_\theta \quad \text{is} \quad n\text{-propre}.$$

Hence by setting $U := R_\theta^{-1}(U_\theta),\ G := R_\theta^{-1}(G_\theta),\ G' := R_\theta^{-1}(G'_\theta)$, we get:

$$(6.22)\qquad \begin{cases} Z \cap U \subset G, \quad Z' \cap U \subset G', \quad U \subset V,\\ G' \setminus G \subset\subset U,\\ G' \setminus G \quad \text{is} \quad n\text{-propre},\end{cases}$$

because R_θ is a biholomorphic mapping between X and Y. The following lemma implies $G \in \mathcal{Z}$ and it completes the proof of the case $n - d = 1$.

LEMMA 6.9. — *One has*

$$G_\theta \cap \big(\{\,|u| < \delta t\} \times \{v_n < \tfrac{1}{4}\rho\,|v'|\}\big) = \emptyset.$$

Proof. — Let $w^* = u^* + iv^*$ be a point in $\{\,|u| < \delta t\} \times \{v_n < \frac{1}{4}\rho\,|v'|\}$. By performing an orthogonal change of coordinates in $\mathbb{R}^{n-1}_{v'}$-space, we may assume there exists $j \in \{1, \dots, n-1\}$ such that,

$$(6.23)\qquad v_n^* < \tfrac{1}{4}\rho\, v_j^* \quad \text{and} \quad |u^*| < \delta t.$$

Take holomorphic functions h and g on $Y = \mathbb{C}^n_w$ as follows:

$$(6.24)\qquad \begin{cases} h(w) := w_n + \sqrt{-1}\,k(w_1^2 + \cdots + w_n^2) - \frac{1}{4}\rho\, w_j \\ g(w) := -h(w - u^*) \end{cases}$$

for $k = (4\delta + 1)\rho/(2(4\delta^2 - 1)t) > 0$. Then it is enough to show:

$$(6.25)\qquad \begin{cases} \operatorname{Im} g(w^*) > 0 \quad \text{and} \\ \operatorname{Im} g(w) \leq 0 \quad \text{on} \quad D \cup E. \end{cases}$$

Since

$$(6.26)\qquad \operatorname{Im} g(w) = -v_n - k\,|u - u^*|^2 + k\,|v|^2 + \tfrac{1}{4}\rho\, v_j,$$

$$(6.27)\qquad \operatorname{Im} g(w^*) = -v_n^* + k\,|v^*|^2 + \tfrac{1}{4}\rho\, v_j^* > k\,|v^*|^2 > 0,$$

for $w \in E$, we have:

$$(6.28)\qquad \begin{aligned} \operatorname{Im} g(w) &\leq -v_n - k\delta^2 t^2 + \tfrac{1}{4}\,kt^2 + \tfrac{1}{8}\,\rho\, t \\ &\leq \delta\rho\,|v| + \tfrac{1}{8}\,\rho\, t - \tfrac{1}{4}\,(4\delta^2 - 1)kt^2 \\ &\leq \tfrac{1}{8}\,(4\delta + 1)\rho\, t - \tfrac{1}{4}\,(4\delta^2 - 1)kt^2 \leq 0 \end{aligned}$$

by the very definition of k. Finally for $w \in D$, the inequality $v_n \geq \frac{1}{2}\rho\,|v'|$ entails $v_n \geq \frac{1}{3}\rho\,|v|$ if ρ is sufficiently small. Therefore we have:

$$(6.29)\qquad \begin{aligned} \operatorname{Im} g(w) &\leq -v_n + k\,|v|^2 + \tfrac{1}{4}\,\rho\, v_j \\ &\leq -\tfrac{1}{3}\,\rho\,|v| + k\,|v|^2 + \tfrac{1}{4}\,\rho\,|v| \\ &= -\tfrac{1}{12}\,\rho\,|v| + k\,|v|^2. \end{aligned}$$

Since $|v| \leq t$, we can take $0 \leq s \leq 1$ such that $|v| = st$ and:

$$(6.30)\qquad \begin{aligned} -\frac{\rho}{12}\,|v| + k\,|v|^2 &= -\frac{\rho}{12}\,st + ks^2t^2 \leq -st\Big[\frac{\rho}{12} - kt\Big] \\ &= -\frac{1}{12}\rho\, st\Big[1 - \frac{6(4\delta + 1)}{4\delta^2 - 1}\Big] \\ &\leq -\frac{1}{12}\rho\, st\Big[1 - \frac{(4\delta + 2)\cdot 6}{(2\delta + 1)(2\delta - 1)}\Big] \\ &= -\frac{1}{12}\rho\, st\Big[1 - \frac{12}{2\delta - 1}\Big] \leq 0 \end{aligned}$$

holds if $\delta \geq \frac{13}{2}$. Thus we have achieved the proof of Lemma 6.9. ∎

Finally we shall treat the case when $n-d > 1$ in the proof of LEMMA 6.7. In this case, we may assume for some $\varepsilon_0 > 0$:

$$Z = Z_0 \cap \{y_n \geq \varepsilon_0 \,|(y_{d+1}, \dots, y_{n-1})|\}, \tag{6.31}$$

with

$$\begin{cases} Z_0 := \mathbb{R}^n_x \times \sqrt{-1}\,\big[(\Gamma'_{\varepsilon_0} \times \{0\}) + \Gamma_{\varepsilon_0}\big], \\ \Gamma'_{\varepsilon_0} := \{y_1 \geq \varepsilon_0 |(y_2, \dots, y_{n-1})|\} \subset \mathbb{R}^{n-1} \\ \Gamma_{\varepsilon_0} := \{y_n \geq \varepsilon_0 |(y_1, \dots, y_{n-1})|\} \subset \mathbb{R}^n_y, \end{cases} \tag{6.32}$$

and

$$Z' = Z'_0 \cap \{y_n \geq \varepsilon_0 \,|(y_{d+1}, \dots, y_{n-1})|\}, \tag{6.33}$$

for a closed subset Z'_0 of $X = \mathbb{C}^n_w$ such that:

$$\begin{cases} Z'_0 \subset \mathbb{R}^n_x \times \sqrt{-1}\,\{y_n \geq 0\} \\ Z'_0 \cap \{y_n = 0\} \subset \mathbb{R}^n_x \times \sqrt{-1}\,(\Gamma'_{\varepsilon_0} \times \{0\}). \end{cases} \tag{6.34}$$

Then it follows from the proof of the case $n - d = 1$ that we can find an open neighborhood U of the origin and compact holomorphically convex subsets G_0 and G'_0 of $X = \mathbb{C}^n$ which satisfy:

$$\begin{cases} Z_0 \cap U \subset G_0, \quad Z'_0 \cap U \subset G'_0, \quad U \subset V, \\ G'_0 \setminus G_0 \subset\subset U \end{cases} \tag{6.35}$$

and

$$G_0 \subset \mathbb{R}^n_x \times \sqrt{-1}\,\big[(\Gamma'_\varepsilon \times \{0\}) + \Gamma_\varepsilon\big] \tag{6.36}$$

for some $\varepsilon > 0$ in an open neighborhood of the origin. It is easy to see:

$$\begin{cases} G := G_0 \cap \{y_n \geq \varepsilon_0 \,|(y_{d+1}, \dots, y_{n-1})|\} \quad \text{and} \\ G' := G'_0 \cap \{y_n \geq \varepsilon_0 \,|(y_{d+1}, \dots, y_{n-1})|\} \end{cases} \tag{6.37}$$

are holomorphically convex subsets of X and G is an element of $\mathcal{Z}$, because the intersection of two holomorphically convex subsets is again holomorphically convex. Since $G' \setminus G \subset\subset U$ is n-propre by Martineau's theorem, U, G and G' satisfy all requirements in the statement of LEMMA 6.7. ∎

Now we finished the proof of the main theorem.

REMARK 6.10. — Making use of Remark 3.4, we can replace A_0 in the statement of THEOREM 6.1 by any closed convex cone A of $\mathbb{R}^d$. In fact, our proof of Theorem 6.1 can be directly applied to the closed convex cone:

$$A_\varepsilon := \{x_1 \geq \varepsilon^{-1} |(x_2, \ldots, x_d)|\} \subset \mathbb{R}^d \tag{6.38}$$

for any $\varepsilon > 0$ without help of Remark 3.4. The proof for this case is a little delicate, but it proceeds similarly. We only have to change the proportion of the open set U in the proof of LEMMA 6.7. For example, set:

$$U := \{\, |x| < 4\delta\ell t\} \times \{\, |y| < t\}$$

for sufficiently large real number $\ell > 0$ which depends on $\varepsilon > 0$.

7. Relation with second microlocal analysis

As we have seen in the previous sections, the structure of the complex $\mathcal{C}_{\Omega|X\,|\,N\times_M T^*_M X}$ for an open convex cone with the edge N has nice relations with the theory of second microlocal analysis. In this section, we will discuss on these relations and give some applications. Originally, the sheaf $\mathcal{C}^2_\Lambda$ of second microfunctions was introduced as a repetition of the Sato's microlocalization by M. Kashiwara, but this construction was far from the direct calculus of the sheaf $\mathcal{C}_M$ of microfunctions (*cf.* [3] and [7]). Around 1988, K. Kataoka and N. Tose [12] found the sheaf $\tilde{\mathcal{C}}^2_\Lambda$ which enjoys the exact sequence:

$$0 \to \mathcal{CO}_{|\Lambda} \longrightarrow \mathcal{C}_{M|\Lambda} \longrightarrow \dot{\pi}_* \tilde{\mathcal{C}}^2_\Lambda \to 0 \tag{7.1}$$

and admits the boundary value representations by holomorphic functions. They defined the sheaf $\tilde{\mathcal{C}}^2_\Lambda$ by using the second comonoidal transformation. Refer to [12] and [13] for the details on this sheaf. Recently in [24], P. Schapira and the author constructed the same sheaf which is called $\mathcal{C}_{ML}$ in a completely different way by applying a new functor μ_{ML} to the sheaf $\mathcal{O}_X$ of holomorphic functions. This construction enables us to establish various functorial properties of the second microfunction $\mathcal{C}_{ML}$ and treat $\mathcal{D}_X$-modules. In order to explain the relation of the theory of $\mathcal{C}_{\Omega|X}$ with that of second microlocal analysis, set:

$$\begin{cases} M := \mathbb{R}^n_x = \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''}, \\ N := \{0\} \times \mathbb{R}^{n-d}_{x''} \subset M \end{cases} \tag{7.2}$$

and let $X = \mathbb{C}^n_z$, $z = x + iy$ and $Y = \mathbb{C}^{n-d}_{z''}$ be their complexifications. Furthermore, let $L = \mathbb{C}^d_{x'} \times \mathbb{R}^{n-d}_{x''} \supset M$ be a partial complexification of M

in X. In [24], P. Schapira and the author constructed the bimicrolocalization functor:

$$\mu_{ML} : \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(T^*_M L \times_L T^*_L X), \tag{7.3}$$

and they defined as follows.

DEFINITION 7.1. — The complex $\mathcal{C}_{ML} := \mu_{ML}(\mathcal{O}_X)[n]$ is a sheaf on $T^*_M L \times_L T^*_L X$ and is called the *sheaf of second microfunctions along L.*

Let us take a coordinate system

$$\big(x; \sqrt{-1}\,\eta''\,\mathrm{d}x''; \sqrt{-1}\,\eta'\,\mathrm{d}x'\big)$$

of $T^*_M L \times_L T^*_L X$ in which $(x; \sqrt{-1}\,\eta''\,\mathrm{d}x'')$ belongs to $M \times_L T^*_L X$. Next assume d and $(n-d)$ are strictly positive, and define for $\varepsilon > 0$ an open convex proper cone with the edge N by:

$$\Omega_\varepsilon := \big\{x_1 > \varepsilon^{-1}\,|(x_2, \dots, x_d)|\big\} \times \mathbb{R}^{n-d}_{x''} \subset M. \tag{7.4}$$

Now take a point $p_0 := (0; \sqrt{-1}\,\eta\,\mathrm{d}x) \in N \times_M T^*_M X$ such that $\eta_n > 0$. Then by virtue of the Legendre transformation in the proof of THEOREM 3.1, we have (see the formula (3.15)):

$$\varinjlim_{\varepsilon>0} \mathcal{C}_{\Omega_\varepsilon|X\,|p_0} \simeq \varinjlim_{Z} \mathcal{H}^n_Z(\mathcal{O}_X)_{(u+i0)} \tag{7.5}$$

for some $u \in \mathbb{R}^n_x$, where Z ranges through the closed convex subsets of X of the type:

$$Z = \mathbb{R}^n_x \times \sqrt{-1}\,\big[(\gamma'_\varepsilon \times \{0\}) + \gamma_\varepsilon\big], \tag{7.6}$$

$$\begin{cases} \gamma'_\varepsilon := \big\{y_1 \leq -\varepsilon\,|(y_2, \dots, y_d)|\big\} \subset \mathbb{R}^d_{y'}, \\ \gamma_\varepsilon := \big\{y_n \geq \varepsilon\,|(y_1, \dots, y_{n-1})|\big\} \subset \mathbb{R}^n_y \end{cases} \tag{7.7}$$

for some $\varepsilon > 0$. By the stalk formula of the functor μ_{ML} in Theorem 3.2 (i) of [24], if we set $q_0 := (u; \sqrt{-1}\,\mathrm{d}x_n; -\sqrt{-1}\,\mathrm{d}x_1) \in T^*_M L \times_L T^*_L X$, we obtain the following theorem:

THEOREM 7.2. — *There is an isomorphism:*

$$\varinjlim_{\varepsilon>0} \mathcal{C}_{\Omega_\varepsilon|X\,|p_0} \simeq \mathcal{C}_{ML\,|q_0}. \tag{7.8}$$

Roughly speaking, it means that the inductive limit sheaf

$$\varinjlim \mathcal{C}_{\Omega_\varepsilon|X\,|N \times_M T^*_M X}$$

is transformed to the sheaf $\mathcal{C}_{ML}$ of second microfunctions by the Legendre transformation.

REMARK 7.3. — Even if in the case when $d = 1$, that is, when the codimension of N in M equals to one, this correspondence clarifies the structure of the sheaf $\mathcal{C}_{\Omega|X|N\times_M T^*_M X}$ for $\Omega := \{x_1 > 0\}$. For example, the theorem on the flabbiness of $\mathcal{C}_{\Omega|X|N\times_M T^*_M X}$ due to [25] can be interpreted to the flabbiness of the sheaf $\mathcal{C}_{ML}$ shown by Kataoka–Tose–Okada [13].

REMARK 7.4. — The use of the sheaf $\mathcal{CO}$ of microfunctions with holomorphic parameters in the study of the sheaf $\mathcal{C}_{M_+|X}$ of Kataoka is already implicit in Theorem 4.2.12 of [10], which supports THEOREM 7.2.

We shall apply the results on the boundary value problems to the study of the second microlocal analysis. Let

$$N := \{x_1 = 0\} \subset M = \mathbb{R}^n$$

be a submanifold of codimension one, and define open subsets

$$\Omega_\pm := \{\pm x_1 > 0\}.$$

Denote by P the H. Lewy operator $D_1 + \sqrt{-1}\, z_1 D_n$ and consider a left $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X P$. Recall the vanishing theorem of Tose–Uchida [28] and D'Agnolo–Zampieri [1]:

PROPOSITION 7.5 (Tose–Uchida, D'Agnolo–Zampieri). — *Let*

$$p_0 := (0; \sqrt{-1}\, \mathrm{d}x_n) \in N \times_M T^*_M X.$$

Then:

$$RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega_\pm|X})_{p_0} \simeq 0. \tag{7.9}$$

Since the Legendre transformation sends the point p_0 to itself and preserves the symbol of the operator P, we have:

COROLLARY 7.6. — *Let $L := \mathbb{C}^1_{z_1} \times \mathbb{R}^{n-1}_{x''} \supset M$ be a partial complexification of M in X. Then*:

$$RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{ML}) \simeq 0 \tag{7.10}$$

at $(0; \sqrt{-1}\, \mathrm{d}x_n; \pm\sqrt{-1}\, \mathrm{d}x_1) \in T^*_M L \times_L T^*_L X$.

Therefore, on account of the exact sequence:

$$0 \to \mathcal{CO}_{z_1} \longrightarrow \mathcal{C}_M \longrightarrow \dot{\pi}_* \mathcal{C}_{ML} \to 0 \tag{7.11}$$

on $\Lambda := M \times_L T_L^*X$, there exists an isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{CO}_{z_1})_{p_0} \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M)_{p_0}, \tag{7.12}$$

where $\mathcal{CO}_{z_1}$ denotes the sheaf of microfunctions with holomorphic parameter in z_1−variable. In particular, it follows from the structure theorem of S–K–K (Theorem 2.3.6 of [18]) that:

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{CO})_{p_0} \simeq \mathcal{C}_{N|p_0}. \tag{7.13}$$

It means that the H. Lewy operator $P = D_1 + \sqrt{-1}\, z_1 D_n$ is not solvable in $\mathcal{CO}_{z_1}$ at $p_0 := (0; \sqrt{-1}\, dx_n)$.

REMARK 7.7. — The micro-symbol (see Definition 2.4.1 of Laurent [15]) of P at $(0;\ \sqrt{-1}\ dx_n; \pm\sqrt{-1}\ dx_1)$ is $\sqrt{-1}\, z_1\zeta_n$ ($\zeta_n \neq 0$) and the second characteristic variety of $\mathcal{M}$ is smooth there. However all cohomology groups of the solution complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{ML})$ vanish as in COROLLARY 7.6. When we consider the microfunction solutions, at least one of the cohomology groups of $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M)$ survives on the characteristic variety of $\mathcal{M}$ in general (refer to the structure theorems in S–K–K [18]). It seems for us that such an interesting phenomenon is familiar in the second microlocal analysis.

8. Applications

Let N be a real analytic submanifold of M of codimension d, Y its complexification in X, and $\mathcal{M}$ a coherent $\mathcal{D}_X$-module on X for which Y is non-characteristic. Recall the following lemma.

LEMMA 8.1 (Proposition 2.3 of [29]). — *For any locally closed subset A of N, we have a canonical isomorphism:*

$$\rho_*\varpi^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{A|X}) \otimes \mathrm{or}_{N|M}[d] \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{C}_{A|Y}), \tag{8.1}$$

where

$$T^*Y \xleftarrow{\rho} Y \times_X T^*X \xrightarrow{\varpi} T^*X$$

are the morphisms associated to the inclusion $Y \hookrightarrow X$ and $\mathcal{M}_Y$ is the induced system of $\mathcal{M}$ on Y.

In this section, whenever we consider the boundary value from an open subset $\Omega \subset M$ to a submanifold N such that $\overline{\Omega} \supset N$, Ω is always assumed to satisfy the cohomological triviality due to Schapira [22]. For

$u \in \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$ a hyperfunction solution of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on an open subset Ω of M, a closed conic subset $\mathrm{SS}^{\mathcal{M}}_\Omega(u)$ of T^*X is defined by

$$\mathrm{SS}^{\mathcal{M}}_\Omega(u) := \mathrm{supp}\, \alpha(u),$$

where α is a natural morphism:

$$\alpha : \pi_X^{-1}\Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}), \tag{8.2}$$

and $\pi_X : T^*X \to X$ is the projection. This is a wave front set introduced for the study of the boundary value problems in [22].

The next proposition is a direct consequence of LEMMA 8.1 and the structure theorem of the complex $\mathcal{C}_{\Omega|X}$ for open quadrants Ω in Section 4.

PROPOSITION 8.2. — *Let $\Omega := \{x_1 > 0, \dots, x_d > 0\}$ be an open quadrant in $M = \mathbb{R}^n_x$ with the edge $N := \{x_1 = \cdots = x_d = 0\} \subset M$ and Y a complexification of N in X. Then for any coherent $\mathcal{D}_X$-module $\mathcal{M}$ for which Y is non-characteristic, we have for every $j < 0$*

$$H^j\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) = 0. \tag{8.3}$$

From now on, we assume that the open subset $\Omega \subset M = \mathbb{R}^n_x$ contains an open convex cone with the edge $N := \{x_1 = \cdots = x_d = 0\}$ and the open neighborhoods of the origin 0 whose intersection with Ω are connected form a fundamental system of neighborhood of 0. In this case, we can take linear functionals $f_1, \dots, f_d$ on $M = \mathbb{R}^n$ which vanish on N such that $f_1 \wedge \cdots \wedge f_d \neq 0$ and $\Omega_0 := \{f_1 > 0, \dots, f_d > 0\} \subset \Omega$. We also assume that the submanifold N is non-characteristic with respect to the $\mathcal{D}_X$-module $\mathcal{M}$.

PROPOSITION 8.3. — *Assume a hyperfunction solution*

$$u \in \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_0$$

of $\mathcal{M}$ on Ω near 0 satisfies the condition:

$$\mathrm{SS}^{\mathcal{M}}_\Omega(u) \cap \pi_X^{-1}(0) \subset T^*_X X. \tag{8.4}$$

Then $u \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)_0$, i.e. u extends through N as a real analytic solution of $\mathcal{M}$ near 0.

Proof. — Since there exists a microlocal restriction morphism:

$$\mathcal{C}_{\Omega|X} \longrightarrow \mathcal{C}_{\Omega_0|X},$$

we can replace the problem of Ω by that of the open quadrant Ω_0. Now, by PROPOSITION 8.2, we have an exact sequence (see the proof of Proposition 3.3 of Uchida [29]):

$$(8.5)\quad 0 \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)_{\overline{\Omega}_0} \longrightarrow \Gamma_{\Omega_0}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow \dot{\pi}_{X*}H^0\,R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega_0|X}) \to$$

and the conclusion of the proposition is clear. □

REMARK 8.4. — When Ω is an open subset of M whose complement is a closed convex set, PROPOSITION 8.3 is proved by Uchida (Proposition 3.3 of [29]) under a much weaker non-characteristic condition.

Now denote by $\mathrm{Car}\mathcal{M}$ the characteristic variety of $\mathcal{M}$ and consider a subset S of $\mathrm{Car}\mathcal{M} \cap T_N^*X$. By the Kashiwara–Kawaï's division theorem (Lemma 8.1 in [6]), we have for every point p of $\dot{T}_N^*Y$ a projection $R_{S,p}$:

$$(8.6)\quad \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)_p \simeq \bigoplus_{q\in \mathrm{Car}\mathcal{M}\cap\rho^{-1}(p)} \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})_q \xrightarrow[R_{S,p}]{} \bigoplus_{q\in S\cap\rho^{-1}(p)} \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})_q,$$

where $\rho : T_N^*X \to T_N^*Y$ and $\mathcal{C}_N$ denotes the sheaf of microfunctions on N.

DEFINITION 8.5. — Let $v \in \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$ be a hyperfunction solution of the induced system $\mathcal{M}_Y$ of $\mathcal{M}$ on N. We say that v is *micro-analytic at* S if $R_{S,p}(v) = 0$ for every point $p \in \dot{T}_N^*Y$.

The next theorem naturally generalizes the extension theorem due to Kashiwara–Kawai (Theorem 1 of [6]) to non elliptic equations.

THEOREM 8.6 (Extension of real analytic solutions). — *Assume that the boundary value* b.v.$(u) \in \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)_0$ *of a real analytic solution* $u \in \Gamma_\Omega\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)_0$ *of* $\mathcal{M}$ *on* Ω *near* 0 *is micro-analytic at*

$$S := \mathrm{Car}\mathcal{M} \cap \big[\mathrm{SS}\,\mathbb{C}_\Omega \cap \dot{T}_N^*X\big].$$

Then u extends through N as a real analytic solution of $\mathcal{M}$ near 0. *In particular, if* b.v.(u) *is real analytic, the conclusion holds.*

Proof. — We may assume from the first $u \in \Gamma_{\Omega_0}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_0$. First take a closed convex cone $A = A_1 \times \mathbb{R}^{n-d}_{x''} \subset \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''} = M$ such that $N \subset A$ and $A \setminus N \subset \Omega_0$. We can assume A_1 is C^ω-diffeomorphic to:

$$(8.7)\qquad A_\varepsilon := \big\{x_1 \geq \varepsilon^{-1}\,|(x_2, \dots, x_d)|\big\} \subset \mathbb{R}^d_{x'}$$

for some $\varepsilon > 0$ as in Remark 6.10. Next choose again linear functionals $g_1, \ldots, g_d$ on $M = \mathbb{R}^n$ which vanish on N such that

$$(8.8)\qquad \begin{cases} g_1 \wedge \cdots \wedge g_d \neq 0 \quad \text{and} \\ \Omega_1 := \{g_1 > 0, \ldots, g_d > 0\} \subset A. \end{cases}$$

Then it follows from the condition of the micro-analyticity of b.v.(u) that the image of u by the chain of the morphisms:

$$(8.9)\quad \pi_X^{-1}\Gamma_\Omega\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \xrightarrow{\alpha} H^0\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) \longrightarrow \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X})$$

is zero on $\pi_X^{-1}(0) \setminus T^*_X X$. Therefore the microlocal injectivity of the boundary value morphism (Theorem 6.1 and Remark 6.10) implies:

$$(8.10)\qquad \mathrm{SS}^{\mathcal{M}}_{\Omega_1}(u_{|\Omega_1}) \cap \pi_X^{-1}(0) \subset T^*_X X.$$

To complete the proof, it suffices to apply PROPOSITION 8.3 to the open quadrant Ω_1. □

The Oaku's theorem in [17] is now a particular case of our main theorem (THEOREM 6.1) as follows.

COROLLARY 8.7 (Local uniqueness of boundary value problems [17]). *The boundary value morphism*:

$$(8.11)\qquad \text{b.v.}: \varinjlim_{\Omega} \Gamma_\Omega\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_N \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

is injective, where Ω *ranges through open convex cones with the edge* N *such that* $\Omega \supset A_0 \times \mathbb{R}^{n-d} \setminus N$.

Let us interpret our main theorem (THEOREM 6.1) in terms of microfunctions.

COROLLARY 8.8 (Propagation of singularities up to the boundary). — *Let* Ω *be an open convex cone with the edge* N. *Assume that the boundary value* b.v.$(u) \in \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$ *of* $u \in \Gamma_\Omega\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$ *is micro-analytic at a point* $q \in \mathit{Car}\mathcal{M} \cap [N \times_M T^*_M X]$. *Then for any open cone* $\Omega_0 \subset\subset \Omega$ *there exists a neighborhood* U *of* q *in* $T^*_M X$, *such that* $u = 0$ *on* $\pi_M^{-1}(\Omega_0) \cap U$ *as a microfunction, where* $\pi_M : T^*_M X \longrightarrow M$ *denotes the natural projection.*

The next example explains that on $T^*_N X \setminus (N \times_M T^*_M X)$ there exist non-trivial contributions to the singularities of the boundary value hyperfunction (*cf.* THEOREM 3.1 (ii)).

EXAMPLE 8.9. — Let $M = \mathbb{R}^n_x$ and $X = \mathbb{C}^n_z = \mathbb{R}^n_x \times \sqrt{-1}\,\mathbb{R}^n_y$ be its complexification. Consider the underlying real analytic manifold $X^{\mathbb{R}}$ of X and its complexification $X \times \overline{X}$. Take an open convex proper cone $\Omega_y \subset \mathbb{R}^n_y$, and we set $\Omega := \mathbb{R}^n_x \times \sqrt{-1}\,\Omega_y \subset X^{\mathbb{R}}$. Let us construct «the boundary value» of holomorphic functions on Ω to M. For this purpose, notice that the induced system of the Cauchy–Riemann system $\mathcal{D}_X \boxtimes \mathcal{O}_{\overline{X}}$ on X, *i.e.* the complexification of M in $X \times \overline{X}$, coincides with $\mathcal{D}_X$. Then we have by Kashiwara [4]:

$$\text{(8.12)}\qquad \begin{aligned} R\Gamma_\Omega(\mathcal{O}_X)_{|M} &\simeq R\mathcal{H}om_{\mathcal{D}_{X\times\overline{X}}}(\mathcal{D}_X \boxtimes \mathcal{O}_{\overline{X}}, \Gamma_\Omega \mathcal{B}_{X^{\mathbb{R}}})_{|M} \\ &\longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{B}_M) \simeq \mathcal{B}_M. \end{aligned}$$

Now we take coordinates (z,w) of $X \times \overline{X}$ by complexifying the coordinates (x,y) of $X^{\mathbb{R}}$ and set the morphisms associated to the inclusion $X \hookrightarrow X \times \overline{X}$:

$$T^*X \xleftarrow{\ \rho\ } X \times_{(X\times\overline{X})} T^*(X \times \overline{X}) \xrightarrow{\ \varpi\ } T^*(X \times \overline{X}),$$

where the morphism ρ is described by

$$(z,0;\zeta\,\mathrm{d}z + \theta\,\mathrm{d}w) \longmapsto (z;\zeta\,\mathrm{d}z)$$

using the associated coordinates $(z;\zeta\,\mathrm{d}z)$ (resp. $(z,w;\zeta\,\mathrm{d}z + \theta\,\mathrm{d}w)$) of T^*X (resp. $T^*(X \times \overline{X})$). Then microlocal version of the above boundary value morphism is

$$\text{(8.13)}\qquad \rho_*\varpi^{-1}\, R\mathcal{H}om_{\mathcal{D}_{X\times\overline{X}}}(\mathcal{D}_X \boxtimes \mathcal{O}_{\overline{X}}, \mathcal{C}_{\Omega|X\times\overline{X}}) \longrightarrow \mathcal{C}_M.$$

Since the characteristic variety of $\mathcal{M} = \mathcal{D}_X \boxtimes \mathcal{O}_{\overline{X}}$ is $\{\theta = i\zeta\}$ in $T^*(X \times \overline{X})$, ρ induces a one to one correspondence between $\varpi^{-1}\mathrm{Car}\mathcal{M}$ and T^*X. One can easily obtain the following formula.

$$\text{(8.14)}\qquad \begin{aligned} T^*_M X \cap \rho\varpi^{-1}\{\mathrm{Car}\mathcal{M} \cap \operatorname{supp}\mathcal{C}_{\Omega|X\times\overline{X}}\} \\ = \{(x;\eta\,\mathrm{d}y) \in T^*_M X^{\mathbb{R}}\,;\ \eta \in (\Omega^0_y)^a\}. \end{aligned}$$

This means that the singular spectrum of the hyperfunction defined by the boundary value of a holomorphic function on Ω is contained in $\mathbb{R}^n_x \times (\Omega^0_y)^a \subset T^*_M X^{\mathbb{R}}$. The next corollary, which is well-known, is a very special case of THEOREM 8.6.

COROLLARY 8.10. — *If the boundary value* b.v.$(f) \in \mathcal{B}_M$ *of* $f \in \mathcal{O}_X(\Omega)$ *is real analytic, then* f *extends through* M *as a holomorphic function.*

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