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ANDREA D'AGNOLO
GIUSEPPE ZAMPIERI

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MICROLOCAL DIRECT IMAGES OF SIMPLE SHEAVES WITH APPLICATIONS TO SYSTEMS WITH SIMPLE CHARACTERISTICS

BY

ANDREA D'AGNOLO and GIUSEPPE ZAMPIERI

RÉSUMÉ. — Dans ce papier, nous établissons un résultat d'hypoellipticité dans le cadre des problèmes aux limites microlocaux (à comparer aux résultats analogues de [SKK], [KS2]). Plus précisément, soit $\mathcal{M}$ un système d'équations microdifférentielles à caractéristiques simples sur une variété complexe X, et soit Λ_i $(i = 1, 2)$ un couple de sous-variétées Lagrangiennes réelles de T^*X. On note $\mathcal{C}_{\Lambda_i}$ les complexes des microfonctions associés. Si le couple (Λ_1, Λ_2) est « positif », nous prouvons l'injectivité du morphisme naturel de « restriction »

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_1})$$

entre les faisceaux de solutions, où j est le premier degré de cohomologie éventuellement non nul.

ABSTRACT. — In this paper we state a hypoellipticity result in the framework of microlocal boundary value problems (which has to be compared with the analogous results of [SKK], [KS2] at the interior). More precisely, let $\mathcal{M}$ be a system of microdifferential equations with simple characteristics on a complex manifold X, and let Λ_i $(i = 1, 2)$ be a pair of real Lagrangian submanifolds of T^*X. Denote by $\mathcal{C}_{\Lambda_i}$ the associated complexes of microfunctions. If the pair (Λ_1, Λ_2) is « positive », we prove the injectivity of the natural « restriction » morphism

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_1})$$

between solution sheaves, where j is the first possibly non-vanishing degree of the cohomology.

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A. D'AGNOLO, Mathématiques, Tour 46–0, 5e étage, Université Paris VI, 4, place Jussieu, 75252 Paris CEDEX 05, France.
Email: dagnolo@mathp6.jussieu.fr
G. ZAMPIERI, Dipartimento di Matematica, Università di Padova, via Belzoni 7, 35131 Padova, Italy.
Email: zampieri@pdmat1.math.unipd.it.
AMS classification: 58G17.

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Introduction

Let M be a real analytic manifold, let X be a complexification of M, and let $\mathcal{M}$ be a system of microdifferential equations with simple characteristics along a germ of smooth regular involutive submanifold V of the cotangent bundle T^*X to X.

A classical result of SATO, KAWAI and KASHIWARA [SKK] asserts that the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ of microfunction solutions to $\mathcal{M}$ has vanishing cohomology in degree smaller than $s^-(M,V)$, where $s^-(M,V)$ denotes the number of negative eigenvalues of the generalized Levi form of V with respect to $T^*_M X$.

The aim of this paper is to prove a similar result « up to the boundary ». More precisely, let $\Omega \subset M$ be an open subset with real analytic boundary S, and consider the complex $\mathcal{C}_\Omega$ of microfunctions at the boundary introduced by SCHAPIRA [S2] as a framework for the study of boundary value problems. Under some cleanness hypotheses, we get in this paper vanishing for the cohomology of the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_\Omega)$ in degree smaller then $s^-(M,V)$: this is a criterion of hypoellipticity for microlocal boundary value problems. Moreover, we give geometrical conditions that ensure the vanishing of the whole complex of solutions $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_\Omega)$.

Our method of proof is similar to that of KASHIWARA and SCHAPIRA [KS2] where, assuming the intersection $V \cap T^*_M X$ is clean and regular, they recover the above mentioned results of [S-K-K], and obtain new vanishing theorems. Let us examine their proof in some details.

First, they use a complex contact transformation χ which replaces $T^*_M X$ with the set Λ of exterior conormals to a strictly pseudoconvex open subset $U \subset X$, and which «straightens» V, i.e. which replaces V with $X \times_Y T^*Y$, where the fiber product is taken over a smooth holomorphic map $f\colon X \to Y$. A quantization of χ reduces $\mathcal{M}$ to a partial de Rham system, and interchanges the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ with $\mu_{\partial U}(f^{-1}\mathcal{O}_Y)$, where $\mu_{\partial U}$ denotes the Sato microlocalization functor along ∂U. Finally, an adjunction formula reduces the study of the complex $\mu_{\partial U}(f^{-1}\mathcal{O}_Y)$ at $p \in (X \times_Y T^*Y) \cap \Lambda$ to the study of $\mu_{\partial U'}(\mathcal{O}_Y)$ at $f_\pi(p)$ (with $T^*_{\partial U'}Y = f_\pi(T^*_{\partial U}X \cap V)$). One is then reduced to the analysis of the microlocal direct image $f_!^{\mu,p} A_{\partial U}$ of the constant sheaf $A_{\partial U}$. The correspondence of cotangent bundles

$$T^*X \xleftarrow{{}^t f'} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y$$

associates to Λ a germ of Lagrangian manifold $f_\pi {}^t f'^{-1}(\Lambda) \subset T^*Y$ at $f_\pi(p)$. The sheaf $A_{\partial U}$ is simple along Λ with shift $\frac{1}{2}$, and it is thus possible to apply the following result of [KS3, Ch. 7]: if F is simple along Λ with shift d, then the microlocal direct image $f_!^{\mu,p}F$ is simple along $f_\pi {}^t f'^{-1}(\Lambda)$ with a shift which is calculated using generalized Levi forms. The vanishing theorems now easily follow.

Let $\Omega \subset M$ be an open subset with real analytic boundary S and consider the distinguished triangle $A_\Omega \to A_M \to A_S \xrightarrow{+1}$. In order to apply the above scheme of proof to boundary value problems, we have to deal simultaneously with the two Lagrangian manifolds $T^*_M X$ and $T^*_S X$. To this end, we first introduce a notion of «positive» pair of Lagrangian manifolds (to be compared with the one of [S1]) which allows us to give a microlocal meaning to the restriction morphism $A_M \to A_S$. In particular, if (Λ_1, Λ_2) is a pair of positive Lagrangian submanifolds of T^*X, there is a natural «restriction» morphism $\mathcal{C}_{\Lambda_2} \to \mathcal{C}_{\Lambda_1}$ between the associated complexes of generalized Sato microfunctions. Then, we give a criterion of faithfulness for the functor $f_!^{\mu,p}$ acting on simple sheaves. This allows us to state a very general result of injectivity for the the natural morphism

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{\Lambda_1})$$

between solution sheaves to a simply characteristic system $\mathcal{M}$, where j is the first possibly non vanishing degree of the cohomology. In particular, this implies the criterion of hypoellipticity for microlocal boundary value problems we were looking for.

1. Symplectic geometry and Levi form

For the notions reviewed in this section, we refer to [SKK], [S1], [KS1], and [D'AZ2].

1.1. Generalized Levi forms. — Let (E,σ) be a complex symplectic vector space. A complex subspace $\rho \subset E$ is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if $\rho^\perp \supset \rho$ (resp. $\rho^\perp = \rho$, resp. $\rho^\perp \subset \rho$), where $(\,\cdot\,)^\perp$ denotes the orthogonal with respect to σ. A real subspace $\lambda \subset E$ is called $\mathbb{R}$-*Lagrangian* if it is Lagrangian in the underlying real symplectic vector space $(E^{\mathbb{R}},\sigma^{\mathbb{R}})$, where $\sigma^{\mathbb{R}} = 2\operatorname{Re}\sigma$.

If $\rho \subset E$ is isotropic, the form σ induces a symplectic structure on the space $\rho^\perp/\rho$, which is denoted by (E^ρ,σ^ρ). For a subset $\lambda \subset E$, let

$$\lambda^\rho = \big((\lambda \cap \rho^\perp) + \rho\big)/\rho \subset E^\rho$$

and recall that if λ is an $\mathbb{R}$-Lagrangian subspace of E then λ^ρ is an $\mathbb{R}$-Lagrangian subspace of E^ρ (as it follows from the formula $(\lambda^\rho)^\perp = (\lambda^\perp)^\rho$).

The following definition slightly generalizes that of [S1] and [KS2, Lemma 3.4].

Definition 1.1. — Let $\rho \subset E$ be isotropic, $\lambda \subset E$ be $\mathbb{R}$-Lagrangian, and set $\mu = \lambda \cap i\lambda$ (an isotropic space). The *generalized Levi form* $L_{\lambda/\rho}$ is the hermitian form on ρ^μ defined by setting for $v,w \in \rho^\mu$

$$L_{\lambda/\rho}(v,w) = \sigma^\mu(v,w^c),$$

where $(\,\cdot\,)^c$ denotes the conjugate with respect to the isomorphism $E^\mu \cong \mathbb{C}\otimes_{\mathbb{R}}\lambda^\mu$. We will denote by $s^\pm(\lambda,\rho)$ the numbers of positive (resp. negative) eigenvalues of $L_{\lambda/\rho}$.

The *inertia index* $\tau(\lambda_1,\lambda_2,\lambda_3)$ of three $\mathbb{R}$-Lagrangian subspaces $\lambda_j \subset E$, $j = 1,2,3$, is defined as the signature of the quadratic form q on $\lambda_1\oplus\lambda_2\oplus\lambda_3$ given by

$$q(x_1,x_2,x_3) = \sigma^{\mathbb{R}}(x_1,x_2) + \sigma^{\mathbb{R}}(x_2,x_3) + \sigma^{\mathbb{R}}(x_3,x_1).$$

Let $\widetilde{\lambda}^\rho$ denote the $\mathbb{R}$-Lagrangian subspace $(\lambda \cap \rho^\perp) + \rho \subset E$.

PROPOSITION 1.2. — *The following equalities hold:*

$$\begin{aligned}\operatorname{sgn} L_{\lambda/\rho} &= -\tfrac{1}{2}\tau(\lambda, i\lambda, \widetilde{\lambda}^{\rho}),\\ \operatorname{rk} L_{\lambda/\rho} &= \dim^{\mathbb{C}}(\rho\cap\mu^{\perp}) - \dim^{\mathbb{C}}(\rho\cap[(\rho^{\perp}\cap\lambda)+(\rho^{\perp}\cap i\lambda)])\\ &= \dim^{\mathbb{C}}\rho - \dim^{\mathbb{C}}(\lambda^{\rho}\cap i\lambda^{\rho}) - \dim^{\mathbb{C}}(\lambda\cap i\lambda)\\ &\qquad\qquad + 2\dim^{\mathbb{C}}(\rho^{\perp}\cap\mu) - \dim^{\mathbb{R}}(\lambda\cap\rho).\end{aligned}$$

For a proof, cf. [KS1, Lemma 3.4], [D'AZ2, Prop. 3.3].

As we will recall (cf. Lemmas 1.5, 1.7 below), the generalized Levi form of Definition 1.1 agrees with classical concepts.

1.2. Homogeneous symplectic spaces. — Let X be a complex analytic manifold of dimension n, and denote by $\tau_X : TX \to X$, $\pi_X : T^*X \to X$ its tangent and cotangent bundle respectively. Denote by $\dot{T}^*X$ the space T^*X with the zero section removed, and more generally, for $V \subset T^*X$, let $\dot{V} = V \cap \dot{T}^*X$. Let α_X be the canonical one-form on T^*X, and $\sigma_X = d\alpha_X$ the symplectic two-form.

Let $H : T^*T^*X \xrightarrow{\sim} TT^*X$ be the Hamiltonian isomorphism. The vector field $-H(\alpha_X)$, called *Euler vector field*, is the infinitesimal generator of the action of $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$ on T^*X. In this paper all submanifolds that we consider are locally conic (i.e. tangent to $H(-\alpha_X)$).

A submanifold $V \subset T^*X$ is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if at each $p \in V$, the space T_pV has the corresponding property in the symplectic vector space $(T_pT^*X, \sigma_X(p))$. The submanifold V is called *regular* if $\alpha_{X|V} \neq 0$.

Let Λ be a submanifold of the underlying real analytic manifold $T^*X^{\mathbb{R}}$ to T^*X. The submanifold Λ is called $\mathbb{R}$-*Lagrangian* if at each $p \in \Lambda$, the space $T_p\Lambda$ is Lagrangian in the real symplectic vector space $(T_pT^*X^{\mathbb{R}}, 2\operatorname{Re}\sigma_X(p))$. For $p \in \Lambda$, set

$$(1.1)\qquad \begin{cases} \big(E(p),\sigma(p)\big) = \big(T_pT^*X, \sigma_X(p)\big),\\ \nu(p) = \mathbb{C}H\big(\alpha_X(p)\big),\\ \lambda_{0X}(p) = T_p\big(\pi_X^{-1}\pi_X(p)\big),\\ \lambda(p) = T_p\Lambda,\\ c\big(\lambda(p),\lambda_{0X}(p)\big) = \dim^{\mathbb{R}}\big(\lambda(p)\cap\lambda_{0X}(p)\big),\\ \gamma\big(\lambda(p),\lambda_{0X}(p)\big) = \dim^{\mathbb{C}}\big(\lambda(p)\cap i\lambda(p)\cap\lambda_{0X}(p)\big).\end{cases}$$

If no confusion may arise, we will drop the point p in the previous notations.

DEFINITION 1.3. — An $\mathbb{R}$-Lagrangian manifold $\Lambda \subset \dot{T}^*X$ is called of *real type at* $p \in \Lambda$ if $\dim^{\mathbb{R}}(\nu \cap \lambda) = 1$.

Notice that Λ is of real type if and only if it exists a complex contact transformation $\chi: T^*X \to T^*X$ at p such that $\chi(\Lambda)$ is the conormal bundle to a real hypersurface of X.

Let M be a real analytic submanifold of the underlying real analytic manifold $X^{\mathbb{R}}$ to X. Via the natural isomorphism $T(X^{\mathbb{R}}) \cong (TX)^{\mathbb{R}}$ the complex tangent plane to M at $x \in M$ is defined as

$$T_x^{\mathbb{C}}M = T_xM \cap iT_xM.$$

Moreover, we will identify the conormal bundle $T_M^*X^{\mathbb{R}}$ to an $\mathbb{R}$-Lagrangian manifold of T^*X, that we will denote by T_M^*X. For $\Lambda = T_M^*X$ and $p \in \dot{T}_M^*X$ we simplify the above notations (1.1), as well as those of Definition 1.1, by writing

$$\begin{cases} \lambda_M(p) = T_pT_M^*X, \\ s^{\pm}(M,p) = s^{\pm}(\lambda_M, \lambda_{0X}), \\ c_M(p) = c\big(\lambda(p), \lambda_{0X}(p)\big), \\ \gamma_M(p) = \gamma\big(\lambda(p), \lambda_{0X}(p)\big), \end{cases} \tag{1.2}$$

eventually dropping the point p.

1.3. Levi forms. — Let us now recall the classical definition of Levi form of a real submanifold.

DEFINITION 1.4. — Let $M \subset X$ be a real analytic submanifold. Let $p \in \dot{T}_M^*X$, and set $x = \pi_X(p)$. The *Levi form* $L_M(p)$ of M at p is the restriction to $T_x^{\mathbb{C}}M$ of the Hermitian form $\overline{\partial}\partial\phi(x)$, where ϕ is a real analytic function vanishing on M such that $\mathrm{d}\phi(x) = p$.

The independence of Definition 1.4 from the choice of the local equation ϕ for M may be deduced from the next Lemma which states the relation between the Levi forms of Definition 1.1 and Definition 1.4.

LEMMA 1.5. — *Let* $p \in \dot{T}_M^*X$, *and set* $x = \pi_X(p)$. *Then*

$$L_M(p) \sim L_{\lambda_M/\lambda_{0X}},$$

where $\sim$ *means that the two forms have same signature and same rank.*

For a proof refer to [KS1, Prop. 11.2.7] in the case $\operatorname{cod}_X^{\mathbb{R}} M = 1$, and to [D'AZ1, Prop. 1.1 and 1.3] for the general case.

Assume now that X is a complexification of a real analytic manifold M.

DEFINITION 1.6 (see [SKK]). — Let $M \subset X$ be a real analytic manifold, X a complexification of M, and let $V \subset \dot{T}^*X$ be a regular involutive submanifold. Let $p \in T^*_MX \cap V$. The *generalized Levi form* $L_{T^*_MX}(V,p)$ is the Hermitian form on $(T_pV)^\perp$ of matrix

$$\big(\{f_j, f_k^c\}(p)\big)_{1 \le j,k \le \ell},$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket, where $f_1 = \cdots = f_\ell = 0$ is a system of holomorphic equations for V at p such that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_\ell \wedge \alpha_X(p) \neq 0,$$

and where f_j^c denotes the holomorphic conjugate to f_j (i.e. the holomorphic function at p satisfying $f^c_{j|T^*_MX} = \bar{f}_{j|T^*_MX}$).

LEMMA 1.7 (see [KS2, Lemma 3.4]). — *Let $p \in \dot{T}^*_MX \cap V$ and set $\rho = (T_pV)^\perp$. Then*

$$L_{T^*_MX}(V,p) = -L_{\lambda_M/\rho}.$$

Notice that the more general case where M is a real analytic submanifold of X is treated in [D'AZ2].

NOTATIONS 1.8. — Let $\Lambda \subset T^*X$ be a real type Lagrangian manifold. On the line of LEMMA 1.7 we will use the following notations

$$\tag{1.3} \begin{cases} L_\Lambda(V,p) = -L_{\lambda/\rho}, \\ s^\pm(\Lambda,V,p) = s^\pm(\lambda,\rho), \\ s^\pm(M,V,p) = s^\pm(\Lambda,V,p), \end{cases}$$

where in the last equality we assumed $\Lambda = T^*_MX$ for a real submanifold $M \subset X$.

2. Review on sheaves

In this section all manifolds and morphisms of manifolds will be real of class C^∞. We collect here some definitions and results from [KS3].

2.1. General notations. — Let X be a manifold, and denote by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of A-modules, where A is a fixed ring with finite global dimension (e.g. $A = \mathbb{Z}$). As a general notation, for $K \subset X$ locally closed, A_K denotes the sheaf which is zero on $X \setminus K$ and which is the constant sheaf on K with stalk A.

The *micro-support* $SS(F)$ of a complex $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ is a closed conic involutive subset of T^*X which describes the directions of non-propagation for the cohomology of F. For $p \in T^*X$ the category $\mathrm{D}^b(X;p)$ is the localization of $\mathrm{D}^b(X)$ with respect to the null system

$$\{F \in \mathrm{Ob}(\mathrm{D}^b(X))\,;\; p \notin SS(F)\}.$$

Recall that the objects of $\mathrm{D}^b(X;p)$ are the same as those of $\mathrm{D}^b(X)$ and that a morphism $F \to G$ of $\mathrm{D}^b(X)$ becomes an isomorphism in $\mathrm{D}^b(X;p)$ if and only if $p \notin SS(H)$, H being the third term of a distinguished triangle

$$F \to G \to H \xrightarrow{+1}.$$

The bifunctor $\mu\mathrm{hom}(\cdot\,,\cdot)$ is well defined in the category $\mathrm{D}^b(X;p)$, i.e. induces a functor (still denoted by $\mu\mathrm{hom}$) from $\mathrm{D}^b(X;p)^\circ \times \mathrm{D}^b(X;p)$ to the category $\mathrm{D}^b(T^*X;p)$ (where $\mathrm{D}^b(X;p)^\circ$ denotes the opposite category to $\mathrm{D}^b(X;p)$). Recall the relations

$$\mathrm{R}\pi_{X*}\mu\mathrm{hom}(F,G) \cong \mathrm{R}\mathcal{H}om(F,G), \tag{2.1}$$

$$H^0\mu\mathrm{hom}(F,G)_p \cong \mathrm{Hom}_{\mathrm{D}^b(X;p)}(F,G). \tag{2.2}$$

2.2. Functors in localized categories. — The usual functors of sheaf theory are not generally well defined in the microlocal category. Let us consider the two following cases.

(a) Let $f\colon X \to Y$ be a morphism of manifolds and consider the associated correspondence of cotangent bundles

$$T^*X \xleftarrow{\;{}^t f'\;} X \times_Y T^*Y \xrightarrow{\;f_\pi\;} T^*Y.$$

Let $p \in X \times_Y \dot{T}^*Y$ and set $p_X = {}^t f'(p)$, $p_Y = f_\pi(p)$. The functor

$$\mathrm{R}f_!\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

is not well defined in $\mathrm{D}^b(X;p_X)$, in general. The microlocal proper direct image functor

$$f_!^{\mu,p}\colon \mathrm{D}^b(X,p_X) \longrightarrow \mathrm{D}^b(Y;p_Y)^\vee$$

takes value in the category $\mathrm{D}^b(Y;p_Y)^\vee$ of contravariant functors from the category $\mathrm{D}^b(Y;p_Y)$ to the category of sets. (Notice that $\mathrm{D}^b(Y;p_Y)$ is a full subcategory of $\mathrm{D}^b(Y;p_Y)^\vee$.) For $F \in \mathrm{Ob}(\mathrm{D}^b(X;p_X))$, $f_!^{\mu,p}F$ is by definition the pro-object

$$f_!^{\mu,p}F = \text{“}\varprojlim_{F'\to F}\text{”}\, \mathrm{R}f_!F', \tag{2.3}$$

where $F' \to F$ ranges over the category of isomorphisms in $\mathrm{D}^b(X;p_X)$.

(b) Let X, Y, Z be real manifolds and let q_{ij} $(i,j = 1,2,3)$ be the projections from $X \times Y \times Z$ to the corresponding factor (e.g. $q_{13}: X \times Y \times Z \to X \times Z$). The composition functor

$$\begin{aligned} \cdot \circ \cdot : \mathrm{D}^b(X \times Y) \times \mathrm{D}^b(Y \times Z) &\longrightarrow \mathrm{D}^b(X \times Z) \\ (K_1, K_2) &\longmapsto \mathrm{R}q_{13!}\big(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2\big) \end{aligned}$$

has no microlocal meaning, in general. For $p_W \in \dot{T}^*W$ $(W = X,Y,Z)$, the microlocal composition functor

$$\begin{aligned} \cdot \circ_\mu \cdot : \mathrm{D}^b\big(X \times Y;(p_X,p_Y)\big) \times \mathrm{D}^b\big(Y \times Z;(p_Y,p_Z)\big) \\ \longrightarrow \mathrm{D}^b\big(X \times Z;(p_X,p_Z)\big)^\vee \end{aligned}$$

is then introduced as

$$K_1 \circ_\mu K_2 = \text{“}\varprojlim_{\substack{K_1'\to K_1\\ K_2'\to K_2}}\text{”}\; \mathrm{R}q_{13!}\big(q_{12}^{-1}K_1' \otimes q_{23}^{-1}K_2'\big), \tag{2.4}$$

where $K_1' \to K_1$ (resp. $K_2' \to K_2$) ranges over the category of isomorphisms in $\mathrm{D}^b(X \times Y;(p_X,p_Y))$ (resp. $\mathrm{D}^b(Y \times Z;(p_Y,p_Z))$).

2.3. Simple sheaves and contact transformations. — Let $\Lambda \subset \dot{T}^*X$ be a (real C^∞) Lagrangian submanifold and let $p \in \Lambda$. Let F be an element of $\mathrm{Ob}(\mathrm{D}^b(X))$ which satisfies $\mathrm{SS}(F) \subset \Lambda$ at p. The concept of such an F being simple along Λ with shift $d \in \frac{1}{2}\mathbb{Z}$ at p is introduced in [KS3, § 7.5]. Recall for instance that if $M \subset X$ is a submanifold, then A_M is simple along T_M^*X with shift $\frac{1}{2}\operatorname{cod}_X^{\mathbb{R}} M$.

Let $\chi: T^*X \to T^*X$ be a complex contact transformation at p and let $p' = \chi(p)$. Denote by Λ_χ^a the antipodal of its graph,

$$\Lambda_\chi^a = \big\{(q,q') \in T^*(X \times X)\,;\; q' = -\chi(q)\big\},$$

a Lagrangian submanifold of $T^*(X \times X)$.

Recall the notations (1.1), (1.2) and (1.3). In the following we will write for short λ^χ instead of $\chi'(\lambda(p))$.

The next proposition, that collects some results of [KS3, Ch. 7], shows how it is possible to define an action of χ on microlocal categories.

PROPOSITION 2.1. — *Let $d_\chi \in \frac{1}{2}\mathbb{Z}$ be such that*

$$d_\chi \equiv \frac{1}{2} \dim^{\mathbb{R}}\big(\lambda_{0\,X\times X}(p', -p) \cap T_{(p',-p)}\Lambda^a_{\chi^{-1}}\big) \quad \operatorname{mod} \mathbb{Z}.$$

Then:

(i) *there exists a complex $K \in \mathrm{Ob}(\mathrm{D}^b(X \times X))$ simple along $\Lambda^a_{\chi^{-1}}$ at $(p', -p)$ with shift d_χ.*

Moreover, this K enjoys the following properties:

(ii) *for $F \in \mathrm{Ob}(\mathrm{D}^b(X;p))$, the pro-object $K \circ_\mu F$ is representable in $\mathrm{D}^b(X;p')$;*

(iii) *the functor $K \circ_\mu \cdot : \mathrm{D}^b(X;p) \longrightarrow \mathrm{D}^b(X;p')$ is an equivalence of categories;*

(iv) *for $F, G \in \mathrm{Ob}(\mathrm{D}^b(X;p))$, the following isomorphism hold*

$$\chi_* \mu\mathrm{hom}(F,G)_{p'} \cong \mu\mathrm{hom}\big(K \circ_\mu F, K \circ_\mu G\big)_{p'};$$

(v) *if F is simple along Λ at p with shift d, then $K \circ_\mu F$ is simple along $\chi(\Lambda)$ at p' with shift $d + d_\chi - \frac{1}{2}\big[\dim^{\mathbb{R}} X + \tau(\lambda_{0X}, \lambda^\chi, \lambda_{0X}{}^\chi)\big]$.*

3. Levi-simple sheaves

From now on all manifolds and morphisms of manifolds will be complex analytic.

Let X be a complex manifold of dimension n. On the line of [KS1, § 11.2], due to the complex structure of X, it is possible to associate a shift to simple sheaves. More precisely, let

$$d(\lambda, \lambda_{0X}) = -\frac{1}{2}\big[n - \dim^{\mathbb{C}}(\lambda \cap i\lambda) - \operatorname{sgn} L_{\lambda/\lambda_{0X}}\big], \tag{3.1}$$

and notice that by PROPOSITION 1.2

$$\begin{aligned} d(\lambda, \lambda_{0X}) &= -\tfrac{1}{2} \dim^{\mathbb{R}}(\lambda \cap \lambda_{0X}) + \gamma(\lambda, \lambda_{0X}) - s^-(\lambda, \lambda_{0X}) \\ &= -\tfrac{1}{2}\big[2n - \dim^{\mathbb{R}}(\lambda \cap \lambda_{0X})\big] \\ &\qquad + \dim^{\mathbb{C}}(\lambda \cap i\lambda) - \gamma(\lambda, \lambda_{0X}) + s^+(\lambda, \lambda_{0X}). \end{aligned}$$

DEFINITION 3.1. — Let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$, and let $\Lambda \subset \dot{T}^*X$ be an $\mathbb{R}$-Lagrangian manifold of real type at $p \in \Lambda$. We call F *Levi-simple* along Λ at p if F is simple along Λ at p with shift $d(\lambda, \lambda_{0X})$.

The following result is essentially due to [KS1, § 11.2].

PROPOSITION 3.2. — *Let $\Lambda \subset \dot{T}^*X$ be an $\mathbb{R}$-Lagrangian manifold of real type at $p \in \Lambda$. Then*

(i) *there exists $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ Levi-simple along Λ at p.*

Let $\chi: T^*X \to T^*X$ be a complex contact transformation and let $K \in \mathrm{Ob}(\mathrm{D}^b(X \times X))$ be simple along $\Lambda^a_{\chi^{-1}}$ at $(\chi(p), -p)$ with shift n. Then

(ii) *if $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ is Levi-simple along Λ at p, then $K \circ_\mu F$ is Levi-simple along $\chi(\Lambda)$ at $\chi(p)$.*

Proof. — We follow here the same line of the proof of [KS1, Prop. 11.2.8].

Let us first recall that if $\lambda_j \subset E$ $(j = 1,2,3,4)$ are $\mathbb{R}$-Lagrangian subspaces, then

$$(3.2)\quad \tau(\lambda_1,\lambda_2,\lambda_3) - \tau(\lambda_1,\lambda_2,\lambda_4) + \tau(\lambda_1,\lambda_3,\lambda_4) - \tau(\lambda_2,\lambda_3,\lambda_4) = 0,$$

$$(3.3)\qquad \tau(i\lambda_1, i\lambda_2, i\lambda_3) = -\tau(\lambda_1,\lambda_2,\lambda_3).$$

By PROPOSITION 2.1, $K \circ_\mu F \in \mathrm{Ob}(\mathrm{D}^b(X;\chi(p)))$ is simple along $\chi(\Lambda)$ at $\chi(p)$ with shift

$$d(\lambda, \lambda_{0X}) - \tfrac{1}{2}\tau(\lambda_{0X}, \lambda^\chi, \lambda_{0X}{}^\chi).$$

Setting $\lambda_1 = \lambda_{0X}$, $\lambda_2 = \lambda^\chi$, $\lambda_3 = i\lambda^\chi$, $\lambda_4 = \lambda_{0X}{}^\chi$ in (3.2) and using (3.3) we get

$$\begin{aligned}\tau(\lambda_{0X}{}^\chi, \lambda^\chi, \lambda_{0X}) &= \tfrac{1}{2}\tau(\lambda, i\lambda, \lambda_{0X}) - \tfrac{1}{2}\tau(\lambda^\chi, i\lambda^\chi, \lambda_{0X}) \\ &= \operatorname{sgn} L_{\lambda^\chi/\lambda_{0X}} - \operatorname{sgn} L_{\lambda/\lambda_{0X}}.\end{aligned}$$

Noticing that $\dim^{\mathbb{C}}(\lambda \cap i\lambda) = \dim^{\mathbb{C}}(\lambda^\chi \cap i\lambda^\chi)$ we get

$$(3.4)\qquad d(\lambda,\lambda_{0X}) - \tfrac{1}{2}\tau(\lambda_{0X}, \lambda^\chi, \lambda_{0X}{}^\chi) = d(\lambda^\chi, \lambda_{0X}),$$

which proves (ii).

As for (i), since Λ is of real type, we may find a complex contact transformation $\psi: T^*X \to T^*X$ such that $\psi(\Lambda) = T^*_M X$ with $\operatorname{cod}^{\mathbb{R}}_X M = 1$ and $s^-(M, \psi(p)) = 0$. Let $L \in \mathrm{Ob}(\mathrm{D}^b(X \times X))$ be simple along Λ^a_ψ at $(p, -\psi(p))$ with shift n. Since $A_M[-1]$ is Levi-simple along $T^*_M X$ at $\psi(p)$, it follows from (ii) that $L \circ_\mu A_M[-1]$ is Levi-simple along Λ at p. $\square$

COROLLARY 3.3. — *Let $M \subset X$ be a real analytic submanifold. Let $p \in \dot{T}^*_M X$ and assume $ip \notin T^*_M X$ (i.e. $T^*_M X$ is of real type at p). Then F is Levi-simple along $T^*_M X$ at p if and only if*

$$F \cong A_M\big[-\operatorname{cod}^{\mathbb{R}}_X M + \gamma(M,p) - s^-(M,p)\big].$$

In particular, if $\operatorname{cod}^{\mathbb{R}}_X M = 1$ and $s^-(M) = 0$, then $A_M[-1]$ is Levi-simple at p.

We shall need the following technical result.

LEMMA 3.4. — *Let $\rho \subset E$ be a complex ℓ-dimensional isotropic subspace. Then*

$$d(\lambda,\lambda_{0X}) = d(\lambda^\rho,\lambda_{0X}{}^\rho) - d(\lambda,\lambda_{0X},\rho) + \delta(\lambda,\rho) + s^+(\lambda,\rho),$$

where we set

$$\text{(3.5)} \qquad \begin{cases} \delta(\lambda,\rho) = \operatorname{cod}^{\mathbb{C}}_\rho\big(\rho \cap \big(\lambda^\perp + i\lambda^\perp\big)\big), \\ d(\lambda,\lambda_{0X},\rho) = \frac{1}{2}\big[\tau(\lambda_{0X},\widetilde{\lambda^\rho},\lambda) + \ell - \dim^{\mathbb{C}}(\rho \cap \lambda)\big]. \end{cases}$$

For a proof cf. [KS2, Lemma 3.3], [D'AZ2, Prop. 3.5].

As for the notations, if $\rho = (T_pV)^\perp$ for $V \subset T^*X$ an involutive subset, we will write

$$\text{(3.6)} \qquad \begin{cases} \delta(\Lambda, V) = \delta(\lambda, \rho), \\ s^\pm(\Lambda, V) = s^\pm(\lambda, \rho). \end{cases}$$

Moreover, to agree with classical notations, if $M \subset X$ is a real analytic submanifold and $\Lambda = T^*_M X$, we will write

$$\text{(3.7)} \qquad \begin{cases} \delta(M, V) = \delta(\Lambda, V), \\ s^\pm(M, V) = s^\pm(\Lambda, V). \end{cases}$$

4. Positivity

Let $\Lambda \subset \dot{T}^*X$ be an $\mathbb{R}$-Lagrangian manifold at $p \in \Lambda$ and assume Λ is $\mathbb{I}$-symplectic (i.e. $\lambda \cap i\lambda = \{0\}$). Let $\Gamma \subset T^*X$ be a $\mathbb{C}^\times$-conic subset. The concept of positive pair (Λ, Γ) is introduced in [S1] after the work of [MSj1], [MSj2]. This is a notion invariant by complex contact transformations which ensures that if Λ is the set of exterior conormals to a strictly pseudoconvex open subset $\Omega \subset X$, then the projection $\pi_X(\Gamma)$ is contained in $X \setminus \Omega$.

In order to treat the case of real type Lagrangian manifolds (not necessarily $\mathbb{I}$-symplectic) we adopt the following point of view which is motivated by the next LEMMA 4.3 that Pierre SCHAPIRA pointed out to us.

DEFINITION 4.1. — Let Λ_j $(j = 1,2)$ be $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2$, and let F_j be Levi-simple along Λ_j at p. The pair (Λ_1,Λ_2) is called *positive* (and we write $\Lambda_1 \succ \Lambda_2$) if and only if there exists a non zero morphism $F_1 \to F_2$ in $\mathrm{D}^b(X;p)$.

By (2.2) this is equivalent to

$$H^0\mu\mathrm{hom}(F_1,F_2)_p \neq 0. \tag{4.1}$$

Let us notice that, by (iv) of PROPOSITION 2.1, this concept of positivity is stable by complex contact transformations. In other words, if $\chi: T^*X \to T^*X$ is a complex contact transformation, then $\Lambda_1 \succ \Lambda_2$ if and only if $\chi(\Lambda_1) \succ \chi(\Lambda_2)$.

NOTATION 4.2. — Let $M \subset X$ be a real analytic hypersurface. Let $p \in \dot{T}^*_M X$ and set $x = \pi_X(p)$. As a general notation, the point p being implicit in our discussion, we denote by M^+ the germ of closed half space at x with boundary M and inner conormal p (i.e. such that $p \in \mathrm{SS}(A_{M^+})$).

LEMMA 4.3. — *Let $M_1, M_2 \subset X$ be real hypersurfaces. Let p be a point of $\dot{T}^*_{M_1}X \cap \dot{T}^*_{M_2}X$ and set $x = \pi_X(p)$. The following assertions are equivalent:*

(i) $H^0(\mu\mathrm{hom}(A_{M_1^+}, A_{M_2^+}))_p \neq 0$,

(ii) $M_1^+ \supset M_2^+$ *at* x.

Proof. — Set for short $C = \mu\mathrm{hom}(A_{M_1^+}, A_{M_2^+})$, and consider the distinguished triangle

$$(\mathrm{R}\pi_{X!}C)_x \to (\mathrm{R}\pi_{X*}C)_x \to (\mathrm{R}\dot{\pi}_{X*}C)_x \xrightarrow{+1}.$$

One has

$$\begin{aligned}(\mathrm{R}\pi_{X!}C)_x &\cong A_{\mathrm{Int}(M_1^+)} \otimes A_{M_2^+},\\ (\mathrm{R}\pi_{X*}C)_x &\cong \mathrm{R}\Gamma_{M_1^+}(A_{M_2^+})_x,\\ (\mathrm{R}\dot{\pi}_{X*}C)_x &\cong \mu\mathrm{hom}(A_{M_1^+}, A_{M_2^+})_p,\end{aligned}$$

where the first isomorphism follows from [KS3, Prop. 4.4.2], the second from (2.1), and the third from $\dot{\pi}_X^{-1}(x) \cap \mathrm{supp}\, C \subset \mathbb{R}^+p$. Since x is not in $\mathrm{Int}(M_1^+)$, $(\mathrm{R}\pi_{X!}C)_x = 0$ and hence

$$\mu\mathrm{hom}(A_{M_1^+}, A_{M_2^+})_p \cong \mathrm{R}\Gamma_{M_1^+}(A_{M_2^+})_x.$$

One easily concludes. $\square$

Recall the following proposition of [D'AZ4] (where the notion of positivity was not explicit).

PROPOSITION 4.4. — *Let* $\Lambda_j \subset \dot{T}^*X$ *(for* $j = 1{,}2$*) be* $\mathbb{R}$*-Lagrangian manifolds of real type at* $p \in \Lambda_1 \cap \Lambda_2$*. Then* $\Lambda_1 \succ \Lambda_2$ *if and only if there exists a complex contact transformation* $\chi\colon T^*X \to T^*X$ *such that*

$$\begin{cases} \chi(\Lambda_j) = T^*_{M_j}X, \ \textit{with } \operatorname{cod}^{\mathbb{R}}_X M_j = 1,\ s^-(M_j,p') = 0,\ (j = 1{,}2), \\ M_1^+ \supset M_2^+ \ \textit{at } x', \end{cases}$$

where $p' = \chi(p)$, $x' = \pi_X(p')$.

Proof. — It is possible to choose a (complex) Lagrangian plane $\lambda_0 \subset E^\nu$ satisfying:

(i) λ_0 is transversal to λ_j $(j = 1, 2)$,

(ii) $s^-(\lambda_1^\nu, \lambda_0) = 0$.

In fact, the set of λ_0's satisfying (i) is open and dense in the Lagrangian Grassmannian of E^ν, and the set satisfying (ii) is open and non empty. Let $\chi\colon T^*X \to T^*X$ be a complex contact transformation such that $(\lambda_{0X}{}^\chi)^\nu = \lambda_0$. Then

$$\chi(\Lambda_j) = T^*_{M_j}X, \quad \operatorname{cod}^{\mathbb{R}}_X M_j = 1, \quad (j = 1, 2)$$

and $s^-(M_1, p') = 0$. Since $T^*_{M_1}X \succ T^*_{M_1}X$ at p', one has

$$H^0\mu\mathrm{hom}(A_{M_1^+}, A_{M_2^+})_p\big[-s^-(M_2, p')\big] \neq 0,$$

and, by the proof of LEMMA 4.3, this is equivalent to

$$H^{-s^-(M_2,p')}\mathrm{R}\Gamma_{M_1^+}(A_{M_2^+})_x \neq 0.$$

The functor $\mathrm{R}\Gamma_{M_1^+}(\,\cdot\,)$ being left exact, it follows that $s^-(M_2, p') = 0$. Moreover, the relation $\Gamma_{M_1^+}(A_{M_2^+})_x \neq 0$ implies that $M_1^+ \supset M_2^+$ at x'. □

REMARK 4.5.

(i) Using PROPOSITION 4.4 it is easy to prove that $\succ$ is a preorder relation. In other words, if Λ_j $(j = 1, 2, 3)$ are $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ satisfying $\Lambda_1 \succ \Lambda_2$, $\Lambda_2 \succ \Lambda_3$, then $\Lambda_1 \succ \Lambda_3$.

(ii) Again by PROPOSITION 4.4 it follows that if $\Lambda_1 \succ \Lambda_2$, then $\lambda_1 \cap i\lambda_1 \subset \lambda_2 \cap i\lambda_2$.

PROPOSITION 4.6. — *Let Λ_j $(j = 1,2)$ be $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2$, and let F_j be Levi-simple along Λ_j at p. Then $\Lambda_1 \succ \Lambda_2$ if and only if $\mu\mathrm{hom}(F_1,F_2)_p = A$. In particular, taking the zero-th cohomology we get*

$$\mathrm{Hom}_{\mathrm{D}^b(X;p)}(F_1,F_2) \cong A. \tag{4.2}$$

Proof. — Notice that the statement is invariant by complex contact transformations. We may then assume by PROPOSITION 4.4 that $\Lambda_j = T^*_{M_j}X$ for $\mathrm{cod}^{\mathbb{R}}_X M_j = 1$, $s^-(M_j) = 0$ $(j = 1,2)$ and $M_1^+ \supset M_2^+$. In this case $\mathrm{R}\Gamma_{M_1^+}(A_{M_2^+})_x = A$, and one concludes by noticing that $\mu\mathrm{hom}(A_{M_1}, A_{M_2})_p \cong \mathrm{R}\Gamma_{M_1^+}(A_{M_2^+})_x$ (cf. the proof of LEMMA 4.3). ☐

Let us call «restriction morphism» the morphism which corresponds to $1 \in A$ via (4.2). Notice that if Λ_i $(i = 1,2)$ are the exterior conormals to some strictly pseudoconvex open subsets $\Omega_i \subset X$, the restriction morphism is nothing but the morphism

$$A_{X\setminus\Omega_1}[-1] \longrightarrow A_{X\setminus\Omega_2}[-1]$$

(or, equivalently, $A_{\Omega_1} \to A_{\Omega_2}$) induced by the inclusion $\Omega_1 \subset \Omega_2$.

REMARK 4.7.

(i) Let Λ_j $(j = 1,2)$ be $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2$, and assume Λ_1 is $\mathbb{I}$-symplectic (i.e. $\lambda_1 \cap i\lambda_1 = \{0\}$). If the pair (Λ_1, Λ_2) is positive in the sense of [S] and $d(\lambda_1, \lambda_{0X}) = d(\lambda_2, \lambda_{0X})$, then $\Lambda_1 \succ \Lambda_2$. The converse implication is not clear to us.

(ii) Let $S \subset M \subset X$ be real analytic submanifolds. Then $T^*_M X \succ T^*_S X$ at $p \in S \times_M \dot{T}^*_M X$ if and only if $d(\lambda_M, \lambda_{0X}) = d(\lambda_S, \lambda_{0X})$. In fact one easily checks that in this case $\mu\mathrm{hom}(A_M, A_S) \cong A_{S \times_M T^*_M X}$.

5. Generalized microfunctions

Let X be a complex analytic manifold of dimension n, and let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on X.

DEFINITION 5.1. — Let $\Lambda \subset T^*X$ be an $\mathbb{R}$-Lagrangian manifold of real type at $p \in \dot{T}^*X$, and let F be Levi-simple along Λ at p. The *complex of microfunctions* along Λ is the object of $\mathrm{D}^b(T^*X;p)$ defined by:

$$\mathcal{C}_\Lambda = \mu\mathrm{hom}(F,\mathcal{O}_X). \tag{5.1}$$

To be rigorous, we should incorporate F and p in the notation of $\mathcal{C}_\Lambda$. We will not be that precise since, in any case, we will restrict our study to a neighborhood of p.

REMARK 5.2. — Let $M \subset X$ be a submanifold, $p \in \dot{T}^*_M X$, $ip \notin T^*_M X$. By COROLLARY 3.3 one has

$$(\mathcal{C}_{T^*_M X})_p \cong \mu\mathrm{hom}(A_M, \mathcal{O}_X)_p\big[\mathrm{cod}^{\mathbb{R}}_X M - \gamma(M,p) + s^-(M,p)\big].$$

If M is a real analytic manifold and X is a complexification of M, we recover the sheaf of Sato microfunctions up to the orientation sheaf $\mathrm{or}_{M/X}$ (which is locally isomorphic to the constant sheaf A_M).

PROPOSITION 5.3 (see [KS3, Ch. 11]). — *Let $\chi : T^*X \to T^*X$ be a complex contact transformation. Then, by choosing a quantization of χ, the following isomorphism holds*

$$\chi_*(\mathcal{C}_\Lambda)_{\chi(p)} \cong \big(\mathcal{C}_{\chi(\Lambda)}\big)_p.$$

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. The complex $\mathcal{C}_\Lambda$ is well defined in the derived category of $\pi_X^{-1}\mathcal{D}_X$-modules. (To see this, just use the left $\mathcal{D}_X$-module structure of the sheaf $\mathcal{O}_X$.)

Let $\mathcal{E}_X$ denote the sheaf of finite order microdifferential operators on T^*X. Let $\Omega \subset X$ be a strictly pseudoconvex subset with real analytic boundary M, and denote by Λ the set of exterior conormals to Ω. In this case, it is known (cf [K]) that $\mathcal{C}_\Lambda$ is well defined in the derived category of $\mathcal{E}_X$-modules near $p \in \Lambda$, but we do not know if this result still holds for general $\mathbb{R}$-Lagrangian manifolds. However, we have a partial result.

PROPOSITION 5.4. — *Let $M \subset X$ be a (real analytic) generic submanifold of X. Then the complex $\mathcal{C}_{T^*_M X}$ is well defined in the category of left $\mathcal{E}_X$-modules*

Proof. — Consider the diagram

$$\begin{array}{ccc} M & \longrightarrow & X \\ \big\downarrow & \nearrow_{\phi} & \\ Z & & \end{array}$$

where Z is a complexification of M, and ϕ is the induced complex analytic smooth map. The cotangent map ${}^t\phi'$ to ϕ gives an identification between $T^*_M X$ and $(Z \times_X T^*X) \cap T^*_M Z$. Moreover, one has the isomorphism

$$(5.2) \qquad {}^t\phi'_* \mathcal{C}_{T^*_M X} \cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Z}\big(\mathcal{E}_{Z\to X}, \mathcal{C}_{M/Z}\big),$$

where $\mathcal{C}_{M/Z} \cong \mathcal{C}_{T^*_M Z}$ are the classical Sato microfunctions on M, and where $\mathcal{E}_{Z\to X}$ denotes the transfer module. The action of $\mathcal{E}_X$ on $\mathcal{C}_{T^*_M X}$ is then induced by the natural right $\phi_\pi^{-1}\mathcal{E}_X$-module structure of $\mathcal{E}_{Z\to X}$, via the isomorphism (5.2). $\square$

PROPOSITION 5.5. — *The complex $\mathcal{C}_{\Lambda,p}$ is concentrated in degree ≥ 0, i.e. $(H^j\mathcal{C}_\Lambda)_p = 0$ for every $j < 0$.*

Proof. — Let $\chi: T^*X \to T^*X$ be a complex contact transformation at p as in PROPOSITION 4.4 such that $\chi(\Lambda) = T^*_M X$, with $\operatorname{cod}^{\mathbb{R}}_X M = 1$ and $s^-(M,p') = 0$ for $p' = \chi(p)$. By quantizing χ, we get

$$\begin{aligned}\chi_*(\mathcal{C}_\Lambda)_{p'} &\cong \mu\mathrm{hom}(A_M, \mathcal{O}_X)_{p'}[1]\\ &\cong \mathrm{R}\Gamma_{M^+}(\mathcal{O}_X)_{x'}[1],\end{aligned}$$

where $x' = \pi_X(p')$. The claim follows since the complex $\mathrm{R}\Gamma_{M^+}(\mathcal{O}_X)_{x'}$ is in degree ≥ 1 by analytic continuation. $\square$

THEOREM 5.6. — *Let Λ_j $(j = 1,2)$ be $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2$, and let F_j be Levi-simple along Λ_j at p. Assume $\Lambda_1 \succ \Lambda_2$. Then the natural morphism*

$$H^0\mathcal{C}_{\Lambda_2,p} \longrightarrow H^0\mathcal{C}_{\Lambda_1,p} \tag{5.3}$$

induced by the restriction morphism is injective.

Proof. — By the contact transformation χ at p as in PROPOSITION 4.4, the morphism (5.3) reduces to

$$\mathcal{H}^1_{M_2^+}(\mathcal{O}_X)_{x'} \longrightarrow \mathcal{H}^1_{M_1^+}(\mathcal{O}_X)_{x'},$$

which is injective by analytic continuation (here $x' = \pi_X(\chi(p))$). $\square$

6. Microlocal direct images of simple sheaves

Let $f: X \to Y$ be a smooth map of real analytic manifolds and consider the associated maps

$$T^*X \xleftarrow{\ {}^tf'\ } X \times_Y T^*Y \xrightarrow{\ f_\pi\ } T^*Y.$$

Set $V = X \times_Y \dot{T}^*Y$, and let $\Lambda \subset T^*X$ be a (real) Lagrangian manifold at $p \in \Lambda \cap V$. If the intersection $\Lambda \cap V$ is clean at p (which means that $V \cap \Lambda$ is smooth and $T_p(V \cap \Lambda) = T_pV \cap T_p\Lambda$ for every $p \in V \cap \Lambda$), it is known (cf. e.g. [KS2]) that, for a sufficiently small open neighborhood $U \subset T^*X$ of p,

$$f_\pi {}^tf'^{-1}(\Lambda \cap U) = f_\pi(V \cap \Lambda \cap U)$$

is a Lagrangian manifold in T^*Y.

Via the functor $f_!^{\mu,p}$, it is possible to establish a similar correspondence between simple sheaves in X along Λ and simple sheaves in Y along $f_\pi {}^tf'^{-1}(\Lambda \cap U)$. To explain this correspondence, let us first reduce by contact transformation, to the case where Λ and $f_\pi {}^tf'^{-1}(\Lambda \cap U)$ are conormal bundles to hypersurfaces. In this frame, one has the following proposition that collects some results of [KS3, § 7.4].

Proposition 6.1. — *Let $f:X \to Y$ be a smooth morphism of real analytic manifolds. Let $M \subset X$, $N \subset Y$ be real analytic hypersurfaces. Let $p \in \dot{T}^*_M X \cap V$, $x = \pi_X(p)$, $q = f_\pi(p)$, $y = \pi_Y(q)$. Assume*

(i) *the intersection $T^*_M X \cap V$ is clean at p,*

(ii) *for an open neighborhood U of p, $f_\pi {}^t f'^{-1}(T^*_M X \cap U) = T^*_N Y$ at q.*

Then:

(a) *there exist local systems of coordinates $(x) = (y,t)$ on X and (y) on Y, at x and y respectively, such that:*

$$p = (0;\mathrm{d}y_1), \quad f(y,t) = y,$$
$$N = \{y \in Y; y_1 = 0\}, \quad M = \{x \in X; y_1 = Q(t)\},$$

where $Q(t)$ is a quadratic form;

(b) *there exists a morphism in the category of pro-objects of* $\mathrm{D}^b(Y)$:

$$\text{“}\varprojlim_{U \ni x}\text{''}\ \mathrm{R}f_! A_{M^+ \cap U} \longrightarrow A_{N^+}[-\delta],$$

where U ranges over the category of open neighborhoods of x and, with the notation (3.5),

$$\begin{aligned} \text{(6.1)} \qquad \delta &= \#\{\text{non positive eigenvalues of } Q(t)\} \\ &= \tfrac{1}{2}\left(\dim^{\mathbb{R}} X - \dim^{\mathbb{R}} Y\right) - d\left(\lambda_M, \lambda_{0X}, \rho\right), \end{aligned}$$

(c) $f_!^{\mu,p} A_{M^+} \cong \text{“}\varprojlim_{U \ni x}\text{''}\mathrm{R}f_! A_{M^+ \cap U}$ *exists in* $\mathrm{D}^b(Y;y)$ *and the morphism in* (b) *is an isomorphism in* $\mathrm{D}^b(Y;y)$.

Remark 6.2. — The previous proposition is due to [KS3, § 7.4], in which (c) reads

(c)′ $f_!^{\mu,p} A_{M^+} \cong \text{“}\varprojlim_{U \ni x}\text{''}\mathrm{R}f_! A_{M^+ \cap U}$ *exists in* $\mathrm{D}^b(Y;q)$ *and the morphism in* (b) *is an isomorphism in* $\mathrm{D}^b(Y;q)$.

Since M^+ and N^+ are closed half-spaces, it is easy to prove that (c)′ is equivalent to (c).

Assume now that $f:X \to Y$ is a morphism in the category of complex analytic manifolds. We will need the following result:

LEMMA 6.3. — *Let $f\colon X \to Y$ be a smooth map of complex analytic manifolds. Under the same hypotheses as in Proposition* 6.1, *one has*:

$$s^{\pm}(N,q) \le s^{\pm}(M,p).$$

Proof. — Let $L = \pi_X(T^*_M X \cap V)$ and consider the diagram

$$\begin{tikzcd} T^*_M X & T^*_M X \cap V \arrow[l] \arrow[r, "f_\pi|T^*_M X\cap V"] \arrow[d] & T^*_N Y \arrow[d] \\ M \arrow[from=u] & L \arrow[l] \arrow[r, "f|L"] & N \end{tikzcd}$$

One proves as in [KS2] that $L \subset X$ is a smooth submanifold, and $f_{|L}$ is a smooth map. Let ϕ be a real analytic function on Y, vanishing on N, and such that $\mathrm{d}\phi = q$. It follows that $\phi \circ f$ is a real analytic function on X vanishing on L and such that $\mathrm{d}(\phi \circ f)(x) \in T^*_L X \cap V = T^*_M X \cap V$. Hence $\mathrm{d}(\phi \circ f)(x) = p$.

Since f is holomorphic, $f^*(\partial\overline{\partial}\phi) = \partial\overline{\partial}(\phi \circ f)$. Moreover, $f'(T^{\mathbb{C}}_x L) = T^{\mathbb{C}}_y N$ and hence $\partial\overline{\partial}(\phi \circ f)_{|T^{\mathbb{C}}_x L} \sim \partial\overline{\partial}\phi_{|T^{\mathbb{C}}_y N}$. By LEMMA 1.5 this implies

$$s^{\pm}(L,p) = s^{\pm}(N,q).$$

Finally, since $L \subset M$ and $p \in L \times_M \dot{T}^*_M X$, one has (cf. [DAZ1])

$$s^{\pm}(L,p) \le s^{\pm}(M,p). \quad \square$$

7. Faithfulness of the microlocal direct image functor

Let $f\colon X \to Y$ be a smooth map of complex analytic manifolds and let $n = \dim^{\mathbb{C}} X$, $n - \ell = \dim^{\mathbb{C}} Y$. The manifold $V = X \times_Y \dot{T}^*Y$ is then identified to an l-codimensional involutive submanifold of T^*X.

Let $\Lambda \subset \dot{T}^*X$ be an $\mathbb{R}$-Lagrangian manifold of real type at $p \in \Lambda \cap V$ and notice that if the intersection $\Lambda \cap V$ is clean at p, ${}^t f' f_\pi^{-1}(\Lambda \cap U)$ is again an $\mathbb{R}$-Lagrangian manifold of real type at $q = f_\pi(p)$ in T^*Y, for a sufficiently small open neighborhood $U \subset T^*X$ of p. We will use notations (1.1), (1.2), (1.3), (3.5), (3.6).

THEOREM 7.1. — *Let Λ_j $(j = 1,2)$ be germs of $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2 \cap V$. Assume the following*:

(a) *the intersection $\Lambda_j \cap V$ is clean at p*;

(b) $\Lambda_1 \succ \Lambda_2$ *at p*;

(c) $s^-(\Lambda_1,V) + \delta(\Lambda_1,V) = s^-(\Lambda_2,V) + \delta(\Lambda_2,V)$.

Let F_j $(j=1,2)$ be Levi-simple along Λ_j at p. Then

(i) *the natural morphism*

$$f_!^{\mu,p}\colon \operatorname{Hom}_{\mathrm{D}^b(X;p)}(F_1,F_2) \longrightarrow \operatorname{Hom}_{\mathrm{D}^b(Y;q)}\big(f_!^{\mu,p}F_1, f_!^{\mu,p}F_2\big) \tag{7.1}$$

is an isomorphism;

(ii) ${}^t f' f_\pi^{-1}(\Lambda_1\cap U) \succ {}^t f' f_\pi^{-1}(\Lambda_2 \cap U)$.

It is also possible to state a slightly more general (and more symmetrical) version of THEOREM 7.1 as follows.

Let X be a complex analytic manifold of dimension n. Let $V\subset \dot{T}^*X$ be a germ at p of ℓ-codimensional complex analytic regular involutive submanifold. Let $j\colon V\to T^*X$ be the embedding and let $b\colon V\to [V]$ be the (locally defined) projection along the bicharacteristic leaves of V. The quotient space $[V]$ inherits from T^*X a structure of complex homogeneous symplectic manifold. We may then choose an identification of $[V]$ near $b(p)$ to an open subset of $\dot{T}^*Y$, where Y is a complex manifold of dimension $(n-\ell)$.

If $\Lambda\subset \dot{T}^*X$ is a germ of $\mathbb{R}$-Lagrangian manifold of real type at $p\in V\cap\Lambda$ such that the intersection $V\cap\Lambda$ is clean, it is easy to prove that $b(V\cap\Lambda)\subset T^*Y$ is a germ of $\mathbb{R}$-Lagrangian manifold of real type at $q=b(p)$.

In order to rephrase the correspondence

$$\Lambda \longmapsto b(V\cap\Lambda) \tag{7.2}$$

in terms of simple sheaves, let us rewrite (7.2) as the correspondence associated to the Lagrangian manifold $\Gamma := T^*Y\times_{T^*Y} V^a \subset T^*(Y\times X)$ (here a denotes the antipodal map),

$$\begin{tikzcd}[column sep=small] & \Gamma\subset T^*(Y\times X) \arrow[dl, "p_1|\Gamma"'] \arrow[dr, "p_2^a|\Gamma"] & \\ T^*Y & & T^*X \end{tikzcd} \tag{7.3}$$

i.e. $b(V\cap\Lambda)=p_1(\Gamma\cap p_2^{a\,-1}\Lambda) =: \Gamma\circ\Lambda$, where p_1, p_2 are the projections from $T^*(Y\times X)$ to the corresponding factor.

THEOREM 7.2. — *Let Λ_j $(j=1,2)$ be germs of $\mathbb{R}$-Lagrangian manifolds of real type at $p\in\Lambda_1\cap\Lambda_2\cap V$. Assume the following:*

(a) *the intersection $\Lambda_j\cap V$ is clean at p;*

(b) $\Lambda_1 \succ \Lambda_2$ *at* p;

(c) $s^-(\Lambda_1,V) + \delta(\Lambda_1,V) = s^-(\Lambda_2,V) + \delta(\Lambda_2,V)$.

Let F_j $(j = 1,2)$ *be Levi-simple along* Λ_j *at* p *and let* K *be simple along* Γ *at* (q,p) *with shift* $(n-\ell)$. *Then*

(o) *the pro-object* $K \circ_\mu F_j$ *is representable in* $\mathrm{D}^b(Y;q)$, *and*

$$K \circ_\mu F_j\big[s^-(\Lambda_j,V) + \ell - \delta(\lambda_j,\rho)\big]$$

is Levi-simple along $\Gamma \circ \Lambda_j$ *at* p;

(i) *the natural morphism*

$$K \circ_\mu : \mathrm{Hom}_{\mathrm{D}^b(X;p)}(F_1,F_2) \longrightarrow \mathrm{Hom}_{\mathrm{D}^b(Y;q)}\big(K \circ_\mu F_1, K \circ_\mu F_2\big) \tag{7.4}$$

is an isomorphism;

(ii) $\Gamma \circ \Lambda_1 \succ \Gamma \circ \Lambda_2$.

The proof will be performed by reducing to the case where the maps $T^*X \xleftarrow{\ i\ } V \xrightarrow{\ b\ } T^*Y$ are induced by a smooth map $f\colon X \to Y$, thus entering the hypotheses of THEOREM 7.1.

8. Proof of the faithfulness

Reduction of Theorem 7.2. — In the framework of this Theorem, let $\chi_X\colon T^*X \to T^*X$ be a contact transformation such that

$$\left\{\begin{array}{ll} \chi_X(\Lambda_1) = T^*_{M_1}X, & \operatorname{cod}^{\mathbb{R}}_X M_1 = 1, \quad s^-(M_1,p') = 0, \\ \chi_X(\Lambda_2) = T^*_{M_2}X, & \operatorname{cod}^{\mathbb{R}}_X M_2 = 1, \\ \chi_X(V) = X \times_Y T^*Y, & \text{for a smooth map} \quad f\colon X \to Y, \\ f_\pi {}^tf'^{-1}(T^*_{M_1}X) = T^*_{N_1}Y, & \operatorname{cod}^{\mathbb{R}}_Y N_1 = 1, \\ f_\pi {}^tf'^{-1}(T^*_{M_2}X) = T^*_{N_2}Y, & \operatorname{cod}^{\mathbb{R}}_Y N_2 = 1 \end{array}\right. \tag{8.1}$$

(where we set $p' = \chi_X(p)$), and let $\chi_Y\colon T^*Y \to T^*Y$ be the complex contact transformation induced by χ_X via the diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{\quad i \quad} & V & \xrightarrow{\quad p \quad} & T^*Y \\ \Big\downarrow{\scriptstyle \chi_X} & & \Big\downarrow{\scriptstyle \chi_X} & & \Big\downarrow{\scriptstyle \chi_Y} \\ T^*X & \xleftarrow{\quad {}^tf' \quad} & X \times_Y T^*Y & \xrightarrow{\quad f_\pi \quad} & T^*Y, \end{array}$$

i.e. $\chi_Y(\widetilde{q}) = f_\pi(\chi_X(\widetilde{p}))$ for $\widetilde{p}$ such that $\widetilde{q} = b(\widetilde{p})$ (the definition is independent of the choice of $\widetilde{p}$ due to the third equality in (8.1)).

Let $q = b(p)$, $q' = f_\pi(p') = \chi_Y(q)$. By (b), $T^*_{M_1}X \succ T^*_{M_2}X$ at p', and hence, by PROPOSITION 4.4, $s^-(M_2, p') = 0$ and $M_1^+ \supset M_2^+$. By LEMMA 6.3, $s^-(N_1, q') = s^-(N_2, q') = 0$. We are then in the following frame:

$$(8.1)' \quad \begin{cases} \chi_X(\Lambda_j) = T^*_{M_j}X, & \operatorname{cod}^{\mathbb{R}}_X M_j = 1, \quad s^-(M_j, p') = 0, \\ M_1^+ \supset M_2^+, & \\ \chi_X(V) = X \times_Y T^*Y, & \text{for a smooth map } \ f\colon X \to Y, \\ f_\pi {}^t f'^{-1}(T^*_{M_j}X) = T^*_{N_j}Y, & \operatorname{cod}^{\mathbb{R}}_Y N_j = 1, \quad s^-(N_j, q') = 0, \end{cases}$$

Notice that, by construction

$$(8.2) \qquad \Gamma \circ \Lambda_j = \chi_Y^{-1}\big(f_\pi {}^t f'^{-1}(\chi_X(\Lambda_j))\big).$$

Let K_X be simple along $\Lambda^a_{\chi_X^{-1}}$ at $(p', -p)$ with shift n, and let K^*_Y be simple along $\Lambda^a_{\chi_Y}$ at $(q, -q')$ with shift $(n - \ell)$.

In terms of simple sheaves, the equality (8.2) reads

$$(8.3) \qquad K \circ_\mu F_j \cong K^*_Y \circ_\mu \big(f_!^{\mu,p}(K_X \circ_\mu F_j)\big)$$

(recall that $K^*_Y \circ_\mu \cdot$ and $K^*_X \circ_\mu \cdot$ are equivalences of categories) and one has

$$\begin{aligned} K^*_Y \circ_\mu \big(f_!^{\mu,p}(K_X \circ_\mu F_j)\big) &\cong K^*_Y \circ_\mu \big(f_!^{\mu,p}(A_{M_j}[-1])\big) \\ &\cong K^*_Y \circ_\mu A_{N_j}\big[-1 - l - d(\lambda_{M_j}, \lambda_{0X}, \rho)\big], \end{aligned}$$

where the first isomorphism follows from PROPOSITION 3.2 (ii), and the second from PROPOSITION 6.1. By LEMMA 3.4 we have

$$d\big(\lambda_{M_j}, \lambda_{0X}, \rho\big) = \delta(\lambda_{M_j}, \rho) - s^-(M_j, V)$$

and hence $K \circ_\mu F_j[s^-(M_j, V) + \ell - \delta(\lambda_{M_j}, \rho)]$ is Levi-simple. Since the numbers appearing in the above shift are stable by contact transformation, (o) follows.

In order to prove (i) and (ii) let us note that we are now in the hypotheses of THEOREM 7.1, that may then be applied. $\square$

Proof of Theorem 7.1. — Using the same techniques as in the previous reduction, by applying a complex contact transformation we may assume from the beginning to be in the following frame

$$\begin{cases} \Lambda_j = T^*_{M_j}X, & \operatorname{cod}^{\mathbb{R}}_X M_j = 1, \quad s^-(M_j, p') = 0, \\ M_1^+ \supset M_2^+, & \\ f_\pi {}^t f'^{-1}(T^*_{M_j}X) = T^*_{N_j}Y, & \operatorname{cod}^{\mathbb{R}}_Y N_j = 1, \quad s^-(N_j, q') = 0, \end{cases}$$

and we are then left to prove the following claims:

(i)′ *the natural morphism*

$$f_!^{\mu,p}\colon \operatorname{Hom}_{\mathrm{D}^b(X;p)}(A_{M_1}, A_{M_2}) \longrightarrow \operatorname{Hom}_{\mathrm{D}^b(Y;q)}\big(f_!^{\mu,p}A_{M_1}, f_!^{\mu,p}A_{M_2}\big)$$

is an isomorphism;

(ii)′ $\operatorname{Hom}_{\mathrm{D}^b(Y;q)}(A_{N_1}, A_{N_2}) \neq 0$.

We recall that, by PROPOSITION 6.1,

$$f_!^{\mu,p}A_{M_j} \cong A_{N_j}\big[s^-(M_j,V) - l + \delta(\lambda_{M_j},\rho)\big]. \tag{8.4}$$

By hypothesis (c), we also know that the shifts appearing in (8.4) for $j = 1, 2$ are equal, and hence (ii)′ will follow from (i)′.

Before proving (i)′, let us state some preliminary results.

9. An analytic lemma

The authors wish to thank Jean-Pierre SCHNEIDERS for useful discussions related to the contents of this section.

Let $T \subset \mathbb{R}^m$ be a germ of open neighborhood of 0 and let h be a real analytic function on T. We say that h is C^ω*-quadratic* if, for a system of real analytic coordinates $(t) = (u, v, w) \in T$, $h(t) = u^2 - v^2$.

LEMMA 9.1. — *Let h_j $(j = 1,2)$ be C^ω-quadratic in T. Let s^+ (resp. s^-) be the number of positive (resp. negative) eigenvalues of* $\operatorname{Hess} h_1$. *Assume*

(i) $h_1 \geq h_2$;

(ii) $\operatorname{Hess} h_2$ *has s^+ positive eigenvalues.*

Then, there exists a local system of coordinates $(t) = (u,v,w) \in T$, with $(u) = (u_1, \ldots, u_{s^+})$, $(v) = (v_1, \ldots, v_{s^-})$, such that at 0:

$$h_1(t) = Q_1(u) - v^2 + O(u)O(v) + O\big(|u|^3\big) + O\big(|v|^3\big), \tag{9.1}$$

$$\begin{aligned} h_2(t) = u^2 - \big[v^2 + Q_2(v,w)\big] - R_2(v,w) + O(u)O(v,w) \\ + O\big(|u|^3\big) + O\big(|v|^3\big), \end{aligned} \tag{9.2}$$

where Q_1, Q_2 are quadratic forms with

$$Q_1 > 0, \quad Q_2 \geq 0, \quad R_2(v,w) = O(w)O(v,w) \geq 0.$$

Proof. — Since h_1 is C^ω-quadratic, we can choose coordinates

$$(\tilde{t}\,) = (\tilde{u}, \tilde{v}, \widetilde{w}) \in T, \quad (\tilde{u}) = (\tilde{u}_1, \dots, \tilde{u}_{s^+}), \quad (\tilde{v}) = (\tilde{v}_1, \dots, \tilde{v}_{s^-}),$$

such that $h_1(\tilde{t}\,) = \tilde{u}^2 - \tilde{v}^2$. Write

$$\operatorname{Hess} h_2 = Q_3(\tilde{u}, \tilde{v}) + Q_4(\tilde{u}, \tilde{v}, \widetilde{w}) + Q_5(\widetilde{w}),$$

where Q_3, Q_4 and Q_5 are quadratic forms and $Q_4(\tilde{u}, \tilde{v}, \widetilde{w}) = O(\tilde{u}, \tilde{v}) O(\widetilde{w})$. The hypothesis (i) and the independence of h_1 from $\widetilde{w}$ imply that $Q_4(\tilde{u}, \tilde{v}, \widetilde{w}) = 0$ and $Q_5(\widetilde{w}) \leq 0$.

By (ii) and the inequality $h_{1|\tilde{u}=\widetilde{w}=0} < 0$ it follows that there exists a change of coordinates

$$\tilde{u} = \tilde{u}(u), \quad \tilde{v} = \tilde{v}(u, v), \quad \widetilde{w} = \widetilde{w}(w), \tag{9.3}$$

such that

$$h_{2|(v=w=0)} = u^2.$$

On the other hand

$$h_{1|(u=0)} = -Q_6(v) + O(|v|^3),$$

where $Q_6 > 0$ is a quadratic form. It is not restrictive to assume $Q_6(v) = v^2$, and so, by (i),

$$h_{2|(u=0)} = -\big[v^2 + Q_7(v, w)\big] - R(v, w) + O(|v|^3),$$

where $Q_7 \geq 0$ is a quadratic form and $0 \leq R(v, w) = O(w) O(v, w)$. The claim easily follows. $\square$

For h_j as above and $\delta, \varepsilon > 0$, let

$$Z_j = \{t \in T\,;\ h_j(t) \leq 0\}$$

$$B_{\delta,\varepsilon} = \{t \in T\,;\ |u| < \varepsilon\ \ |v, w| < \delta\varepsilon\}.$$

PROPOSITION 9.2. — *Let h_2 and h_1 be as in Lemma* 9.1, *and set $d = m - s^+$. Then there exists $\delta > 0$ such that for every $0 < \varepsilon \ll 1$ one has*

(i) $\mathrm{R}\Gamma_c(Z_j \cap B_{\delta,\varepsilon}; A_T) \cong A[d]$;

(ii) *the morphism*

$$\mathrm{R}\Gamma_c(Z_1 \cap B_{\delta,\varepsilon}; A_T) \longrightarrow \mathrm{R}\Gamma_c(Z_2 \cap B_{\delta,\varepsilon}; A_T)$$

induced by the restriction morphism $A_{Z_1} \to A_{Z_2}$ $(Z_1 \supset Z_2)$ *is an isomorphism.*

Denote by $a_T : T \to \{0\}$ the projection. Recall that one has

$$\mathrm{R}\Gamma_c\big(Z_j \cap B_{\delta,\varepsilon}; A_T\big) \cong \mathrm{R}a_{T!}A_{Z_j \cap B_{\delta,\varepsilon}}. \tag{9.4}$$

Proof. — By LEMMA 9.1, we may find a local system of coordinates $(t) = (u, v, w)$ on T such that $h_1(t)$ and $h_2(t)$ are given by (9.1) and (9.2). It follows that, for some $0 < \delta < 1$, the small roots of the equations $h_1(u, v, w) = 0$ and $h_2(u, v, w) = 0$ satisfy

$$\delta|u| \leq |v, w|.$$

Consider the projections

$$T \xrightarrow{q} V \xrightarrow{a_V} \{0\},$$

where q is the restriction to T of the projection $\mathbb{R}^m_{u,v,w} \to \mathbb{R}^d_{v,w}$, and $V = q(T)$ (notice that it is not restrictive to assume $T = B_{a,b}$ for some $a, b > 0$). If we prove that for $0 < \varepsilon \ll 1$, $|v, w| < \varepsilon$, the set $q^{-1}(v, w) \cap Z_j \cap B_{\delta,\varepsilon}$ is (non-empty) compact and contractible, then

$$\mathrm{R}q_!A_{Z_j \cap B_{\delta,\varepsilon}} \cong A_{B'_{\delta,\varepsilon}},$$

where

$$B'_{\delta,\varepsilon} = q(B_{\delta,\varepsilon}) = \big\{(v, w)\, ;\ |v, w| < \delta\varepsilon\big\}.$$

Since $\mathrm{R}a_{T!} = \mathrm{R}a_{V!} \circ \mathrm{R}q_!$ and $\mathrm{R}a_{V!}A_{B'_{\delta,\varepsilon}} \cong A[d]$, by (9.4) the claim (i) will follow.

Let us prove that $q^{-1}(v, w) \cap Z_j \cap B_{\delta,\varepsilon}$ is compact and contractible. For $u_0 \in \mathbb{R}^{s^+}_u$, $\lambda > 0$, let $u = \lambda u_0$. Considering for example $j = 1$, we have $h_1(\lambda u_0, v) = 0$ if and only if

$$\big[1 + O(\lambda)\big]\lambda^2 + a\lambda - \big[v^2 + O(|v|^3)\big] = 0$$

which admits two small roots λ, since the discriminant is positive. This proves (i).

Set $C^j_{(v,w)} = q^{-1}(v, w) \cap Z_j \cap B_{\delta,\varepsilon}$ and notice that

$$\big(\mathrm{R}q_!A_{Z_j \cap B_{\delta,\varepsilon}}\big)_{(v,w)} = \mathrm{R}\Gamma_c\big(C^j_{(v,w)}; \mathbb{R}^{m-d}_u\big).$$

By the previous argument, in order to prove (ii) it is enough to prove that the morphism

$$\mathrm{R}\Gamma_c\big(C^1_{(v,w)}; \mathbb{R}^{m-d}_u\big) \longrightarrow \mathrm{R}\Gamma_c\big(C^2_{(v,w)}; \mathbb{R}^{m-d}_u\big)$$

induced by the restriction morphism $A_{C^1_{(v,w)}} \to A_{C^2_{(v,w)}}$ is an isomorphism for every $(v, w) \in B'_{\delta,\varepsilon}$. This is the case since since $C^1_{(v,w)}$, $C^2_{(v,w)}$ are contractible to a common point. $\square$

10. End of proof of Theorem 7.1

In order to end the proof of THEOREM 7.1 we have to show that the morphism

$$\text{“}\varprojlim_{U\ni x}\text{”}\, \mathrm{R}f_! A_{M_1^+\cap U} \longrightarrow \text{“}\varprojlim_{U\ni x}\text{”}\, \mathrm{R}f_! A_{M_2^+\cap U}$$

induced by $A_{M_1^+} \to A_{M_2^+}$ is non zero.

By PROPOSITION 6.1, the above inverse limits exist in $\mathrm{D}^b(Y;y)$ and hence it is possible to calculate the germs by:

$$\big(\text{“}\varprojlim_{U\ni x}\text{”}\, \mathrm{R}f_! A_{M_j^+\cap U}\big)_y \cong \text{“}\varprojlim_{U\ni x}\text{”}\big(\mathrm{R}f_! A_{M_j^+\cap U}\big)_y$$

It is then enough to prove that the induced morphism

$$\text{(10.1)}\qquad \text{“}\varprojlim_{U\ni x}\text{”}\big(\mathrm{R}f_! A_{M_1^+\cap U}\big)_y \longrightarrow \text{“}\varprojlim_{U\ni x}\text{”}\big(\mathrm{R}f_! A_{M_2^+\cap U}\big)_y$$

is an isomorphism.

Let us choose real analytic systems of local coordinates (y,t) on X and $(y)=(y_1,y')$ on Y so that

$$p=(0;\mathrm{d}y_1),\quad f(y,t)=y,\quad M_j=\{y_1=h_j(y',t)\}$$

with $h_j = dh_j = 0$ at x $(j=1,2)$. Since

$$(\mathrm{R}f_! A_{M_2^+\cap U})_y \cong \mathrm{R}\Gamma_c\big(f^{-1}(y)\cap M_2^+\cap U; A_{f^{-1}(0)}\big),$$

in these coordinates (10.1) reads

$$\text{(10.1)}'\qquad \text{“}\varprojlim_{V\ni 0}\text{”}\, \mathrm{R}\Gamma_c\big(V\cap\{h_1(0,t)\le 0\}; A_{\mathbb{R}_t^{2n-2\ell}}\big) \longrightarrow \text{“}\varprojlim_{V\ni 0}\text{”}\, \mathrm{R}\Gamma_c\big(V\cap\{h_2(0,t)\le 0\}; A_{\mathbb{R}_t^{2n-2\ell}}\big),$$

where V is a neighborhood of 0 in $\mathbb{R}_t^{2n-2\ell}$. By PROPOSITION 6.1 (a), we know that $h_j(0,t)$ are C^ω-quadratic. The inclusion $M_1^+ \supset M_2^+$ implies that $h_1 \ge h_2$. Moreover, by hypothesis (c) of THEOREM 7.1 and by (6.1), we see that $\operatorname{Hess} h_1(0,t)$ and $\operatorname{Hess} h_2(0,t)$ have the same number of positive eigenvalues. The injectivity of (10.1)$'$ is then a consequence of PROPOSITION 9.2 for $m=2n-2\ell$, $h_j(t)=h_j(0,t)$. $\square$

11. Applications to systems with simple characteristics

11.1 A general statement. — Let X be a complex analytic manifold of dimension n. Let $V \subset \dot{T}^*X$ be a germ of l-codimensional complex analytic regular involutive submanifold at $p \in V$. If $\Gamma \subset V$ is a real submanifold, we will denote by $\widetilde{\Gamma}$ the union of the (complex) bicharacteristic leaves of V issued from Γ.

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module (i.e. a system of linear partial differential equations) with simple characteristics along V.

THEOREM 11.1. — *Let Λ be a germ of $\mathbb{R}$-Lagrangian manifold of real type at $p \in \Lambda \cap V$. Assume that:*

(a) *the intersection $\Lambda \cap V$ is clean at p;*

(b) $\alpha_{X|\Lambda\cap V} \neq 0$.

Then, for any $k < s^-(\Lambda,V) + \delta(\Lambda,V)$,
$$H^k\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_\Lambda)_p = 0.$$

THEOREM 11.2. — *Let Λ_j $(j = 1,2)$ be germs of $\mathbb{R}$-Lagrangian manifolds of real type at $p \in \Lambda_1 \cap \Lambda_2 \cap V$. Assume the following:*

(a) *the intersection $\Lambda_j \cap V$ is clean at p;*

(b) $\alpha_{X|\Lambda_1\cap\Lambda_2\cap V} \neq 0$;

(c) $\Lambda_1 \succ \Lambda_2$ *at p.*

Then:

(i) $s^-(\Lambda_1,V) + \delta(\Lambda_1,V) \leq s^-(\Lambda_2,V) + \delta(\Lambda_2,V)$

and the natural morphism
$$H^k\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_2})_p \longrightarrow H^k\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_1})_p$$
is injective for $k = s^-(\Lambda_1,V) + \delta(\Lambda_1,V)$.

Assume moreover:

(d) $s^-(\Lambda_1,V) + \delta(\Lambda_1,V) = s^-(\Lambda_2,V) + \delta(\Lambda_2,V)$;

(e) $\widetilde{V\cap\Lambda_1} = \widetilde{V\cap\Lambda_2}$.

Then:

(ii) *the natural morphism*
$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_2})_p \longrightarrow \mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_1})_p$$
is an isomorphism.

We notice that THEOREM 11.1 was proved by [SKK] and [KS2] when $\Lambda = T^*_{\mathbb{R}^n}\mathbb{C}^n$, and by [D'AZ2] in the general case. A particular case of Theorem 11.2 (i) is treated in [TU] by different methods.

In proving these results we follow here the same line as in [KS2].

Proof of Theorems 11.1 and 11.2. — Denote by $\mathcal{E}_X$ the sheaf of finite order microdifferential operators on T^*X. Let $\chi : T^*X \to T^*X$ be a complex contact transformation as in $(8.1)'$. Since the left coherent $\mathcal{E}_X$-module $\mathcal{E}_X\mathcal{M} = \mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$ is simply characteristic along V, it follows from [SKK], [Tr] that $\chi_*(\mathcal{E}_X\mathcal{M})$ is the $\mathcal{E}_X$-module associated to the de Rham system relative to f, i.e.

$$\chi_*(\mathcal{E}_X\mathcal{M}) \cong \mathcal{E}_X/\mathcal{E}_X(D_1,\ldots,D_l)$$

for a choice of local complex coordinates $(z) = (z_1,\ldots,z_n)$ on X at $x' = \pi_X(\chi(p))$ such that $f(z) = (z_{l+1},\ldots,z_n)$ (where D_j denotes the holomorphic derivative with respect to z_j).

LEMMA 11.3. — *With the previous notations, there is the following chain of isomorphisms*

$$\begin{aligned}(11.1)\qquad \chi_*\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_j})_p &\\ \cong \mu\mathrm{hom}(A_{M_j},f^{-1}\mathcal{O}_Y)_p[1]&\\ \cong \mu\mathrm{hom}(f_!^{\mu,p}A_{M_j},\mathcal{O}_Y)_q[1]&\\ \cong \mu\mathrm{hom}(A_{N_j},\mathcal{O}_Y)_q\big[1-s^-(M_j,V)-\delta(M_j,V)\big].&\end{aligned}$$

Proof of Lemma 11.3. — We will prove each isomorphism separately.

(i) The third isomorphism is a consequence of PROPOSITION 6.1.

(ii) The second isomorphism in (11.1) is proved by the following chain of isomorphisms (for $k \in \mathbb{Z}$):

$$\begin{aligned}H^k\mu\mathrm{hom}(A_{M_j},f^{-1}\mathcal{O}_Y)_p &\cong \mathrm{Hom}_{\mathrm{D}^b(X,p)}\big(A_{M_j},f^{-1}\mathcal{O}_Y[k]\big)\\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X,p)^\vee}\big(A_{M_j},f_{\mu,p}^{-1}\mathcal{O}_Y[k]\big)\\ &\cong \mathrm{Hom}_{\mathrm{D}^b(Y,q)^\wedge}\big(f_!^{\mu,p}A_{M_j},\mathcal{O}_Y[k]\big)\\ &\cong H^k\mu\mathrm{hom}\big(f_!^{\mu,p}A_{M_j},\mathcal{O}_Y\big)_q,\end{aligned}$$

where the microlocal inverse image $f_{\mu,p}^{-1}$, which is defined similarly to $f_!^{\mu,p}$, is in this case isomorphic to f^{-1}, since f is smooth. The third isomorphism here is the adjunction isomorphism for microlocal images (cf. [KS3, Prop. 6.1.8]).

(iii) We will prove the first isomorphism in two different ways, interesting on their own.

a) By the remark preceding PROPOSITION 5.4, we know that $\mathcal{C}_{T^*_{M_j}X}$ is well defined in the derived category of $\mathcal{E}_X$-modules, and hence so is $\mathcal{C}_{\Lambda_j} \cong \chi^{-1}\mathcal{C}_{T^*_{M_j}X}$ (even if this structure is not canonical). We may then write

$$\begin{aligned}\chi_* \mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_j}) &\cong \chi_* \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X\mathcal{M},\mathcal{C}_{\Lambda_j})\\ &\cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X\mathcal{M}_0,\mathcal{C}_{T^*_{M_j}X})\\ &\cong \mu\mathrm{hom}(A_{M_j}[-1],\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}_0,\mathcal{O}_X))\\ &\cong \mu\mathrm{hom}(A_{M_j},f^{-1}\mathcal{O}_Y)[1].\end{aligned}$$

b) To explain the second approach, let us assume that

$$\mathcal{M} \cong \mathcal{D}_X/\mathcal{D}_X\,P,$$

where P is a single partial differential operator (and hence $\ell = 1$). The general case of a coherent module $\mathcal{M}$ is treated similarly by taking a local free resolution.

Recall (*cf.* [KS2, Ch. 10]) that there is a natural ring homomorphism

$$\mathcal{E}_{X,p} \longrightarrow \mathrm{Hom}_{\mathrm{D}^b(X;p)}(\mathcal{O}_X,\mathcal{O}_X) \tag{11.2}$$

which is compatible with quantized contact transformations.

Applying the functor $\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\pi_X^{-1}(\cdot),\mathcal{C}_{\Lambda_j})$ to the exact sequence

$$0 \to \mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X \longrightarrow \mathcal{M} \longrightarrow 0,$$

we get the distinguished triangle

$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_j}) \longrightarrow \mathcal{C}_{\Lambda_j} \xrightarrow{P\cdot} \mathcal{C}_{\Lambda_j} \xrightarrow{+1}$$

in the derived category of $\pi_X^{-1}\mathcal{D}_X$-modules. The complex

$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{\Lambda_j})$$

may then be considered as the third term of a distinguished triangle for the morphism $\mathcal{C}_{\Lambda_j} \xrightarrow{P\cdot} \mathcal{C}_{\Lambda_j}$ in $\mathrm{D}^b(T^*X;p)$.

Assume that the operator $P \in \mathcal{D}_X$ is transformed by χ to a differential operator (*i.e.* assume $\chi_*(\mathcal{E}_X/\mathcal{E}_X\,P) \cong \mathcal{E}_X/\mathcal{E}_X\,Q$ for a differential operator

$Q \in \mathcal{D}_X$). Then the third term of a distinguished triangle associated to $\mathcal{C}_{T^*_{M_j}X} \xrightarrow{Q\cdot} \mathcal{C}_{T^*_{M_j}X}$ is given by the complex

$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X\, Q, \mathcal{C}_{T^*_{M_j}X}).$$

This last complex, as claimed, is in turn isomorphic to the complex

$$\chi_* \,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_j}). \quad \square$$

End of the proof of Theorem 11.1. — This is an immediate consequence of LEMMA 11.3. $\square$

End of the proof of Theorem 11.2. — By Remark 4.5 (ii) we know that $\lambda_1 + i\lambda_1 \supset \lambda_2 + i\lambda_2$. Since

$$\delta(M_2, V) - \delta(M_1, V) = \mathrm{cod}^{\mathbb{C}}_{\rho\cap(\lambda_1+i\lambda_1)}\, \rho \cap (\lambda_2 + i\lambda_2),$$

the estimate $s^-(\Lambda_1, V) + \delta(\Lambda_1, V) \leq s^-(\Lambda_2, V) + \delta(\Lambda_2, V)$ follows from LEMMA 1.7 and PROPOSITION 1.2. If the inequality is strict, the injectivity of the morphism in (i) of THEOREM 11.2 is trivial; otherwise, if equality holds, it is a consequence of THEOREM 7.1.

(ii) follows also from THEOREM 7.1 by observing that hypothesis (e) implies (with the usual notations) $T^*_{N_1}Y = T^*_{N_2}Y$. $\square$

REMARK 11.4. — It is clear that, if for the given Lagrangian manifolds Λ_j ($j = 1, 2$) the complexes $\mathcal{C}_{\Lambda_j}$ are well defined in the derived category of $\mathcal{E}_X$-modules, then in THEOREMS 11.1, 11.2 we can consider simply characteristic $\mathcal{E}_X$-modules instead of $\mathcal{D}_X$-modules (i.e. systems of microdifferential equations).

11.2 Microlocal boundary value problems. — We shall now apply our general theorem to the case of boundary value problems.

DEFINITION 11.5. — Let $M \subset X$ be a real analytic submanifold, and let $\Omega \subset M$ be an open subset with real analytic boundary S. Let $p \in S \times_M \dot{T}^*_M X$ and assume that $ip \notin T^*_S X$, and that $d(\lambda_M, \lambda_{0X}) = d(\lambda_S, \lambda_{0X}) =: d$ (in other words, assume that $T^*_M S$ and $T^*_S X$ are real type Lagrangian manifolds with $T^*_M X \succ T^*_S X$). According to [S2], consider the complex in $\mathrm{D}^b(T^*X; p)$ of microfunctions at the boundary

$$\mathcal{C}_{\Omega/X} = \mu\mathrm{hom}(A_\Omega[d], \mathcal{O}_X) \otimes or_{M/X}.$$

Write also $\mathcal{C}_{M/X}$ and $\mathcal{C}_{S/X}$ instead of $\mathcal{C}_{T^*_M X}$ and $\mathcal{C}_{T^*_S X}$ respectively.

Notice that $H^k(\mathcal{C}_{\Omega/X})_p = 0$ for $k < 0$. In fact this follows by applying THEOREM 5.6 to the distinguished triangle

$$\mathcal{C}_{S/X} \longrightarrow \mathcal{C}_{M/X} \longrightarrow \mathcal{C}_{\Omega/X} \xrightarrow{+1} .$$

THEOREM 11.6. — *Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module with simple characteristics along an ℓ-codimensional complex analytic regular involutive submanifold V of T^*X. Let Ω be as in Definition* 11.5, *and assume*

(a) *the intersections $T^*_MX \cap V$ and $T^*_SX \cap V$ are clean at p;*

(b) $\alpha_{X|T^*_MX \cap T^*_SX \cap V} \neq 0$.

Then, with notations (3.7), *for $k < s^-(M,V) + \delta(M,V)$ we have:*

$$H^k\, \mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}\big(\mathcal{M},\mathcal{C}_{\Omega/X}\big)_p = 0.$$

If moreover

(d) $s^-(M,V) + \delta(M,V) = s^-(S,V) + \delta(S,V)$,

(e) $T^*_M\widetilde{X \cap V} = T^*_S\widetilde{X \cap V}$,

then:

$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}\big(\mathcal{M},\mathcal{C}_{\Omega/X}\big)_p = 0.$$

Proof. — Setting $\Lambda_1 = T^*_MX$ and $\Lambda_2 = T^*_SX$, this is a corollary of THEOREM 11.2. ∎

REMARK 11.7. — If M is a real analytic manifold, X a complexification of M, and $S \subset M$, then the hypothesis $d(\lambda_M, \lambda_{0X}) = d(\lambda_S, \lambda_{0X})$ is satisfied, and $\delta(M,V) = 0$. In particular, THEOREM 11.6 applies to the geometrical setting mentioned in the introduction. Moreover in this case

(i) if $s^-(M,V) = \mathrm{cod}^{\mathbb{C}}_{T^*X} V$ then (d) of THEOREM 11.6 is satisfied;

(ii) if $s^-(M,V,p')$ is constant for $p' \in T^*_MX \cap V$ near p, then (d) is a necessary condition.

EXAMPLE 11.8. — Let M be an open neighborhood of 0 in $\mathbb{R}^n$, and let $X \subset \mathbb{C}^n$ be a complexification of M. Take local coordinates $(z) = (x+iy)$ in X and (x) in M at 0, and let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_XP$ be the $\mathcal{D}_X$-module associated to the operator

$$P = D_1 + iz_1D_2.$$

Let $\Omega \subset M$ be an open subset, and assume $S = \partial\Omega$ to be defined by the analytic equation $\phi = 0$ with, $\mathrm{d}\phi(0) \neq \mathrm{d}x_2$. We will consider boundary value problems at the microlocal point $p = (0; i\,\mathrm{d}x_2)$.

In this case, the intersections $V \cap T^*_M X$ and $V \cap T^*_S X$ are clean, and one easily checks that $d(\lambda_M, \lambda_{0X}) = d(\lambda_S, \lambda_{0X}) = n$. Moreover $s^-(M, V) = 1$ and $\delta(M, V) = 0$, and so we may apply COROLLARY 11.5 getting

$$\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}\big(\mathcal{M}, \mathcal{C}_{\Omega/X}\big)_p = 0. \tag{11.3}$$

If we make the particular choice of S being defined by the equation $x_1 = 0$, then $V \cap T^*_M X = V \cap T^*_S X$, and so (e) of THEOREM 11.2 is fulfilled. Since (d) is also fulfilled, we get in this case

$$\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}\big(\mathcal{M}, \mathcal{C}_{\Omega/X}\big)_p = 0. \tag{11.4}$$

Notice that (11.3) for S defined by $x_1 = 0$ was already stated in [T-U].

We observe that (11.4) is equivalent to the statements

(i) $\qquad \mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega/X})_p = 0\,;$

(ii) $\qquad \mathcal{E}xt^1_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{S/X})_p \xrightarrow{\sim} \mathcal{E}xt^1_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M/X})_p.$

Moreover, since $\mathrm{char}(\mathcal{M}) \cap T^*_M X = \emptyset$ over $x_1 \neq 0$ near p, one has by Sato's fundamental theorem

$$\begin{aligned} &\mathrm{R}\Gamma_{\{x_1=0\}}\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M/X})_p \\ &\qquad\xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M/X})_p. \end{aligned} \tag{11.5}$$

It follows that (11.4) is equivalent to

$$\mathrm{R}\Gamma_{\{x_1=0\}}\,\mathrm{R}\mathcal{H}om_{\pi_X^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega/X})_p = 0. \tag{11.6}$$

Notice that (11.6) should be recovered using the theory of micro-support of [KS1], [KS3]. (On the other hand, (11.5) and (11.6) should provide an alternative proof of (11.4).)

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