BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 123, nº 4 (1995), p. 477-491

<http://www.numdam.org/item?id=BSMF_1995__123_4_477_0>

Bull. Soc. math. France,
123, 1995, p. 477–491.

ORBIT THEOREMS FOR SEMIGROUP OF REGULAR MORPHISMS AND NONLINEAR DISCRETE TIME SYSTEMS

BY

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ABSTRACT. — Let S be a semigroup generated by a parametrized family of bijective regular morphisms on an algebraic variety W and G the group generated by S; we prove that for any x in W, the orbits Sx and Gx have the same dimension. We give a description of Gx and an orbit theorem for nonlinear discrete time systems.

RÉSUMÉ. — Soient S le semi-groupe engendré par une famille paramétrisée de morphismes réguliers bijectifs sur une variété algébrique W et G le groupe engendré par S; on montre que pour tout x dans W, les orbites Gx et Sx ont même dimension. On donne une description de Gx et un théorème d'orbite pour les systèmes non-linéaires en temps discret.

1. Introduction

We consider a discrete time system defined by the following state equation:

$$(1) \qquad X_{n+1} = \varphi(X_n, u_n) \qquad X_n \in W,\ u_n \in E,$$

where W is a real algebraic variety, E is a subset of a real algebraic variety V and φ is a regular morphism; in the present paper a real algebraic variety is a real irreducible algebraic set and a regular morphism is a map with rational components P_i/Q_i where Q_i has no zero in

(*) Texte reçu le 12 mai 1993.
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AMS classification: 93B27, 93C55, 14P10.

Keywords: controllability, algebraic varieties, regular morphisms, nonlinear discrete time systems.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037–9484/1995/477/$ 5.00

$W \times V$ (our reference for the notions of algebraic geometry used here, is BOCHNACK, COSTE and ROY (1987)).

In the sequel the morphism $\varphi(\cdot\,, u)$ is noted φ_u and the Zariski closure of a set A is noted $Z(A)$. We always assume that:

(H1) E contains a nonempty open subset of the regular part of V;

(H2) For each u in E, φ_u is bijective and φ_u^{-1} is continuous.

It would be noted that (H1) implies $V = Z(E)$ and in (H2) we do not assume that φ_u is a diffeomorphism.

Let S be the semigroup generated by the maps φ_u, $u \in E$ (*i.e.* S is the set of maps $\varphi_{u_1} \circ \varphi_{u_2} \circ \cdots \circ \varphi_{u_k}$ with $(u_1, \ldots, u_k)$ in E^k, where $k > 0$; note that the identity map is not always in S). Let G be the group generated by S. If x is in W, we note Sx (resp. Gx) its orbit by S (resp. G).

Our first objective is to prove an analogue of the positive form of Chow's lemma; we obtain:

THEOREM 1. — *For any x in W, we have* $\dim Gx = \dim Sx$.

(A precise definition of $\dim Gx$ and of $\dim Sx$ will be given in the next section.) This theorem is important in control theory. It is true for the analytic continuous time systems (see KRENER, 1974) and fails for general discrete time systems.

However in a recent work JAKUBCZYK and SONTAG (1989) have proved THEOREM 1 for discrete time systems when x is an equilibrium point, φ is analytic, the control value set E is connected and for each fixed u, φ_u is a global diffeomorphism; in SONTAG (1986), it is proved that the assumption that x is an equilibrium point cannot be relaxed in the analytic case; this shows the great qualitative difference between the analytic and the algebraic situation.

THEOREM 1 is related to our paper MOKKADEM (1989), where we proved the following:

THEOREM 2. — *If S is a semigroup of bijective bicontinuous regular morphisms of a real algebraic variety W and G is the group generated by S, then for any $x \in W$ we have $Z(Sx) = Z(Gx)$.*

In MOKKADEM (1989), we used THEOREM 2 to prove THEOREM 1 in some particular cases ($x \in Sx$ or $\varphi_u^{-1}(x)$ is a regular morphism) and asked about the general case; the present paper gives a positive response to the general case.

One consequence of THEOREM 1 for the system (1) is the following. Let us call the system *accessible in x* if $\dim Sx = \dim W$ and *weakly controllable in x* if $\dim Gx = \dim W$ (SONTAG, 1979), then:

COROLLARY 1. — *The system* (1) *is weakly controllable in x if and only if it is accessible in x.*

Our second objective is to give a description of the orbits Sx and Gx: PROPOSITION 3 in section 3 (a dichotomy proposition) asserts that there is only two kinds of orbits Sx, periodic or foliation and that if $Gx = Gy$ then Sx and Sy are of the same type; THEOREM 3 in section 3 asserts that Gx is a countable union of semialgebraic sets and this union is disjoint if the φ_u are diffeomorphisms: this theorem is an algebraic version of the «orbit theorem» in the discrete time case. The orbit theorem is important in understanding nonlinear systems and many papers deal with this theorem, see for example NAGANO (1966), SUSSMANN and JURDJEVIC (1972) and SUSSMAN (1973), for the continuous time systems, SONTAG (1986) and JAKUBCZYK and SONTAG (1989) for the discrete time systems.

2. Preliminary results and notations

We start by introducing some notations and definitions. Let $k \geq 1$ and $\ell = (e_1, e_2, \dots, e_k)$ where $e_i = \pm 1$; the map

$$\varphi_{u_k}^{e_k} \circ \cdots \circ \varphi_{u_1}^{e_1}(x)$$

defined on $W \times E^k$ is denoted by $\varphi^{\ell,k}$. When x is fixed, we obtain a map defined on E^k and denoted by $\varphi_x^{\ell,k}$. For $\ell = (+1, +1, \dots, +1)$ and $\ell = (-1, -1, \dots, -1)$, we write respectively φ_x^k and φ_x^{-k}.

We shall denote by $D_k(x)$ the set $\varphi_x^k(E^k)$ of states accessible from x in k steps, and by $D_{-k}(x)$ the set $\varphi_x^{-k}(E^k)$ of states controllable to x in k steps. The set $\varphi_x^{\ell,k}(E^k)$ is denoted by $F_{\ell,k}(x)$ and we put:

$$F_k(x) = \bigcup_{\ell=(e_1,\dots,e_k)} F_{\ell,k}(x). \tag{2}$$

The Zariski closures of these sets are denoted W or $\mathcal{W}$ with appropriate index:

$$Z\big(D_k(x)\big) = W_k(x), \qquad Z\big(D_{-k}(x)\big) = W_{-k}(x),$$

$$Z\big(F_{\ell,k}(x)\big) = \mathcal{W}_{\ell,k}(x), \qquad Z\big(F_k(x)\big) = \bigcup_{\ell} \mathcal{W}_{\ell,k}(x) = \mathcal{W}_k(x),$$

(note that the $W_k(x)$ are irreducible). Clearly,

$$Sx = \bigcup_{k>0} D_k(x) \quad \text{and} \quad Gx = \bigcup_{k>0} F_k(x).$$

We can state the following definitions (used implicitely in MOKKADEM 1989):

DEFINITION 1.

$$\dim D_k(x) = \dim W_k(x), \qquad \dim D_{-k}(x) = \dim W_{-k}(x),$$
$$\dim F_{\ell,k}(x) = \dim \mathcal{W}_{\ell,k}(x), \qquad \dim F_k(x) = \dim \mathcal{W}_k(x).$$

DEFINITION 2.

$$\dim Sx = \sup_{k>0} \dim W_k(x), \quad \dim Gx = \sup_{k>0} \dim \mathcal{W}_k(x).$$

Now we give some properties of the above maps and sets.

PROPOSITION 1. — *The sequences* $\dim D_k(x)$ *and* $\dim F_k(x)$ *are increasing sequences.*

This proposition is a consequence of the inclusions

$$\varphi_u\big(W_k(x)\big) \subset W_{k+1}(x), \quad \varphi_u\big(\mathcal{W}_k(x)\big) \subset \mathcal{W}_{k+1}(x)$$

and the bijectivity of the regular morphism φ_u. □

In the following proposition we prove that the maps $\varphi^{\ell,k}$ are continuous semialgebraic maps; this is not used in the sequel but the consequence is that the set $F_{\ell,k}(x)$ is a subset of the semialgebraic set $\varphi_x^{\ell,k}(V^k)$ and contains an open subset of the regular part of $\varphi_x^{\ell,k}(V^k)$, according to (H1); then the dimensions given in Definition 1 and Definition 2 are also the topological dimensions.

PROPOSITION 2. — *For any map* $\varphi_x^{\ell,k}$ *there exists a continuous semialgebraic map* Φ *defined on a semialgebraic set* $\mathcal{E}_k \supset E^k$, *such that* $\varphi_x^{\ell,k}$ *is the restriction of* Φ *to*E^k. *We can consider then* $\varphi_x^{\ell,k}$ *as a semialgebraic map on* $\mathcal{E}_k \supset E^k$.

We prove the following result: $\varphi^{\ell,k}$ is the restriction of a continuous semialgebraic map defined on a semialgebraic set $\mathcal{R}_k \supset W \times E^k$.

The PROPOSITION 2 is an immediate consequence of this result.

For $k = 1$ and $\ell = (1)$ the result is obvious and $\mathcal{R}_1 = W \times V$; for $k = 1$ and $\ell = (-1)$,

$$\varphi^{\ell,k}(x,u) = \varphi_u^{-1}(x).$$

Let us put:

$$\psi(x,u) = \big(\varphi(x,u),u\big).$$

Then ψ is a regular morphism on $W \times V$ and is injective on $W \times E$; by the semialgebraic triviality theorem (see BOCHNACK, COSTE and ROY 1987, thm. 9.3.1, p. 195), there exists a finite partition of $W \times V$ by semialgebraic sets T_i, for $1 \le i \le r$, such that in each T_i, all the fibers are homeomorphic. Thus, for any T_i such that $T_i \cap \psi(W \times E) \neq \emptyset$, ψ is injective on $\psi^{-1}(T_i)$; denoting by $\mathcal{R}_1$ the union of such T_i, it is easy to see that ψ is a homeomorphism between $\psi^{-1}(\mathcal{R}_1)$ and $\mathcal{R}_1$; the inverse is a semialgebraic map on $\mathcal{R}_1$; clearly

$$\mathcal{R}_1 \supset \psi(W \times E) \supset W \times E$$

(the last inclusion comes from the bijectivity of the maps φ_u for $u \in E$); now, taking π the projection of $W \times V$ on W, it is easy to see that $\varphi_u^{-1}(x)$ is the restriction of $\pi \circ \psi^{-1}$ to $W \times E$.

Assume the result true for $n \le k$ (*i.e.* $\varphi^{\ell,n}$ is continuous and semialgebraic on $\mathcal{R}_n$); we can write in $W \times E^{k+1}$

$$\varphi^{\ell,k+1}\big(x,u_1,\dots,u_{k+1}\big) = \varphi^{e,1}\big(\varphi^{h,k}(x,u_1,\dots,u_k),u_{k+1}\big) \tag{3}$$

with $h = (e_1,\dots,e_k), e = \pm 1$ and $\ell = (e,h)$. Now, define:

$$\Lambda\big(x,u_1,\dots,u_{k+1}\big) = \big(\varphi^{h,k}(x,u_1,\dots,u_k),u_{k+1}\big). \tag{4}$$

Clearly Λ is a semialgebraic map on $\mathcal{R}_k \times V$ and its range is $W \times V$. Denote the semialgebraic set $\Lambda^{-1}(\mathcal{R}_1)$ by $\mathcal{R}_{k+1}$; it is easy to see that

$$\Lambda\big(W \times E^{k+1}\big) = W \times E$$

(because the maps $\varphi_{u_k}^{e_k} \circ \cdots \circ \varphi_{u_1}^{e_1}$ are bijective) and then $W \times E^{k+1} \subset \mathcal{R}_{k+1}$. The map $\varphi^{e,1} \circ \Lambda$ is continuous and semialgebraic on $\mathcal{R}_{k+1}$ and it follows from (3) and (4) that $\varphi^{\ell,k+1} = \varphi^{e,1} \circ \Lambda$ on $W \times E^k$; the result is then proved.

Other Properties. — Because φ_x^k is a regular morphism and E^k contains an open subset of the regular part of V^k, it follows that $D_k(x)$ contains an open subset of the regular part of $W_k(x)$. This property holds for $F_{\ell,k}(x)$ by PROPOSITION 2 and is more precise if we add one of the following assumptions (used also in JACKUBCZYK and SONTAG (1989) and in MOKKADEM 1989):

(i) $\varphi_u^{-1}(x)$ is a regular morphism;

(ii) E is semialgebraic;

(iii) E is contained in the closure of its interior (*i.e.* $E \subset \mathrm{clos}(\mathrm{int}\, E)$).

In the case (i) $\mathcal{W}_{\ell,k}(x)$ is irreducible; for (ii) from PROPOSITION 2, $\varphi_x^{\ell,k}(E^k)$ is semialgebraic and then contains an open subset of the regular part of its Zariski closure $\mathcal{W}_{\ell,k}(x)$; for (iii) we use Proposition 2 and Corollary 9.3.2 in p. 198 of BOCHNACK, COSTE and ROY (1987); $\varphi_x^{\ell,k}$ is semialgebraic on $\mathcal{E}_k$, then $S = (\varphi_x^{\ell,k})^{-1}(\mathcal{W}_{\ell,k}(x))$ is semialgebraic; denote by T_i, for $i = 1, \dots, r$, the partition of $\mathcal{W}_{\ell,k}(x)$ given by the Corollary 9.3.2. and denote by S_i their inverse images; clearly $E^k \cap (\cup S_i) \neq \emptyset$ (otherwise $Z(F_{\ell,k}) \neq \mathcal{W}_{\ell,k}$); now, because S_i is open, by (iii) E^k contains an open subset of some S_i, $Z(T_i) = \mathcal{W}_{\ell,k}(x)$ and $\varphi_x^{\ell,k}$ is an open map on S_i.

3. Main results

The lemmas and propositions in this section give information about the orbits and are used to prove THEOREM 1.

LEMMA 1. — *If* $y \in Gx$ *then* $\dim Sy = \dim Sx$, *i.e. the dimension of* Sy *is the same for all* y *in* Gx.

Proof. — Let
$$n_0 = \max_{y \in Gx} \dim Sy$$
and y_0 be such that $\dim Sy_0 = n_0$. The set
$$N = \{y \in Z(Gx)\,;\ \dim Sy < n_0\}$$
is the set of zeros of a family of regular morphisms (minors of jacobians of the morphisms φ_y^k); then N is an algebraic subset of $Z(Gx)$. It is a proper subset because $y_o \notin N$. Assume that $N \cap Gx \neq \emptyset$ and pick $y \in N \cap Gx$; clearly $Sy \subset N$ because $y' \in Sy$ implies $Sy' \subset Sy$ and then $\dim Sy' < n_0$; it follows that $Z(Sy) \subset N$; there is a contradiction because by THEOREM 2, $Z(Sy) = Z(Gx)$. ▯

Now we can state our dichotomy result.

PROPOSITION 3. — *In any orbit* Gx *there are only two exclusive possibilities*:

1) *the periodic case*: *for any* y *in* Gx *the sequence* $W_k(y)$ *is periodic* (*i.e. there exist integers* k_0 *and* r, *such that* $W_k(y) = W_{k+r}(y)$ *for* $k \geq k_0$).

2) *the foliation case*: *for any* y *in* Gx, *the sets*
$$M_k(y) = W_k(y) \cap Gx, \qquad k \in \mathbb{N},$$
are disjoint (*here* $W_0(y) = \{y\}$).

Proof. — We note that if the sequence $W_k(y)$ is periodic then

$$\dim Sy = \dim Z(Sy) \tag{5}$$

because

$$Z(Sy) = \bigcup_{0\leq j\leq k_0+r} W_j(y). \tag{6}$$

Note also that if $W_{k_0}(y) = W_{k_0+r}(y)$ for some k_0 and r, then the sequence $W_k(y)$ is periodic: this follows from the construction of the varieties $W_k(y)$.

Now we claim that if there exists y_0 such that the sequence $W_k(y_0)$ is periodic, then we are in the periodic case. Let y in Gx; using LEMMA 1, THEOREM 2 and (5) we obtain

$$\dim Sy = \dim Z(Sy)\,;$$

by PROPOSITION 1 there exists k_1 such that $\dim W_k(y) = \dim Z(Sy)$ for $k \geq k_1$; but $Z(Sy)$ has a finite number of components and then $W_{k_0}(y) = W_{k_0+r}(y)$ for some integers k_0 and r; the claim is proved.

We conclude the proof of the PROPOSITION 3. Let y in Gx; assume that we are not in the periodic case and that for some $(r, k_0) \in \mathbb{N}^2$,

$$M_{k_0}(y) \cap M_{k_0+r}(y) \neq \emptyset \tag{7}$$

It follows that:

$$Z_{k_0} = W_{k_0}(y) \cap W_{k_0+r}(y) \neq 0 \quad \text{and} \quad \dim Z_{k_0} < \dim Sy. \tag{8}$$

Let us denote respectively by N_k and Z_k the sets $M_k(y) \cap M_{k+r}(y)$ and $W_k(y) \cap W_{k+r}(y)$; using (8) and the non periodicity we obtain:

$$Z_k \neq \emptyset \quad \text{and} \quad \dim Z_k < \dim Sy \ \text{ for any } k \geq k_0. \tag{9}$$

Now we pick y_0 in N_{k_0}; it is easy to see that $Sy_0 \subset \bigcup_{k\geq k_0} N_k$ and then from (9), $\dim Sy_0 < \dim Sy$; this contradicts the LEMMA 1. □

REMARK 1. — We notice in the above proof, that Gx satisfies the periodic case if and only if $\dim Sx = \dim Z(Sx)$. Note also that, in the foliation case, the sets $D_k(x)$ are disjoints.

LEMMA 2. — *Let x in W; there exists an integer n_0 such that for any y in Gx,* $\dim D_{n_0}(y) = \dim Sy$.

Proof. — Let m be the common dimension of the orbits Sy, $y \in Gx$; we define:

$$T_n = \{z \in Z(Gx)\,;\ \operatorname{rank} \varphi_z^n < m\}. \tag{10}$$

Clearly:

$$T_n = \{z \in Z(Gx)\,;\ \dim D_n(z) < m\}. \tag{11}$$

From (10) it follows that T_n is an algebraic set; PROPOSITION 1 and formula (11) imply:

$$T_{n+1} \subset T_n. \tag{12}$$

By the Hilbert's basis theorem (see e.g. BRÖCKER 1975), there exists an integer n_0 such that:

$$T_n = T_{n_0} \quad \text{for } n \geq n_0. \tag{13}$$

Using (11) and (13) we obtain

$$T_{n_0} = \{z \in Z(Gx)\,;\ \dim Sz < m\} \tag{14}$$

and LEMMA 1 implies that $T_{n_0} \cap Gx = \emptyset$; LEMMA 2 is then proved. □

REMARK 2. — From LEMMAS 1 and 2 it follows that: for any x in W, there exists n_0 and m such that $\dim W_{k+n_0}(z) = m$ for any z in Gx and $k \in \mathbb{N}$.

Proof of Theorem 1. — Let x in W; there are two cases.

- *The periodic case.* In this case $\dim Sx = \dim Z(Sx)$; THEOREM 1 is then a consequence of THEOREM 2.
- *The foliation case.* For h and k in $\mathbb{N}$, we define:

$$\begin{cases} W_{h,k}(x) = Z\Big\{ \bigcup\limits_{(u_1,\dots,u_h)\in E^h} \varphi_{u_1}^{-1} \circ \cdots \circ \varphi_{u_h}^{-1}\big(W_k(x)\big)\Big\}, \\ W_{0,k}(x) = W_k(x), \end{cases} \tag{15}$$

and

$$\begin{cases} M_{h,k}(x) = W_{h,k}(x) \cap Gx, \\ M_{0,k}(x) = M_k(x). \end{cases} \tag{16}$$

We begin by proving the following claim:

$$(*)\qquad \begin{cases} \textit{Let } n_0 \textit{ be the integer of the Lemma 2, then} \\ \textit{for any } (u_1, ..., u_{h+n_0}) \textit{ in } E^{h+n_0}, \\ \quad \varphi_{u_{h+n_0}} \circ ... \circ \varphi_{u_1}\big(M_{h,k}(x)\big) \subset M_{k+n_0}(x). \\ \textit{In particular, for any } u \textit{ in } E \textit{ we have:} \\ \qquad \varphi_u\big(M_{h,k}(x)\big) \subset M_{h+n_0-1,k+n_0}(x). \end{cases}$$

We set:

$$A = \bigcup_{(u_1,\dots,u_h)\in E^h} \varphi_{u_1}^{-1} \circ \cdots \circ \varphi_{u_h}^{-1}\big(W_k(x)\big).$$

Let y in A; there exists $(u_1, ..., u_h)$ such that

$$(17)\qquad z = \varphi_{u_h} \circ \cdots \circ \varphi_{u_1}(y) \in W_k(x)\,;$$

then

$$W_{n_0}(z) \subset W_{k+n_0}(x)$$

and by the Remark 2

$$(18)\qquad W_{n_0}(z) = W_{k+n_0}(x)\,;$$

but $W_{n_0}(z) \subset W_{h+n_0}(y)$ and then $W_{h+n_0}(y) = W_{k+n_0}(x)$; this proves that $D_{h+n_0}(y) \subset W_{h+n_0}(x)$. Then for any $(u_1, ..., u_{h+n_0})$ in E^{h+n_0},

$$\varphi_{u_{h+n_0}} \circ \cdots \circ \varphi_{u_1}(A) \subset W_{k+n_0}(x)$$

and this leads to

$$\varphi_{u_{h+n_0}} \circ \cdots \circ \varphi_{u_1}\big(Z(A)\big) \subset W_{k+n_0}(x)$$

it is now easy to conclude the proof of the first part of the claim (*).

For the second part of the claim, we have according to the first part:

$$\varphi_{u_{h+n_0}} \circ \cdots \circ \varphi_{u_2} \circ \varphi_u\big(M_{h,k}(x)\big) \subset M_{k+n_0}(x).$$

Therefore

$$\varphi_u\big(M_{h,k}(x)\big) \subset \varphi_{u_2}^{-1} \circ \cdots \circ \varphi_{u_{h+n_0}}^{-1}\big(M_{k+n_0}(x)\big) \subset W_{h+n_0-1,k+n_0}(x)$$

as $M_{h,k}(x) \subset Gx$, we conclude the proof of the claim. □

From the claim (*) and the bijectivity of the morphisms φ_u it follows that:

$$\dim Z(M_{h,k}(x)) \leq \dim Sx. \tag{19}$$

To conclude we prove the following assertion: each $F_{\ell,k}(x)$ is contained in some $M_{h,k'}(x)$.

We proceed by induction on k. For $k = 1$; if $\ell = (1), F_{\ell,k}(x) \subset M_1(x)$ and if $\ell = (-1), F_{\ell,k}(x) \subset M_{1,0}(x)$. Now assume the assertion true for the integers smaller than k and write

$$F_{\ell,k}(x) = \bigcup_{u\in E} \varphi_u^{e_k}\big(F_{\ell',k-1}(x)\big), \tag{20}$$

where $\ell = (e_1, \ldots, e_k)$ and $\ell' = (e_1, \ldots, e_{k-1})$; by the induction hypothesis, there exists $M_{h,k'}(x)$ such that:

$$F_{\ell',k-1}(x) \subset M_{h,k'}(x). \tag{21}$$

If $e_k = 1$, by the claim (*) and (20) we have

$$F_{\ell,k}(x) \subset M_{h+n_0-1,k'+n_0}(x).$$

If $e_k = -1$, the definition of the $M_{h,k}(x)$ and (20) give:

$$F_{\ell,k}(x) \subset M_{h+1,k'}(x).$$

The assertion is then proved and Theorem 1 follows from (19).

Orbit Theorem

We give now more precision on the structure of the orbit Gx. We consider the two cases.

1. The periodic case.

In this case $\dim Gx = \dim Z(Gx)$ and clearly, Gx contains an open subset of the regular part of $Z(Gx)$. Because the morphisms φ_u are homeomorphisms, it follows that Gx is an open subset of $Z(Gx)$ with dimension $\dim Z(Gx)$ at each of its points (we do not know if $Z(Gx)$ contains or not a branch with smaller dimension than $\dim Z(Gx)$). One can see also (using PROPOSITION 2) that Gx is locally semialgebraic; we can prove that it is semialgebraic; it is easy to see that $Z(Gx)$ is invariant by G and that the set L of $z \in Z(Gx)$ such that $\dim Gz < \dim Gx$ is a proper algebraic subset of $Z(Gx)$; we define $U = Z(Gx) - L$, (U is the set of locally weakly controllable points of the system (1) restricted to $Z(Gx)$). The set U is an open semialgebraic subset of $Z(Gx)$ with a finite number of semialgebraic connected components; because any such component is contained in an orbit by G, it follows that Gx is the union of some of this components.

If the φ_u are diffeomorphisms and W is smooth (*i.e.* without singularities) then Gx is an open subset of the regular part of $Z(Gx)$ and consequently Gx is an embedded analytic subvariety of W.

2. The foliation case.

We have the following.

a) If $k - h \neq k' - h'$ then $M_{h,k}(x) \cap M_{h',k'}(x) = \emptyset$. Assume that $y \in M_{h,k}(x) \cap M_{h',k'}(x)$; then the claim (*) in the proof of the THEOREM 1 implies:

$$D_{h+n_0}(x) \subset M_{k+n_0}(x) \quad \text{and} \quad D_{h'+n_0}(x) \subset M_{k'+n_0}(x). \tag{22}$$

If $h' < h$ write $h = h' + \eta$, we obtain

$$D_{h+n_0}(x) \subset M_{k'+n_0+\eta}(x)$$

and because we are in the foliation case, it follows that $k+n_0 = k'+n_o+\eta$; this gives a).

We shall denote by

$$\mathcal{M}_\alpha = \bigcup_{k-h=\alpha} M_{h,k}(x)$$

and notice that for fixed α,

$$M_{h,k}(x) \subset M_{h',k'}(x) \quad \text{if } k \leq k' \text{ and } k - h = k' - h' = \alpha$$

(the same inclusion holds for the $W_{h,k}$). If $\alpha \neq \alpha'$, then $\mathcal{M}_\alpha \cap \mathcal{M}_{\alpha'} = \emptyset$ and as each $F_{\ell,k}(x)$ is contained in some $M_{h,k'}(x)$, we have:

$$Gx = \bigcup_{\alpha \in \mathbb{Z}} \mathcal{M}_\alpha.$$

b) Let us call Y_n, where $n \in \mathbb{Z}$, the countable family of irreducible components of the $W_{h,k}$ whose dimension is $\dim Gx$; we claim that Gx is contained in $\bigcup Y_n$. To prove this, assume that some y in Gx is contained only in components of dimension smaller than $\dim Gx$; we pick z in Gx

such that z is also in the regular part of Y_n (this is always possible). There is φ in G, such that $\varphi(z) = y$; φ is an homeomorphism of a neighborhood of z in Y_n into a space of smaller dimension; this is impossible by the Theorem on the invariance of domain (see HUREWICZ and WALLMAN 1941).

Now arguing like in the periodic case, it is not difficult to see that the properties obtained without the diffeomorphism hypothesis continue to hold in the foliation case for $Y_n \cap Gx$ in Y_n (instead of Gx in $Z(Gx)$).

c) Now we assume that the morphisms φ_u are diffeomorphisms and W is smooth.

We take Y_n; Gx contains an open subset U of the regular part of Y_n. We claim that there exists y in U, such that y is not in a proper intersection of Y_n with some Y_m, otherwise Y_n would be a countable union of proper algebraic subsets; this is impossible.

Now we fix such y. Let z in Gx such that z is in a proper intersection; there exists $\varphi \in G$ such $\varphi(y) = z$. Let B be an open neighborhood of y such that $H = Y_n \cap B$ is smooth; denote by B' and H' the sets $\varphi(B)$ and $\varphi(H)$; because φ is a diffeomorphism it follows that H' is included in one Y_m. Let Y_k such that z is in the proper intersection $Y_m \cap Y_k$; then $\varphi^{-1}(Y_k \cap B')$ is also contained in an $H'' = Y_s \cap B$; clearly $H'' \neq H$ and $H'' \cap H \neq \emptyset$ (because $Y_m \cap Y_k) \neq \emptyset$); $y \notin H'' \cap H$ but $\varphi^{-1}(z) \in H'' \cap H$, there is a contradiction. We conclude that the algebraic varieties $Y_n, n \in \mathbf{Z}$, are disjoint. Now, it is easy to see that $Y_n \cap Gx$ is an open subset of the regular part of Y_n.

We summarize the above discussion in the following theorem.

THEOREM. — *Let x in W and $m = \dim Gx$.*

• *In the periodic case, Gx is an open semialgebraic subset of $Z(Gx)$ with pure dimension m (i.e. the dimension is the same in each point).*

If W is smooth and the morphisms φ_u are diffeomorphisms, then Gx is a smooth semialgebraic subset; in particular, Gx is an embedded analytic subvariety of W.

• *In the foliation case, Gx is a countable union of semialgebraic subsets with pure dimension m and each of these subsets is open in its Zariski closure.*

If W is smooth and the morphisms φ_u are diffeomorphisms, then Gx is a countable disjoint union of smooth semialgebraic subsets (with dimension m); in particular, Gx is an embedded analytic subvariety of W.

Examples

Example 1 (periodic case):

$$\begin{aligned} x_{n+1} &= x_n(z_n^2+1),\\ y_{n+1} &= y_n(z_n^2+1),\\ z_{n+1} &= (z_n-u)^3\,; \end{aligned}$$

$$W=\mathbb{R}^3,\quad E=\mathbb{R}.$$

Note that ϕ_u is not a diffeomorphism. Let $\tilde{x}=(x_0,y_0,z_0)$.

- If $x_0=y_0=0$ then

$$S\tilde{x}=G\tilde{x}=D_1(\tilde{x})=Z(S\tilde{x})=\{x=y=0\}.$$

- Otherwise it is easy to see that

$$Z(G\tilde{x})=Z(S\tilde{x})=Z\big(D_2(\tilde{x})\big)=\{y_0x-x_0y=0\}$$

and that

$$\begin{aligned} G\tilde{x} &= \big\{y_0x-x_0y=0\,;\ x^2+y^2>0,\ x_0x\geq 0\,;\ y_0y\geq 0\big\},\\ S\tilde{x} &= \big\{y_0x-x_0y=0\,;\ x_0x\geq 0,\ y_0y\geq 0,\\ &\qquad\qquad x^2\geq x_0^2,\ y^2\geq y_0^2,\ x^2+y^2>0\big\}. \end{aligned}$$

Example 2 (foliation case):

$$\begin{aligned} x_{n+1} &= ux_n^3,\\ y_{n+1} &= y_n+1\,; \end{aligned}$$

$$W=\mathbb{R}^2,\quad E=\mathbb{R}-\{0\}.$$

Let $\tilde{x}=(x_0,y_0)$.

- If $x_0=0$, then

$$\begin{aligned} &D_n(\tilde{x})=\big\{(0,y_0+n)\big\},\quad Z\big(D_n(\tilde{x})\big)=D_n(\tilde{x}),\quad \dim D_n(\tilde{x})=0,\\ &S\tilde{x}=\big\{(0,y_0+k)\,;\ k\in\mathbb{N}^\star\big\},\\ &G\tilde{x}=\big\{(0,y_0+k)\,;\ k\in\mathbb{Z}\big\},\\ &Z(S\tilde{x})=Z(G\tilde{x})=\{x=0\}. \end{aligned}$$

- If $x_0 \neq 0$, then

$$\begin{aligned} D_n(\tilde{x}) &= \{(x, y_0 + n)\,;\ x \neq 0\},\\ S\tilde{x} &= \{(x, y_0 + k)\,;\ k \in \mathbb{N}^\star,\ x \neq 0\},\\ G\tilde{x} &= \{(x, y_0 + k)\,;\ k \in \mathbb{Z},\ x \neq 0\},\\ Z(G\tilde{x}) &= Z(S\tilde{x}) = \mathbb{R}^2. \end{aligned}$$

Example 3 (periodic and foliation case):

$$\begin{aligned} x_{n+1} &= ux_n + vy_n,\\ y_{n+1} &= vx_n - uy_n\,; \end{aligned}$$

$$W = \mathbb{R}^2, \quad E = \{u^2 + v^2 = \alpha^2\},$$

where α is a constant different from 1 and 0. Let $\tilde{x} = (x_0, y_0)$, then $D_n(\tilde{x})$ is the circle

$$x^2 + y^2 = \alpha^{2n}(x_0^2 + y_0^2).$$

If $x_0^2 + y_0^2 = 0$ the orbit is $\{0\}$, if $x_0^2 + y_0^2 \neq 0$ the orbit and forward orbit are disjoint union of circles.

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