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# EXTENSION OPERATORS FOR ANALYTIC FUNCTIONS DEFINED ON CERTAIN CLOSED SUBVARIETIES OF A STEIN SPACE

BY

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RÉSUMÉ. — Soient $M$ un espace de Stein irréductible et $V$ une sous-variété fermée de $M$ telle que $\mathcal{O}(V)$ soit un espace de séries de puissance. Dans cet article, nous donnons des conditions nécessaires et suffisantes pour l'existence d'un opérateur d'extension linéaire et continu de $\mathcal{O}(V)$ dans $\mathcal{O}(M)$ en termes de functions plurisous-harmoniques définies sur ces variétés. En fait, nous obtenons ces résultats en résolvant un problème d'extension plus général. Nous considérons aussi quelques conséquences de ces résultats.

ABSTRACT. — Let $M$ be an irreducible Stein space and let $V$ a closed subvariety of $M$ with the property that $\mathcal{O}(V)$ is a power series space. In this paper we give a necessary and sufficient condition for the existence of a continuous linear extension operator from $\mathcal{O}(V)$ into $\mathcal{O}(M)$ in terms of plurisubharmonic functions defined on these varieties. Actually we obtain these results by solving a general lifting problem. We also consider some consequences of these results.

**0.** — Let $M$ be an irreducible Stein space and $V$ a closed subvariety of $M$. One of the consequences of the Oka-Cartan theory is that every analytic function on $V$ can be extended to an analytic function on $M$. The question as to whether this extension process can be achieved by a continuous linear extension operator was studied by various authors.

Such a continuous operator if it exists, will imbed the Fréchet space of all analytic functions on $V, \mathcal{O}(V)$, into $\mathcal{O}(M)$ as a closed complemented subspace. In some cases this simple observation exhibits an obstruction, for the existence of a continuous linear extension operator. This situation

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occurs for example, when $\mathcal{O}(V)$ has no continuous norm (i.e. when $V$ has infinite number of irreducible components) or when every continuous linear mapping from $\mathcal{O}(V)$ into $\mathcal{O}(M)$ is compact (see [9]). On the other hand positive answers in the cases :

(i) when $M$ is a strictly pseudoconvex domain in a Stein manifold and $V$ is of the form $V = M \cap \widetilde{V}$ where $\widetilde{V}$ is a closed submanifold near $\overline{M}$ intersecting $\partial M$ transversally, and

(ii) when $M = \mathbb{C}^n$ and $V$ a closed submanifold for which $\mathcal{O}(V)$, is isomorphic to $\mathcal{O}(\mathbb{C}^d)$ for some $d$ as Fréchet spaces, e.g. when $V$ is a smooth algebraic variety (see [17]),

were obtained in [10] by using $\overline{\partial}$-methods. In both of the cases considered above, the spaces $\mathcal{O}(V)$ turns out to belong to a well studied and well understood class of Fréchet spaces. A *power series space* is a sequence space of the form

$$\Lambda_R(\alpha) = \Big\{ x = \{x_n\}_{n=1}^{\infty} \,;\ \ \|x\|_r \doteq \sum_{k=1}^{\infty} |x_k|\, r^{\alpha_k} < +\infty \quad \text{for all}\ \ 0 < r < R \Big\}$$

where $0 < R \leq +\infty$ and $\alpha = \{\alpha_n\}$ is an increasing unbounded sequence of positive numbers. The space $\Lambda_R(\alpha)$ equipped with the norms $\|\ \ \|_r$, for $0 < r < R$ is a Fréchet space. It is easy to see that for a fixed $\alpha$, the spaces $\Lambda_R(\alpha)$, for $0 < R < +\infty$, are all isomorphic to each other and so we have two types of power series spaces; the ones that are isomorphic to $\Lambda_1(\alpha)$, (finite type), and the ones that are isomorphic to $\Lambda_\infty(\alpha)$ (infinite type). A large number of Fréchet function spaces occuring in analysis are actually power series spaces [14]. In the case (i) considered above, $\mathcal{O}(V)$ is (isomorphic to) $\Lambda_1(n^{1/d})$ and in the case (ii) is $\Lambda_\infty(n^{1/d})$ where in both cases $d$ is the dimension of $V$.

In this article we shall investigate the above mentioned question in the case when $\mathcal{O}(V)$ is a power series space. More generally we consider for a given data $(M, V, W, T)$ consisting of a irreducible Stein space $M$, a subvariety $V$ of $M$, a Stein space $W$ for which $\mathcal{O}(W)$ is a power series space and a continuous linear operator $T$ from $\mathcal{O}(W)$ into $\mathcal{O}(V)$, the problem of finding a continuous linear operator $\widetilde{T}$ such that the following diagram commutes

$$\begin{tikzcd} & \mathcal{O}(M) \arrow[d, "T"] \\ \mathcal{O}(W) \arrow[ur, dashed, "\widetilde{T}"] \arrow[r, "R"] & \mathcal{O}(V) \end{tikzcd}$$

where $R$ is the restriction operator. Observe that in the special case

$W = V$ and $T = I$ the identity of $\mathcal{O}(V), \tilde{I}$ if it exists, is a continuous linear extension operator. The obstruction to finding $\widetilde{T}$ for an arbitrary $T$ in the above set up is due to the non vanishing of the first derived functor $\operatorname{Ext}^1(\cdot,\cdot)$ of the functor Pro in the terminology of the locally convex homological algebra developed by PALAMADOV [11] (cf. [15]). Indeed in the above set up, denoting by $I(V)$ the ideal of the variety $V$, the short exact sequence

$$O \to I(V) \longrightarrow \mathcal{O}(M) \xrightarrow{R} \mathcal{O}(V) \to O$$

gives rise to the exact sequence

$$\begin{aligned} 0 \to L\big(\mathcal{O}(W), I(V)\big) &\longrightarrow L\big(\mathcal{O}(W), \mathcal{O}(M)\big) \\ &\longrightarrow L\big(\mathcal{O}(W), \mathcal{O}(V)\big) \xrightarrow{\delta} \operatorname{Ext}^1\big(\mathcal{O}(W), I(V)\big) \\ &\longrightarrow \operatorname{Ext}^1\big(\mathcal{O}(W), \mathcal{O}(M)\big) \longrightarrow \operatorname{Ext}^1\big(\mathcal{O}(W), \mathcal{O}(V)\big) \to 0 \end{aligned}$$

where $L(E,F)$ denotes the space of all continuous linear operators from $E$ into $F$ (see [15]). For a nuclear Fréchet space $E$, $\operatorname{Ext}^1(E, I(V))$ can be identified with the first Čech cohomology group of the sheaf $I^{E^*}(V)$, of germs of $E^*$ valued analytic functions on $M$ that vanish on $V$ (see for example [1]). Hence the possible non vanishing of $\operatorname{Ext}^1$ in this case reflects the failure of the Cartan theorem $(B)$ for $E^*$ valued coherent analytic sheaves on $M$. Various conditions on the pair of Fréchet spaces which assure the vanishing of this derived functor are given in [15] (see also [1]). In particular the vanishing of $\operatorname{Ext}^1(\mathcal{O}(W), I(V))$ when $\mathcal{O}(W)$ is a power series space of infinite type follows from these general considerations (see also Remark 1). Hence in the above mentioned set up we will restrict our attention to Stein spaces $W$ for which $\mathcal{O}(W)$ is isomorphic to a finite type power series space.

We shall use the standard terminology and notation of complex analysis as in [6], [7] except perhaps in our usage of the term Stein space. In this note by a Stein space we mean a reduced, irreducible Stein space in the sense of [6] which has a Hausdorff, separable topology.

Some results of this work was announced in [3].

**1.** — Returning to our problem, let us fix a Stein space $M$, a closed subvariety $V$ of $M$ and a Stein space $W$ for which $\mathcal{O}(W)$ is a power series space. Since we will be investigating the extendibility of continuous linear operators from $\mathcal{O}(W)$ into $\mathcal{O}(V)$, we can, without loss of generality take $W$ to be either $\Delta^d$, the unit polydisc in $\mathbb{C}^d$, or $\mathbb{C}^d$ itself depending upon the type of the power series space $\mathcal{O}(W)$, where $d = \dim W$. In both case a continuous linear operator $T$ from $\mathcal{O}(W)$ into $\mathcal{O}(V)$ induces a plurisubharmonic function $\rho_T$ on $V$ via the formula

$$\rho_T(z) = \varlimsup_{\xi\to z} \varlimsup_{|n|\to\infty} \frac{\ln |T(z^n)(\xi)|}{|n|}$$

where we have used the multi index notation $z^n = z_1^{n_1} \cdots z_d^{n_d}$ for $n = (n_1, \ldots, n_d)$ and $|n| = n_1 + \cdots + n_d$. In the case when $W = \Delta^d$, it is readily seen that this plurisubharmonic function takes negative values. With the above notation we have :

THEOREM 1. — *For a continuous linear operator $T$ from some $\mathcal{O}(\Delta^d)$ into $\mathcal{O}(V)$ the following conditions are equivalent :*

(i) *There exists a continuous linear operator $\widetilde{T} : \mathcal{O}(\Delta^d) \to \mathcal{O}(M)$ such that $R \circ \widetilde{T} = T$ where $R$ is the restriction operator from $\mathcal{O}(M)$ onto $\mathcal{O}(V)$.*

(ii) *There exists a negative plurisubharmonic function $\rho$ on $M$ such that $\rho_T \leq p|V$ on $V$.*

*Proof.*
(i) $\Rightarrow$ (ii). Let

$$\rho(z) = \varlimsup_{\xi\to z} \varlimsup_{|n|\to\infty} \frac{\ln |\widetilde{T}(z^n)(\xi)|}{|n|}.$$

Then $\rho$ is a plurisubharmonic function on $M$ and in view of the fact that $\widetilde{T}$ is an extension of $T$ one has

$$\rho_T(z) \leq \rho(z) \quad \text{for } z \in V.$$

(ii) $\Rightarrow$ (i). Using multi index notation we set $e_n \doteq z_1^{n_1} \cdots z_n^{n_d}$, $f_n \doteq T(e_n) \in \mathcal{O}(V)$ for $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Now choose a negative plurisubharmonic function $\Phi : M \to \mathbb{R}$ with the property that

$$\varlimsup_{\xi\to z} \varlimsup_{|n|\to\infty} \frac{\ln |f_n(\xi)|}{|n|} = \rho_T(z) < \Phi(z)$$

for all $z \in V$. Let

$$\Omega_V = \Big\{ (z, w) \in M \times \mathbb{C}^d ;\ z \in V,\ \max_{1\leq i\leq d} |w_i| \doteq \|w\| < \mathrm{e}^{-\Phi(z)} \Big\}.$$

Fix $(z_0, w_0) \in \Omega_V$ with $\rho_T(z_0) \neq -\infty$, say $\|w_0\| < \mathrm{e}^{-\Phi(z_0)-\delta}$ for some $\delta > 0$. We choose an $\epsilon > 0$ with $2\epsilon < \delta$ and find a neighborhood $\widehat{U}_1$ of $z_0$ in $V$ such that

(i) $\sup_{\xi\in\widehat{U}_1} \rho_T(z_0) < -\epsilon$,

(ii) $\sup_{\xi\in\widehat{U}_1} \Phi(\xi) \leq \Phi(z_0) + \epsilon$.

Now Hartog's lemma ([8, p. 21], cf. [12]) implies the existence of a neighborhood $U_1 \subset\subset \widehat{U}_1$ of $z_0$ in $V$ such that

$$\sup_{\xi\in U_1} \frac{\ln|f_n(\xi)|}{|n|} \leq \rho_T(z_0) + \epsilon \quad \text{for } n \text{ large.}$$

Fix a neighborhood $U_2$ of $w_0$ in $\mathbb{C}^d$ such that $\sup_{w\in U_2} \|w\| < \mathrm{e}^{-\Phi(z_0)-\delta_\epsilon}$. Now let $U = U_1 \times U_2 \subseteq M \times \mathbb{C}^d$. For $(\xi, w) \in U$ we have

$$\|w\| < \mathrm{e}^{-\Phi(z_0)-\delta+\epsilon} \leq \mathrm{e}^{-\Phi(\xi)+\epsilon-\delta+\epsilon} < \mathrm{e}^{-\Phi(\xi)}$$

so $U \subseteq \Omega_V$. Moreover for large $n$, we have :

$$\sup_{(\xi,w)\in U} \big|f_n(\xi)\big|\, |w_1^{n_1}\cdots w_d^{n_d}| \leq \mathrm{e}^{|n|\{\rho_T(z_0)-\Phi(z_0)+2\epsilon-\delta\}}.$$

An estimate of this kind can also be easily obtained in the case when $\rho_T(z_0) = -\infty$. It follows that the function $F$ defined by a locally uniformly convergent infinite series via the formula

$$F(z,w) \doteq \sum_{n\in\mathbb{N}^d} f_n(z) w^n$$

is an analytic function on $\Omega_V$. We set :

$$\Omega_M = \big\{(z,w) \in M \times \mathbb{C}^d\, ;\ \|w\| < \mathrm{e}^{-\Phi(z)}\big\}.$$

Then $\Omega_M$ is a Stein space (see [5, Thm 5.4]) and $\Omega_V$ is a closed analytic subvariety of $\Omega_M$.

In view of Cartan theorem B, there exists an analytic function $G$ on $\Omega_M$ such that $G$ restricted to $\Omega_V$ is $F$. This function can be represented in the usual way, as a convergent (uniformly on compacta of $\Omega_M$) infinite series via the formula

$$G(z,w) = \sum_{n\in\mathbb{N}^d} a_n(z)\, w_1^{n_1}\cdots w_d^{n_d}$$

where

$$a_n(z) = \frac{1}{(2\pi i)^d} \int\!\cdots\!\int_{|\xi|=r} \frac{G(z,\xi_1,\ldots,\xi_d)}{\prod \xi_j^{n_j+1}}\, \mathrm{d}\xi_1\cdots \mathrm{d}\xi_d,$$

with $0 < r < e^{-\Phi(z)}$ and $n \in \mathbb{N}^d$. Since for $z \in V$, one has

$$\sum_n a_n(z)\, w^n = \sum_n f_n(z) w^n$$

on the polydisc $\Delta(0, e^{-\Phi(z)})$, we conclude that $a_n(z) = f_n(z)$ for all $z \in V$ and $n \in \mathbb{N}^d$; in other words the analytic function $a_n \in \mathcal{O}(M)$ is an extension of $f_n \in \mathcal{O}(V)$ for each $n \in \mathbb{N}^d$.

Moreover, in view of the Cauchy inequalities applied to $G(z,\cdot)$, $z \in M$ we have :

$$\overline{\lim_{|n|\to\infty}} \frac{\ln |a_n(z)|}{|n|} \le \Phi(z). \tag{1}$$

Now fix a compact set $K$ of $M$ and choose another compact subset $\widehat{K}$ of $M$, such that $K \subset \widehat{K}$. Set

$$\max_{z\in\widehat{K}} \Phi(z) \doteq -\alpha.$$

We fix an $\beta > 0$, with $\beta < \alpha$. In view of Hartog's lemma and (1) above for $|n|$ large enough we have :

$$\sup_{z\in K} \frac{\ln |a_n|}{|n|} \le -\alpha + \beta.$$

It follows that for every compact subset $K$ of $M$ there exists an $R(K) < 1$ and a $C > 0$ such that :

$$\sup_{z\in K} \bigl|a_n(z)\bigr| \le C \sup_{\|z\|\le R(K)} \bigl|e_n(z)\bigr|. \tag{2}$$

But this means that the linear operator $\widetilde{T}$ defined from $\mathcal{O}(\Delta^d)$ into $\mathcal{O}(M)$ by the formula $\widetilde{T}(e_n) \doteq a_n$, for $n \in \mathbb{N}^d$, is a continuous operator satisfying $R \circ \widetilde{T} = T$. This finishes the proof of the THEOREM 1. $\square$

The above result can also be interpreted as giving a description of the kernel of the operator $\delta$ appearing in the long exact sequence (1). Our next result gives a necessary and sufficient condition for this operator to be the zero operator. But first we need a lemma on the structure of plurisubharmonic functions on Stein spaces.

LEMMA 1. — *Let $X$ be a Stein space and $\rho$ a plurisubharmonic function on $X$. Then there exists a sequence $\{f_n\}_n$ of holomorphic functions on $X$ and a sequence of integers $\{c_n\}_n$ such that*

$$\rho(z) = \overline{\lim_n} \frac{\ln|f_n(z)|}{c_n}, \qquad z \in X.$$

*Proof.* — First we will show that the possibility of approximating a continuous plurisubharmonic function on compact subsets by Hartog's type functions, which is well known for domains of holomorphy in $\mathbb{C}^N$, (see [9, p. 55]), is also valid for Stein spaces. To this end let us fix a continuous plurisubharmonic function $\psi$ on $X$, and a holomorphically convex compact subset $K \subseteq X$. Choose a Oka-Weil domain $\mathcal{P}$, such that $K \subseteq \mathcal{P} \subset\subset X$, and fix a holomorphic mapping $\Phi : X \to \mathbb{C}^N$ such that $\Phi$ restricted to $\mathcal{P}$ is a biholomorphism onto a closed subvariety $V$ of the unit polydisc $\Delta^N \subseteq \mathbb{C}^N$. We can think of $\psi$ as a plurisubharmonic function on $V$. Arguing as in the proof of Theorem 5.3.1 of [5] we find a Stein domain $\Omega$ of $\Delta^N$ containing $V$ and a plurisubharmonic function $\widetilde{\psi}$ on $\Omega$ such that $\widetilde{\psi}_{|V} = \psi$. Although $\widetilde{\psi}$ need not be continuous on $\Omega$ representing it on compacta as a pointwise limit of a decreasing sequence of continuous plurisubharmonic functions and observing that on $K_1 \doteq \psi(K)$ the convergence is uniform, in view of [9, p. 55] for a given $\epsilon > 0$, we can find analytic functions $f_1, \dots, f_s$ near $K$, and integers $c_1, \dots, c_s$ such that :

$$\psi(z) - \epsilon \le \max_{1 \le i \le s} \frac{\ln|f_i(z)|}{c_i} \le \psi(z) + \epsilon, \qquad \forall z \in K.$$

Now fix a point $z_0 \in K$ and choose an $f_j$ and $c_j$ such that :

$$\psi(z_0) - \epsilon \le \frac{\ln|f_j(z_0)|}{c_i} \le \psi(z_0) + \epsilon.$$

Since $\psi$ is continuous we can find a ball $U$ around $z_0$ such that :

$$\text{(3)} \qquad e^{c_j(\psi(z) - 2\epsilon)} < \left|f_j(z)\right| \qquad \text{for } z \in U.$$

By approximating $f_j$ on the holomorphically convex compact set $K \cup \overline{U}$ uniformly by global analytic functions we can find an $F \in \mathcal{O}(X)$ such that (3) holds with $f_j$ replaced by $F$ and also

$$\psi(z) + 2\epsilon \ge \log \frac{|F(z)|}{c_j}, \qquad z \in K.$$

Now cover $K$ with balls constructed above to get for a given $\epsilon > 0$ analytic functions $F_1, \dots, F_k$ on $X$ and integers $c_1, \dots, c_k$ such that :

$$\psi(z) - 2\epsilon < \max_{1 \le j \le k} \left\{ \frac{\ln |F_j(z)|}{c_j} \right\} \le \psi(z) + 2\epsilon, \qquad z \in K$$

Hence Proposition 2 of [9] is valid also for Stein spaces.

Now let $\rho$ be a given plurisubharmonic function on $X$. In view of Theorem 5.5 of [5] there exists a sequence of continious plurisubharmonic functions $\{\rho_n\}$ that decrease pointwise to $\rho$. Choose an exhaustion of $X$ by holomorphically convex compact sets $\{K_n\}_n$. Fix a sequence of positive numbers $\{\epsilon_n\}_n$ that decrease to zero. For each $n$ there exists analytic functions $F_1^n, \dots, F_{\rho(n)}^n$ and integers $c_1^n, \dots c_{\rho(n)}^n$ such that :

$$\rho_n(z) - \epsilon_n \le \max_{1 \le i \le \rho(n)} \frac{\ln |F_i^n(z)|}{c_i^n} \le \rho_n(z) + \epsilon_n \qquad \forall z \in K_n.$$

We enumerate $\{F_i^n\}_{i,n}$ (similarly $\{c_i^n\}_{i,n}$) as

$$\{F_1', \dots, F_{\rho(1)}', \dots, F_1^n, \dots, F_{\rho(n)}^n \cdots\}$$

and denote the resulting sequence by $\{G_\alpha\}_\alpha$, (similarly $\{c_\alpha\}_\alpha$). Set :

$$\gamma_\alpha(z) = \frac{\ln |G_\alpha(z)|}{c_\alpha}.$$

Now fix a point $z \in X$, say $z \in K_N$. Let $n > N$ and

$$k = \sum_{i=1}^{n-1} \rho(i) + 1.$$

Since $K_N \subset K_n$ we have

$$\rho_n(z) - \epsilon_n \le \max_{1 \le i \le p(n)} \frac{\ln |F_i^n(z)|}{c_i^n} \le \rho_n(z) + \epsilon_n.$$

Hence

$$\rho(z) - \epsilon_n \le \sup_{\alpha > k} \gamma_\alpha(z)$$

and so

$$\rho(z) - \epsilon_n \le \inf_s \sup_{\alpha > s} \gamma_\alpha(z). \tag{4}$$

On the other hand choose any $\alpha$ with $\alpha > k$, with $k$ as above, then

$$\gamma_\alpha(z) = \frac{\ln|F_i^s(z)|}{c_i^s}$$

for some $s \geq n$. So we have $\gamma_\alpha(z) \leq \rho_s(z) + \epsilon_s = \rho_n(z) + \epsilon_n$; hence $\sup_{\alpha>k} \gamma_\alpha(z) \leq \rho_n(z) + \epsilon_n$. It follows that :

$$\inf_t \sup_{\alpha>t} \gamma_\alpha(z) \leq \inf_n(\rho_n(z) + \epsilon_n) = \rho(z). \tag{5}$$

So combining (4) and (5) and setting $f_n \doteq G_n$ we get :

$$\rho(z) = \overline{\lim}_n \frac{\ln|f_n(z)|}{c_n}.$$

This finishes the proof of the Lemma. □

COROLLARY 1. — *Let $M$ be a Stein space and $V$ a closed subvariety of $M$. Then the following are equivalent :*

(i) *For every Stein space $W$ for which $\mathcal{O}(W)$ is a finite type power series space and for every continuous linear operator $T : \mathcal{O}(W) \to \mathcal{O}(V)$ there exists a continuous linear operator $\widehat{T} : \mathcal{O}(W) \to \mathcal{O}(M)$ such that $R \circ \widehat{T} = T$ where $R$ is the restriction operator from $\mathcal{O}(M)$ into $\mathcal{O}(V)$.*

(ii) *For every negative plurisubharmonic function $\rho$ on $V$ there exists a negative plurisubharmonic function $\hat{\rho}$ on $M$ such that $\rho \leq \hat{\rho}_{|V}$.*

*Proof.* — In view of THEOREM 1 we only need to prove the implication (i) ⇒ (ii). To this end we fix a negative plurisubharmonic function $\rho$ on $V$. In view of the LEMMA we can find a sequence $\{f_n\}_n$ of analytic functions on $V$, and a sequence of positive integers $\{c_n\}_n$, with $c_n \uparrow \infty$ such that :

$$\rho(z) = \overline{\lim}_n \frac{\ln|f_n(z)|}{c_n}.$$

In view of Hartog's lemma for every compact set $K$ in $V$ there exists a negative number $\alpha$ and a constant $c > 0$ such that, for all $n$,

$$\sup_{z\in K} |f_n(z)| \leq c\, e^{\alpha c_n}. \tag{6}$$

Hence the assignment

$$T(z^n) = \begin{cases} 0 & \text{if } n \notin \{c_k\}_k, \\ f_{c_s} & \text{if } n = c_s \text{ for some } s \end{cases}$$

defines, in view of (6), a continuous linear operator $T : \mathcal{O}(\Delta) \to \mathcal{O}(V)$. We fix a $\widehat{T} : \mathcal{O}(\Delta) \to \mathcal{O}(M)$ with $\widehat{T}_{|V} = T$ and let as usual

$$\rho_{\widehat{T}}(z) = \overline{\lim_{\xi \to z}}\ \overline{\lim_{n}}\ \frac{\ln |\widehat{T}(z^n)(\xi)|}{n}.$$

Since $\rho = \rho_T$, the argument given in (i) $\Rightarrow$ (ii) of THEOREM 1 shows that $\rho \leq \rho_{\widehat{T}|V}$. This finishes the proof of COROLLARY 1. $\square$

The above corollary can be used to characterize among the *hyperconvex varieties* $V$ of a Stein space $M$ (i.e. the varieties $V$ such that $\mathcal{O}(V)$ is a finite type power series space, see [2]) the ones which admit a continuous linear extension operator $\mathcal{E} : \mathcal{O}(V) \to \mathcal{O}(M)$. Recall that for a Stein space $X$ and a compact set $K \subset X$ the plurisubharmonic function :

$$w_K^X(z) \doteq \overline{\lim}_{\xi \to z} \sup\Big\{ u(\xi) : u \in \mathrm{PSH}(X),\ u \leq 0 \text{ on } X \text{ and } u \leq -1 \text{ on } K \Big\}$$

is called the *plurisubharmonic measure* ($\mathcal{P}$-measure) of $K$ relative to $X$ (see eg. [4], [13], [18]). These functions are natural complex counterparts of harmonic measures of classical potential theory. Since any negative plurisubharmonic function on a Stein space is dominated by a constant multiple of a $\mathcal{P}$-measure one can reexpress the condition (ii) above using $\mathcal{P}$-measures to obtain :

COROLLARY 2. — *Let $M$ be a Stein space and $V$ a hyperconvex subvariety of $M$. Then the following conditions are equivalent* :

(i) *There exists a continuous linear extension operator*

$$\mathcal{E} : \mathcal{O}(V) \longrightarrow \mathcal{O}(M).$$

(ii) *There exists compact sets $K \subseteq V$, $S \subseteq M$ with non empty interiors and a constant $C > 0$ such that* :

$$w_K^V \leq C\, w_S^M{}_{|V}.$$

REMARKS.

(i) Although we have chosen to treat the case when $\mathcal{O}(W)$ is isomorphic to an infinite type power series space by making use of some general considerations, we note that the line of reasoning given in the proof of THEOREM 1 can also be used in this case. Indeed the existence of an operator $\widehat{T} : \mathcal{O}(W) \to \mathcal{O}(M)$ with $R\circ\widehat{T} = T$ for any $T : \mathcal{O}(W) \to \mathcal{O}(V)$ can be deduced, in this case, from the fact that for any plurisubharmonic function $\rho$ on $V$ there exists a plurisubharmonic function $\hat{\rho}$ on $M$ such that $\rho \leq \hat{\rho}_{|V}$.

(ii) In the case when $\mathcal{O}(M)$ is isomorphic to an infinite type power series space and when $W$ is hyperconvex, THEOREM 1 characterizes the operators $T$ for which such a $\widehat{T}$ exists as the ones for which $\sup_{z\in V} \rho_T(z) < 0$. This family is precisely the family of all *compact operators* from $\mathcal{O}(W)$ into $\mathcal{O}(V)$. This can also be derived from the general extention properties of compact operators and the fact that every continuous operator from a finite type power series space into an infinite type power series space is compact.

(iii) For a smoothly bounded relatively compact domain $D$ with $C^2$ boundary in a Stein manifold and a negative plurisubharmonic function $\rho$ on $D$ one has that

$$\rho(z) < C\big\{-d(z, \partial D)\big\}, \qquad z \in D$$

for some constant $C > 0$ where $d(z, \partial D)$ is the distance of $z$ from $\partial D$ (see [10, Lemma 3.2]). Hence in the case when $D$ is given by $D = \{z : u(z) < 0\}$, for some $C^2$ plurisubharmonic function $u$ defined in a neighborhood of $\overline{D}$, we have that any negative plurisubharmonic function on $D$ is dominated by a positive constant multiple of $u$, since $-d(\,\cdot\,, \partial D)$ is dominated by a positive constant multiple of $u$. This property remains valid for submanifolds of $D$ of the form $D \cap M'$ where $M'$ is a closed complex submanifold in a neighborhood of $\overline{D}$ which intersects $\partial D$ transversally since in this case $D \cap M' = \{z \in M' : u(z) < 0\}$. Now combining Corollary 5 of [2] with Corollary 2 above we obtain the following slight generalization of Theorem 4.2 of [10].

COROLLARY 3. — *Let $M$ be a Stein manifold and $D \subset\subset M$ a smoothly bounded domain in $M$ of the form $D = \{z : u(z) < 0\}$ for some $C^2$ plurisubharmonic function defined in a neighborhood of $\overline{D}$. For a complex manifold $M'$ in a neighborhood of $\overline{D}$ which intersects $\partial D$ transversally there exists a continuous linear extension operator $\mathcal{E} : \mathcal{O}(D\cap M') \to \mathcal{O}(D)$.*

Even if we drop the transversality condition in the above corollary we can still get some information about the class of continuous linear operators $T : \mathcal{O}(\Delta^d) \to \mathcal{O}(D \cap M')$ which admit a continuous linear extension operator, namely these are preciselsy the operators for which $\rho_T \leq Cu$ on $D \cap M'$ for some $C > 0$. This observation can be used in constructing concrete operators for which no such $\widehat{T}$ exists. For example following

Example 5.3 of [10], let

$$D = \{(z\,;w) \in \mathbb{C}^2\,;\ |z|^2 + |w-1|^2 < 1\}$$

and

$$M' = \{(z,w) \in \mathbb{C}^2\,;\ w = z^2\}.$$

Then the operator $T : \mathcal{O}(\Delta) \to \mathcal{O}(D \cap M')$ defined as $T(f)(z,w) \doteq f(\mathrm{e}^{-z^3})$ admits no extension operator $\widehat{T} : \mathcal{O}(\Delta) \to \mathcal{O}(D)$, since, an easy computation shows the impossibility of finding a $C > 0$ satisfying

$$\rho_T(z,w) = \ln|\mathrm{e}^{-z^3}| \le C\{|z|^2 + |w-1|^2 - 1\}.$$

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