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Extension operators for analytic functions defined on certain closed subvarieties of a Stein space


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EXTENSION OPERATORS FOR ANALYTIC FUNCTIONS DEFINED ON CERTAIN CLOSED SUBVARIETIES OF A STEIN SPACE

BY

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ABSTRACT. — Let $M$ be an irreducible Stein space and let $V$ a closed subvariety of $M$ with the property that $\mathcal{O}(V)$ is a power series space. In this paper we give a necessary and sufficient condition for the existence of a continuous linear extension operator from $\mathcal{O}(V)$ into $\mathcal{O}(M)$ in terms of plurisubharmonic functions defined on these varieties. Actually we obtain these results by solving a general lifting problem. We also consider some consequences of these results.

0. — Let $M$ be an irreducible Stein space and $V$ a closed subvariety of $M$. One of the consequences of the Oka-Cartan theory is that every analytic function on $V$ can be extended to an analytic function on $M$. The question as to whether this extension process can be achieved by a continuous linear extension operator was studied by various authors.

Such a continuous operator if it exists, will imbed the Fréchet space of all analytic functions on $V, \mathcal{O}(V)$, into $\mathcal{O}(M)$ as a closed complemented subspace. In some cases this simple observation exhibits an obstruction, for the existence of a continuous linear extension operator. This situation

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occurs for example, when \( \mathcal{O}(V) \) has no continuous norm (i.e. when \( V \) has infinite number of irreducible components) or when every continuous linear mapping from \( \mathcal{O}(V) \) into \( \mathcal{O}(M) \) is compact (see [9]). On the other hand positive answers in the cases:

(i) when \( M \) is a strictly pseudoconvex domain in a Stein manifold and \( V \) is of the form \( V = M \cap \tilde{V} \) where \( \tilde{V} \) is a closed submanifold near \( M \) intersecting \( \partial M \) transversally, and

(ii) when \( M = \mathbb{C}^n \) and \( V \) a closed submanifold for which \( \mathcal{O}(V) \), is isomorphic to \( \mathcal{O}(\mathbb{C}^d) \) for some \( d \) as Fréchet spaces, e.g. when \( V \) is a smooth algebraic variety (see [17]),

were obtained in [10] by using \( \bar{\partial} \)-methods. In both of the cases considered above, the spaces \( \mathcal{O}(V) \) turns out to belong to a well studied and well understood class of Fréchet spaces. A power series space is a sequence space of the form

\[
\Lambda_R(\alpha) = \left\{ x = \{x_n\}_{n=1}^{\infty} ; \|x\|_r = \sum_{k=1}^{\infty} |x_k| r^{\alpha_k} < +\infty \right\}
\]

where \( 0 < R \leq +\infty \) and \( \alpha = \{\alpha_n\} \) is an increasing unbounded sequence of positive numbers. The space \( \Lambda_R(\alpha) \) equipped with the norms \( \| \cdot \|_r \), for \( 0 < r < R \) is a Fréchet space. It is easy to see that for a fixed \( \alpha \), the spaces \( \Lambda_R(\alpha) \), for \( 0 < R < +\infty \), are all isomorphic to each other and so we have two types of power series spaces; the ones that are isomorphic to \( \Lambda_1(\alpha) \), (finite type), and the ones that are isomorphic to \( \Lambda_\infty(\alpha) \) (infinite type). A large number of Fréchet function spaces occurring in analysis are actually power series spaces [14]. In the case (i) considered above, \( \mathcal{O}(V) \) is (isomorphic to) \( \Lambda_1(n^{1/d}) \) and in the case (ii) is \( \Lambda_\infty(n^{1/d}) \) where in both cases \( d \) is the dimension of \( V \).

In this article we shall investigate the above mentioned question in the case when \( \mathcal{O}(V) \) is a power series space. More generally we consider for a given data \((M, V, W, T)\) consisting of a irreducible Stein space \( M \), a subvariety \( V \) of \( M \), a Stein space \( W \) for which \( \mathcal{O}(W) \) is a power series space and a continuous linear operator \( T \) from \( \mathcal{O}(W) \) into \( \mathcal{O}(V) \), the problem of finding a continuous linear operator \( \tilde{T} \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{O}(M) & \xrightarrow{T} & \mathcal{O}(V) \\
\mathcal{O}(W) & \xrightarrow{\tilde{T}} & \mathcal{O}(V) \\
R & \downarrow & \\
\mathcal{O}(V) & \end{array}
\]

where \( R \) is the restriction operator. Observe that in the special case
$W = V$ and $T = I$ the identity of $\mathcal{O}(V)$, if it exists, is a continuous linear extension operator. The obstruction to finding $\tilde{T}$ for an arbitrary $T$ in the above set up is due to the non vanishing of the first derived functor $\text{Ext}^1(\cdot, \cdot)$ of the functor $\text{Pro}$ in the terminology of the locally convex homological algebra developed by Palamadov [11] (cf. [15]). Indeed in the above set up, denoting by $I(V)$ the ideal of the variety $V$, the short exact sequence

$$0 \to I(V) \to \mathcal{O}(M) \xrightarrow{R} \mathcal{O}(V) \to 0$$

gives rise to the exact sequence

$$0 \to L(\mathcal{O}(W), I(V)) \to L(\mathcal{O}(W), \mathcal{O}(M))$$

$$\to L(\mathcal{O}(W), \mathcal{O}(V)) \xrightarrow{\delta} \text{Ext}^1(\mathcal{O}(W), I(V))$$

$$\to \text{Ext}^1(\mathcal{O}(W), \mathcal{O}(M)) \to \text{Ext}^1(\mathcal{O}(W), \mathcal{O}(V)) \to 0$$

where $L(E, F)$ denotes the space of all continuous linear operators from $E$ into $F$ (see [15]). For a nuclear Fréchet space $E$, $\text{Ext}^1(E, I(V))$ can be identified with the first Čech cohomology group of the sheaf $I^E(V)$, of germs of $E^*$ valued analytic functions on $M$ that vanish on $V$ (see for example [1]). Hence the possible non vanishing of $\text{Ext}^1$ in this case reflects the failure of the Cartan theorem (B) for $E^*$ valued coherent analytic sheaves on $M$. Various conditions on the pair of Fréchet spaces which assure the vanishing of this derived functor are given in [15] (see also [1]). In particular the vanishing of $\text{Ext}^1(\mathcal{O}(W), I(V))$ when $\mathcal{O}(W)$ is a power series space of infinite type follows from these general considerations (see also Remark 1). Hence in the above mentioned set up we will restrict our attention to Stein spaces $W$ for which $\mathcal{O}(W)$ is isomorphic to a finite type power series space.

We shall use the standard terminology and notation of complex analysis as in [6], [7] except perhaps in our usage of the term Stein space. In this note by a Stein space we mean a reduced, irreducible Stein space in the sense of [6] which has a Hausdorff, separable topology.

Some results of this work was announced in [3].

1. — Returning to our problem, let us fix a Stein space $M$, a closed subvariety $V$ of $M$ and a Stein space $W$ for which $\mathcal{O}(W)$ is a power series space. Since we will be investigating the extendibility of continuous linear operators from $\mathcal{O}(W)$ into $\mathcal{O}(V)$, we can, without loss of generality take $W$ to be either $\Delta^d$, the unit polydisc in $\mathbb{C}^d$, or $\mathbb{C}^d$ itself depending
upon the type of the power series space $\mathcal{O}(W)$, where $d = \text{dim } W$. In both case a continuous linear operator $T$ from $\mathcal{O}(W)$ into $\mathcal{O}(V)$ induces a plurisubharmonic function $\rho_T$ on $V$ via the formula

$$\rho_T(z) = \lim_{\xi \to z} \lim_{|n| \to \infty} \frac{\ln |T(z^n)(\xi)|}{|n|}$$

where we have used the multi index notation $z^n = z_1^{n_1} \cdots z_d^{n_d}$ for $n = (n_1, \ldots, n_d)$ and $|n| = n_1 + \cdots + n_d$. In the case when $W = \Delta^d$, it is readily seen that this plurisubharmonic function takes negative values.

With the above notation we have:

**Theorem 1.** For a continuous linear operator $T$ from some $\mathcal{O}(\Delta^d)$ into $\mathcal{O}(V)$ the following conditions are equivalent:

(i) There exists a continuous linear operator $\tilde{T} : \mathcal{O}(\Delta^d) \to \mathcal{O}(M)$ such that $R \circ \tilde{T} = T$ where $R$ is the restriction operator from $\mathcal{O}(M)$ onto $\mathcal{O}(V)$.

(ii) There exists a negative plurisubharmonic function $\rho$ on $M$ such that $\rho_T \leq p|V|$ on $V$.

**Proof.**

(i) $\Rightarrow$ (ii). Let

$$\rho(z) = \lim_{\xi \to z} \lim_{|n| \to \infty} \frac{\ln |T(z^n)(\xi)|}{|n|}.$$  

Then $\rho$ is a plurisubharmonic function on $M$ and in view of the fact that $\tilde{T}$ is an extension of $T$ one has

$$\rho_T(z) \leq \rho(z) \quad \text{for } z \in V.$$

(ii) $\Rightarrow$ (i). Using multi index notation we set $e_n = z_1^{n_1} \cdots z_d^{n_d}$, $f_n = T(e_n) \in \mathcal{O}(V)$ for $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Now choose a negative plurisubharmonic function $\Phi : M \to \mathbb{R}$ with the property that

$$\lim_{\xi \to z} \lim_{|n| \to \infty} \frac{\ln |f_n(\xi)|}{|n|} = \rho_T(z) < \Phi(z)$$

for all $z \in V$. Let

$$\Omega_V = \left\{ (z, w) \in M \times \mathbb{C}^d ; z \in V, \max_{1 \leq i \leq d} |w_i| = \|w\| < e^{-\Phi(z)} \right\}.$$  

Fix $(z_0, w_0) \in \Omega_V$ with $\rho_T(z_0) \neq -\infty$, say $\|w_0\| < e^{-\Phi(z_0) - \delta}$ for some $\delta > 0$. We choose an $\epsilon > 0$ with $2\epsilon < \delta$ and find a neighborhood $\widehat{U}_1$ of $z_0$ in $V$ such that

$$\text{TOME 123 — 1995 — N° 2}$$
(i) \( \sup_{\xi \in \hat{U}_1} \rho_T(z_0) < -\epsilon, \)

(ii) \( \sup_{\xi \in \hat{U}_1} \Phi(\xi) \leq \Phi(z_0) + \epsilon. \)

Now Hartog’s lemma ([8, p. 21], cf. [12]) implies the existence of a neighborhood \( U_1 \subset \subset \hat{U}_1 \) of \( z_0 \) in \( V \) such that

\[
\sup_{\xi \in U_1} \frac{\ln |f_n(\xi)|}{|n|} \leq \rho_T(z_0) + \epsilon \quad \text{for } n \text{ large.}
\]

Fix a neighborhood \( U_2 \) of \( w_0 \) in \( \mathbb{C}^d \) such that \( \sup_{w \in U_2} \|w\| < e^{-\Phi(z_0) - \delta}. \)

Now let \( U = U_1 \times U_2 \subset M \times \mathbb{C}^d \). For \( (\xi, w) \in U \) we have

\[
\|w\| < e^{-\Phi(z_0) - \delta + \epsilon} \leq e^{-\Phi(\xi) + \epsilon - \delta + \epsilon} < e^{-\Phi(\xi)}
\]

so \( U \subset \Omega_V \). Moreover for large \( n \), we have:

\[
\sup_{(\xi, w) \in U} |f_n(\xi)| |w_1^{n_1} \cdots w_d^{n_d}| \leq e^{\ln \{\rho_T(z_0) - \Phi(z_0) + 2\epsilon - \delta\}}.
\]

An estimate of this kind can also be easily obtained in the case when \( \rho_T(z_0) = -\infty \). It follows that the function \( F \) defined by a locally uniformly convergent infinite series via the formula

\[
F(z, w) = \sum_{n \in \mathbb{N}^d} f_n(z) w^n
\]

is an analytic function on \( \Omega_V \). We set:

\[
\Omega_M = \{(z, w) \in M \times \mathbb{C}^d ; \|w\| < e^{-\Phi(z)}\}.
\]

Then \( \Omega_M \) is a Stein space (see [5, Thm 5.4]) and \( \Omega_V \) is a closed analytic subvariety of \( \Omega_M \).

In view of Cartan theorem B, there exists an analytic function \( G \) on \( \Omega_M \) such that \( G \) restricted to \( \Omega_V \) is \( F \). This function can be represented in the usual way, as a convergent (uniformly on compacta of \( \Omega_M \)) infinite series via the formula

\[
G(z, w) = \sum_{n \in \mathbb{N}^d} a_n(z) w_1^{n_1} \cdots w_d^{n_d}
\]

where

\[
a_n(z) = \frac{1}{(2\pi i)^d} \int_{|\xi|=r} \cdots \int_{|\xi|=r} \frac{G(z, \xi_1, \ldots, \xi_d)}{\prod \xi_j^{n_j+1}} \, d\xi_1 \cdots d\xi_d,
\]

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with $0 < r < e^{-\Phi(z)}$ and $n \in \mathbb{N}^d$. Since for $z \in V$, one has

$$\sum_n a_n(z) w^n = \sum_n f_n(z)w^n$$

on the polydisc $\Delta(0, e^{-\Phi(z)})$, we conclude that $a_n(z) = f_n(z)$ for all $z \in V$ and $n \in \mathbb{N}^d$; in other words the analytic function $a_n \in \mathcal{O}(M)$ is an extension of $f_n \in \mathcal{O}(V)$ for each $n \in \mathbb{N}^d$.

Moreover, in view of the Cauchy inequalities applied to $G(z, \cdot)$, $z \in M$ we have:

$$\lim_{|n| \to \infty} \frac{\ln |a_n(z)|}{|n|} \leq \Phi(z).$$

Now fix a compact set $K$ of $M$ and choose another compact subset $\hat{K}$ of $M$, such that $K \subseteq \hat{K}$. Set

$$\max_{z \in \hat{K}} \Phi(z) \doteq -\alpha.$$

We fix an $\beta > 0$, with $\beta < \alpha$. In view of Hartog’s lemma and (1) above for $|n|$ large enough we have:

$$\sup_{z \in K} \frac{\ln |a_n|}{|n|} \leq -\alpha + \beta.$$

It follows that for every compact subset $K$ of $M$ there exists an $R(K) < 1$ and a $C > 0$ such that:

$$\sup_{z \in K} |a_n(z)| \leq C \sup_{\|z\| \leq R(K)} |e_n(z)|.$$

But this means that the linear operator $\tilde{T}$ defined from $\mathcal{O}(\Delta^d)$ into $\mathcal{O}(M)$ by the formula $\tilde{T}(e_n) \doteq a_n$, for $n \in \mathbb{N}^d$, is a continuous operator satisfying $R \circ \tilde{T} = T$. This finishes the proof of the Theorem 1.

The above result can also be interpreted as giving a description of the kernel of the operator $\delta$ appearing in the long exact sequence (1). Our next result gives a necessary and sufficient condition for this operator to be the zero operator. But first we need a lemma on the structure of plurisubharmonic functions on Stein spaces.
LEMMA 1. — Let $X$ be a Stein space and $\rho$ a plurisubharmonic function on $X$. Then there exists a sequence $\{f_n\}_n$ of holomorphic functions on $X$ and a sequence of integers $\{c_n\}_n$ such that

$$\rho(z) = \lim_{n \to \infty} \frac{\ln |f_n(z)|}{c_n}, \quad z \in X.$$ 

Proof. — First we will show that the possibility of approximating a continuous plurisubharmonic function on compact subsets by Hartog's type functions, which is well known for domains of holomorphy in $\mathbb{C}^N$, (see [9, p. 55]), is also valid for Stein spaces. To this end let us fix a continuous plurisubharmonic function $\psi$ on $X$, and a holomorphically convex compact subset $K \subseteq X$. Choose a Oka-Weil domain $\mathcal{P}$, such that $K \subseteq \mathcal{P} \subset X$, and fix a holomorphic mapping $\Phi : X \to \mathbb{C}^N$ such that $\Phi$ restricted to $\mathcal{P}$ is a biholomorphism onto a closed subvariety $V$ of the unit polydisc $\Delta^N \subseteq \mathbb{C}^N$. We can think of $\psi$ as a plurisubharmonic function on $V$. Arguing as in the proof of Theorem 5.3.1 of [5] we find a Stein domain $\Omega$ of $\Delta^N$ containing $V$ and a plurisubharmonic function $\tilde{\psi}$ on $\Omega$ such that $\tilde{\psi}|_V = \psi$. Although $\psi$ need not be continuous on $\Omega$ representing it on compacta as a pointwise limit of a decreasing sequence of continuous plurisubharmonic functions and observing that on $K = \psi(K)$ the convergence is uniform, in view of [9, p. 55] for a given $\epsilon > 0$, we can find analytic functions $f_1, \ldots, f_s$ near $K$, and integers $c_1, \ldots, c_s$ such that:

$$\psi(z) - \epsilon \leq \max_{1 \leq i \leq s} \frac{\ln |f_i(z)|}{c_i} \leq \psi(z) + \epsilon, \quad \forall z \in K.$$ 

Now fix a point $z_0 \in K$ and choose an $f_j$ and $c_j$ such that:

$$\psi(z_0) - \epsilon \leq \frac{\ln |f_j(z_0)|}{c_j} \leq \psi(z_0) + \epsilon.$$ 

Since $\psi$ is continuous we can find a ball $U$ around $z_0$ such that:

$$e^{c_j(\psi(z) - 2\epsilon)} < |f_j(z)| \quad \text{for} \quad z \in U.$$ 

By approximating $f_j$ on the holomorphically convex compact set $K \cup \overline{U}$ uniformly by global analytic functions we can find an $F \in \mathcal{O}(X)$ such that (3) holds with $f_j$ replaced by $F$ and also

$$\psi(z) + 2\epsilon \geq \log \frac{|F(z)|}{c_j}, \quad z \in K.$$ 

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Now cover $K$ with balls constructed above to get for a given $\epsilon > 0$ analytic functions $F_1, \ldots, F_k$ on $X$ and integers $c_1, \ldots, c_k$ such that:

$$\psi(z) - 2\epsilon < \max_{1 \leq j \leq k} \left\{ \frac{\ln|F_j(z)|}{c_j} \right\} \leq \psi(z) + 2\epsilon, \quad z \in K$$

Hence Proposition 2 of [9] is valid also for Stein spaces.

Now let $\rho$ be a given plurisubharmonic function on $X$. In view of Theorem 5.5 of [5] there exists a sequence of continuous plurisubharmonic functions $\{\rho_n\}$ that decrease pointwise to $\rho$. Choose an exhaustion of $X$ by holomorphically convex compact sets $\{K_n\}_n$. Fix a sequence of positive numbers $\{\epsilon_n\}_n$ that decrease to zero. For each $n$ there exists analytic functions $F^n_1, \ldots, F^n_{\rho(n)}$ and integers $c^n_1, \ldots, c^n_{\rho(n)}$ such that:

$$\rho_n(z) - \epsilon_n \leq \max_{1 \leq i \leq \rho(n)} \frac{\ln|F^n_i(z)|}{c^n_i} \leq \rho_n(z) + \epsilon_n \quad \forall z \in K_n.$$ 

We enumerate $\{F^n_{i,n}\}$ (similarly $\{c^n_{i,n}\}$) as

$$\{F'_1, \ldots, F'_{\rho(1)}, \ldots, F^n_1, \ldots, F^n_{\rho(n)}, \ldots\}$$

and denote the resulting sequence by $\{G_{\alpha}\}_{\alpha}$ (similarly $\{c_{\alpha}\}_{\alpha}$). Set:

$$\gamma_{\alpha}(z) = \frac{\ln|G_{\alpha}(z)|}{c_{\alpha}}.$$ 

Now fix a point $z \in X$, say $z \in K_N$. Let $n > N$ and

$$k = \sum_{i=1}^{n-1} \rho(i) + 1.$$ 

Since $K_N \subset K_n$ we have

$$\rho_n(z) - \epsilon_n \leq \max_{1 \leq i \leq \rho(n)} \frac{\ln|F^n_i(z)|}{c^n_i} \leq \rho_n(z) + \epsilon_n.$$ 

Hence

$$\rho(z) - \epsilon_n \leq \sup_{\alpha > k} \gamma_{\alpha}(z)$$

and so

$$(4) \quad \rho(z) - \epsilon_n \leq \inf_s \sup_{\alpha > s} \gamma_{\alpha}(z).$$
On the other hand choose any $\alpha$ with $\alpha > k$, with $k$ as above, then
\[ \gamma_\alpha(z) = \frac{\ln |F^s(z)|}{c^s} \]
for some $s \geq n$. So we have $\gamma_\alpha(z) \leq \rho_s(z) + \epsilon_s = \rho_n(z) + \epsilon_n$; hence $\sup_{\alpha > k} \gamma_\alpha(z) \leq \rho_n(z) + \epsilon_n$. It follows that:
\[ \inf_t \sup_{\alpha > t} \gamma_\alpha(z) \leq \inf_n (\rho_n(z) + \epsilon_n) = \rho(z). \]
So combining (4) and (5) and setting $f_n = G_n$ we get:
\[ \rho(z) = \lim_n \frac{\ln |f_n(z)|}{c_n}. \]
This finishes the proof of the Lemma. 

**Corollary 1.** Let $M$ be a Stein space and $V$ a closed subvariety of $M$. Then the following are equivalent:

(i) For every Stein space $W$ for which $\mathcal{O}(W)$ is a finite type power series space and for every continuous linear operator $T : \mathcal{O}(W) \to \mathcal{O}(V)$ there exists a continuous linear operator $\hat{T} : \mathcal{O}(W) \to \mathcal{O}(M)$ such that $R \circ \hat{T} = T$ where $R$ is the restriction operator from $\mathcal{O}(M)$ into $\mathcal{O}(V)$.

(ii) For every negative plurisubharmonic function $\rho$ on $V$ there exists a negative plurisubharmonic function $\hat{\rho}$ on $M$ such that $\rho \leq \hat{\rho}|_V$.

**Proof.** In view of Theorem 1 we only need to prove the implication (i) $\Rightarrow$ (ii). To this end we fix a negative plurisubharmonic function $\rho$ on $V$. In view of the Lemma we can find a sequence $\{f_n\}_n$ of analytic functions on $V$, and a sequence of positive integers $\{c_n\}_n$ with $c_n \uparrow \infty$ such that:
\[ \rho(z) = \lim_n \frac{\ln |f_n(z)|}{c_n}. \]
In view of Hartog's lemma for every compact set $K$ in $V$ there exists a negative number $\alpha$ and a constant $c > 0$ such that, for all $n$,
\[ \sup_{z \in K} |f_n(z)| \leq c e^{\alpha c_n}. \]
Hence the assignment
\[ T(z^n) = \begin{cases} 0 & \text{if } n \not\in \{c_k\}_k, \\ c_s & \text{if } n = c_s \text{ for some } s \end{cases} \]
defines, in view of (6), a continuous linear operator $T : \mathcal{O}(\Delta) \to \mathcal{O}(V)$. We fix a $\tilde{T} : \mathcal{O}(\Delta) \to \mathcal{O}(M)$ with $\tilde{T}|_{V} = T$ and let as usual

$$\rho_{\tilde{T}}(z) = \lim_{\xi \to z} \lim_{n} \ln |\tilde{T}(z^n)(\xi)|.$$  

Since $\rho = \rho_{T}$, the argument given in (i) $\Rightarrow$ (ii) of Theorem 1 shows that $\rho \leq \rho_{\tilde{T}}|_{V}$. This finishes the proof of Corollary 1. 

The above corollary can be used to characterize among the hyperconvex varieties $V$ of a Stein space $M$ (i.e. the varieties $V$ such that $\mathcal{O}(V)$ is a finite type power series space, see [2]) the ones which admit a continuous linear extension operator $E : \mathcal{O}(V) \to \mathcal{O}(M)$. Recall that for a Stein space $X$ and a compact set $K \subset X$ the plurisubharmonic function :

$$w_{K}^{X}(z) = \lim_{\xi \to z} \sup \left\{ u(\xi) : u \in \text{PSH}(X), \right. \\
\left. u \leq 0 \text{ on } X \text{ and } u \leq -1 \text{ on } K \right\}$$

is called the plurisubharmonic measure ($\mathcal{P}$-measure) of $K$ relative to $X$ (see eg. [4], [13], [18]). These functions are natural complex counterparts of harmonic measures of classical potential theory. Since any negative plurisubharmonic function on a Stein space is dominated by a constant multiple of a $\mathcal{P}$-measure one can reexpress the condition (ii) above using $\mathcal{P}$-measures to obtain :

**Corollary 2.** — Let $M$ be a Stein space and $V$ a hyperconvex subvariety of $M$. Then the following conditions are equivalent :

(i) There exists a continuous linear extension operator

$$E : \mathcal{O}(V) \to \mathcal{O}(M).$$

(ii) There exists compact sets $K \subset V$, $S \subset M$ with non empty interiors and a constant $C > 0$ such that :

$$w_{K}^{V} \leq C w_{S|M}^{M}.$$  

**Remarks.**

(i) Although we have chosen to treat the case when $\mathcal{O}(W)$ is isomorphic to an infinite type power series space by making use of some general considerations, we note that the line of reasoning given in the proof of Theorem 1 can also be used in this case. Indeed the existence of
an operator $\hat{T} : \mathcal{O}(W) \to \mathcal{O}(M)$ with $R\circ \hat{T} = T$ for any $T : \mathcal{O}(W) \to \mathcal{O}(V)$ can be deduced, in this case, from the fact that for any plurisubharmonic function $\rho$ on $V$ there exists a plurisubharmonic function $\hat{\rho}$ on $M$ such that $\rho \leq \hat{\rho}|_V$.

(ii) In the case when $\mathcal{O}(M)$ is isomorphic to an infinite type power series space and when $W$ is hyperconvex, Theorem 1 characterizes the operators $T$ for which such a $\hat{T}$ exists as the ones for which $\sup_{z \in V} \rho_T(z) < 0$. This family is precisely the family of all compact operators from $\mathcal{O}(W)$ into $\mathcal{O}(V)$. This can also be derived from the general extension properties of compact operators and the fact that every continuous operator from a finite type power series space into an infinite type power series space is compact.

(iii) For a smoothly bounded relatively compact domain $D$ with $C^2$ boundary in a Stein manifold and a negative plurisubharmonic function $\rho$ on $D$ one has that

$$\rho(z) < C \{ -d(z, \partial D) \}, \quad z \in D$$

for some constant $C > 0$ where $d(z, \partial D)$ is the distance of $z$ from $\partial D$ (see [10, Lemma 3.2]). Hence in the case when $D$ is given by $D = \{ z : u(z) < 0 \}$, for some $C^2$ plurisubharmonic function $u$ defined in a neighborhood of $\bar{D}$, we have that any negative plurisubharmonic function on $D$ is dominated by a positive constant multiple of $u$, since $-d( \cdot, \partial D)$ is dominated by a positive constant multiple of $u$. This property remains valid for submanifolds of $D$ of the form $D \cap M'$ where $M'$ is a closed complex submanifold in a neighborhood of $\bar{D}$ which intersects $\partial D$ transversally since in this case $D \cap M' = \{ z \in M' : u(z) < 0 \}$. Now combining Corollary 5 of [2] with Corollary 2 above we obtain the following slight generalization of Theorem 4.2 of [10].

**Corollary 3.** — Let $M$ be a Stein manifold and $D \subset M$ a smoothly bounded domain in $M$ of the form $D = \{ z : u(z) < 0 \}$ for some $C^2$ plurisubharmonic function defined in a neighborhood of $\bar{D}$. For a complex manifold $M'$ in a neighborhood of $\bar{D}$ which intersects $\partial D$ transversally there exists a continuous linear extension operator $\mathcal{E} : \mathcal{O}(D \cap M') \to \mathcal{O}(D)$.

Even if we drop the transversality condition in the above corollary we can still get some information about the class of continuous linear operators $T : \mathcal{O}(D^d) \to \mathcal{O}(D \cap M')$ which admit a continuous linear extension operator, namely these are precisely the operators for which $\rho_T \leq Cu$ on $D \cap M'$ for some $C > 0$. This observation can be used in constructing concrete operators for which no such $\hat{T}$ exists. For example following...
Example 5.3 of [10], let
\[ D = \{ (z; w) \in \mathbb{C}^2 ; |z|^2 + |w - 1|^2 < 1 \} \]
and
\[ M' = \{ (z, w) \in \mathbb{C}^2 ; w = z^2 \}. \]
Then the operator \( T : \mathcal{O}(\Delta) \to \mathcal{O}(D \cap M') \) defined as \( T(f)(z, w) = f(e^{-z^2}) \) admits no extension operator \( \tilde{T} : \mathcal{O}(\Delta) \to \mathcal{O}(D) \), since, an easy computation shows the impossibility of finding a \( C > 0 \) satisfying
\[ \rho_T(z, w) = \ln|e^{-z^2}| \leq C\{|z|^2 + |w - 1|^2 - 1\}. \]

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