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# PROJECTIONS FROM A VON NEUMANN ALGEBRA ONTO A SUBALGEBRA

BY

GILLES PISIER (*)

RÉSUMÉ. — Cet article est principalement consacré à la question suivante : soient $M, N$ deux algèbres de Von Neumann avec $M \subset N$. S'il existe une projection complètement bornée $P : N \to M$, existe-t-il automatiquement une projection contractante $\widetilde{P} : N \to M$ ? Nous donnons une réponse affirmative sous la seule restriction que $M$ soit semi-finie. La méthode consiste à identifier isométriquement l'espace d'interpolation complexe $(A_0, A_1)_\theta$ associé au couple $(A_0, A_1)$ défini comme suit : $A_0$ (resp. $A_1$) est l'espace de Banach des $n$-uples $x = (x_1, \dots, x_n)$ d'éléments de $M$ muni de la norme $\|x\|_{A_0} = \| \sum x_i^* x_i \|_M^{1/2}$ (resp. $\|x\|_{A_1} = \| \sum x_i x_i^* \|_M^{1/2}$) .

ABSTRACT. — This paper is mainly devoted to the following question : let $M, N$ be Von Neumann algebras with $M \subset N$. If there is a completely bounded projection $P : N \to M$, is there automatically a contractive projection $\widetilde{P} : N \to M$ ? We give an affirmative answer with the only restriction that $M$ is assumed semi-finite. The main point is the isometric identification of the complex interpolation space $(A_0, A_1)_\theta$ associated to the couple $(A_0, A_1)$ defined as follows : $A_0$ (resp. $A_1$) is the Banach space of all $n$-tuples $x = (x_1, \dots, x_n)$ of elements in $M$ equipped with the norm $\|x\|_{A_0} = \| \sum x_i^* x_i \|_M^{1/2}$ (resp. $\|x\|_{A_1} = \| \sum x_i x_i^* \|_M^{1/2}$).

## Introduction

This paper is mainly devoted to the following question. Let $M, N$ be von Neumann algebras with $M \subset N$; if there is a completely bounded (*c.b.* in short) projection $P : N \to M$, is there automatically a contractive projection $\widetilde{P} : N \to M$ ?

We give an affirmative answer with the only restriction that $M$ is assumed semi-finite. At the time of this writing, the case when the

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subalgebra $M$ is a type III factor seems unclear, although this might be not too hard to deduce from our results using crossed product techniques from the Tomita-Takesaki theory with which we are not familiar (see the final remark).

If $N = B(H)$, a positive answer (without any restriction on $M$) was given in [P1], [P2] (and independently in [CS]). I am grateful to Eberhard KIRCHBERG for mentioning to me that a more general statement might be true. It should be mentioned that the above question seems open if «completely bounded» is replaced by «bounded» in the assumption on the projection $P$. For more results in this direction, see [P3] and [HP2]. We should recall that, by a classical result of TOMIYAMA [T], every norm one projection $P$ from $N$ onto $M$ necessarily is a conditional expectation and in particular is completely positive. In the second part of the paper we give an interpolation theorem which generalizes a result in [P1], as follows. Let $N$ be a von Neumann algebra equipped with a normal semi-finite faithful trace $\varphi$. Let us denote by $L_p(\varphi)$ the noncommutative $L_p$-space associated to $(N, \varphi)$ in the usual way. Fix $n \geq 1$. Let us denote by $A_0$ (resp. $A_1$) the space $N^n$ equipped with the norms

$$\|(x_1, \dots, x_n)\|_{A_0} = \Big\| \Big(\sum x_i x_i^*\Big)^{1/2} \Big\|_N ,$$

$$\|(x_1, \dots, x_n)\|_{A_1} = \Big\| \Big(\sum x_i^* x_i\Big)^{1/2} \Big\|_N .$$

We prove in section 2 that the complex interpolation space $(A_0, A_1)_\theta$ is the space $N^n$ equipped with the norm

$$\|(x_1, \dots, x_n)\|_\theta = \Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_p(\varphi))}^{1/2}$$

where we have denoted by $L_x$ (resp. $R_x$) the operator of left (resp. right) multiplication by $x$ on $L_p(\varphi)$, and where $p = \theta^{-1}$. Note that the case $\theta = 0$ corresponds to $L_\infty(\varphi)$ identified with $N$ and $\theta = 1$ corresponds to $L_1(\varphi)$ identified with $N_*$ in the usual way. Again in the particular case $N = B(H)$ this result was proved in [P1].

We refer to [Ta1] for background on von Neumann algebras and to [Pa] for complete boundedness.

We will use several times the following elementary fact.

LEMMA 0.1. — *Let $M \subset N$ be von Neumann algebras. Let $(p_i)_{i \in I}$ be a directed net of projections in $M$ such that, for all $x$ in $M$, $p_i x p_i$ tends to $x$ in the $\sigma(M, M_*)$ topology. Assume that for each $i$ there is a norm one projection $P_i : N \to p_i M p_i$. Then there is a norm one projection $P$ from $N$ onto $M$.*

*Proof.* — Let $\mathcal{U}$ be a nontrivial ultrafilter refining the net. For any $x$ in $N$, we define

$$P(x) = \lim_{\mathcal{U}} P_i(p_i x p_i)$$

where the limit is in the $\sigma(M, M_*)$ sense. Then $P(x) \in M$ and $\|P(x)\| \leq \|x\|$. Moreover, for any $x$ in $M$ we have

$$P_i(p_i x p_i) = p_i x p_i.$$

Hence $P(x) = x$ for all $x$ in $M$, and we conclude that $P$ is a projection from $N$ to $M$. ∎

## 1. Projections

The main result of this section is the following.

THEOREM 1.1. — *Let $M \subset N \subset B(H)$ be von Neumann algebras with $M$ semi-finite. If there is a completely bounded (c.b. in short) projection $P : N \to M$, then there is a norm one projection $\widetilde{P} : N \to M$.*

Actually, we use less than complete boundedness, we only need to assume that there is a constant $C$ such that for all $x_1, \ldots, x_n$ in $N$ we have

$$\tag{1.1} \begin{cases} \left\| \sum P(x_i)^* P(x_i) \right\| \leq C^2 \left\| \sum x_i^* x_i \right\|, \\ \left\| \sum P(x_i) P(x_i)^* \right\| \leq C^2 \left\| \sum x_i x_i^* \right\|. \end{cases}$$

The proof is given at the end of this section.

NOTATION. — Let $\varphi$ be a normal faithful semi-finite trace on a von Neumann algebra $N$. We denote by $L_2(\varphi)$ the usual associated Hilbert space. For any $a$ in $N$, we denote by $L_a$ (resp. $R_a$) the operator of left (resp. right) multiplication by a in $L_2(\varphi)$, i.e. we set for all $x$ in $L_2(\varphi)$

$$L_a x = ax, \quad R_a x = xa.$$

The key lemma in the proof of THEOREM 1.1 is the next statement.

LEMMA 1.2. — *Let $N$ be a semi-finite von Neumann algebra with a normal faithful semi-finite trace $\varphi$ as above. Consider a finite set $x_1, \ldots, x_n$ in $N$ and assume*

$$\tag{1.2} \left\| \sum\nolimits_1^n L_{x_i} R_{x_i^*} \right\|_{B(L_2(\varphi))} \leq 1,$$

*then there is a decomposition $x_i = a_i + b_i$ with $a_i \in N$, $b_i \in N$ such that*

$$\tag{1.3} \left\| \left( \sum a_i^* a_i \right)^{1/2} \right\| + \left\| \sum b_i b_i^* \right\|^{1/2} \leq 1.$$

More generally, the main idea of this paper seems to be the identification of the expression

$$\|(x_1,\ldots,x_n)\| = \Big\|\sum\nolimits_1^n L_{x_i}R_{x_i^*}\Big\|^{1/2}_{B(L_2(\varphi))}$$

with the norm of a simple interpolation space obtained by the complex interpolation method. See section 2 for further details.

COROLLARY 1.3. — *Let $N$ be as in Lemma* 1.2 *and let $M$ be a finite von Neumann algebra equipped with a normalized finite trace $\tau$. Let $P : N \to M$ be any linear map satisfying* (1.1). *Then for all finite sequences $x_1,\ldots,x_n$ in $N$ we have*

$$\sum_1^n \tau\big(P(x_i)P(x_i)^*\big) = \sum_1^n \tau\big(P(x_i)^*P(x_i)\big) \le C^2\Big\|\sum_1^n L_{x_i}R_{x_i^*}\Big\|_{B(L_2(\varphi))}.$$

*Proof.* — Assume $\|\sum L_{x_i}R_{x_i^*}\| \le 1$. Let $a_i, b_i$ be as in LEMMA 1.2. Let us denote $\|x\|_2 = (\tau(x^*x))^{1/2}$ for all $x$ in $M$. Then we have

$$\begin{aligned}\Big\{\sum\|P(x_i)\|_2^2\Big\}^{1/2} &\le \Big\{\sum\|P(a_i)\|_2^2\Big\}^{1/2} + \Big\{\sum\|P(b_i)\|_2^2\Big\}^{1/2}\\ &\le \Big\|\sum P(a_i)^*P(a_i)\Big\|^{1/2} + \Big\|\sum P(b_i)P(b_i)^*\Big\|^{1/2}\\ &\le C. \quad \square\end{aligned}$$

LEMMA 1.4.. — *Let $N$ be as in Lemma 1.2 and let $M \subset N$ be a finite von Neumann subalgebra. Assume that there is a projection $P : N \to M$ satisfying* (1.1). *Then for all nonzero projection $p$ in the center of $M$ and for all unitary operators $u_1,\ldots,u_n$ in $M$ we have*

$$n = \Big\|\sum_1^n L_{pu_i}R_{(pu_i)^*}\Big\|_{B(L_2(\varphi))}. \tag{1.4}$$

*Proof.* — Fix $p$ as in LEMMA 1.4. By [Ta1, p. 311 ] there is a finite trace $\tau$ on $M$ with $\tau(p) \neq 0$. By COROLLARY 1.3 applied to the normalized trace $x \mapsto \tau(p)^{-1}\tau(x)$ on $pMp = pM$ we have

$$n = \sum \|pu_i\|_2^2 \le C^2\Big\|\sum_1^n L_{pu_i}R_{(pu_i)^*}\Big\|_{B(L_2(\varphi))}.$$

To replace $C^2$ by 1 in this inequality, we use the same trick as HAAGERUP in [H1]. Let

$$T_n = \sum_1^n L_{pu_i}R_{(pu_i)^*}.$$

We have for each $k$

$$T_n^k = \sum_{1\le m\le n^k} L_{x_m} R_{x_m^*}$$

where each $x_m$ is of the form $pu$ with $u$ unitary in $M$. It follows that

$$n^k \le C^2\|T_n^k\| \le C^2\|T_n\|^k$$

hence $n \le C^{2/k}\|T_n\|$. Letting $k$ tend to infinity we obtain (1.4) (since the other direction is trivial by the triangle inequality.) □

*Proof of Lemma 1.2.* — We will use the duality between $N^n$ and $N_*^n$.

Let $C$ be the set of elements $(x_i)_{i\le n}$ in $N^n$ which admit a decomposition $x_i = a_i + b_i$ in $N$ satisfying (1.3). We will show that if (1.2) holds, then necessarily $(x_i)$ lies in the bipolar $C^{oo}$ of $C$ in the duality between $N^n$ and $N_*^n$. This is enough to conclude. Indeed since the set $C$ is clearly convex and $\sigma(N^n, N_*^n)$ closed we have $C = C^{oo}$, so we obtain that $(x_i)$ is in $C$ if $(x_i)$ satisfies (1.2).

Hence assume given $(x_i)$ satisfying (1.2). Consider $(\xi_i)_{i\le n}$ in $N_*^n$ and assume $(\xi_i)_{i\le n} \in C^o$. This means that for any $a_i$ in $N$ such that

$$\text{(1.5)} \qquad \text{either} \quad \Big\|\sum_1^n a_i a_i^*\Big\|^{1/2} \le 1 \quad \text{or} \quad \Big\|\sum a_i^* a_i\Big\|^{1/2} \le 1,$$

we have

$$\Big|\sum \xi_i(a_i)\Big| \le 1.$$

We use the classical identification $N_* = L_1(\varphi)$ and we use the density of $N\cap L_1(\varphi)$ in $L_1(\varphi)$. By these well known properties of $L_1(\varphi)$ for each $\varepsilon > 0$ we can find a projection $p$ in $N$ with $\varphi(p) < \infty$ and elements $b_1, \dots, b_n$ in $pNp$ such that

$$\text{(1.6)} \qquad \|\xi_i - b_i\|_{N_*} < \varepsilon.$$

It follows that for any $(a_i)$ satisfying (1.5) we have

$$\Big|\sum \langle b_i, a_i\rangle\Big| \le 1 + n\varepsilon.$$

So that replacing $b_i$ by $b_i/(1+n\varepsilon)$ we may as well assume (since $\varepsilon > 0$ is arbitrary) that, for any $(a_i)$ satisfying (1.5) we have

$$\text{(1.7)} \qquad \Big|\sum \varphi(b_i a_i)\Big| \le 1.$$

We first claim that this implies

$$\varphi\Big\{\Big(\sum b_i^* b_i\Big)^{1/2}\Big\} \le 1 \quad \text{and} \quad \varphi\Big\{\Big(\sum b_i b_i^*\Big)^{1/2}\Big\} \le 1. \tag{1.8}$$

Indeed, let $r$ (resp. $c$) be the element of $M_n(N)_*$ corresponding to the $n \times n$ matrix which has coefficients equal to $b_1, \ldots, b_n$ on the first row (resp. column) and zero elsewhere. Then by (1.7) $r$ and $c$ are in the unit ball of $M_n(N)_*$. From this (1.8) immediately follows by the identification between $M_n(N)_*$ and $L_1(\widetilde{\varphi})$ where $\widetilde{\varphi}$ is the semi-finite trace defined on $M_n(N)$ by

$$\widetilde{\varphi}\big((a_{ij})\big) = \sum \varphi(a_{ii}).$$

Secondly, we claim that, for any $\delta > 0$, $b_i$ can be written as $b_i = \alpha y_i \beta$ with $\alpha, y_i, \beta$ in $pNp$ such that

$$\varphi\big(|a|^4\big) \le 1 + \delta\varphi(p), \quad \varphi\big(|\beta|^4\big) \le 1 + \delta\varphi(p), \quad \sum \varphi\big(|y_i|^2\big) \le 1.$$

Let

$$\begin{cases} \alpha = \Big\{\Big(\sum b_i b_i^*\Big)^{1/2} + \delta p\Big\}^{1/4}, \\ \beta = \Big\{\Big(\sum b_i^* b_i\Big)^{1/2} + \delta p\Big\}^{1/4}. \end{cases}$$

Note that we clearly have

$$\begin{cases} \alpha^{-2}\Big(\sum b_i b_i^*\Big)\alpha^{-2} \le \Big(\sum b_i b_i^*\Big)^{1/2}, \\ \beta^{-2}\Big(\sum b_i^* b_i\Big)\beta^{-2} \le \Big(\sum b_i^* b_i\Big)^{1/2}. \end{cases} \tag{1.9}$$

We also note that

$$\varphi(\beta^4) \le 1 + \delta\varphi(p) \quad \text{and} \quad \varphi(\alpha^4) \le 1 + \delta\varphi(p). \tag{1.10}$$

Now in the von Neumann algebra $pNp$ (with unit $p$) we introduce the analytic $pNp$ valued functions $f_k$ (for $k = 1, \ldots, n$) defined on the strip $S = \{z \in \mathbb{C}\,;\ 0 < \mathrm{Re}(z) < 1\}$ by

$$f_k(z) = \alpha^{-2(1-z)} b_k \beta^{-2z}.$$

We have

$$\begin{cases} f_k(it) = \alpha^{2it}\alpha^{-2} b_k \beta^{-2it}, \\ f_k(1+it) = \alpha^{2it} b_k \beta^{-2-2it}. \end{cases}$$

Since $\alpha^{2it}$ and $\beta^{-2it}$ are unitary in $pNp$, it follows that for all real $t$

$$\sum \|f_k(it)\|^2_{L_2(\varphi)} = \varphi\Big\{\alpha^{-2}\sum b_k b_k^* \alpha^{-2}\Big\},$$

hence by (1.9) and (1.8)

$$\leq \varphi\Big\{\Big(\sum b_k b_k^*\Big)^{1/2}\Big\} \leq 1.$$

Note that $f_k$ is bounded on $\bar{S}$ since $\alpha, \beta$ are bounded below in $pNp$. We now invoke the three lines lemma (cf. [BL, p. 4]). Note that, as is well known, this lemma remains valid for bounded analytic functions on $S$, not necessarily continuous on $\bar{S}$, using the nontangential boundary values to extend the functions to $\bar{S}$. Using this, we conclude that, for all $z$ in the strip $S$, we have

$$\sum \|f_k(z)\|^2_{L_2(\varphi)} \leq 1.$$

In particular this holds for $z = \frac{1}{2}$ and we can define $y_k = f_k(\frac{1}{2})$. Then we have $b_k = \alpha y_k \beta$ and all the announced properties hold. We now return to our original $n$-tuple $x_1, \ldots, x_n$ in $N$.

We have by (1.6)

$$\begin{aligned}\Big|\sum \langle \xi_k, x_k\rangle\Big| &\leq \Big|\sum \langle b_k, x_k\rangle\Big| + n\varepsilon \\ &\leq \Big|\sum \varphi(\alpha y_k \beta x_k)\Big| + n\varepsilon\end{aligned}$$

hence by Cauchy-Schwarz and by (1.10)

$$\begin{aligned}&\leq \Big\{\sum \|\beta x_k \alpha\|^2_{L_2(\varphi)}\Big\}^{1/2} + n\varepsilon \\ &= \Big\{\varphi\Big(\sum \beta x_k \alpha\alpha^* x_k^* \beta^*\Big)\Big\}^{1/2} + n\varepsilon \\ &= \Big\langle \beta^*\beta, \sum L_{x_k} \alpha\alpha^* R_{x_k^*}\Big\rangle^{1/2}_{L_2(\varphi)} + n\varepsilon \\ &\leq \Big\|\sum L_{x_k} R_{x_k^*}\Big\|^{1/2}_{B(L_2(\varphi))} \big(1 + \delta\varphi(p)\big)^{1/2} + n\varepsilon.\end{aligned}$$

Since $\varepsilon, \delta > 0$ are arbitrary, we conclude that if (1.2) holds we have $\big|\sum\langle \xi_k, x_k\rangle\big| \leq 1$ for all $(\xi_k)$ in $C^o$. Hence we have $(x_k) \in C^{oo}$ and the proof is complete. $\square$

To prove THEOREM 1.1, we will combine LEMMA 1.2 with a rather straightforward extension of some results of Haagerup in [H1] on injective von Neumann algebras. HAAGERUP's work is based on CONNES' ideas on injective factors [Co].

DEFINITION 1.5. — *Let $M \subset N$ be von Neumann algebras. We will say that a state $\omega$ on $N$ is an $M$-hypertrace on $N$ if we have*

$$\forall a \in M,\ \forall x \in N, \quad \omega(ax) = \omega(xa).$$

THEOREM 1.6. — *Let $M \subset N$ be von Neumann algebras with $N$ semi-finite and $\varphi$ a faithful normal semi-finite trace on $N$. The following are equivalent*

(i) *$M$ is finite and there is a norm one projection $P$ from $N$ onto $M$.*

(ii) *For any finite set $u_1, \ldots, u_n$ of unitaries in $M$ and any nonzero central projection $p$ in $M$ we have*

$$n = \Big\| \sum_1^n L_{pu_i} R_{(pu_i)^*} \Big\|_{B(L_2(\varphi))}. \tag{1.11}$$

(iii) *For every nonzero central projection $p$ in $M$ there is an $M$-hypertrace $\omega$ on $N$, such that $\omega(1-p) = 0$.*

(iv) *For every state $\omega_0$ on the center of $M$ there is an $M$-hypertrace $\omega$ on $N$ extending $\omega_0$.*

*Proof.* — The proof of LEMMA 2.2 in [H1] extends word for word. We simply replace there $B(H)$ by $N$ and we denote by $M$ the subalgebra. □

REMARK. — For the convenience of the reader, we recall the key idea which is behind the preceding statement. This is best described in the case when $M$ is a factor. In that case the implication (ii) ⇒ (i) is proved as follows : using the uniform convexity of $L_2(\varphi)$ one shows that (ii) implies the existence of a net $(z_\alpha)$ in the unit sphere of $L_2(\varphi)$ such that $\|u z_\alpha u^* - z_\alpha\|_{L_2(\varphi)} \to 0$ for all $u$ unitary in $M$. Then if we define on $N$

$$\omega(x) = \lim_{\mathcal{U}} \langle x z_\alpha, z_\alpha \rangle_{L_2(\varphi)} = \lim_{\mathcal{U}} \varphi(x z_\alpha^* z_\alpha)$$

we find that $\omega$ is an $M$-hypertrace on $N$. Moreover since $M$ is a factor, $\omega$ restricted to $M$ is the trace of $M$. It is then easy to conclude that there is a norm one projection $P : N \to M$ which is built exactly like a conditional expectation.

*Proof of Theorem 1.1.* — By a well known crossed product argument, (cf. [Ta2]) there is a semi-finite algebra $\widetilde{N}$ with $N \subset \widetilde{N}$ and a completely contractive projection $Q : \widetilde{N} \to N$. Hence, replacing $N$ by $\widetilde{N}$ we may assume that $N$ is semi-finite.

Let $P$ be a projection satisfying (1.1). We first assume $M$ finite. Then by LEMMA 1.4, the second assertion in THEOREM 1.6 holds. Therefore, by (ii) $\Rightarrow$ (i) in THEOREM 1.6 there is a norm one projection from $N$ onto $M$.

Now if $M$ is semi-finite, we can write

$$M = \overline{\bigcup p_i M p_i} \qquad \text{(weak-$*$ closure)}$$

where $p_i$ is an increasing net of finite projections in $M$ such that $p_i x p_i \to x$ in the $\sigma(M, M_*)$-sense for all $x$ in $M$. Clearly $x \mapsto p_i P(x) p_i$ is a projection from $p_i N p_i$ onto $p_i M p_i$ which satisfies (1.1) hence by the first part of the proof, there is a norm one projection from $p_i N p_i$ onto $p_i M p_i$. A fortiori there is a norm one projection from $N$ onto $p_i M p_i$, hence we conclude by LEMMA 0.1 that there is a norm one projection $\widetilde{P}$ from $N$ onto $M$. ∎

## 2. An interpolation theorem

Let $N$ be a semi-finite von Neumann algebra, let $1 \le p < \infty$ and let $L_p(\varphi)$ be the classical non-commutative $L_p$-space associated to a faithful normal semi-finite trace $\varphi$ on $N$. For the construction and the basic properties of $L_p(\varphi)$, the classical references are [D], [S], [Ku], [Sti]. For a more concise and recent exposition, see [N].

The following statement extends a result proved in [P1] in the particular case $N = B(H)$.

THEOREM 2.1. — *Fix an integer $n \ge 1$. Let $A_0$ (resp. $A_1$) be the space $N^n$ equipped with the norm*

$$\big\|(x_1, \ldots, x_n)\big\|_{A_0} = \Big\|\sum_1^n x_i x_i^*\Big\|^{1/2}$$

$$\Big(\textit{resp. } \|(x_1, \ldots, x_n)\|_{A_1} = \Big\|\sum x_i^* x_i\Big\|^{1/2}\Big).$$

*Then for $0 < \theta < 1$, the complex interpolation space $(A_0, A_1)_\theta$ is the space $N^n$ equipped with the norm*

$$\big\|(x_1, \ldots, x_n)\big\|_\theta = \Big\|\sum L_{x_i} R_{x_i^*}\Big\|^{1/2}_{B(L_p(\varphi))}$$

*where $\theta = 1/p$.*

REMARK. — Note that THEOREM 2.1 implies LEMMA 1.2 by a well known property of the interpolation spaces, namely the (norm one) inclusion $(A_0, A_1)_\theta \subset A_0 + A_1$, (see [BL] for more details). But actually, the proof of THEOREM 2.1 is quite similar to that of LEMMA 1.2, although slightly more technical.

We will use Szegö's classical factorization theorem which says that under a nonvanishing condition, a positive function $W$ in $L_1(\mathbb{T})$ can always be written as $W = |F|^2$ ($W = \overline{F}F$ is more suggestive in view of the non-commutative case) for some $F$ in $H^2$. Moreover, this can be done with $F$ «outer», so that $z \mapsto 1/F(z)$ is analytic inside the disc, and if we additionally require $F(0) > 0$ then $F$ is unique. Actually, we will need an extension of this theorem (due to Devinatz) valid for $B(H)$-valued functions. The following consequence of Devinatz's theorem will be enough for our purposes (cf. [D], [He]).

THEOREM 2.2. — *Let $H$ be a separable Hilbert space and let $W : \mathbb{T} \to B(H)$ be a function such that, for all $x, y$ in $H$, the function*

$$t \longmapsto \langle W(t)x, y\rangle$$

*is in $L_1(\mathbb{T})$. Assume that there is $\delta > 0$ such that $W(t) \geq \delta I$ for all $t$. Then there is a unique analytic function $F:\ D \to B(H)$ such that :*

(i) *for all $x$ in $H$, $z \mapsto F(z)x$ is in $H^2(H)$ and its boundary values satisfy almost everywhere on $\mathbb{T}$*

$$\langle W(t)x, y\rangle = \langle F(t)x, F(t)y\rangle,$$

(ii) $F(0) \geq 0$,

(iii) $z \mapsto F(z)^{-1}$ *exists and is bounded analytic on $D$.*

The following corollary was pointed out to me by Uffe HAAGERUP during our collaboration on [HP1] (we ended up not using it in our paper).

COROLLARY 2.3. — *Consider a von Neumann subalgebra $N \subset B(H)$. Then in the situation of Theorem 2.2, if $W$ is $N$-valued, $F$ necessarily also is $N$-valued.*

*Proof.* — Indeed, for any unitary $u$ in the commutant $N'$, the function $z \mapsto u^*F(z)u$ still satisfies the conclusions of THEOREM 2.2, hence (by uniqueness) we must have $F = u^*Fu$, which implies by the bicommutant theorem that

$$F(z) \in N'' = N. \quad \square$$

*Proof of Theorem 2.1.* — By well known results, $N$ can be written as a direct sum of $\sigma$-finite semi-finite algebras. Hence we can assume that $N$ is $\sigma$-finite and that $H = L_2(\varphi)$ is separable.

Let $\theta = 1/p$. Let us denote $L_\infty(\varphi) = N$. Then it is well known that we have isometrically

$$\big(L_\infty(\varphi), L_1(\varphi)\big)_\theta = L_p(\varphi).$$

Clearly if $x_1, \ldots, x_n$ in $N$ are such that $\|(x_1, \ldots, x_n)\|_0 \leq 1$, then we have

$$\Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_\infty(\varphi))} \leq 1.$$

Similarly, it is easy to check by transposition that if $\|(x_1, \ldots, x_n)\|_1 \leq 1$, then

$$\Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_1(\varphi))} \leq 1.$$

Hence, if $(x_1, \ldots, x_n)$ is in the unit ball of $(A_0, A_1)_\theta$, we have necessarily by classical interpolation theory (where $1/p = \theta$)

$$\Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_p(\varphi))} \leq 1.$$

This is the easy direction. To prove the converse, we assume that

$$\Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_p(\varphi))} \leq 1. \tag{2.1}$$

We will proceed by duality as in the proof of LEMMA 1.2. Let $B$ denote the open unit ball in the space $(A_{0*}, A_{1*})_\theta$. Note that $A_{0*}$ (resp. $A_{1*}$) coincides with $N_*^n$ equipped with the norm

$$\|(\xi_1, \ldots, \xi_n)\| = \varphi\Big\{ \Big( \sum \xi_i^* \xi_i \Big)^{1/2} \Big\}$$
$$\Big(\text{resp. } \|(\xi_1, \ldots, \xi_n)\| = \varphi\Big\{ \Big( \sum \xi_i \xi_i^* \Big)^{1/2} \Big\}\Big).$$

Let $B^o$ be the polar of $B$ in the duality between $N^n$ and $N_*^n$. By a well known duality property of interpolation spaces (cf. [BL], [Be]) the polar $B^o$ coincides with the unit ball of $(A_0, A_1)_\theta$. Hence to conclude it suffices to show that (2.1) implies $(x_1, \ldots, x_n) \in B^o$. Equivalently, to complete the proof it suffices to show that, if (2.1) holds, then for any $(\xi_1, \ldots, \xi_n)$ in $B$ we have $\left| \sum \xi_i(x_i) \right| \leq 1$. The rest of the proof is devoted to the verification of this. By density, if we identify again $N_*$ with $L_1(\varphi)$ in the usual way, we may assume that $\xi_i$ is of the form $\xi_i(x) = \varphi(b_i x)$ for some $b_i$ in $qMq$ where $q$ is a finite projection in $M$, i.e. a projection with $\varphi(q) < \infty$.

In that case we have $\xi_i(x_i) = \xi_i(q x_i q)$. Note that (2.1) remains true if we replace $(x_i)$ by $(q x_i q)$. Therefore, at this point we may as well replace $N$ by the finite von Neumann algebra $qNq$ (with unit $q$) so that we are reduced to the finite case. Hence, for simplicity, we assume in the rest of the proof that $N$ is finite with unit $I$ and that $\xi_i$ lies in $N$ viewed as a subspace of $L_1(\varphi)$ (i.e. that the elements $b_i$ above are in $N$ and $q = I$). By definition of $(A_{0*}, A_{1*})_\theta$, since $(\xi_i)$ is in $B$ there are functions $f_i : \bar{S} \to L_1(\varphi)$ which are bounded, continuous on $\bar{S}$ and analytic on $S$ such that denoting

$$\partial_0 = \{z \in \mathbb{C}\,;\ \mathrm{Re} z = 0\}, \quad \partial_1 = \{z \in \mathbb{C}\,;\ \mathrm{Re} z = 1\}$$

we have $\xi_i = f_i(\theta)$ for $i = 1, \ldots, n$, with

$$\text{(2.2)} \qquad \begin{cases} \sup_{z\in\partial_0} \varphi\Big\{\Big(\sum f_i(z)^* f_i(z)\Big)^{1/2}\Big\} < 1, \\ \sup_{z\in\partial_1} \varphi\Big\{\Big(\sum f_i(z) f_i(z)^*\Big)^{1/2}\Big\} < 1. \end{cases}$$

Since $\xi_i$ is in $N \subset L_1(\varphi)$ and $N^n$ is dense in $N_*^n$, we may as well assume by a well known fact (cf. [St]) that the functions $f_1, \ldots, f_n$ take their values into a fixed finite dimensional subspace of $N \subset L_1(\varphi)$. We are then in a position to use THEOREM 2.2 and its corollary.

Let $\delta > 0$ to be specified later. We define functions $W_1$ and $W_2$ on $\partial S = \partial_0 \cup \partial_1$ by setting

$$\begin{aligned} &\forall z \in \partial_1, \quad W_1(z) = \Big\{\Big(\sum f_i(z) f_i(z)^*\Big)^{1/2} + \delta I\Big\}^{1/2} \\ &\forall z \in \partial_0, \quad W_1(z) = I, \\ &\forall z \in \partial_1, \quad W_2(z) = I, \\ &\forall z \in \partial_0, \quad W_2(z) = \Big\{\Big(\sum f_i(z)^* f_i(z)\Big)^{1/2} + \delta I\Big\}^{1/2}. \end{aligned}$$

By (2.2) we can choose $\delta$ small enough so that

$$\text{(2.3)} \qquad \sup_{z\in\partial_1} \varphi(W_1^2) < 1 \quad \text{and} \quad \sup_{z\in\partial_0} \varphi(W_2^2) < 1.$$

By THEOREM 2.2 and COROLLARY 2.3, using a conformal mapping from $S$ onto $D$, we find bounded $N$-valued analytic functions $F$ and $G$ on $S$ with (nontangential) boundary values satisfying

$$\text{(2.4)} \qquad FF^* = W_1^2 \quad \text{and} \quad G^*G = W_2^2.$$

Moreover, $F^{-1}$ and $G^{-1}$ are analytic and bounded on $S$. Therefore we can write

$$f_i(z) = F(z) g_i(z) G(z)$$

where

$$\text{(2.5)} \qquad g_i(z) = F(z)^{-1} f_i(z) G(z)^{-1}.$$

We claim that

$$\forall z \in S, \quad \sum \|g_i(z)\|^2_{L_2(\varphi)} \leq 1. \tag{2.6}$$

By the three lines lemma, to verify this it suffices to check it on the boundary of $S$. (Note that we know a priori that $\sup_{z\in S} \|g_i(z)\|_{L_2(\varphi)} < \infty$ since $\|F^{-1}\| < \delta^{-1/2}$ and $\|G^{-1}\| \leq \delta^{-1/2}$, hence $g_i$ is an $H^\infty$ function with values in $L_2(\varphi)$, and its nontangential boundary values still satisfy (2.5) a.e. on the boundary of $S$.) We have

$$\begin{aligned}\forall z \in \partial_1, \quad \sum \|g_i(z)\|^2_{L_2(\varphi)} &= \varphi\Big\{\sum g_i(z)g_i(z)^*\Big\}\\ &= \varphi\Big\{F(z)^{-1}\sum f_i(z)f_i(z)^*F(z)^{-1*}\Big\}\\ &= \varphi\{(FF^*)^{-1}(W_1^2 - \varepsilon^2 I)^2\}\\ &\leq \varphi(W_1^2) < 1 \quad \text{(by (2.4) and (2.3))}.\end{aligned}$$

Similarly, we find

$$\forall z \in \partial_0, \quad \sum \|g_i(z)\|^2_{L_2(\varphi)} \leq \varphi(W_2^2) < 1.$$

This proves our claim (2.6). Finally, if $\theta = 1/p$ we have

$$L_{2p}(\varphi) = \big(N, L_2(\varphi)\big)_\theta \quad \text{and} \quad L_{2p'}(\varphi) = \big(L_2(\varphi), N\big)_\theta.$$

Hence by definition of the latter complex interpolation spaces, since $\|F(z)\|_N = \|W_1\|_N \leq 1$ on $\partial_0$ and $\|F(z)\|_{L_2(\varphi)} < 1$ on $\partial_1$ (by (2.3)), we have

$$\|F(\theta)\|_{L_{2p}(\varphi)} \leq 1$$

and similarly

$$\|G(\theta)\|_{L_{2p'}(\varphi)} \leq 1.$$

Therefore we can conclude as in section 1 : we have

$$\xi_i = f_i(\theta) = F(\theta)g_i(\theta)G(\theta),$$

hence if $(x_i)$ satisfies (2.1) we have by (2.6) (and Cauchy-Schwarz)

$$\begin{aligned}\Big|\sum \xi_i(x_i)\Big| &= \Big|\sum \varphi\{F(\theta)g_i(\theta)G(\theta)x_i\}\Big|\\ &\leq \Big\{\sum \|G(\theta)x_iF(\theta)\|^2_{L_2(\varphi)}\Big\}^{1/2}\\ &\leq \Big\|\sum x_iF(\theta)F(\theta)^*x_i^*\Big\|^{1/2}_{L_p(\varphi)}\\ &\leq \Big\|\sum L_{x_i}R_{x_i^*}\Big\|^{1/2}_{B(L_p(\varphi))} \leq 1.\end{aligned}$$

Thus we have verified that (2.1) implies $(x_i) \in B^o$. This concludes the proof of THEOREM 2.1. $\square$

REMARK (Added 26 February 1994).

In september 93, after this paper had been submitted, Uffe HAAGERUP extended the proof of THEOREM 1.1 to the general case (without the assumption that $M$ is semi-finite). The same improvement was obtained simultaneously but by a completely different method by Erik CHRISTENSEN and Allan SINCLAIR (to appear.)

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