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SMOOTHNESS AND IRREDUCIBILITY OF VARIETIES OF PLANE CURVES WITH NODES AND CUSPS

BY

EUGENII SHUSTIN (*)

RÉSUMÉ. — Soit $V(d, m, k)$ la variété des courbes projectives planes irréductibles de degré d n'ayant pour singularités que m nodes et k cusps. Nous montrons que $V(d, m, k)$ est non vide, lisse et irréductible quand $m + 2k < \alpha d^2$ où α est une constante absolue explicite. Cette inégalité est optimale quant à l'exposant de d .

ABSTRACT. — Let $V(d, m, k)$ be the variety of plane projective irreducible curves of degree d with m nodes and k cusps as their only singularities. We prove that $V(d, m, k)$ is non-empty, non-singular and irreducible when $m + 2k < \alpha d^2$, where α is some absolute explicit constant. This estimate is optimal with respect to the exponent of d .

0. Introduction

In the present article we deal with plane projective algebraic curves over an algebraically closed field of characteristic 0.

It is well-known that the variety of irreducible curves of a given degree with a given number of nodes is non-singular [9], irreducible [2], and that each germ of this variety is a transversal intersection of germs of equisingular strata corresponding to all singular points [9] (from now on, speaking of a variety with the last property, we shall write *T-variety*, or *variety with property T*).

Our goal is a similar result for curves with nodes and ordinary cusps. Let $V(d, m, k)$ denote the set of irreducible curves of degree d with m nodes and k cusps as their only singularities.

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- It is known (see [5], [6]) that $V(d, m, k) = \emptyset$ if

$$\frac{9}{8}m + 2k > \frac{5}{8}d^2.$$

- On the other hand (see [13]), $V(d, m, k) \neq \emptyset$ when

$$(0.1) \quad m + 2k \leq \frac{1}{2}d^2 + O(d).$$

Our result is

THEOREM 0.2.

- If $m + 2k \leq \alpha_0 d^2$, where

$$(0.3) \quad \alpha_0 = \frac{7 - \sqrt{13}}{81} \approx 0.0419,$$

then $V(d, m, k)$ is a non-empty non-singular T-variety of dimension

$$\frac{1}{2}d(d+3) - m - 2k.$$

- If $m + 2k \leq \alpha_1 d^2$, where

$$(0.4) \quad \alpha_1 = \frac{2}{225} \approx 0.0089,$$

then $V(d, m, k)$ is irreducible.

Let us make some comments.

First, (0.3) implies (0.1), and then $V(d, m, k) \neq \emptyset$.

Let $\mathbb{P}^N$, with $N = \frac{1}{2}d(d+3)$, be the space of plane curves of degree d . Let z be a singular point of $F \in \mathbb{P}^N$. It is well-known (see [2, [9]]) that :

(1) if z is a node then the germ at F of the variety of curves $\Phi \in \mathbb{P}^N$, having a node in some neighbourhood of z , is smooth, has codimension 1, and its tangent space is open in $\{\Phi \in \mathbb{P}^N \mid z \in \Phi\}$;

(2) if z is a cusp then the germ at F of the variety of curves $\Phi \in \mathbb{P}^N$, having a cusp in some neighbourhood of z , is smooth, has codimension 2, and its tangent space is open in $\{\Phi \in \mathbb{P}^N \mid (\Phi \cdot F)(z) \geq 3\}$ (here and further on the notation $(F \cdot G)(z)$ means the intersection number of the curves F, G at the point z).

Hence the property T implies the smoothness of $V(d, m, k)$ and the expected value of its dimension given in THEOREM 0.2. Further, it is well-known [15] that $V(d, m, k)$ is a non-singular T-variety, when

$$(0.5) \quad k < 3d.$$

Generalizations of this fact to arbitrary singularities, given in [1], [12] are based — in fact — on the same idea. The following conditions are sufficient for the smoothness of $V(d, m, k)$ and the property T :

$$m = 0, \quad 2k < \frac{(7 - \sqrt{13})d^2}{81} \approx 0.0418 d^2 \quad (\text{see [10], [11]}),$$

and for the irreducibility of $V(d, m, k)$:

$$m + 2k < \frac{3}{2}d \quad (\text{see [10], [11]}),$$

$$k \leq 3 \quad (\text{see [7]}),$$

$$\frac{1}{2}(d^2 - 4d + 1) \leq m \leq \frac{1}{2}(d^2 - 3d + 2) \quad (\text{see [8]}).$$

The main idea of our proof is as follows. We have to prove the property T and the irreducibility for $V(d, m, k)$. To any curve $F \in V(d, m, k)$ with nodes $z_1, \dots, z_m$ and cusps $w_1, \dots, w_k$ we assign two linear systems of curves of degree n :

$$\Lambda_1(n, F) = \{\Phi \mid z_1, \dots, z_m \in \Phi, (\Phi \cdot F)(w_i) \geq 3, i = 1, \dots, k\},$$

$$\Lambda_2(n, F) = \{\Phi \mid z_1, \dots, z_m \in \text{Sing}(\Phi), (\Phi \cdot F)(w_i) \geq 6, i = 1, \dots, k\}.$$

First we show the non-speciality of $\Lambda_1(d, F)$ for any $F \in V(d, m, k)$, which means according to the Riemann-Roch theorem that

$$(0.6) \quad \dim \Lambda_1(d, F) = \frac{1}{2}d(d+3) - m - 2k.$$

On the other hand, $\Lambda_1(d, F)$ is the intersection of the tangent spaces to germs of equisingular strata at F in the space of curves of degree d , and (0.6) gives us the transversality of this intersection, or the desired property T. Then we show that, for any F from some open dense subset $U \subset V(d, m, k)$, the system $\Lambda_2(d, F)$ is non-special. That implies the irreducibility. Indeed, first we show that an open dense subset of $\Lambda_2(d, F)$ is contained in $V(d, m, k)$; more precisely, it consists of curves of degree d having m nodes in a fixed position and k cusps in a fixed position with fixed tangents. Then from the non-speciality we derive that $\dim \Lambda_2(d, F) = \text{const}$, $F \in U$, and that conditions imposed by fixed singular points on curves of degree d are independent. Afterwards we represent U as an open dense subset of the space of some linear bundle, whose fibres are $\Lambda_2(d, F)$, $F \in U$, and whose base is an open dense subset of $\text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2))$, where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane.

The text is divided into five parts : in section 1 there are some preliminary notions and results; in section 2 we present examples of reducible varieties or ones without property T; in section 3 we construct irreducible curves in $\Lambda_1(n, F)$, where $n < d$; in section 4 we prove the property T; and in section 5 — the irreducibility.

1. Preliminaries

Here we shall recall some notions and well-known classical results [3], [14], and also present some simple technical results needed below. Namely, we introduce a certain class of linear systems of plane curves and show how to compute their dimensions by means of linear series on curves.

Let $\Sigma = \bigoplus_{t \geq 0} \Sigma(t)$ be the graded ring of polynomials in three homogeneous variables over the base field. We think of the space of plane curves of degree t as the projectivization $P(\Sigma(t))$. A linear system of plane curves of degree t is a subspace of $P(\Sigma(t))$. Let

$$I = \bigoplus_{t \geq 0} I(t) \subset \Sigma$$

be a homogeneous ideal, defining a zero-dimensional subscheme $Z \subset \mathbb{P}^2$. This ideal determines a sequence of linear systems $\Lambda(t) = P(I(t))$, $t \geq 1$. Denote this class of linear systems by $\mathcal{C}$. In other words, these are linear systems defined by linear conditions associated to finite many base points. It is well-known [3] that

$$(1.1) \quad \dim \Lambda(t) = \dim P(\Sigma(t)) - \deg Z + i(\Lambda(t)),$$

where $i(\Lambda(t)) \geq 0$ is called the *speciality index* of $\Lambda(t)$. If $i(\Lambda(t)) = 0$ then the linear system $\Lambda(t)$ is called *non-special*. For a given ideal I , $\Lambda(t)$ is non-special when t is big enough (see [3]).

PROPOSITION 1.2. — *Let $\Lambda(t), \Lambda'(t)$ belong to $\mathcal{C}$, and, for all $t \geq 1$,*

$$\Lambda(t) \subset \Lambda'(t).$$

If, for some $n \geq 1$, the system $\Lambda(n)$ is non-special then $\Lambda'(n)$ is non-special.

Proof. — The systems $\Lambda(t), \Lambda'(t)$ are non-special for t big enough. Take a straight line $L \in P(\Sigma(1))$ not intersecting the zero-dimensional schemes Z, Z' associated to our linear systems. Let us embed the space $P(\Sigma(n))$ into $P(\Sigma(t))$, multiplying by L^{t-n} . Then :

$$\Lambda(n) = \Lambda(t) \cap P(\Sigma(n)), \quad \Lambda'(n) = \Lambda'(t) \cap P(\Sigma(n)).$$

So, the non-speciality of $\Lambda(n)$ means the transversality of the intersection of $\Lambda(t)$ and $P(\Sigma(n))$ in $P(\Sigma(t))$. But this implies that $\Lambda'(t)$ and $P(\Sigma(n))$ intersect transversally in $P(\Sigma(t))$, hence

$$\text{codim}(\Lambda'(n), P(\Sigma(n))) = \text{codim}(\Lambda'(t), P(\Sigma(t))) = \deg Z',$$

what is equivalent to the desired non-speciality. $\square$

From now on, divisor will always mean an effective Cartier divisor on a curve.

Let F be a reduced plane curve. For any divisor D on F and any component $H \subset F$ the symbol $D|_H$ means the restriction of D on H . For any curve G the symbol G_F means the formal expression $\sum n(P) \cdot P$, where the sum is taken over all local branches of F and $n(P)$ is the intersection number of P and G . If F, G have no common components, G_F is the divisor on F cut out by G , otherwise we admit infinite coefficients in the above expression.

By $D(F)$ we denote the double point divisor of the curve F . We omit its exact definition (see, for example, [14]), but only list the properties used in the sequel.

PROPOSITION 1.3 (see [14]).

(1) *The divisor $D(F)$ can be expressed as*

$$D(F) = \sum n(P) \cdot P,$$

where P runs through all the local branches of F centered at singular points, and the coefficients $n(P)$ are positive integers. In particular, $n(P) = 1$ for both branches centered at a node, and $n(P) = 2$ for a branch centered at a cusp.

(2) *Let z be a singular point of the curve F and a non-singular point of some curve G , then*

(i) *for any singular local branch P of F centered at z ,*

$$(G \cdot P)(z) \leq n(P) + 1,$$

(ii) *for any pair P_1, P_2 of local branches of F centered at z ,*

$$(G \cdot P_1) \leq n(P_1), \quad \text{or} \quad (G \cdot P_2) \leq n(P_2).$$

(3) *If F is an irreducible curve of degree d and geometric genus $g(F)$ then :*

$$\deg D(F) = d(d-3) + 2 - 2g(F).$$

(4) If a reduced curve G has no common components with F then :

$$D(FG)|_F = D(F) + G_F.$$

For any divisor D on F , the symbol $\mathcal{L}_F(n, D)$ denotes the linear system of plane curves of degree n

$$\{\Phi \mid \Phi_F \geq D + D(F)\}.$$

It is clear from the definition and PROPOSITION 1.3 that $\Lambda_1(n, F)$, $\Lambda_2(n, F)$ belong to this class. Also these systems belong to $\mathcal{C}$.

THEOREM 1.4 (Brill-Noether (see [14])). — *If F is irreducible then curves from $\mathcal{L}_F(n, D)$ cut out on F the linear series $|nL_F - D - D(F)|$, where L is a general straight line.*

THEOREM 1.5 (Noether (see [14])). — *Let $F_1, \dots, F_k$ be different irreducible curves of degrees $n_1, \dots, n_k$, and $F = F_1 \cdots F_k$, $\deg F = d$. Then :*

$$(1.6) \quad \mathcal{L}_F(n, D) = \sum_{i=1}^k \mathcal{L}_{F_i}(n + n_i - d, D|_{F_i}) \cdot F_1 \cdots F_{i-1} F_{i+1} \cdots F_k.$$

THEOREM 1.7 (Riemann-Roch for curves (see [3], [14])). — *For any divisor D on an irreducible curve F the dimension of the linear series $|D|$ is*

$$\dim |D| = \deg D - g(F) + i(D),$$

where $i(D)$ is non-negative. If $\deg D > 2g(F) - 2$ then $i(D) = 0$.

PROPOSITION 1.8. — *For any reduced curve F of degree $d \leq n$,*

$$(1.9) \quad \dim \mathcal{L}_F(n, D) \geq \frac{1}{2}n(n+3) - \frac{1}{2}\deg D(F) - \deg D.$$

The non-speciality of $\mathcal{L}_F(n, D)$ is equivalent to the equality in (1.9).

Proof. — Assume that F is irreducible. Representing $\mathcal{L}_F(n, D)$ as the span of $|nL_F - D - D(F)|$ and $F \cdot P(\Sigma(n-d))$, we obtain

$$\dim \mathcal{L}_F(n, D) = \dim |nL_F - D - D(F)| + \dim \Sigma(n-d),$$

hence according to THEOREM 1.7 and PROPOSITION 1.3,

$$\begin{aligned} \dim \mathcal{L}_F(n, D) &\geq nd - \deg D - \deg D(F) - g(F) \\ &\quad + \frac{1}{2}(n-d+1)(n-d+2) \\ &= nd - \frac{1}{2}d(d-3) - 1 - \deg D - \frac{1}{2}\deg D(F) \\ &\quad + \frac{1}{2}(n-d+1)(n-d+2), \end{aligned}$$

which is equivalent to (1.9). Also we obtain that the equality in (1.9) means $i(D) = 0$. Therefore, for all $t \geq d$,

$$(1.10) \quad \text{codim}(\mathcal{L}_F(t, D), P(\Sigma(t))) = \frac{1}{2} \deg D(F) + \deg D.$$

On the other hand, for t big enough, $\mathcal{L}_F(t, D)$ is non-special. Comparing this with (1.1) and (1.10), we get that the equality in (1.9) means the non-speciality of $\mathcal{L}_F(n, D)$.

If F is reducible, combine the previous computation with (1.6).

PROPOSITION 1.11. — *Let $F \in V(d, m, k)$.*

(1) *If $G \in \Lambda_1(n, F)$ is reduced, then there is a divisor D on G of degree $\leq m + 2k$ such that, for all $t \geq 1$,*

$$(1.12) \quad \Lambda_1(t, F) \supset \mathcal{L}_G(t, D).$$

(2) *Let $G \in \Lambda_1(n, F)$ be irreducible, let S be a subset of $\text{Sing}(F)$, and let H be a reduced curve containing S but not G . Let $\Lambda_3(t, F, S)$ be a linear system of curves $\Phi \in \Lambda_1(t, F)$ such that $S \subset \text{Sing}(\Phi)$, and Φ meets F at each cusp from S with multiplicity ≥ 5 . Then there is a divisor D on GH such that*

$$\deg D|_G \leq m + 2k, \quad \deg D|_K \leq \text{card}(S \cap K),$$

for each component $K \subset H$, and, for all $t \geq 1$,

$$\Lambda_3(t, F, S) \supset \mathcal{L}_{GH}(t, D).$$

Proof. — We will construct the divisor $D = \sum n(P) \cdot P$ explicitly.

(1) We have to find a divisor D on G such that any curve from $\mathcal{L}_G(t, D)$ goes through each node and each cusp of F , and intersects a tangent line to F at any cusp with multiplicity ≥ 2 .

Let z be a node of F . Since $G \in \Lambda_1(n, F)$, then G goes through z . If G is non-singular at z we can put $n(P) = 1$ for the local branch P of G centered at z . If G is singular at z then we can put $n(P) = 0$ for all local branches of G centered at z , because in this case, according to PROPOSITION 1.3, curves from $\mathcal{L}_G(t, 0)$ go through z .

Let z be a cusp of F . Analogously, G goes through z . If G is non-singular at z , then the local branch P of G at z is tangent to the tangent line L to the curve F at z . Put $n(P) = 2$. Now, since any curve from $\mathcal{L}_G(p, D)$ intersects P with multiplicity ≥ 2 , the same holds for L . If G is singular

at z , then either there is a singular local branch P of G centered at z , or there are at least two local branches P_1, P_2 of G centered at z . In the first case we put $n(P) = 2$, in the second case we put $n(P_1) = n(P_2) = 1$. According to PROPOSITION 1.3 any curve from $\mathcal{L}_G(t, D)$ is singular at z , and thereby intersects L with multiplicity ≥ 2 .

(2) We can obtain the second statement easily by combining the previous arguments with the Noether theorem. $\square$

2. Non-transversality and reducibility

The upper bounds in the sufficient conditions (0.3), (0.4) are the best possible as far as the exponent of d is concerned. The slightly modified classical examples [15] presented below give an upper bound for the allowable coefficient of d^2 in (0.3), (0.4).

THEOREM 2.1. — *The set $V(6p, 0, 6p^2)$ is reducible if $p = 1, 2$, and has components with different dimensions if $p \geq 3$.*

Proof. — The case $p = 1$ is well-known [15]. Let $p \geq 2$. It is easy to see that the curves

$$H = F_{2p}^3 + G_{3p}^2$$

belong to $V(6p, 0, 6p^2)$, where F_{2p}, G_{3p} are general curves of degrees $2p, 3p$ respectively. A simple computation gives us :

$$(2.2) \quad \begin{aligned} \dim\{H \in V(6p, 0, 6p^2) \mid H = F_{2p}^3 + G_{3p}^2\} \\ = \frac{1}{2}6p(6p+3) - 12p^2 + \frac{1}{2}(p-1)(p-2). \end{aligned}$$

According to [13], for $p \geq 2$, there is a component of $V(6p, 0, 6p^2)$ with dimension :

$$\frac{1}{2}6p(6p+3) - 12p^2.$$

If $p \geq 3$ we obtain at least two components of $V(6p, 0, 6p^2)$ with different dimensions.

Let $p = 2$. According to (0.5), $V(12, 0, 24)$ is a T-variety, and hence has dimension 42. According to (2.2) curves $H = F_4^3 + G_6^2$ form a component $\tilde{V}$ of $V(12, 0, 24)$. Assume that $\tilde{V} = V(12, 0, 24)$.

Let J be an irreducible curve of degree 12 with 28 cusps constructed in [13]. Since $V(12, 0, 28)$ is a T-variety (see (0.5)), we can smooth out any four cusps of J , preserving the others, by means of a variation of J in the space $P(\Sigma(12))$. Indeed, since all 28 equisingular strata intersect transversally at J , we can leave four of them by moving J along the intersection of the others. So we obtain that J belongs to the closure of $\tilde{V}$,

and hence to any set s_{24} of 24 cusps of J there correspond a quartic F_4 and a sextic G_6 , passing through s_{24} . Distinct 24-tuples of cusps correspond to distinct quartics, because, according to Bézout's theorem, a quartic cannot contain more than 24 cusps of J . On the other hand, two 24-tuples s_{24}, s'_{24} with 23 common cusps give quartics F_4, F'_4 with 23 common points. Therefore F_4, F'_4 have a common component C_i of degree $i=1, 2$, or 3. If $i=3$ then $F_4 = C_3C_1, F'_4 = C_3C'_1$. Since C_3 passes through at most 18 cusps of J , then the straight lines C_1, C'_1 have at least 5 common points, that means they coincide. The cases $i=1$ or 2 lead analogously to contradictions, which prove that $V(12, 0, 24)$ is reducible.

THEOREM 2.3. — *The set $V(7p-3, 0, 6p^2)$ contains a component without property T when $p \geq 3$.*

Proof. — Obviously, the curve $H = A_{p-3}F_{2p}^3 + B_{p-3}G_{3p}^2$ belongs to $V(7p-3, 0, 6p^2)$, if $A_{p-3}, F_{2p}, B_{p-3}, G_{3p}$ are general curves of degrees $p-3, 2p, p-3, 3p$ respectively. The property T is equivalent to the non-speciality of $\Lambda_1(7p-3, H)$. From THEOREM 1.5 it is not difficult to deduce that

$$\Lambda_1(7p-3, H) = \{ \Phi \mid \Phi = R_{3p-3}F_{2p}^2 + S_{4p-3}G_{3p} \}$$

with arbitrary curves R_{3p-3}, S_{4p-3} of degrees $3p-3, 4p-3$. Further, a trivial computation gives

$$\dim \Lambda_1(7p-3, H) = \frac{1}{2}(7p-3) \cdot 7p - 12p^2 + 1,$$

that means $\Lambda_1(7p-3, H)$ is special.

COROLLARY 2.4. — *The allowable coefficient at d^2 in the right hand side of (0.3) cannot exceed $\frac{12}{49}$, and in the right hand side of (0.4) cannot exceed $\frac{1}{3}$.*

3. Main lemma

LEMMA 3.1. — *For any curve $F \in V(d, m, k)$ and real $\alpha \geq (m+2k)/d^2$, there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, where $n = [(\sqrt{2\alpha} + 2/3)d]$.*

Proof. — Let $z_1, \dots, z_m$ be the nodes of F , and let $w_1, \dots, w_k$ be the cusps of F . Let h be the minimal integer such that $\Lambda_1(h, F) \neq \emptyset$. Then, $\Lambda_1(h-1, F) = \emptyset$ implies

$$m+2k > \frac{1}{2}(h-1)(h+2),$$

and hence

$$(3.2) \quad h < \sqrt{2(m+2k)} \leq \sqrt{2\alpha}d.$$

Take a general curve $H \in \Lambda_1(h, F)$. Assume that $H = H_1^{i_1} \cdots H_r^{i_r}$, where $H_1, \dots, H_r$ are irreducible components of degrees $h_1, \dots, h_r$ respectively. Since h is minimal,

$$\max\{i_1, \dots, i_r\} \leq 2.$$

We shall construct the curve Φ as follows. First we will construct, for each $s = 1, \dots, r$, a curve C_s of degree $\ell_s \leq i_s h_s + \frac{2}{3}d$ such that C_s does not contain H_s and the curve

$$(3.3) \quad R_s \stackrel{\text{def}}{=} H_1^{i_1} \cdots H_{s-1}^{i_{s-1}} C_s H_{s+1}^{i_{s+1}} \cdots H_r^{i_r}$$

belongs to $\Lambda_1(h + \ell_s - i_s h_s, F)$. After that we obtain the desired curve Φ in the form

$$G_0 H + G_1 R_1 + \cdots + G_r R_r,$$

where $G_0, G_1, \dots, G_r$ are generic curves of suitable degrees.

The rest of the proof is divided into five steps : in steps 1, 2, 3 and 4 we construct the curves $C_1, \dots, C_r$, in the fifth step we construct the curve Φ .

Let us do the construction of C_1 . Let H_1 pass through $z_1, \dots, z_p, w_1, \dots, w_q$ and meet F at $w_{q+1}, \dots, w_{q+t}$ with multiplicities ≥ 3 . Let $\deg H_1 = h_1$. The Bézout theorem gives :

$$(3.4) \quad 2p + 2q + 3t \leq h_1 d.$$

Step 1. — Assume $i_1 = 1$. Let us find a curve C_1 passing through $z_1, \dots, z_p, w_1, \dots, w_q$, meeting F at $w_{q+1}, \dots, w_{q+t}$ with multiplicities ≥ 3 , and not containing H_1 . This can be done under the following sufficient condition on $\ell_1 = \deg C_1$

$$\frac{1}{2}\ell_1(\ell_1 + 3) - \frac{1}{2}(\ell_1 - h_1)(\ell_1 - h_1 + 3) > p + q + 2t,$$

which is equivalent to

$$\ell_1 > \frac{1}{2}h_1 - \frac{3}{2} + \frac{p + q + 2t}{h_1},$$

and, using (3.4), we can take

$$(3.5) \quad \ell_1 \leq \frac{1}{2}h_1 + \frac{p + q + 2t}{h_1} \leq \frac{1}{2}h_1 + \frac{p + q + 2t}{2p + 2q + 3t}d \leq \frac{1}{2}h_1 + \frac{2}{3}d.$$

Step 2. — From now on assume $i_1 = 2$. First we look for a curve C'_1 of degree ℓ'_1 passing through $z_1, \dots, z_p$, meeting F at $w_{q+1}, \dots, w_{q+t}$ with

multiplicities ≥ 3 and not containing H_1 . As in the first step we have the following sufficient condition for the existence of such a curve

$$\ell'_1 > \frac{1}{2}h_1 - \frac{3}{2} + \frac{p+2t}{h_1},$$

and then we can take

$$(3.6) \quad \ell'_1 \leq \frac{1}{2}h_1 + \frac{p+2t}{h_1}.$$

Step 3. — Assume that there is a curve $H'_1 \neq H_1$ of degree $h'_1 \leq h_1$, passing through $w_1, \dots, w_q$, and that h'_1 is the minimal such degree. This minimality implies :

$$(3.7) \quad h_1'^2 \leq 2q.$$

Then we put $C_1 = C'_1(H'_1)^2$. According to (3.6)

$$\ell_1 = \deg C_1 \leq \frac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h'_1.$$

Hence, using (3.4), (3.7) and the initial assumption $h'_1 \leq h_1$, it is easy to compute

$$(3.8) \quad \begin{aligned} \ell_1 &\leq \frac{1}{2}h_1 + \frac{p+q+2t}{2p+2q+3t}d - \frac{q}{h_1} + 2h'_1 \\ &\leq \frac{1}{2}h_1 + \frac{2}{3}d - \frac{q}{h_1} + 2h'_1 \leq \frac{2}{3}d + 2h_1. \end{aligned}$$

Step 4. — Now assume that H_1 is the unique curve of degree $\leq h_1$, passing through $w_1, \dots, w_q$. In particular, that means

$$(3.9) \quad \frac{1}{2}h_1(h_1 + 3) \leq q.$$

Let C'_1 be the curve of degree ℓ'_1 constructed in step 2. Consider two situations.

- Assume first $q \leq h_1^2$. Then $2q < 2h_1(2h_1 + 3)/2$. That means the set M of curves of degree $2h_1$, meeting F at $w_1, \dots, w_q$ with multiplicities ≥ 3 , is infinite. The only curve in M containing H_1 is H_1^2 . Now take $H'_1 \in M$ different from H_1^2 , and put $C_1 = C'_1H'_1$. Here :

$$\ell_1 = \deg C_1 \leq \frac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h_1.$$

Finally, from (3.4), (3.9) we get :

$$(3.10) \quad \ell_1 \leq \frac{2}{3}d + 2h_1.$$

• Now assume :

$$(3.11) \quad q \geq h_1^2 + 1.$$

Introduce the integer

$$h'_1 = \max\{\nu \in \mathbb{N} \mid h_1(\nu - h_1) + 1 \leq q\}.$$

According to (3.11), $h'_1 \geq 2h_1$. Denote by M the set of curves of degree h'_1 meeting F with multiplicity ≥ 3 at each point $w_1, \dots, w_\pi$, where $\pi = h_1(h'_1 - h_1) + 1$. If $G \in M$ contains H_1 then $G = G_1 H_1$, and G_1 goes through $w_1, \dots, w_\pi$, because

$$(G_1 \cdot F)(w_i) = (G \cdot F)(w_i) - (H_1 \cdot F)(w_i) \geq 3 - 2 = 1, \quad i = 1, \dots, \pi.$$

Hence G_1 contains H_1 , because these curves meet at $\pi > \deg H_1 \cdot \deg G_1$ points. Then there is a curve $H'_1 \in M$ not containing H_1 as component, because the sufficient condition for this existence is

$$\frac{1}{2} h'_1 (h'_1 + 3) - 2(h_1(h'_1 - h_1) + 1) > \frac{1}{2} (h'_1 - 2h_1)(h'_1 - 2h_1 + 3),$$

which is equivalent to

$$3h_1 > 2.$$

Finally, let H''_1 be a curve of the smallest degree h''_1 meeting F at w_i , $i = h_1(h'_1 - h_1) + 2, \dots, q$, with multiplicities ≥ 3 . By definition of h'_1 , the number of these points is $\leq h_1 - 1$. If $h_1 \leq 3$ then, obviously, $h''_1 \leq h_1 - 1$. If $h_1 \geq 4$, since

$$\frac{1}{2} (h''_1 - 1)(h''_1 + 2) \leq 2h_1 - 2,$$

we have :

$$(3.12) \quad h''_1 \leq \frac{5}{6} h_1.$$

The last inequality is true in the case $h_1 \leq 3$ too. In particular, H''_1 does not contain H_1 . Now put $C_1 = C'_1 H'_1 H''_1$. Here we have from (3.4), (3.6), and (3.12)

$$\begin{aligned} \ell_1 = \deg C_1 &\leq \frac{1}{2} h_1 + \frac{p+2t}{h_1} + h'_1 + h''_1 \\ (3.13) \quad &= \frac{1}{2} h_1 + \frac{p + \frac{4}{3}q + 2t}{h_1} + h'_1 + h''_1 - \frac{4q}{3h_1} \\ &\leq \frac{1}{2} h_1 + \frac{2}{3}d + h'_1 + h''_1 - \frac{4}{3}(h'_1 - h_1) \\ &\leq \frac{8}{3} h_1 + \frac{2}{3}d - \frac{1}{3}h'_1 \leq 2h_1 + \frac{2}{3}d, \end{aligned}$$

because $h'_1 \geq 2h_1$ as it was mentioned above.

Step 5. — Inequalities (3.2), (3.5), (3.8), (3.10), (3.13) and the definitions of n and α imply that the degree of any curve R_j , $j = 1, \dots, r$, defined by (3.3), is less than n . Now consider the linear system

$$(3.14) \quad \lambda_0 G_0 H + \lambda_1 G_1 R_1 + \dots + \lambda_r G_r R_r, \quad (\lambda_0, \dots, \lambda_r) \in \mathbb{P}^r,$$

where $G_0, \dots, G_r$ are generic curves of *positive* degrees $n - h$, $n - \deg R_1$, $\dots$, $n - \deg R_r$ respectively. According to the construction of $H, C_1, \dots, C_r$, this is a subsystem of $\Lambda_1(n, F)$. Also note that the curves $G_0, \dots, G_r$ do not go through base points of our linear system. Then we obtain immediately from the Bertini theorem (see [3], [14]) that a generic member in the linear system (3.14) is reduced and irreducible. Thereby we can take this generic member as the desired curve Φ .

4. The property T

Let $F \in V(d, m, k)$. As said above, the property T and the smoothness of $V(d, m, k)$ at F follow from :

PROPOSITION 4.1. — *Under condition (0.3) the linear system $\Lambda_1(d, F)$ is non-special.*

Proof. — According to LEMMA 3.1, under condition (0.3) there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, $n = \lfloor \sqrt{2\alpha_0} + \frac{2}{3}d \rfloor$. According to PROPOSITION 1.11 there is a divisor D on Φ of degree

$$(4.2) \quad \deg D \leq m + 2k$$

such that

$$\Lambda_1(p, F) \supset \mathcal{L}_\Phi(p, D), \quad p \geq 1.$$

Therefore, according to PROPOSITION 1.2, it is enough to establish the non-speciality of $\mathcal{L}_\Phi(d, D)$, which will follow from

$$(4.3) \quad \deg(G_\Phi - D(\Phi) - D) > 2g(\Phi) - 2$$

where $G \in \mathcal{L}_\Phi(d, D)$, and $g(\Phi)$ is the geometric genus of Φ . Indeed, we have by PROPOSITION 1.3 and (4.2) :

$$(4.4) \quad \begin{aligned} \deg(G_\Phi - D(\Phi) - D) &= nd - n(n - 3) - 2 + 2g(\Phi) - \deg D \\ &\geq n(d - n + 3) - (m + 2k) + 2g(\Phi) - 2 \\ &> n(d - n + 3) - \alpha_0 d^2 + 2g(\Phi) - 2. \end{aligned}$$

Since $n = \lfloor (\sqrt{2\alpha_0} + \frac{2}{3})d \rfloor$ and α_0 is the positive root of the equation

$$(\sqrt{2\alpha} + \frac{2}{3})\left(\frac{1}{3} - \sqrt{2\alpha}\right) = \alpha,$$

hence

$$n(d - n + 3) \geq \alpha_0 d^2.$$

Then (4.4) implies (4.3) and completes the proof.

5. Irreducibility

We prove the irreducibility along the plan mentioned in introduction.

PROPOSITION 5.1. — *For any $F \in V(d, m, k)$, the intersection of $V(d, m, k)$ with $\Lambda_2(d, F)$ contains an open dense subset of $\Lambda_2(d, F)$, and consists exactly of curves from $V(d, m, k)$ with the same nodes, and the same cusps with the same tangents as F .*

Proof. — Since $F \in \Lambda_2(d, F)$ and any curve $G \in \Lambda_2(d, F)$ is singular at $z_1, \dots, z_m$, then the Bertini theorem implies that almost all curves in $\Lambda_2(d, F)$ have nodes at $z_1, \dots, z_m$, and are non-singular outside $\text{Sing}(F)$. Consider the cusp $w_1 \in F$. In some affine neighbourhood of w_1 we fix an affine coordinate system (x, y) such that $w_1 = (0; 0)$, and $y = 0$ is a tangent to F at w_1 . Then in this neighbourhood, F is defined by polynomial

(5.2)
$$Ay^2 + Bx^3 + \sum_{2i+3j>6} A_{ij}x^i y^j, \quad AB \neq 0,$$

and can be locally parametrized analytically (see [3], [14]) by

(5.3)
$$x = \tau^2, \quad y = \lambda \tau^3 + O(\tau^4).$$

To determine $(G \cdot F)(w_1)$ we plug (5.3) into the affine equation

$$\sum a_{ij}x^i y^j = 0$$

of G , and then compute the order of vanishing at $\tau = 0$ (see [14]). Thus we obtain for curves $G \in \Lambda_2(d, F)$ that :

$$a_{00} = a_{01} = a_{10} = a_{11} = a_{20} = 0 .$$

Since $F \in \Lambda_2(d, F)$, almost all curves in $\Lambda_2(d, F)$ have affine equations like (5.2), that means they have a cusp at w_1 with the tangent $y = 0$. Now to complete the proof we should note that any curve $G \in V(d, m, k)$ with nodes $z_1, \dots, z_m$, cusps $w_1, \dots, w_k$ and tangents $T_{w_1}F, \dots, T_{w_k}F$, belongs to $\Lambda_2(d, F)$.

PROPOSITION 5.4. — *Under condition (0.4), $V(d, m, k)$ contains an open dense subset $\tilde{V}$ consisting of curves F with non-special linear system $\Lambda_2(d, F)$.*

REMARK 5.5. — Even under condition (0.4), $V(d, m, k)$ may contain curves G with special system $\Lambda_2(d, G)$. This holds for $V(2p, 3p, 0)$, $p \geq 3$ (see [12]).

Proof of Proposition 5.4. — It is enough to prove the statement for any irreducible component V_0 of $V(d, m, k)$. For any curve $G \in V_0$, let $\mathcal{H}(G)$ denote the linear system of curves of the smallest degree h , passing through $\text{Sing}(G)$. Evidently, any $H \in \mathcal{H}(G)$ is reduced, and

$$(5.6) \quad h = \deg H \leq \sqrt{2(m+k)} < \sqrt{2\alpha_1} d.$$

Denote by V_1 the set of curves $G \in V_0$ with maximal h . This is an open dense irreducible subset of V_0 (see [3]). Now denote by V_2 the set of curves $G \in V_1$ with minimal $\dim \mathcal{H}(G)$. Similarly this is an open dense irreducible subset of V_1 . Then

$$W = \bigcup_{G \in V_2} \mathcal{H}(G)$$

is irreducible as the image in $P(\Sigma(h))$ of the space of a projective bundle with base V_2 and fibres $\mathcal{H}(G)$, $G \in V_2$. Denote by W_0 the set of curves $H \in W$ with minimal number of irreducible components. It is irreducible, open and dense in W . In particular, that means all the curves $H \in W_0$ determine the same sequence of degrees of their components (up to permutation). Moreover, if H runs through W_0 , then any of its irreducible components K runs through some irreducible set

$$W(K) \subset P(\Sigma(\deg K)).$$

For $H \in W_0 \cap \mathcal{H}(G)$ define $N(H)$ to be $\sum N(K, H)$, where K runs through all components of H , and $N(K, H) = \text{card}(K \cap \text{Sing}(G))$. Denote by W_1 the set of curves $H \in W_0$ with minimal $N(H)$. First, it is an open dense irreducible subset of W_0 , and, second, for any component K of $H \in W_1$,

$$N(K, H) = \min_{K' \in W(K)} N(K', H').$$

At last, introduce

$$V_3 = \{G \in V_2 \mid \mathcal{H}(G) \cap W_1 \neq \emptyset\}.$$

According to the above construction, this is an open dense subset of V_0 . Now we will show that V_3 contains an open subset consisting of curves satisfying the conditions of PROPOSITION 5.4, in three steps.

Step 1. — Fix $G \in V_3$ and $H \in \mathcal{H}(G) \cap W_1$. Show that any component K of H of degree δ contains at most $\frac{1}{2}\delta(\delta + 3)$ points from $\text{Sing}(G)$.

Indeed, let K contain $\frac{1}{2}\delta(\delta + 3) + 1$ points from $\text{Sing}(G)$. Denote the set of these points by S , and consider the linear system $\Lambda_3(d, G, S)$. Let us take an irreducible curve $\Phi \in \Lambda_1(n, G)$, $n = [(\frac{2}{3} + \sqrt{2\alpha_1})d]$, from LEMMA 3.1. According to PROPOSITION 1.11, for all $t \geq 1$,

$$\Lambda_3(t, G, S) \supset \mathcal{L}_{\Phi K}(t, D),$$

where

$$(5.7) \quad \deg D|_{\Phi} \leq m + 2k, \quad \deg D|_K \leq \frac{1}{2}\delta(\delta + 3) + 1.$$

We shall show that $\mathcal{L}_{\Phi K}(d, D)$ is non-special. According to PROPOSITION 1.8 and arguments from its proof, this is equivalent to

$$i((d - \delta)L_{\Phi} - D(\Phi) - D|_{\Phi}) = i((d - n)L_K - D(K) - D|_K) = 0.$$

According to THEOREM 1.7 these equalities follow from :

$$(5.8) \quad (d - \delta)n - \deg D(\Phi) - \deg D|_{\Phi} > 2g(\Phi) - 2,$$

$$(5.9) \quad \delta(d - n) - \deg D(K) - \deg D|_K > 2g(K) - 2.$$

According to PROPOSITION 1.3 and (5.7), inequality (5.8) follows from :

$$m + 2k < n(d - \delta - n + 3).$$

This inequality can be easily deduced from the definition of n and (5.6), because $\alpha_1 = \frac{2}{225}$ satisfies the inequality :

$$\alpha_1 < \left(\frac{2}{3} + \sqrt{2\alpha_1}\right)\left(\frac{1}{3} - 2\sqrt{2\alpha_1}\right).$$

Analogously, by PROPOSITION 1.3 and (5.7) the inequality (5.9) follows from :

$$d - n \geq \frac{3}{2}\delta.$$

This can be easily deduced from the definition of n and (5.6), because α_1 is the root of the equation

$$1 - \left(\frac{2}{3} + \sqrt{2\alpha}\right) = \frac{3}{2}\sqrt{2\alpha}.$$

So, according to PROPOSITION 1.2, $\Lambda_3(d, G, S)$ is non-special, and according to PROPOSITIONS 1.3 and 1.8

$$(5.10) \quad \dim \Lambda_3(d, G, S) = \frac{1}{2}d(d + 3) - m - 2k - 2 \cdot \text{card } S.$$

If G runs through V_3 , then S runs through some subset of

$$\mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

thereby defining a morphism

$$\nu : \tilde{V}_3 \longrightarrow \mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

where $\tilde{V}_3$ is the finite covering of V_3 corresponding to different choices of $\frac{1}{2}\delta(\delta+3)+1$ points from the set $\mathrm{Sing}(G) \cap K$ (which might contain more than $\frac{1}{2}\delta(\delta+3)+1$ points). The tangent space to the fibre $\nu^{-1}(S)$ at the point $(G, S) \in \tilde{V}_3$ is contained in $\Lambda_3(d, G, S)$ (see [2], [12]). Therefore (5.10) implies :

$$\dim \nu^{-1}(S) = \dim V_3 - 2\left(\frac{1}{2}\delta(\delta+3)+1\right),$$

hence $\nu(\tilde{V}_3)$ is dense in $\mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2)$. Therefore there is a curve $G \in V_3$ such that, for any set S of $\frac{1}{2}\delta(\delta+3)+1$ points in $\mathrm{Sing}(G) \cap K$, no more than $\frac{1}{2}\delta(\delta+3)$ points of S lie on a curve of degree δ . But this contradicts the definition of the set V_3 and the initial assumption that $K \cap \mathrm{Sing}(G)$ contains more than $\frac{1}{2}\delta(\delta+3)$ points, and thus completes the proof.

Step 2. — Consider the linear system $\Lambda_3(d, G, \mathrm{Sing}(G))$. As in the previous step, for any curve $H \in \mathcal{H}(G) \cap W_1$ and all $t \geq 1$ we have

$$\Lambda_3(t, G, \mathrm{Sing}(G)) \supset \mathcal{L}_{\Phi H}(t, D),$$

where D satisfies (5.7) for any component $K \subset H$, hence $\Lambda_3(d, G, \mathrm{Sing}(G))$ is non-special.

Step 3. — As in the first step, the non-speciality of $\Lambda_3(d, G, \mathrm{Sing}(G))$, $G \in V_3$, implies that the image of V_3 by the morphism

$$\mu : V_3 \rightarrow \mathrm{Sym}^{m+k}(\mathbb{P}^2), \quad \mu(G) = \mathrm{Sing}(G),$$

contains an open dense subset U of $\mathrm{Sym}^{m+k}(\mathbb{P}^2)$. According to [4], under condition (0.4), for any Z from some open subset $U' \subset U$ the linear system $\Lambda(d, Z)$ of curves of degree d , having multiplicity ≥ 3 at each point $z \in Z$, is non-special. It is easy to see that for any $G \in V(d, m, k)$ and $t \geq 1$:

$$\Lambda_2(t, G) \supset \Lambda(t, \mathrm{Sing}(G)).$$

Therefore $\Lambda_2(d, F)$ is non-special for any curve $F \in \mu^{-1}(U') \cap V_3$. $\square$

Now we can finish the proof of the irreducibility of $V(d, m, k)$, showing that $\tilde{V}$ is irreducible. To any curve $F \in \tilde{V}$ we assign the set $\text{Sing}(F)$ and the set of tangents at its cusps. Thereby we obtain a morphism

$$\pi : \tilde{V} \longrightarrow \text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2)),$$

where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane. According to PROPOSITION 5.1 any fibre of π is an open subset of some linear system $\Lambda_2(d, F)$, $F \in \tilde{V}$, hence is irreducible. The non-speciality of these linear systems, PROPOSITIONS 1.2 and 1.8 imply immediately that all the fibres have the same dimension :

$$\frac{1}{2}d(d+3) - 3m - 5k = \dim \tilde{V} - \dim (\text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2))).$$

Finally, this equality means that $\pi(\tilde{V})$ is dense in

$$\text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2)),$$

hence is irreducible. This completes the proof.

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