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FRANÇOIS BLANCHARD

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A DISJOINTNESS THEOREM INVOLVING TOPOLOGICAL ENTROPY

BY

FRANÇOIS BLANCHARD (*)

RÉSUMÉ. — Un recouvrement standard d'un compact X est un recouvrement de celui-ci par deux ouverts non-denses. Dans le carré cartésien d'un flot (X,T), un couple (x,x') hors de la diagonale est appelé couple d'entropie quand tout recouvrement standard (U,V) tel que $(x,x') \in \mathrm{Int}(U^c) \times \mathrm{Int}(V^c)$ a une entropie positive. L'ensemble des couples d'entropie n'est pas vide dès que l'entropie du flot est positive, il est invariant par $T \times T$, et tout couple situé dans sa fermeture est couple d'entropie s'il n'est pas dans la diagonale.

On dit qu'un flot est d'entropie uniformément positive si tout recouvrement standard est d'entropie positive, ce qui revient à dire que tout couple hors diagonale est couple d'entropie. Nous utilisons les propriétés des couples d'entropie pour montrer que les flots d'entropie uniformément positive, et même une classe plus générale de flots, sont disjoints des flots minimaux d'entropie nulle. Nous construisons ensuite un exemple de flot d'entropie uniformément positive contenant une seule orbite périodique.

ABSTRACT. — A cover of some compact set X by two non dense open sets is called a standard cover. In the cartesian square of a flow (X,T), pairs (x,x') outside the diagonal are defined as entropy pairs whenever any standard cover (U,V) such that $(x,x') \in \mathrm{Int}(U^c) \times \mathrm{Int}(V^c)$ has positive entropy. The set of such pairs is nonempty provided $h(X,T) > 0$; it is $T \times T$-invariant, and all pairs in its closure belong either to it or to the diagonal.

A flow is said to have uniform positive entropy if any standard cover has positive entropy (or if all non diagonal pairs are entropy pairs). Properties of entropy pairs are used to show that flows with uniform positive entropy (in fact a wider class) are disjoint from minimal flows with entropy 0. A flow with uniform positive entropy containing only one periodic orbit is constructed.

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F. BLANCHARD, Laboratoire de Mathématiques Discrètes, case 930, 163 av. de Luminy, 13288 Marseille Cedex 09 (France). e-mail : blanchar@lmd.univ-mrs.fr.

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0. Introduction

A flow is a compact metric set X endowed with a homeomorphism T. Two of the most important notions for the study of flows are minimality and topological entropy. A minimal flow is one such that there is no closed T-invariant proper subset in X except the empty set. In 1967 FURSTENBERG [F] introduced disjointness of flows — any common extension factors through the cartesian product — and proved minimal flows to be disjoint from all flows which are transitive for all powers of T, and have a dense set of periodic points (also called F-flows); in this paper we refer to this result as Furstenberg's theorem. It was the first of an impressive series of results about the interplay of minimality and other properties like equicontinuity and mixing (see for instance [Au]). Let us mention another disjointness theorem : flows disjoint from all distal flows are exactly minimal weakly mixing flows [P].

On the other hand, topological entropy was introduced by ADLER, KONHEIM and MCANDREW in 1965 [AKM] as a measure of the quantity of information contained in flows, and also led to many fruitful investigations, but these hardly ever met with those about minimal flows. It is true F-flows with entropy 0 [We] and minimal flows with positive entropy [HK] have long been known to exist. This of course means that Furstenberg's theorem is not in the field of the theory of entropy, but does not shut out possible connections.

In the measure-theoretic setting K-systems play an outstanding role; one of the many reasons is they are disjoint from systems with entropy 0. Would the definition of some « fully positive topological entropy » property favour the interplay between entropy and other properties of topological dynamics?

In [B] two distinct notions of that kind were introduced; the purpose was to test whether they have some mixing implications. The weaker, completely positive entropy (c.p.e.), means that all topological factors have positive entropy. It corresponds to what is called weak disjointness from all zero-entropy flows. Completely positive entropy does not imply any kind of mixing, not even transitivity. The stronger, uniform positive entropy (u.p.e.), is defined by the fact that any cover by two non dense open sets (standard cover) has positive entropy. It implies weak but not strong topological mixing.

The aim of the present article is, as in the previous one, to understand better how topological entropy is woven into the general pattern of topological dynamics. However, here we approach the subject considering the matter of disjointness.

First, a «pairwise» point of view about entropy is developped in section 2. Let us say that $(x, x') \in X^2$, $x \neq x'$, is an *entropy pair* for the homeomorphism T if any standard cover (U, V) such that (x, x') in $\mathrm{Int}(U^c) \times \mathrm{Int}(V^c)$ has positive entropy with respect to T. Entropy pairs exist in any flow with positive entropy : such a flow has a standard cover with positive entropy (PROPOSITION 1), and for any open cover (U, V) with positive entropy, there is an entropy pair $(x, x') \in U^c \times V^c$ (PROPOSITION 2). The closure $E'(X, T)$ of the set of entropy pairs in $(X^2, T \times T)$ is invariant and contains only entropy pairs and points of the diagonal (PROPOSITION 3). Entropy pairs still have other simple but very convenient properties. Let ϕ be a factor map from (X, T) onto (Y, S) : the preimage of an entropy pair contains an entropy pair, and the image of an entropy pair (x, x') is one whenever $\phi(x) \neq \phi(x')$ (PROPOSITION 4); and if X' is a closed invariant subset of X and (x, x') is an entropy pair of (X', T), then it is also an entropy pair of (X, T) (PROPOSITION 5). All this is done rather easily, with the use of compactness, metricity and elementary properties of the entropy of covers.

In section 3 of this article PROPOSITION 4 is used to prove a disjointness theorem of the precise kind the author was looking for, since it involves at the same time entropy and minimality : *diagonal flows*, i.e. those having the property that $\{(x, Tx), x \in X\}$ is contained in $E'(X, T)$, are disjoint from minimal zero-entropy flows (PROPOSITION 6). Diagonality is strictly weaker than u.p.e., since finite unions of flows with u.p.e. are diagonal, but we do not know whether it is *significantly* so. PROPOSITION 6 parallels Furstenberg's theorem, though with completely different assumptions. It seems more closely related to another result, also from [F] : weakly mixing flows are disjoint from minimal distal flows (u.p.e. implies weak mixing and entropy 0 is a consequence of distality).

Finally, some examples are given. Example 7, constructed as a kind of skew product of a full shift and a minimal subshift, has u.p.e. without being an F-flow, which shows that PROPOSITION 6 is not a particular case of Furstenberg's theorem (and also, using results in [Wi], that flows with u.p.e. may have an uncountable set of ergodic measures with maximal entropy). Example 9 is obtained by hybridating a zero-entropy minimal subshift with the full 2-shift in a different way, already described in [B]; it has c.p.e. without being disjoint from its minimal «parent»; so one cannot arbitrarily weaken the assumptions of PROPOSITION 6. Entropy pairs may be used for other purposes : for instance they permit to define a maximal factor with entropy 0 in any flow; this topic will be developped in a forthcoming paper with Y. LACROIX [BL].

This research started after many stimulating talks with J. KWIATKOWSKI on connected topics. I am deeply grateful to him, as well as to P. LIARDET, who suggested the use of minimal joinings for the proof of a previous version of PROPOSITION 6, and B. HOST, who found out the actual, much shorter proof, also obtaining a more general statement. Y. LACROIX, A. MAASS and especially M. LEMANCZYK made several useful remarks.

1. Definitions and background

This section contains several claims; their proofs can be found in [F] for those concerning minimality and disjointness, or in [DGS] for the ones about entropy.

A *transitive* flow (X, T) is one such that for any two non empty open sets U, V in X there is a positive integer n such that $U \cup T^{-n}V \neq \emptyset$; a flow is said to be *weakly mixing* if its cartesian square is transitive, or equivalently if for any choice of nonempty open U, V, U', V' in X one can find $n > 0$ such that $U \cup T^{-n}U' \neq \emptyset$ and $V \cup T^{-n}V' \neq \emptyset$. For $n \in \mathbb{Z}$ put

$$\Delta_n = \big\{(x, T^n x),\ x \in X\big\}\ ;$$

Δ_n is a $T \times T$-invariant subset of the cartesian product $X \times X$, called the *out-diagonal of order* n (the *diagonal* when $n = 0$). Transitivity of (X, T) is equivalent to the fact that the union of all out-diagonals of positive order is dense in $X \times X$.

Suppose (X, T) and (Y, S) are two flows. A *factor map* or *homomorphism* $\phi : (Y, S) \to (X, T)$ is a continuous, onto map changing S to T; in this case (X, T) is said to be a *factor* of (Y, S).

A *joining* of (X, T) and (X', T') is a flow (Y, S) together with two factor maps $\phi : (Y, S) \to (X, T)$ and $\psi : (Y, S) \to (X', T')$ (when $X = X'$ it is called a *self-joining*). The cartesian product $(X \times X', T \times T')$, together with the projections π and π', is always a joining of (X, T) and (X', T').

There are two equivalent definitions of disjointness. The first is more striking; two flows (X, T) and (X', T') are said to be *disjoint* if any common joining (Y, S, ϕ, ψ) factors through the cartesian product : there is a factor map $\chi : (Y, S) \to (X \times X', T \times T')$ such that $\phi = \pi \circ \chi$, $\psi = \pi \circ \chi$. The second is the only one we use in this paper. A *proper subjoining* of the cartesian product is a proper closed invariant subset of $(X \times X', T \times T')$ projecting to (X, T) and (X', T'). Two flows are disjoint iff there exists no proper subjoining, i.e. if the only subjoining of $(X \times X', T \times T')$ is the cartesian product itself. Remember that when (X, T) and (Y, S) are disjoint, at least one of them must be minimal.

Before defining topological entropy, it is necessary to introduce some notations. Let $\mathcal{R}$ and $\mathcal{S}$ be two open covers of X. The cover $\mathcal{S}$ is said to be *finer* than $\mathcal{R}$ (notation $\mathcal{R} \le \mathcal{S}$) if any member of $\mathcal{S}$ is contained in an element of $\mathcal{R}$. Denote by $\mathcal{R} \vee \mathcal{S}$ the cover made up by all sets $R \cap S$, with $R \in \mathcal{R}$, $S \in \mathcal{S}$. For $n \in \mathbb{N}$, write $\mathcal{R}_n = \bigvee_{0 \le i < n} T^{-i}\mathcal{R}$. Let $H(\mathcal{R}) = \inf(\log \#\mathcal{R}')$, where the inf is taken over the set of all finite subcovers of $\mathcal{R}$, and $\#\mathcal{R}'$ denotes the number of non empty members of $\mathcal{R}'$.

The *entropy* of the cover $\mathcal{R}$ is the non negative number

$$h(T, \mathcal{R}) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{R}_n).$$

The limit exists. Some properties of $h(T, \mathcal{R})$ deduced from corresponding properties of H prove very useful, particularly here :

- $h(T, \mathcal{R})$ is increasing for the partial order of covers ($\mathcal{R} \le \mathcal{S}$ implies $h(T, \mathcal{R}) \le h(T, \mathcal{S})$) ;
- it is subadditive, i.e. $h(T, \mathcal{R} \vee \mathcal{S}) \le h(T, \mathcal{R}) + h(T, \mathcal{S})$;
- if $\phi : (Y, S) \to (X, T)$ is a factor map, then the set $\phi^{-1}(\mathcal{R})$ of all preimages of elements of $\mathcal{R}$ satisfies $h(S, \phi^{-1}(\mathcal{R})) = h(T, \mathcal{R})$; of course the same is not true for images.

The *topological entropy* of (X, T) is the (sometimes infinite) number

$$h(X, T) = \sup h(T, \mathcal{R}),$$

where the sup is taken over all finite open covers of X.

Finally, the following definitions are constantly used in the sequel. A *standard cover* of a compact metric set is a cover (U, V) by two non dense open sets. Denote by U^c the complement of the set U. Given x, x' in X, $x \ne x'$, (U, V) is said to *distinguish* x and x' if $x \in \mathrm{Int}(U^c)$ and $x' \in \mathrm{Int}(V^c)$. A pair $(x, x') \in X^2$, $x \ne x'$, is said to be an *entropy pair* if for any standard cover $\mathcal{R} = (U, V)$ of X distinguishing x and x' one has $h(T, \mathcal{R}) > 0$.

When $x \ne x'$ it is easy to construct one such standard cover ; the definition does not make sense for diagonal pairs. Existence of one entropy pair in (X, T) implies $h(X, T) > 0$.

Recall a flow (X, T) is said to have *uniform positive entropy* (u.p.e) if for any standard cover $\mathcal{R}$ one has $h(T, \mathcal{R}) > 0$. It is said to have *completely positive entropy* (c.p.e) if all non trivial topological factors have positive entropy. These two properties are stable under factor maps, and u.p.e. is stronger than c.p.e.. A flow having u.p.e. is easily characterised as one for which any (x, x') in X^2, $x \ne x'$, is an entropy pair.

2. Localising the entropy in the cartesian square of a flow

The first two propositions are trivial in a u.p.e. flow. They permit to understand the significance of entropy pairs in flows not having this property.

In a metric space, denote by $B(z,\eta)$ and $B'(z,\eta)$ the open and closed balls with centre z and radius η.

PROPOSITION 1. — *Any flow (X,T) with positive entropy has a standard cover with positive entropy.*

Proof. — Positive entropy implies there is some finite open cover $\mathcal{S} = (U_1,\dots,U_k)$ of X with $h(T,\mathcal{S}) > 0$, but we want to find first a *two-open-set* cover with positive entropy. For any $i = 1,\dots,k$ one may construct an open set V_i such that $\mathcal{S}_i = (U_i,V_i)$ is a cover of X, and $\bigcap_i V_i = \emptyset$: as X is a metric space, to $x \in U_i$ associate an open ball $B_i(x)$ having closure contained in U_i (it does not matter that x may belong to several U_i's) ; one thus obtains an open cover of X. Extract by compactness a finite subcover $(W_1,\dots,W_p)$, let F_j be the closure of W_j and call V_i the intersection of all F_j^c containing U_i^c. Since U_i contains V_i^c, $\mathcal{S}_i = (U_i,V_i)$ covers X and as the F_j, $j = 1,\dots,p$ cover X one has $\bigcap_i V_i = \emptyset$.

The supremum $\bigvee_{i=1\dots k}\mathcal{S}_i$ is finer than $\mathcal{S}$: any set in it is either contained in some of the U_i's or equal to $\bigcap_i V_i = \emptyset$. One thus gets

$$0 < h(T,\mathcal{S}) \le h\big(T,\textstyle\bigvee_{i=1\dots k}\mathcal{S}_i\big) \le \displaystyle\sum_{i=1\dots k} h(T,\mathcal{S}_i),$$

so that at least one of the $\mathcal{S}_i$'s must have positive entropy. Call it (U,V).

Now it is easy to shrink (U,V) into a standard cover with positive entropy. Suppose U is dense : then V contains some closed ball $F = B'(x,\varepsilon)$ with $\varepsilon > 0$ and $x \in U$. Subtract F from U : one thus obtains an open set U', which cannot be dense since F has non empty interior, but as V contains F, (U',V) is a cover of X ; as this cover is finer than (U,V) it has positive entropy. By doing the same with V in case it is dense, one obtains a cover (U',V') having all suitable properties. $\square$

Suppose a standard cover $\mathcal{R}$ has entropy 0 : by definition no pair (x,x'), $x \neq x'$, distinguished by $\mathcal{R}$, can be an entropy pair. What about the case $h(T,\mathcal{R}) > 0$?

PROPOSITION 2. — *For any open cover $\mathcal{R} = (U,V)$ of X with $h(T,\mathcal{R}) > 0$, there are two distinct points $x \in U^c$, $x' \in V^c$ such that (x,x') is an entropy pair.*

Proof. — Suppose $\mathcal{R} = (U, V)$ with $h(T, \mathcal{R}) > 0$. We show one can find a strictly coarser cover $\mathcal{R}_1 = (U_1, V_1)$ with $h(T, \mathcal{R}_1) > 0$, having the property that U_1^c has diameter at most half this of U^c, and the same for V_1^c. By induction one gets two strictly decreasing sequences of non empty closed sets (U_i^c) and (V_i^c) converging to two points x and x' such that (x, x') is an entropy pair. Here is how we do this.

The closed set U^c cannot be reduced to a singleton $\{x\}$, because in this case $\mathcal{R}$ would have entropy 0 : supposing $U^c = \{x\}$, for any n the cover $\mathcal{R}_n$ contains the open set $U \cap TU \cap \cdots \cap T^{n-1}U$, which is equal to X minus at most n points, so $\mathcal{R}_n$ has a subcover with cardinality at most $n+1$, and $h(T, \mathcal{R}) = 0$. So there exist at least two distinct points y and y' in U^c, and $d(y, y') > 0$. Fix ε_1 such that $0 < \varepsilon_1 < \frac{1}{2} d(y, y')$, and construct a cover of U^c by open balls with radius ε_1 centred in U^c; call it $\mathcal{S}$. Necessarily any finite subcover $(U_1', \ldots, U_k')$ of the compact set U^c by elements of $\mathcal{S}$ has cardinality at least 2. Denoting by F_i' the closure of U_i', i.e. the corresponding closed ball with radius ε_1, put $F_i = U^c \cap F_i'$. By the choice of ε_1 each closed set F_i is a proper subset of U^c. Remark (U, V) is coarser than $\bigvee_{i=1\ldots k}(F_i^c, V)$, so

$$0 < h(T, (U, V)) \leq h\big(T, \textstyle\bigvee_{i=1\ldots k}(F_i^c, V)\big) \leq \displaystyle\sum_{i=1\ldots k} h\big(T, (F_i^c, V)\big) ;$$

this implies at least one of the covers (F_i^c, V) has positive entropy. Call U_1 the corresponding set F_j^c. Then, choosing some suitable ε_1', do the same for V, thus obtaining V_1. Iterate infinitely many times. As (U_i^c) and (V_i^c), $i > 0$, are two decreasing sequences of non empty closed sets they have non empty intersections, reduced to singletons $\{x\}$ and $\{x'\}$ because ε_i and ε_i' decrease exponentially as $i \to \infty$; as $x \in U^c$, $x' \in V^c$ they are distinct.

We claim (x, x') is an entropy pair. Given a standard cover (U, V) of X, distinguishing x and x', one can find $\varepsilon > 0$ with $B'(x, \varepsilon)$ in U^c and $B'(x', \varepsilon)$ in V^c; for i such that ε_i and ε_i' are less than $\frac{1}{2}\varepsilon$, U_i^c is in $B'(x, \varepsilon)$ and V_i^c in $B'(x', \varepsilon)$. Thus (U_i, V_i) is coarser than $(B'^c(x, \varepsilon), B'^c(x', \varepsilon))$, which is in its turn coarser than (U, V); this implies

$$0 < h\big(T, (U_i, V_i)\big) \leq h\big(T, (B'^c(x, \varepsilon), B'^c(x', \varepsilon))\big) \leq h\big(T, (U, V)\big),$$

which being true for any suitable choice of (U, V) finishes the proof. □

Denote by $E(X, T)$ the set of all entropy pairs of X, and by $E'(X, T)$ its closure. An entropy pair is not ordered, therefore $E(X, T)$ and $E'(X, T)$ are symmetric for the exchange of coordinates in X^2.

PROPOSITION 3. — *Let (X,T) be a flow with positive topological entropy. Then $E'(X,T)$ is a non empty closed invariant subset of X^2 containing only entropy pairs and points of Δ_0.*

Proof.

• *$E(X,T)$ is not empty.* As (X,T) has positive topological entropy, by PROPOSITION 1 one can find a cover (U,V) with positive entropy; so by PROPOSITION 2 there is at least one entropy pair in $U^c \times V^c$.

• *$E(X,T)$ and $E'(X,T)$ are $T\times T$-invariant.* For an entropy pair (x,x'), fix $k \in \mathbb{Z}$, and find two non dense open sets U' and V' distinguishing $T^k x$ and $T^k y$; this is possible because $x \neq x'$ implies $T^k x \neq T^k x'$. Continuity of T allows us to choose two open sets U, V such that :

a) $x \in \mathrm{Int}(V^c)$ and $x' \in \mathrm{Int}(U^c)$;

b) U^c is the closure of $\mathrm{Int}(U^c)$, and V^c is the closure of $\mathrm{Int}(V^c)$;

c) $\mathrm{Int}(V'^c)$ contains $T^k\,\mathrm{Int}(V^c)$ and $\mathrm{Int}(U'^c)$ contains $T^k\,\mathrm{Int}(U^c)$.

By a) sets U and V are not dense, by b) and c) (U,V) is coarser than $T^{-k}(U',V')$, hence a cover of X, so :

$$h\big(T,(U',V')\big) \geq h\big(T,(T^k(U,V))\big) = h\big(T,(U,V)\big) > 0.$$

Therefore $(T^k x, T^k x') \in E(X,T)$. To finish the proof remark that the closure of an invariant set is invariant.

• *Any non diagonal point of $E'(X,T)$ is an entropy pair.* Suppose (x,x'), $x \neq x'$, is the limit of a sequence (x_n,x'_n), $n \in \mathbb{Z}$, of entropy pairs, and let (U,V) be a standard cover of X distinguishing x and x'. Choose n such that $x_n \in \mathrm{Int}(U^c)$, $x'_n \in \mathrm{Int}(V^c)$: (U,V) is a standard cover also distinguishing x_n and x'_n, therefore having positive entropy. As this is true for any choice of (U,V), (x,x') is an entropy pair. □

PROPOSITION 4 is basic in the sequel and will be used elsewhere [BL]; invariance of $E(X,T)$ is merely a particular case. The same is true, on a lesser scale, for PROPOSITION 5.

PROPOSITION 4. — *Let $\phi : (Y,S) \to (X,T)$ be a factor map.*

1) *If $(x,x') \in E(X,T)$, there exist y and y' in Y such that $\phi(y) = x$, $\phi(y') = x'$ and $(y,y') \in E(Y,S)$.*

2) *Conversely if (y,y') belongs to $E(Y,S)$ and $\phi(y) \neq \phi(y')$, then $(\phi(y),\phi(y'))$ belongs to $E(X,T)$.*

Proof

1) Any standard cover (U, V) of X distinguishing x and x' has positive entropy. By the classical properties of covers this implies

$$h\big(\phi^{-1}(U), \phi^{-1}(V)\big) > 0.$$

Therefore by PROPOSITION 2 $\phi^{-1}(U)^c \times \phi^{-1}(V)^c$ contains an entropy pair. Now choose standard covers (U_n, V_n) all distinguishing x and x', and such that both U_n^c and V_n^c are closed balls with diameter less than ε_n, where $\varepsilon_n \to 0$ as $n \to \infty$. Again by PROPOSITION 2 associate to (U_n, V_n) an entropy pair (y_n, y_n') of Y. From this sequence of entropy pairs extract by compactness a subsequence converging to some (y, y'). The condition on diameters ensures $(\phi(y), \phi(y')) = (x, x')$, which implies $y \neq y'$, so by PROPOSITION 3 (y, y') is an entropy pair.

2) Suppose (y, y') is an entropy pair of Y with $\phi(y) \neq \phi(y')$, and put $\phi(y) = x$, $\phi(y') = x'$. Choose any standard cover (U, V) of X distinguishing x and x' : the pair of open sets $(\phi^{-1}(U), \phi^{-1}(V))$ covers Y, since its image covers X; by continuity of ϕ, $(\phi^{-1}(U))^c = \phi^{-1}((U)^c)$ contains some open neighbourhood of y, and $\phi^{-1}(V)^c$ contains some open neighbourhood of y' : this implies $(\phi^{-1}(U), \phi^{-1}(V))$ is a standard cover distinguishing y from y'. Therefore

$$0 < h\big(T, (\phi^{-1}(U), \phi^{-1}(V))\big) = h\big(T, (U, V)\big),$$

and (x, x') is an entropy pair of X. $\square$

PROPOSITION 5. — *Suppose W is a closed T-invariant subset of (X, T). Then if (x, x') is an entropy pair of $(W, T_{|W})$ it is also an entropy pair of (X, T).*

Proof. — Suppose $\mathcal{R} = (U, V)$ is a standard cover of X distinguishing x and x'. Then $\mathcal{R}' = (U \cap W, V \cap W)$ is a standard cover of W and distinguishes x and x', so $h(T_{|W}, \mathcal{R}') > 0$.

Take some subcover $\mathcal{S}$ of $\mathcal{R}_n$ with minimal cardinality. It must also cover the subset W : it does so via $\mathcal{S} \cap W$, which is a subcover of $\mathcal{R}_n'$ since W is T-invariant. As some elements of $\mathcal{S} \cap W$ may be empty, one has

$$\inf\big\{\#(\mathcal{S}') \mid \mathcal{S}' \text{ subcover of } \mathcal{R}_n'\big\} \leq \#(\mathcal{S} \cap W) = \#(\mathcal{S});$$

hence $H(T, \mathcal{R}_n) \geq H(T, \mathcal{R}_n')$, and finally $h(T, \mathcal{R}) \geq h(T, \mathcal{R}') > 0$. $\square$

3. The main theorem and some examples

Before stating the main theorem, let us introduce a notation and give two definitions. For any closed $T \times S$-invariant subset J of $X \times Y$, put $J(x) = \{y \in Y \mid (x, y) \in J\}$. As the preimage of a closed singleton $J(x)$ is closed. A simple calculation yields $J(Tx) = S(J(x))$.

A joining $J \subset X \times Y$ is said to be *minimal* if it contains no strictly smaller closed invariant subset with projections X and Y.

A *diagonal flow* is one such that $E'(X, T)$ contains $\Delta_1 = \{(x, Tx) \mid x \in X\}$; in view of PROPOSITION 3 this is equivalent to $\Delta_1 \cap \Delta_0^c$ being contained in $E(X, T)$.

The statement and proof of the next proposition, in their present form, are due to B. HOST.

PROPOSITION 6. — *Minimal zero-entropy flows are disjoint from diagonal flows.*

Proof. — We show that if a minimal (Y, S) is not disjoint from a diagonal (X, T), it has positive entropy. Suppose X and Y have these properties : they possess a non trivial subjoining J of $X \times Y$. Here is the basic idea of the proof. Suppose we can find some element x of X such that $J(x) \cap J(Tx) = \emptyset$. Then of course $x \neq Tx$, so by diagonality $(x, Tx) \in E(X, T)$. Then we can carry over positive entropy from X to Y by way of J : let π and π' be the projections of J onto (X, T) and (Y, S). By PROPOSITION 4.1 applied to the map π there must exist y, y' in Y such that $((x, y), (Tx, y')) \in E(J, T \times S)$; as (x, y) and (Tx, y') belong to J, $y \in J(x)$ and $y' \in J(Tx)$, so $y \neq y'$. But then by PROPOSITION 4.2 $(\pi'(x, y), \pi'(Tx, y')) = (y, y')$ must belong to $E(Y, S)$; hence $h(Y, S) > 0$.

So let us prove our provisional assumption that there is x in X with $J(x) \cap J(Tx) = \emptyset$. We can assume J is minimal. Indeed the intersection of a decreasing family of subjoinings is closed invariant and, by compactness, has projections X and Y, so it is a joining; applying Zorn's lemma we therefore obtain the existence of a minimal non trivial subjoining inside any non trivial joining.

Now suppose $J(x) \cap J(Tx) \neq \emptyset$ for any $x \in X$, and consider the subset of $X \times Y$:

$$\begin{aligned} J' &= \bigcup_{x \in X} \{x\} \times \big(J(x) \cap J(Tx)\big) \\ &= \bigcup_{x \in X} \{x\} \times \big(J(x) \cap S(J(x)\big) \\ &= J \cap (\mathrm{Id} \times S)(J). \end{aligned}$$

It is closed invariant. For x in X one has $J'(x) = J(x) \cap J(Tx) \neq \emptyset$, so $\pi(J') = X$, and since $\pi'(J')$ is a non empty closed invariant subset of the minimal set Y it is Y itself. So J' is a subjoining of J. Were it equal to J one would have $J(x) = J(Tx) = S(J(x))$ for all x : as a non empty closed invariant subset of Y, $J(x)$ would be equal to Y, thus $J = X \times Y$, contradicting the assumption that J is a proper subjoining. So there must exist some x with $J(x) \cap J(Tx) = \emptyset$, which finishes the proof. □

REMARKS

1) The same proof works when (Y, S) is totally minimal, and some off-diagonal Δ_n is contained in $E(X, T)$ instead of Δ_1 (B. HOST).

2) A flow with u.p.e. is diagonal; but so is also a finite union of u.p.e. flows, disjoint or not : as Tx belongs to the same u.p.e. component as x, one can apply PROPOSITION 5.

3) Disjoint unions of u.p.e. flows have a coarse non trivial zero-entropy factor (a union of fixed points). When the union is not disjoint this factor may be trivial; an instance is example 8 in [B]. In fact any zero-entropy factor of a diagonal flow must collapse x and Tx for all x in X, $x \neq Tx$, or else by PROPOSITION 4.2 it would have positive entropy. So it can contain only fixed points. Diagonality and a very slight extra assumption, like transitivity or even weaker, are enough to ensure c.p.e.

Now, what if all transitive diagonal flows were F-flows? This would mean PROPOSITION 6 is essentially contained in Furstenberg's disjointness theorem. Actually this is not the case. We construct a family of subshifts with u.p.e., but not possessing a dense set of periodic points.

Given a finite alphabet A, A^* is the set of words on A, including the empty word ε; the length of the word u is denoted by $|u|$, the number of occurrences of the letter $a \in A$ in u by $|u|_a$, and the cylinder set of A^Z associated with u by $[u]$: $[u] = \{x \in C^Z \mid x(0, |u| - 1) = u\}$. A *factor* of u is a word u' such that $u = vu'w$, $v, w \in C^*$. A *subshift* S is a closed shift-invariant subset of the compact set A^Z, endowed with the shift σ; it is completely defined by the set $L(S)$ of words occurring as sequences of consecutive letters in the coordinates of its elements.

EXAMPLE 7. — Let B be an alphabet such that $\{0, 1\} \cap B = \emptyset$. Consider the semigroup morphism $\psi : (B \cup \{0\})^* \to B^*$ defined by $\psi(0) = \varepsilon$; $\psi(b) = b$, $b \in B$. ψ simply erases symbol 0 and preserves all others; for this reason the corresponding map from $(B \cup \{0\})^Z$ to B^Z does not commute with the shift.

Suppose Y is a subshift on B, and $X = \{0, 1\}^Z$. By convention the empty word ε is included in $L(Y)$. Put

$$L = \{u \in (A \cup \{0\})^* \mid \psi(u) \in L(Y)\}.$$

L is the set of words u on $B \cup \{0\}$ such that when erasing all zeros in u one gets a word in $L(Y)$; L contains in particular any word 0^n, and defines a subshift $X\nabla Y \subset (B \cup \{0\})^Z$ such that $L = L(X\nabla Y)$. $X\nabla Y$ may be thought of as a kind of symbolic skew-product between X and Y; in fact it is a symbolic factor of some topological skew-product *stricto sensu* between them.

PROPOSITION 8. — *If Y is minimal, $X\nabla Y$ has u.p.e.*

Proof. — We must prove that any standard cover of $X\nabla Y$ has positive entropy for the shift : it is sufficient to show this is true for a family F such that given any standard cover $\mathcal{S}$, there is $\mathcal{S}' \in F$ such that $\mathcal{S}' \leq \mathcal{S}$. The following family meets this requirement :

$$F = \Big\{\mathcal{R} = \big\{[u]^c, [u']^c\big\} \mid u, u' \in L(X\nabla Y),\ |u| = |u'|,\ u \neq u'\Big\}.$$

So let us prove $h(\sigma, \mathcal{R}) > 0$ for $\mathcal{R} \in F$. For the sake of simplicity suppose $q = |u|_0 - |u'|_0 \geq 0$.

Since Y is minimal, there exists an integer $k \geq |u|_0$ such that any word $w \in L(Y)$, $|w| \geq k$, contains $\psi(u)$ and $\psi(u')$ as factors. For given n choose $v = v_1 v_2 \cdots v_n \in L(Y)$, with $|v_i| = k$ for $i = 1, \ldots, n$. For each i, v_i has $\psi(u)$ and $\psi(u')$ as factors because it has length k : $v_i = x_i \psi(u) x_i' = y_i \psi(u') y_i'$, with $j(i) = |y_i| - |x_i| < k$.

For $i = 1, \ldots, n$ we construct two distinct words with the same length $v_i(\alpha)$, $v_i(\beta)$ on $B \cup \{0\}$ such that $v_i(\alpha) = s_i u s_i'$, $v_i(\beta) = t_i u' t_i'$, with $|s_i| = |t_i| = \sup(|x_i|, |y_i|)$: this means $v_i(\alpha)$ has an occurrence of u and $v_i(\beta)$ an occurrence of u', both at time $|s_i|$. This is done by first replacing $\psi(u)$ by u or $\psi(u')$ by u' in v_i, and then adding a string of 0's with suitable length at the beginning and/or end of word $x_i u x_i'$ or $y_i u' y_i'$: put

$$v_i(\alpha) = \begin{cases} 0^{j(i)} x_i u x_i' & \text{if } j(i) \geq 0, \\ x_i u x_i' 0^{-j(i)} & \text{if } j(i) < 0\,; \end{cases},$$

$$v_i(\beta) = \begin{cases} y_i u' y_i' 0^{j(i)+q} & \text{if } j(i) \geq 0, \\ 0^{-j(i)} y_i u' y_i' 0^q & \text{if } j(i) < 0. \end{cases}$$

Words $v_i(\alpha)$ and $v_i(\beta)$ have the same length $k + j(i) + |u|_0$. Note that $\psi(v_i(\alpha)) = \psi(v_i(\beta)) = v_i$.

Now, for $p = c_1c_2\cdots c_n \in \{\alpha,\beta\}^n$, put $v(p) = v_1(c_1)\cdots v_n(cn)$. As only 0's have been added to v in order to construct $v(p)$, $\psi(v(p)) = v$, and $v(p) \in L(X\nabla Y)$, which means $[v(p)]$ is a nonempty cylinder set in $X\nabla Y$, containing at least one point $x(p)$. Since $\#(\{\alpha,\beta\}^n) = 2^n$, there are at least 2^n points $x(p)$, $p \in \{\alpha,\beta\}^n$. Given $p \neq p'$, $\sigma^i x(p) \in [u]$ and $\sigma^i x(p') \in [u']$, or the other way round, at least once for $0 \leq i \leq |v(p)|-1$, so that $x(p)$ and $x(p')$ must belong to different open sets of $\mathcal{R}_{|v(p)|}$; thus $H(\mathcal{R}_{|v(p)|}) \geq \log(\#\{\alpha,\beta\}^n) = n$. Now have n tend to infinity : as $|v(p)| = n(k + j(i) + |u|_0) < 3kn$, one has $h(\sigma,\mathcal{R}) \geq \frac{1}{3}k$. $\square$

Suppose Y is still a minimal set, but not reduced to a periodic orbit. Then $X\nabla Y$ cannot contain any periodic orbit, except the fixed point on the letter 0, or else Y would have to contain one (the image of this periodic orbit under the mapping associated to ψ). This implies $X\nabla Y$ cannot be an F-flow.

By the way this example also shows flows with u.p.e. may have an uncountable set of distinct ergodic measures with maximal entropy. To prove this take a Tœplitz minimal flow Y having an uncountable set of distinct ergodic measures [Wi], and construct $X\nabla Y$ as above.

There are two possible converses for PROPOSITION 6 :

1) Characterising flows disjoint from all diagonal flows. Since some non minimal flows, for instance those with the specification property, have u.p.e. [B], any flow disjoint from all of them must be minimal.

2) Characterising flows disjoint from all minimal flows with entropy 0. By Furstenberg's theorem, at least all F-flows belong to this class; now we know all diagonal flows also do. The latter assumption cannot be weakened arbitrarily, as shown by existence of c.p.e. flows which are not disjoint from all minimal zero-entropy flows.

EXAMPLE 9. — Let Y be a subshift of A^Z, $0 \notin A$, and let $\phi : Y\times\{0,1\}^Z \to (A\cup\{0\})^Z$ be defined by the alphabetic map from $A\times\{0,1\}$ to $A\cup\{0\} : (a,1) \mapsto a$ $(a \in A)$, $(a,0) \mapsto 0$. Call Z the image $\phi(Y\times\{0,1\}^Z)$.

It was shown in [B, prop. 10] that Z has c.p.e. whenever Y is minimal (and also that it has not u.p.e. when Y has not). It is very easy to show that $Y\times\{0,1\}^Z$, by definition a common extension of Z and Y, is conjugate to some proper closed invariant subset of $Y\times Z$, which implies Y and Z are not disjoint. Now to obtain the required example it is enough to assume Y is minimal with entropy 0.

REMARK. — Example 9 is another instance of two non disjoint flows having no common factors.

Let us finish with two open questions. An answer to any of them would help situate the results of this paper in the general frame of topological dynamics.

1) Do there exist transitive diagonal flows without u.p.e. ?

2) Do there exist minimal flows with u.p.e. ?

Question 2 have has now been answered positively by E. GLASNER and B. WEISS [GW] : this permits to measure more accurately the difference between PROPOSITION 6 and Furstenberg's disjointness theorem.

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