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Equivariant holomorphic extensions of real analytic manifolds

EQUIVARIANT HOLOMORPHIC EXTENSIONS OF
REAL ANALYTIC MANIFOLDS

BY

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RÉSUMÉ. — Soit G un groupe de Lie agissant proprement et analytiquement sur une variété analytique réelle X. On construit un espace complexe X^*, une action holomorphe du complexifié $G^{\mathbb{C}}$ de G sur X^* et une application analytique G-équivariante $\iota : X \to X^*$ qui possède les propriétés suivantes. Chaque application analytique G-équivariante $\phi : X \to Z$, où Z est l'espace complexe avec l'action holomorphe de $G^{\mathbb{C}}$, est de la forme $\phi = \phi^* \circ \iota$, où ϕ^* est une application holomorphe $G^{\mathbb{C}}$-équivariante définie sur un voisinage $G^{\mathbb{C}}$-invariant de $\iota(X) \subset X^*$. En outre, le quotient Q^* de X^* par l'algèbre $\mathcal{O}(X^*)^{G^{\mathbb{C}}}$ est un espace de Stein que l'on peut considérer comme une complexification naturelle de l'espace semi-analytique réelle X/G.

ABSTRACT. — Let G be a Lie group which acts properly and analytically on a real analytic manifold X. Then there exist a complex space X^*, where the complexified group $G^{\mathbb{C}}$ acts holomorphically and an analytic G-map $\iota : X \to X^*$ such that every analytic G-map ϕ from X into a complex space Z where $G^{\mathbb{C}}$ acts holomorphically can be written as $\phi = \phi^* \circ \iota$ where ϕ^* is a holomorphic $G^{\mathbb{C}}$-map defined on a $G^{\mathbb{C}}$-invariant neighbourhood of $\iota(X)$ in X^*. Moreover, the quotient Q^* of X^* with respect to the algebra $\mathcal{O}(X^*)^{G^{\mathbb{C}}}$ is a Stein space which can be considered as the natural complexification of the real semianalytic space X/G.

Real analytic manifolds can often be studied by using complex analytic methods. One reason for this is the following result (see [W,B], [S]) : every real analytic manifold X of dimension n can be embedded totally real and closed into a complex manifold X^* of dimension n.

Using this result along with his solution of Levi's problem and the embedding theorem of Remmert, GRAUERT [G] showed that every real analytic manifold may be realized as a closed analytic submanifold of some $\mathbb{R}^N$.

In this paper we present analogous results on real analytic manifolds equipped with a real analytic action of a Lie group G. In this «equivariant case» there are several possibilities for defining the notion of an equivariant complexification. We have choosen a definition which is given in categorical terms (section 2). If an equivariant complexification exists in this sense, then it contains information about any other kind of complexification. Another motivation for our definition is the result that the equivariant complexification of a real analytic G-principal bundle is a holomorphic $G^{\mathbb{C}}$-principal bundle over a Stein manifold where $G^{\mathbb{C}}$ denotes the universal complexification of the Lie group G (section 1).

A real analytic G-principal bundle is an example of a real analytic manifold X with a proper G-action. In this context we prove (section 6) the following

THEOREM 1. — *Let G be a Lie group which acts properly and real analytically on a real analytic manifold X. Then there exists an equivariant complexification X^* of X with a holomorphic $G^{\mathbb{C}}$-action.*

In order to study actions of groups on a manifold X it is useful to have information about the orbit space X/G. Under the assumptions of THEOREM 1, we also prove (section 6) the following

THEOREM 2. — *There exists an equivariant complexification X^* such that the quotient Q^* of X^* with respect to the smallest complex analytic equivalence relation given by the G-orbits is a Stein space. Moreover, Q^* can be considered as a natural complexification of the quotient X/G which is a semianalytic subset of Q^*.*

COROLLARY. — *The algebra of G-invariant holomorphic functions on X separates the G-orbits.*

If in addition the group G is assumed to be a linearly reductive Lie group, then X^* can be choosen to be a Stein manifold. An application of this result is the

EMBEDDING THEOREM. — *If G is a linearly reductive Lie group which acts properly on X and the G-orbit type of X is finite, then there exists a linearly equivariant closed embedding of X into some $\mathbb{R}^N$.*

1. Complexifications of Lie groups

To every Lie group G there exists a complex Lie group $G^{\mathbb{C}}$ and a real analytic homomorphism $\iota : G \to G^{\mathbb{C}}$ with the following property

(cf. [Ho]) : to every continuous homomorphism ϕ from G into a complex Lie group H there exists an unique holomorphic homomorphism $\phi^{\mathbb{C}} : G^{\mathbb{C}} \to H$ with $\phi = \phi^{\mathbb{C}} \circ \iota$.

If G is simply connected, then $G^{\mathbb{C}}$ is the unique simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, where $\mathfrak{g}$ denotes the Lie algebra of G. The homomorphism is determinated by the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g} \otimes \mathbb{C}$, $v \mapsto v \otimes 1$. For a connected Lie group G the complexification $G^{\mathbb{C}}$ is a quotient of the complexification of the universal covering group $\widetilde{G}$ of G by an appropriate closed normal subgroup of $\widetilde{G}^{\mathbb{C}}$ (see the proof of the next proposition). If G is general, then the complexification $G^{\mathbb{C}}$ is the complex Lie group $G^{\mathbb{C}} = G \times_{G_1} G_1^{\mathbb{C}}$ where G_1 denotes the connected component of the identity of G.

EXAMPLE. — The complexification of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is $\mathbb{C} \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{-1, 1\}$. The map $\iota : \mathbb{R}^* \to \mathbb{C} \times \mathbb{Z}_2$ is given by :

$$\iota(x) = \big(\log |x|, x/|x|\big).$$

We now collect some properties of the pair (G, ι) which will be used later on.

PROPOSITION.

(i) $G^{\mathbb{C}}$ is the smallest complex Lie subgroup of $G^{\mathbb{C}}$ which contains $\iota(G)$.

(ii) $G^{\mathbb{C}}$ with the inclusion is a complexification of $\iota(G)$.

(iii) $\iota(G)$ is contained in the set of fixed points of an anti-holomorphic group involution $\omega : G^{\mathbb{C}} \to G^{\mathbb{C}}$. In particularly, $\iota(G)$ is a totally real closed submanifold of $G^{\mathbb{C}}$ of maximal possible dimension.

(iv) $G^{\mathbb{C}}$ is a Stein manifold.

Proof. — We may assume that the group G is connected. Let G^* denote the smallest complex Lie subgroup of $G^{\mathbb{C}}$ which contains $\iota(G)$. By definition of $G^{\mathbb{C}}$, there exists a unique holomorphic homomorphism $\psi : G^{\mathbb{C}} \to G^{\mathbb{C}}$ such that $\psi \circ \iota = \iota$ and $\psi(G^{\mathbb{C}}) = G^*$. This implies $\psi = \mathrm{id}_{G^{\mathbb{C}}}$.

Property (ii) is a direct consequence of the definition of $G^{\mathbb{C}}$.

Now we may assume that $G = \iota(G)$. Let $\pi : \widetilde{G} \to G$ be the universal covering of G and $(\widetilde{G}^{\mathbb{C}}, \tilde{\iota})$ the complexification of $\widetilde{G}$. We denote by $\widehat{G}$ the image of $\widetilde{G}$ in $\widetilde{G}^{\mathbb{C}}$ and by $\widehat{\Gamma}$ the image of $\Gamma = \ker \pi$ in $\widetilde{G}^{\mathbb{C}}$. From the construction of $G^{\mathbb{C}}$ it follows that $G^{\mathbb{C}} = \widetilde{G}^{\mathbb{C}}/N$, where N denotes the smallest complex normal closed subgroup of $\widetilde{G}^{\mathbb{C}}$ which contains $\widehat{\Gamma}$. The

injectivity of ι implies that $N \cap \widehat{G} = \tilde{\iota}(\pi^{-1}(\iota^{-1}(1))) = \widehat{\Gamma}$. Since $\widehat{G}$ is a connected component of the set of fixed points of the involutive anti-holomorphic homomorphism $\widetilde{\omega} : \widetilde{G}^{\mathbb{C}} \to \widetilde{G}^{\mathbb{C}}$ which is induced by the involution on $\mathfrak{g} \otimes \mathbb{C}$, it is a closed subgroup of $\widetilde{G}^{\mathbb{C}}$. In particular, $\widehat{\Gamma} = N \cap \widehat{G}$ is a discrete subgroup of $\widetilde{G}^{\mathbb{C}}$. Moreover, since Γ is contained in the center of $\widetilde{G}$, the identity principle shows us that $\widehat{\Gamma}$ lies in the center of $\widetilde{G}^{\mathbb{C}}$. Thus we have $N = \widehat{\Gamma}$. Consequently, $G = \iota(G) = \widetilde{G}/\Gamma = \widehat{G}/\widehat{\Gamma}$. Furthermore, the involution $\widetilde{\omega}$ induces an anti-holomorphic involution $\omega : G^{\mathbb{C}} \to G^{\mathbb{C}}$ and the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G.

It remains to prove that $G^{\mathbb{C}}$ is a Stein manifold. It is sufficient to prove that the center $Z(G^{\mathbb{C}})$ of $G^{\mathbb{C}}$ is a Stein manifold (see [M,M]), i.e. we have to show that the connected component $Z_1(G^{\mathbb{C}})$ of the identity of $Z(G^{\mathbb{C}})$ is isomorphic to $\mathbb{C}^n \times (\mathbb{C}^*)^m = (\mathbb{R} \times \mathbb{R}^m/\mathbb{Z}^m)^{\mathbb{C}}$.

The center $Z(\widetilde{G}^{\mathbb{C}})$ of $\widetilde{G}^{\mathbb{C}}$ is stable with respect to $\widetilde{\omega}$. Since $\widetilde{G}^{\mathbb{C}}$ is simply connected, it follows that $Z_1(\widetilde{G}^{\mathbb{C}})$ is isomorphic to $\mathbb{C}^\ell$. This implies that $Z_1(\widehat{G})^{\mathbb{C}}$ is contained in $Z_1(\widetilde{G}^{\mathbb{C}})$. But $Z_1(\widetilde{G}^{\mathbb{C}})$ is stable with respect to $\widetilde{\omega}$. Consequently, it follows that $Z_1(\widehat{G})^{\mathbb{C}} = Z_1(\widetilde{G}^{\mathbb{C}})$; i.e. $Z_1(\widehat{G})$ is isomorphic to $\mathbb{R}^\ell$. Finally, from $\widehat{\Gamma} \subset \tilde{\iota}(Z(\widetilde{G})) \subset Z(\widetilde{G}^{\mathbb{C}})$ and $Z(G^{\mathbb{C}}) = Z(\widetilde{G}^{\mathbb{C}})/\Gamma$, it follows that :

$$Z_1(G^{\mathbb{C}}) = Z_1(\widetilde{G}^{\mathbb{C}})/\widehat{\Gamma} \cap Z_1(\widetilde{G}^{\mathbb{C}}) = (Z_1(\widehat{G})/\widehat{\Gamma} \cap Z_1(\widehat{G}))^{\mathbb{C}}$$
$$\simeq (\mathbb{R}^n \times (S^1)^m)^{\mathbb{C}} = \mathbb{C}^n \times (\mathbb{C}^*)^m, \quad \ell = n + m. \quad \square$$

REMARK. — It may happen that the kernel of $\iota : G \to G^{\mathbb{C}}$ is of positive dimension. The easiest examples are obtained as quotients of $\widetilde{G} \times \mathbb{R}$, with $G = \mathrm{SL}(\mathbb{R}^2)$, by discrete central subgroups with a dense projection on $\mathbb{R}$.

The situation is simpler if the group G is compact. The complexification of a compact Lie group is a complex reductive group and the given compact Lie group is a maximal compact subgroup of $G^{\mathbb{C}}$. Conversely, every complex reductive group is the complexification of a maximal compact subgroup.

A Lie group G is called *holomorphically extendable* if $\iota : G \to G^{\mathbb{C}}$ is injective. In this case we call $G^{\mathbb{C}}$ a holomorphic extension of the Lie group G.

If G is a Lie subgroup of some complex group H, then G is holomorphically extendable. In particularly, every Lie subgroup of a holomorphically extendable group is holomorphically extendable. Also every solvable or linear Lie group is holomorphically extendable.

2. Equivariant extensions of real analytic manifolds

By definition, a real analytic G-space is a reduced real analytic space X with countable topology equipped with a fixed real analytic action of a Lie group G. If X is a complex space and G acts on X by biholomorphic maps, then we say that X is a complex G-space. A complex space with a holomorphic action of a complex Lie group H is called a *holomorphic H-space*.

For the next definition we fix a continuous homomorphism γ from the Lie group G into a Lie (resp. complex Lie) group H. Note that every H-space is also a G-space via γ.

A complex (resp. holomorphic) H-space X^* is said to be a *H-complexification* (resp. *holomorphic H-complexification*) with respect to γ of a real analytic G-space X if there exists a real analytic G-map $\iota : X \to X^*$ such that :

(i) to every real analytic G-map ϕ from X into a complex (resp. holomorphic) H-space Y there exists an open H-neighborhood T^* of $\iota(X)$ in X^* and a holomorphic H-map $\phi^* : T^* \to Y$ such that $\phi = \phi^* \circ \iota$, and

(ii) if ψ is a holomorphic H-map from an open H-neighborhood of $\iota(X)$ into Y with the property $\phi = \psi \circ \iota$, then $\phi^* = \psi$ in a H-neighborhood of $\iota(X)$.

If in addition the map $\iota : X \to X^*$ is a closed embedding, then X^* is called a *H-extension* (resp. *holomorphic H-extension*) of X.

In this paper we are mainly interested in the case where γ is the map from G into its complexification $G^{\mathbb{C}}$ and in this case we do not mention γ explicitly. Also if $H = G$, then unless otherwise stated, we assume γ to be the identity. There is another important case, namely $H = \{1\}$.

A $\{1\}$-complexification of a real analytic G-space X is called a *complexification of the quotient X/G*.

REMARK. — The notion of a holomorphic $G^{\mathbb{C}}$-complexification makes sense if X is a complex G-space. In this case one require all maps to be holomorphic and $T^* = X^*$. If X^* exists, then it is called the complexification of the complex G-space X (cf. [H2]).

The concept of an extension is more restrictive than that of a complexification.

EXAMPLE. — The group $SL(\mathbb{R}^{n+1})$ acts transitively on the projective space $\mathbb{P}_n(\mathbb{R})$. The usual $2 : 1$ covering $S^n \to \mathbb{P}_n(\mathbb{R})$ for even n is a $SL(\mathbb{R}^{n+1})$-map and the $SL(\mathbb{R}^{n+1})$-complexification as well as the holomorphic $SL(\mathbb{C}^{n+1})$-complexification of S^n and of $\mathbb{P}_n(\mathbb{R})$ is the complex projective space $\mathbb{P}_n(\mathbb{C})$ (see [Ak] for more details).

More generaly, if H is a closed subgroup of a Lie group G, then $G^{\mathbb{C}}/H^*$ is a holomorphic $G^{\mathbb{C}}$-complexification of G/H, where H^* denotes the smallest closed complex subgroup of $G^{\mathbb{C}}$ which contains the image of H in $G^{\mathbb{C}}$.

REMARK. — Every Lie group G is an analytic G-space, where the G-action on G is given by the multiplication on the left. A recent result of Winkelmann shows that there exists a G-extension G^* of G which is a Stein manifold.

3. Actions of compact groups

Let K be a compact Lie group and X a Stein K-space. We denote by $X/\!/K$ the quotient of X with respect to the algebra $\mathcal{O}(X)^K$ of K-invariant holomorphic functions. We denote the quotient map by $\pi_X : X \to X/\!/K$. The quotient $X/\!/K$ is a Stein space whose structure sheaf is given by the presheaf $U \to \mathcal{O}(\pi_X^{-1}(U))^K$ (see [H2]).

We denote by $B(x)$ the smallest analytic K-subset of X which contains a given point $x \in X$ and by $E(x)$ the intersection over all K-invariant analytic K-subsets of $\pi_X^{-1}(\pi_X(x))$. The analytic K-set $E(x)$ depends only on the point $p = \pi_X(x) \in X/\!/K$ and is non-empty.

If $\phi : X \to [0, \infty)$ is a K-invariant differentiable strictly plurisubharmonic exhaustion function on X, then we set :

$$M_\phi = \big\{ x \in X \; ; \; \phi_{|B(x)} \text{ has a minimal value in } x \big\}.$$

The K-subset M_ϕ is closed in X. To see this, note that $K^{\mathbb{C}}$ acts on X in the infinitesimal sense. For an element v of the Lie algebra $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ of $K^{\mathbb{C}}$, denote by $\tilde{v}$ the induced vector field on X. Then one has (cf. [H2]) :

$$M_\phi = \big\{ x \in X \; ; \; \tilde{v}(\phi)(x) = 0 \text{ for all } v \in \mathfrak{k}^{\mathbb{C}} \big\}.$$

Using this description of M_ϕ, it is clear that it is closed.

LEMMA. — *The natural map $M_\phi \to X/\!/K$ is proper and the induced map $M_\phi/K \to X/\!/K$ is an isomorphism of topological spaces.*

Proof. — For $r \in \mathbb{R}$ let $D_\phi(r)$ denote the open K-subset of $x \in X$ such that $\phi(x) < r$. It is proven in [H2] that :

  (i) $\pi_X(D_\phi(r)) = \pi_X(D_\phi(r) \cap M_\phi)$ is open in $X/\!/K$,

  (ii) for every $x \in X$ one has $E(x) \cap M_\phi = K \cdot x_0$ for some $x_0 \in M_\phi$,

  (iii) for every $x \in M_\phi$ one has $E(x) = B(x)$.

For this one uses the fact that X is an orbit convex subset of its complexification. One can then apply the results of sections 5.4 and 6.3 in [H2]. The lemma is an immediate consequence. $\square$

The next result will be used later on.

PROPOSITION. — *Let K be a compact Lie group and X a Stein K-manifold of finite K-orbit type. Let $\omega : X \to X$ be a K-equivariant anti-holomorphic involution on X with a non-empty set X^ω of ω-fixed points. Then the natural map $X^\omega \to X//K$ is proper and induces a closed topological embedding $X^\omega/K \to X//K$.*

Proof. — If we realize X^ω as a closed subset of some M_ϕ, then the conclusion of the proposition follows from the lemma.

Let $X^{\mathbb{C}}$ be the complexification of the K-space X (see [H2]). Since $X^{\mathbb{C}} = K^{\mathbb{C}} \cdot X$, the holomorphic Stein $K^{\mathbb{C}}$-space $X^{\mathbb{C}}$ is of finite K-orbit type. Hence there exists a holomorphic linearly equivariant embedding $f : X^{\mathbb{C}} \to \mathbb{C}^n$. The map $g : X \to \mathbb{C}^{2n}$, defined by $g(x) = (f(x), \overline{f(\omega(x))})$ is a linearly equivariant holomorphic immersion and $g(X^\omega)$ is contained in the totally real K-subspace $V = \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^n \, ; \, w = \bar{z} \right\}$ of $\mathbb{C}^{2n}$. Let $\langle \ , \ \rangle$ be a positive definite Hermitian K-invariant product on $\mathbb{C}^{2n}$ which is an extension of a scalar product on V. Then $V \subset M_\delta$ for the function $\delta : \mathbb{C}^{2n} \to \mathbb{R}$, $\delta(z) = \langle z, z \rangle$. Pulling everything back to X one obtains $X^\omega \subset M_\phi$ with the function $\phi = \delta \circ g$. $\square$

REMARK. — In the proof of proposition we used one part of the following result : a holomorphic Stein $K^{\mathbb{C}}$-manifold X can be embedded linearly equivariant into some $\mathbb{C}^N$ if and only if the K-orbit type of X is finite.

This result is proven in [H1]. The formulation of the statement there is in terms of $K^{\mathbb{C}}$-orbit type. This is not correct since it gives only a sufficient condition.

COROLLARY 1. — *If K is a compact subgroup of a Lie group G, then the image H of $K^{\mathbb{C}}$ in $G^{\mathbb{C}}$ is closed and H is the complexification of the image $\widehat{K}$ of K in $G^{\mathbb{C}}$. The holomorphic $G^{\mathbb{C}}$-space $G^{\mathbb{C}}/H$ is a holomorphic $G^{\mathbb{C}}$-complexification of G/K and a holomorphic $G^{\mathbb{C}}$-extension of $\iota(G)/\widehat{K}$. Moreover, $\iota(G)/\widehat{K}$ is a closed totally real submanifold of the Stein manifold $G^{\mathbb{C}}/H$. In particular, if G is a holomorphically extendable Lie group, then $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a holomorphic $G^{\mathbb{C}}$-extension of G/K.* $\square$

If ϕ is a non-negative differentiable strictly plurisubharmonic K-invariant function on a complex K-space X, then the set of zeros of ϕ has a basis of open Stein K-neighborhoods in X. The proof of this fact

for the trivial K-action and smooth X is given in [H,W] and generalizes with only minor changes to this more general situation. In particular, under the assumptions of the Proposition we have the following

COROLLARY 2. — *Identifying X^ω (resp. X^ω/K) with its image in X (resp. $X//K$), it follows that* :

(i) *X^ω has a basis of open Stein K-neighborhoods in X, and*

(ii) *X^ω/K has a basis of open Stein neighborhoods in $X//K$.*

Proof. — The quotient X^ω/K is a closed subset of $X//K$ and $X//K$ can be identified in a natural way with $X^{\mathbb{C}}//K^{\mathbb{C}}$ (see [H2]). The proof of the proposition shows that X^ω/K is a closed subset of V/K which is a closed subset of $V^{\mathbb{C}}//K^{\mathbb{C}}$, $V^{\mathbb{C}} = \mathbb{C}^{2n}$. Hence we can identify $V^{\mathbb{C}}//K^{\mathbb{C}}$ with a closed subset of some $\mathbb{C}^q$ such that V/K becomes a closed subset of $\mathbb{R}^q$ (cf. [P,S]). If $y_1, \ldots, y_q$ denote the imaginary parts of the coordinates $z_1, \ldots, z_q$ of $\mathbb{C}^q$, then $\phi(z_1, \ldots, z_q) = y_1^2 + \cdots + y_q^2$ defines a strictly plurisubharmonic function on every open neighborhood of X^ω/K in $X//K$ which vanishes on X^ω/K. This proves part (ii) of COROLLARY 2. Part (i) is proven with similar arguments. □

REMARK. — In sections 5 and 6 the following slightly more general version of COROLLARY 2 is needed.

Let K be a compact Lie group and X a complex K-space. Let Q be a K-subset of X and $\{U_\alpha\}$ an open covering of Q. Suppose that for each α the open subset U_α of X can be identified with a locally analytic subset of some $\mathbb{C}^{n_\alpha}$ such that $Q \cap U_\alpha$ becomes a subset of $\mathbb{R}^{n_\alpha}$. Then Q has a basis of open Stein K-neighborhoods in X.

This result is also contained in [H,W] since one can use a partition of unity argument to obtain a non-negative strictly plurisubharmonic K-invariant function ϕ which is defined on a neighborhood U of Q in X with $Q \subset \{x \in U \ ; \ \phi(x) = 0\}$ (cf. [G]).

4. Proper actions

A real analytic action of a Lie group G on a real analytic manifold X is said to be *proper* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$, is proper.

For a proper action the orbit space X/G is a locally compact Hausdorff space with countable topology. In particularly, every G-orbit is closed. Since a G-action on X is proper if and only if $\{g \in G; \ g \cdot C_1 \cap C_2 \neq \emptyset\}$ is compact for any two compact subsets C_1, C_2 of X, each isotropy group G_x of G is compact (cf. [P]).

Let $x \in X$ and let V be a G_x-stable subspace in the tangent space $T_x X$ which is complementary to the tangent space of the orbit $G \cdot x$. Then there exists an open G_x-neighborhood S of $0 \in V$ and a real analytic G-isomorphism from $G \times_{G_x} S$ onto an open G-neighborhood of x in X. In other words, the G-action in a G-neighborhood of x is determinated by the action in a G-neighborhood of the zero section in the normal bundle $\Lambda = G \times_{G_x} V$ over the orbit $G \cdot x \cong G/G_x$. Consequently, the orbit space X/G has locally the structure of the orbit space S/G_x which is an open subset of the semialgebraic set V/G_x (see [Sch]).

EXAMPLE. — Let G act freely on X; i.e. $G_x = \{1\}$ for all $x \in X$. Then X is a G-principal bundle over X/G in the category of real analytic manifolds if and only if the G-action on X is proper.

5. Extensions of homogeneous bundles

The local model of a proper action is given by an open G-subset of a G-vector bundle $\Lambda = G \times_K V$ over G/K where K is a compact subgroup of G which acts linearly on the vector space V. The vector bundle Λ is obtained as the quotient of $G \times V$ with respect to the K-action which is defined by $h \cdot (g, v) = (gh^{-1}, h \cdot v)$ for all $h \in K$, $g \in G$, $v \in V$. This action extends holomorphically to a $K^{\mathbb{C}}$-action on the Stein manifold $G^{\mathbb{C}} \times V^{\mathbb{C}}$. Consequently, $\Lambda^* = (G^{\mathbb{C}} \times V^{\mathbb{C}})//K^{\mathbb{C}}$ is a holomorphic Stein $G^{\mathbb{C}}$-space.

In order to describe the image $\widehat{\Lambda}$ of Λ in Λ^* we introduce the following notation. Let $\widehat{K}$ (resp. $\widehat{G}$) be the image of K (resp. G) in $G^{\mathbb{C}}$ and denote the kernel of the homomorphism $K \to \widehat{K}$ by L. It follows that $L^{\mathbb{C}}$ is the kernel of the homomorphism $K^{\mathbb{C}} \to \widehat{K}^{\mathbb{C}}$. Hence the quotient $\widehat{V}^{\mathbb{C}} = V^{\mathbb{C}}//L^{\mathbb{C}}$ is a holomorphic Stein $\widehat{K}^{\mathbb{C}}$-space and the quotient map $q : V^{\mathbb{C}} \to \widehat{V}^{\mathbb{C}}$ is a holomorphic $K^{\mathbb{C}}$-map. Since $G^{\mathbb{C}}$ is the holomorphic extension of $\widehat{G}$, the group $\widehat{K}^{\mathbb{C}}$ is closed in $G^{\mathbb{C}}$ (section 3, COROLLARY 1). Thus the quotient $(G^{\mathbb{C}} \times \widehat{V}^{\mathbb{C}})//\widehat{K}^{\mathbb{C}}$ is the homogeneous $G^{\mathbb{C}}$-bundle $G^{\mathbb{C}} \times_{\widehat{K}^{\mathbb{C}}} \widehat{V}^{\mathbb{C}}$. Finally, we set $\widehat{V} = V/L$. Note that $\widehat{V}$ is a closed semialgebraic subset of $\widehat{V}^{\mathbb{C}}$ which can be described by inequalities in terms of the L-invariant polynomials on V (see [P,S]). With this notation the proof of the next result is straightforward.

LEMMA 1.

(i) *The natural map from $\widehat{\Lambda}$ into $G^{\mathbb{C}} \times_{\widehat{K}^{\mathbb{C}}} \widehat{V}^{\mathbb{C}}$ is a closed embedding and the natural map from Λ onto $\widehat{\Lambda} \cong \widehat{G} \times_{\widehat{K}} \widehat{V}$ is open.*

(ii) *The natural map from $\Lambda^* = (G^{\mathbb{C}} \times V^{\mathbb{C}})//K^{\mathbb{C}}$ onto $G^{\mathbb{C}} \times_{\widehat{K}^{\mathbb{C}}} \widehat{V}^{\mathbb{C}}$ is an isomorphism of holomorphic $G^{\mathbb{C}}$-spaces.* $\square$

REMARK. — It was already shown that there exists an anti-holomorphic group involution $\omega : G^{\mathbb{C}} \to G^{\mathbb{C}}$ which fixes $\widehat{G}$. From this it follows that $\widehat{K}^{\mathbb{C}}$ is ω-stable. Moreover, if $K^{\mathbb{C}} \to GL(V)$ is the extension of a real representation $K \to GL(V)$, then there is an involution $\sigma : \widehat{V}^{\mathbb{C}} \to \widehat{V}^{\mathbb{C}}$ such that $\sigma(h \cdot v) = \omega(h) \cdot \sigma(v)$ holds for all $h \in \widehat{K}^{\mathbb{C}}$ and $v \in \widehat{V}^{\mathbb{C}}$. In particular, there is an induced involution $\gamma = [\omega, \sigma]$ on $\Lambda^* \cong G^{\mathbb{C}} \times_{\widehat{K}^{\mathbb{C}}} \widehat{V}^{\mathbb{C}}$ such that the image of $\widehat{\Lambda}$ is γ-stable. It can also be shown that the real dimension of the semialgebraic set $\widehat{\Lambda}$ is equal to the complex dimension of Λ^*. Hence $\widehat{\Lambda}$ is a totally real semialgebraic subset of Λ^* of maximal possible dimension.

If X is a complex G-space, then we denote by $X /\!/ G$ the quotient of X with respect to the equivalence relation which is defined by the algebra of invariant holomorphic functions on X. Since $\Lambda^* /\!/ G^{\mathbb{C}}$ is isomorphic to $\widehat{V}^{\mathbb{C}} /\!/ \widehat{K}^{\mathbb{C}}$ the quotient $\Lambda^* /\!/ G^{\mathbb{C}}$ is a Stein space. Moreover, we have the following lemma.

LEMMA 2. — *The image of Λ in $\Lambda^* /\!/ G^{\mathbb{C}}$ is $\widehat{\Lambda}/\widehat{G}$ which is regarded as a semianalytic subset of $\Lambda^* /\!/ G^{\mathbb{C}}$. The involution γ on Λ^* induces an involution $\overline{\gamma}$ on $\Lambda^* /\!/ G^{\mathbb{C}}$ whose set of fixed points contains $\widehat{\Lambda}/\widehat{G}$.* ☐

Let T be an open G-subset of $\Lambda = G \times_K V$ and $\widehat{T}$ the corresponding image in $\widehat{\Lambda} = \widehat{G} \times_{\widehat{K}} \widehat{V}$. Note that $\widehat{T}$ is an open G-subset of the closed semianalytic G-subset $\widehat{\Lambda}$ of $\Lambda^* = G^{\mathbb{C}} \times_{\widehat{K}^{\mathbb{C}}} \widehat{V}^{\mathbb{C}}$. We also identify $\widehat{T}/\widehat{G}$ with an open subset of $\widehat{\Lambda}/\widehat{G}$ which is a closed semianalytic subset of $\Lambda^* /\!/ G^{\mathbb{C}} = V^{\mathbb{C}} /\!/ K^{\mathbb{C}} = \widehat{V}^{\mathbb{C}} /\!/ \widehat{K}^{\mathbb{C}}$.

The following technical lemma is used in the next section.

LEMMA 3. — *Every open Stein $G^{\mathbb{C}}$-neighborhood T^* of $\widehat{T}$ in Λ^* is a holomorphic $G^{\mathbb{C}}$-complexification of T and $T^* /\!/ G^{\mathbb{C}}$ is a complexification of the quotient T/G. A given T^* can be shrink so that :*

(i) *T^* is saturated with respect to the quotient map $\Lambda^* \to \Lambda^* /\!/ G^{\mathbb{C}}$, i.e. the inclusion $T^* \to \Lambda^*$ induces an open embedding of $Q^* = T^* /\!/ G^{\mathbb{C}}$ into $\Lambda^* /\!/ G^{\mathbb{C}}$,*

(ii) *T^* (resp. Q^*) are stable with respect to the involution on Λ^* (resp. $\Lambda^* /\!/ G^{\mathbb{C}}$),*

(iii) *$T^* \cap \widehat{\Lambda} = \widehat{T}$, $\overline{T^*} \cap \widehat{\Lambda} = \overline{\widehat{T}}$,*

(iv) *$Q^* \cap \widehat{\Lambda}/\widehat{G} = \widehat{T}/\widehat{G}$ and $\overline{Q^*} \cap \widehat{\Lambda}/\widehat{G} = \overline{\widehat{T}/\widehat{G}}$,*

where the closure is taken with respect to Λ^* (resp. $\Lambda^* /\!/ G^{\mathbb{C}}$).

Proof. — Since T is an open G-subset of $G \times_K V$, it can be written as $G \times_K S$ where S is an open K-subset of V. Let Φ be a real analytic

G-map from T into a holomorphic $G^{\mathbb{C}}$-space Y. Denote by ϕ the real analytic K-map from S into Y, which is defined by $\phi(x) = \Phi([1, x])$ for all $x \in S$. Note that ϕ is invariant with respect to the kernel L of the homomorphism $K \to \widehat{K}$. The map ϕ can be extend to a holomorphic K-map $\widetilde{\phi}$ from an open K-neighborhood $\widetilde{S}$ of S in $V^{\mathbb{C}}$ which is L-invariant. After shrinking $\widetilde{S}$ we can assume that :

(a) $\widetilde{S} \cap V = S$,

(b) $\widetilde{S}$ is an open Stein K-subset of $V^{\mathbb{C}}$, and

(c) $\widetilde{S}$ is stable with respect to the involution on $V^{\mathbb{C}}$.

The inclusion $\alpha : \widetilde{S} \to V^{\mathbb{C}}$ induces a holomorphic map $\bar{\alpha} : \widetilde{S}/\!/K \to V^{\mathbb{C}}/\!/K^{\mathbb{C}}$. Note that the image of S in $\widetilde{S}/\!/K$ (resp. $V^{\mathbb{C}}/\!/K^{\mathbb{C}}$) can be topologically identified with S/K (section 3, PROPOSITION) and that $\bar{\alpha}$ maps S/K topologically onto S/K. Since $\bar{\alpha}$ is locally biholomorphic in a neighborhood of S/K in $\widetilde{S}/\!/K$ (see [H2, 6.3]), there exists an open neighborhood P of S/K in $\widetilde{S}/\!/K$ which is mapped by $\bar{\alpha}$ biholomorphically onto an open neighborhood of S/K in $V^{\mathbb{C}}/\!/K^{\mathbb{C}}$. Replacing, if necessary, $\widetilde{S}$ by the inverse image of P of the map $\widetilde{S} \to \widetilde{S}/\!/K$, one can assume that α and $\bar{\alpha}$ are open embeddings. Hence, $P = \widetilde{S}/\!/K$ is an open subset of $V^{\mathbb{C}}/\!/K^{\mathbb{C}}$. From this it follows that the complexification of the complex K-space $\widetilde{S}$ is the open $K^{\mathbb{C}}$-subset $S^* = K^{\mathbb{C}} \cdot \widetilde{S}$ of $V^{\mathbb{C}}$ (see [H2, 6.3]). In particular, $\widetilde{\phi}$ extends to a holomorphic $K^{\mathbb{C}}$-map $\phi^* : S^* \to Y$ which is $L^{\mathbb{C}}$-invariant. The holomorphic map $\Psi : G^{\mathbb{C}} \times S^* \to Y$ which is defined by $\Psi(g, x) = g \cdot \phi^*(x)$ is $K^{\mathbb{C}}$-invariant. The induced $G^{\mathbb{C}}$-map Φ^* from the open subset $T^* = (G^* \times S^*)/\!/K^{\mathbb{C}}$ of Λ^* into Y is therefore the complexification of the map Φ. This argument proves that T^* is a holomorphic $G^{\mathbb{C}}$-complexification of T.

If the map Φ is assumed to be a G-invariant map into a complex space Y, then the map Φ^* induces a holomorphic map $\Phi^*/\!/G^{\mathbb{C}}$ from $T^*/\!/G^{\mathbb{C}}$ into Y. From this it follows that $T^*/\!/G^{\mathbb{C}}$ is a complexification of the quotient T/G.

Since $Q^* = T^*/\!/G^{\mathbb{C}}$ is an open subset of $\Lambda^*/\!/G^{\mathbb{C}} = V^{\mathbb{C}}/\!/K^{\mathbb{C}}$ and $T/G = S/K$ is an open subset of $\Lambda/G = V/K$, we can shrink Q^* so that $Q^* \cap \widehat{\Lambda}/\widehat{G} = \widehat{T}/\widehat{G}$ and $\overline{Q^*} \cap \widehat{\Lambda}/\widehat{G} = \overline{\widehat{T}/\widehat{G}}$.

This purely topological fact can be seen by choosing a covering of $\widehat{T}/\widehat{G}$ which consists of open balls $B_{r_\alpha}(x_\alpha)$ (with respect to some continuous metric on $\Lambda^*/\!/G^{\mathbb{C}}$) of radius r_α around points $x_\alpha \in \widehat{T}/\widehat{G}$ such that :

(a) $\lim_{\alpha \to \infty} r_\alpha = 0$,

(b) $\overline{B_{r_\alpha}(x_\alpha)} \cap \widehat{\Lambda}/\widehat{G} \subset Q^*$, and

(c) $A(\alpha_0) = \{\alpha\, ;\, r_\alpha \geq r_{\alpha_0}\}$ is finite for all α_0,

and then by replacing Q^* by $\bigcup_\alpha B_{r_\alpha}(x_\alpha)$.

Shrinking further, if necessary, we can also assume that Q^* is an open Stein subspace of $\Lambda^*/\!/G^{\mathbb{C}}$ (section 3). We may also assume that Q^* is invariant with respect to the involution on $\Lambda^*/\!/G^{\mathbb{C}}$. Since T^* is the inverse image of Q^* with respect to the quotient map $\Lambda^* \to \Lambda^*/\!/G^{\mathbb{C}}$, T^* is an open Stein subset of Λ^*. The properties (i)–(iv) for T^* follow from the corresponding properties of Q^*. $\square$

6. Extensions of proper actions

In the previous section we have complexified the local models for a proper action of a Lie group G on a real analytic manifold X. In order to obtain a global holomorphic $G^{\mathbb{C}}$-complexification of X, we have to glue the local complexifications together. One difficulty is to ensure that the resulting space X^* is a Hausdorff space. This can be carried out as in the non equivariant case (see [W,B]), except that one has to do every step in the proof simultaneously for X and X/G. In order to be complete, we shall write this down in detail. This is the first long step in the proof of the

COMPLEXIFICATION THEOREM. — *Let X be a real analytic manifold endowed with a proper real analytic action of a Lie group G. Then there exist holomorphic $G^{\mathbb{C}}$-complexifications X^* (resp. Q^*) of X (resp. the quotient X/G) such that* :

(i) *Q^* and X^* are normal complex spaces,*

(ii) *Q^* is a Stein space which is isomorphic to $X^*/\!/G^{\mathbb{C}}$,*

(iii) *the image of X (resp. X/G) in X^* (resp. Q^*) is a closed totally real semianalytic subset of X^* (resp. Q^*) of maximal possible dimension,*

(iv) *if G is a holomorphically extendable Lie group, then X^* is smooth and $X \to X^*$ is a closed embedding.*

Proof. — There exist locally finite coverings $\{A_i'\}_{i\in I}$, $\{B_i'\}_{i\in I}$ and $\{C_i'\}_{i\in I}$ of X/G such that :

(a) $C_i' \Subset B_i'$ and $B_i' \Subset A_i'$ for all $i \in I$,

(b) to every $i \in I$ there exists a real analytic open G-embedding $\phi_i : \pi_X^{-1}(A_i') \to G \times_{K_i} \mathbb{R}^{n_i}$ where K_i is a compact subgroup of G which acts linearly on $\mathbb{R}^{n_i}$ and $\pi_X : X \to X/G$ denotes the quotient map.

The following notation will be used :

$$T_i' = \pi_X^{-1}(A_i'), \qquad U_i' = \pi_X^{-1}(B_i'), \qquad V_i' = \pi_X^{-1}(C_i'),$$

$$T_i = \phi_i(T_i'), \qquad U_i = \phi_i(U_i'), \qquad V_i = \phi_i(V_i'),$$

$$T_{ij} = \phi_i(T_i' \cap T_j'), \quad U_{ij} = \phi_i(U_i' \cap U_j'), \quad V_{ij} = \phi_i(V_i' \cap V_j').$$

Furthermore, let Λ_i denote the G-bundle $G \times_{K_i} \mathbb{R}^{n_i}$ and $\pi_i : \Lambda_i \to \Lambda_i/G$ the quotient map. The map $T_i'/G \to \pi_i(T_i)$, which is induced by the open G-embedding $\phi_i : T_i' \to \Lambda_i$, is denoted by ψ_i. We also set :

$$A_i = \psi_i(A_i') = \pi_i(T_i), \qquad B_i = \psi_i(B_i') = \pi_i(U_i),$$

$$C_i = \psi_i(C_i') = \pi_i(V_i), \qquad A_{ij} = \psi_i(A_i' \cap A_j') = \pi_i(T_{ij}),$$

$$B_{ij} = \psi_i(B_i' \cap B_j') = \pi_i(U_{ij}) \quad C_{ij} = \psi_i(C_i' \cap C_j') = \pi_i(V_{ij}).$$

The identity $T_i' \cap T_j' = T_j' \cap T_i'$ induces a real analytic G-isomorphism $\phi_{ij} : T_{ji} \to T_{ij}$ and an isomorphism $\psi_{ij} : A_{ji} \to A_{ij}$. There exists holomorphic $G^{\mathbb{C}}$-complexification T_{ij}^* of T_{ij} and holomorphic $G^{\mathbb{C}}$-isomorphisms $\phi_{ij}^* : T_{ji}^* \to T_{ij}^*$ which extend $\phi_{ij} : T_{ji} \to T_{ij}$.

We can identify T_{ij}^* with an open $G^{\mathbb{C}}$-subset of $\Lambda_i^* = (G^{\mathbb{C}} \times \mathbb{C}^{n_i})//K^{\mathbb{C}}$ which is saturated with respect to the quotient map $\pi_i^* : \Lambda_i^* \to \Lambda_i^*//G^{\mathbb{C}}$. Furthermore, we always assume that :

(1) $\phi_{ji}^{*\,-1} = \phi_{ij}^*$ and $T_{ij}^* = \emptyset$ if and only if $T_{ij} = \emptyset$.

The $G^{\mathbb{C}}$-isomorphism $\phi_{ij}^* : T_{ji}^* \to T_{ij}^*$ induces an isomorphism $\psi_{ij}^* : A_{ji}^* \to A_{ij}^*$, where we set $A_{ij}^* = \pi_i^*(T_{ij}^*)$. After shrinking, if necessary, we may assume that :

(2) $T_{ij}^* \cap \widehat{\Lambda}_i = \widehat{T}_{ij}$, $A_{ij}^* \cap \widehat{\Lambda}_i/\widehat{G} = \widehat{A}_{ij}$ and $\psi_{ij}^{*\,-1} = \psi_{ji}^*$, using notation analogous to that in section 5. For every pair (i,j) we choose an open subset B_{ij}^* of $\Lambda_i^*//G^{\mathbb{C}}$ such that :

(3) $B_{ij}^* \Subset A_{ij}^*$, $\psi_{ji}^*(B_{ij}^*) = B_{ji}^*$, $B_{ij}^* \cap \widehat{\Lambda}_i/\widehat{G} = \widehat{B_{ij}}$, $\overline{B_{ij}^*} \cap \widehat{\Lambda}_i/\widehat{G} = \overline{\widehat{B}_{ij}}$.

Since $\overline{\widehat{C}}_i \cap \psi_{ij}^*(\overline{\widehat{C}}_j \cap \overline{\widehat{B}_{ji}})$ is a compact subset of $\widehat{B}_{ij}$, there exists an open subset D_{ij}^* of $\Lambda_i^*//G^{\mathbb{C}}$ such that :

(4) $D_{ij}^* \Subset B_{ij}^*$, $\psi_{ij}^*(D_{ji}^*) = D_{ij}^*$ and $\overline{\widehat{C}}_i \cap \psi_{ij}^*(\overline{\widehat{C}}_j \cap \overline{\widehat{B}_{ji}}) \subset D_{ij}^*$.

The sets $\overline{\widehat{C}}_i \setminus D_{ij}^*$ and $\psi_{ij}^*(\overline{\widehat{C}}_j \cap \overline{\widehat{B}_{ji}}) \setminus D_{ij}^*$ are compact and disjoint subsets of $\Lambda_i^*//G^{\mathbb{C}}$. Hence there exist open and disjoint subsets E_{ij}^* and F_{ij}^* of $\Lambda_i^*//G^{\mathbb{C}}$ such that :

(5) $\overline{\widehat{C}_i} \subset E_{ij}^* \cup D_{ij}^*$ and $\psi_{ij}(\overline{\widehat{C}_j} \cap \overline{\widehat{B}_{ji}}) \subset D_{ij}^* \cup F_{ij}^*$.

Let E_i^* be an open subset of $\Lambda_i^*//G^{\mathbb{C}}$ such that :

(6) $E_i^* \cap \widehat{\Lambda}_i/\widehat{G} = \widehat{C}_i$, $\overline{E_i^*} \cap \widehat{\Lambda}_i/\widehat{G} = \overline{\widehat{C}}_i$ and $E_i^* \subset E_{ij}^* \cup D_{ij}^*$ for all j with $T_{ij} \neq \emptyset$.

Since $\overline{E_j^*} \cap \overline{B_{ji}^*}$ is a compact subset of A_{ji}^*, it follows that :

$$\overline{\psi_{ij}^*(E_j^* \cap B_{ji}^*)} \subset \psi_{ij}^*(\overline{E_j^*} \cap \overline{B_{ji}^*}).$$

Thus we have :

$$\psi_{ij}^*(\overline{E_j^*} \cap \overline{B_{ji}^*}) \cap \widehat{\Lambda}_i/\widehat{G} = \psi_{ij}^*(\overline{E_j^*} \cap \overline{B_{ji}^*}) \cap \overline{B_{ij}^*} \cap \widehat{\Lambda}_i/\widehat{G}$$
$$= \psi_{ij}^*(\overline{E_j^*} \cap \overline{B_{ji}^*}) \cap \overline{\widehat{B}_{ij}}$$
$$= \psi_{ij}^*(\overline{E_{ji}^*} \cap \overline{B_{ji}^*} \cap \overline{\widehat{B}_{ji}})$$
$$= \psi_{ij}^*(\overline{\widehat{C}_j} \cap \overline{\widehat{B}_{ji}}).$$

This implies that :

(7) $\overline{\psi_{ij}^*(E_j^* \cap B_{ji}^*)} \cap \widehat{\Lambda}_i/\widehat{G} \subset \psi_{ij}^*(\overline{\widehat{C}_j} \cap \overline{\widehat{B}_{ji}})$.

For every point $p \in \widehat{B}_i$ we may choose an open neighborhood B_{ip}^* of p which satisfies the following four conditions :

(C1) $B_{ip}^* \Subset B_{ij}^*$ for all j such that $p \in \widehat{B}_{ij}$,

(C2) $B_{ip}^* \Subset D_{ij}^* \cup F_{ij}^*$ for all j such that $p \in \psi_{ij}^*(\overline{\widehat{C}_j} \cap \overline{\widehat{B}_{ji}})$,

(C3) $B_{ip}^* \cap \psi_{ij}^*(E_j^* \cap B_{ji}^*) = \emptyset$ for all j such that p is not the image of a point in $\widehat{C}_j'$.

Note that this is possible, since $B_{ji}^* \neq \emptyset$ for only finitely many j and the condition on p implies $p \notin \overline{\psi_{ij}^*(E_j^* \cap B_{ji}^*)}$ by virtue of (7).

(C4) $B_{ip}^* \Subset \psi_{ij}^*(B_{ji}^* \cap B_{jk}^*) \cap \psi_{ik}^*(B_{ki}^* \cap B_{kj}^*)$ and $\psi_{ji}^* = \psi_{jk}^* \circ \psi_{ki}^*$ on B_{ip}^* for all pairs (j,k) such that $p \in \widehat{B}_{ij} \cap \widehat{B}_{ik}$.

Let B_i^* denote the union of the sets B_{ip}^* with $p \in \widehat{B}_i$. Let C_i^* be a relatively compact open subset of B_i^* such that $\widehat{C}_i \subset C_i^* \subset E_i^*$. It follows that $C_i^* \cap \widehat{\Lambda}_i/\widehat{G} = \widehat{C}_i$ and $\overline{C_i^*} \cap \widehat{\Lambda}_i/\widehat{G} = \overline{\widehat{C}}_i$. We set $C_{ij}^* = C_i^* \cap \psi_{ij}^*(C_j^* \cap B_{ji}^*)$ and $C_{ijk}^* = C_{ij}^* \cap C_{ik}^*$. Note that $C_{ij}^* \subset B_{ij}^*$ and $\psi_{ij}^*(C_{ji}^*) = C_{ij}^*$. Furthermore, the relation $\sim$ which is given on the disjoint union $\bigsqcup_{i \in I} C_i^*$ by the isomorphisms $\psi_{ij}^* : C_{ji}^* \to C_{ij}^*$ is an equivalence relation. This can be seen as follows.

If $q \in C^*_{ijk}$, then $q \in B^*_{ip}$ for some $p \in \widehat{B}_i$. This implies :

$$B^*_{ip} \cap \psi^*_{ij}(E^*_j \cap B^*_{ji}) \neq \emptyset, \quad B^*_{ip} \cap \psi^*_{ik}(E^*_k \cap B^*_{ki}) \neq \emptyset.$$

Consequently $q \in B^*_{ij} \cap B^*_{ik}$ by (C3) and $\psi^*_{ki}(q) \in B^*_{ki} \cap C^*_k$ and $\psi^*_{jk}(\psi^*_{ki}(q)) = \psi^*_{ji}(q)$. It follows that $\psi^*_{ji}(q) \in C^*_j \cap \psi^*_{ji}(C^*_i \cap B^*_{ij})$ and analogously $\psi^*_{ji}(q) \in \psi^*_{jk}(C^*_k \cap B^*_{kj})$. This shows that $\psi^*_{ji}(q) \in C^*_{jik}$. Hence, $\psi^*_{ji}(C^*_{ijk}) \subset C^*_{jik}$, $\psi^*_{ij}(C^*_{jik}) \subset C^*_{ijk}$ and $\psi^*_{ji} : C^*_{ijk} \to C^*_{jik}$ is an isomorphism.

Let Q^* denote the quotient of $\bigsqcup C^*_i$ with respect to the equivalence relation $\sim$. We have to show that Q^* is a Hausdorff space. In order to do this we first show that $\overline{C^*_{ij}}$ is contained in B^*_{ij}.

Since $\overline{D^*_{ij}} \subset B^*_{ij}$, it suffices to prove $C^*_{ij} \subset D^*_{ij}$. If $q \in C^*_{ij}$ is given, then $C^*_{ij} \subset C^*_i \subset B^*_i$ implies $q \in B^*_{ip}$ for some $p \in \widehat{B}_i$. From $q \in \psi^*_{ij}(C^*_j \cap B^*_{ji})$ it follows that $q \in \psi^*_{ij}(E^*_j \cap B^*_{ji})$ and (C3) implies that p lies in the image of $\overline{C'_j}$. Thus we have $p \in \psi^*_{ij}(\widehat{C}_j \cap \widehat{B}_{ji})$. From condition (C2), we see that $q \in D^*_{ij} \cup F^*_{ij}$. Since $C^*_i \subset E^*_i \subset E^*_{ij} \cup D^*_{ij}$ (5), $E^*_{ij} \cap F^*_{ij} = \emptyset$ implies $q \in D^*_{ij}$.

Let $p', q' \in Q^*$ be two different points. We denote by $p \in C^*_i$ (resp. $q \in C^*_j$) representatives of p' (resp. q'). If it is not possible to separate p' and q' by open subsets of Q^*, then there exist sequences (p_α) in $\Lambda^*_i // G^{\mathbb{C}}$ and (q_α) in $\Lambda^*_j // G^{\mathbb{C}}$ which converges to p (resp. q) such that $p_\alpha \in C^*_{ij}$, $q_\alpha \in C^*_{ji}$ and $p_\alpha = \psi^*_{ij}(q_\alpha)$. Since $\overline{C^*_{ij}} \subset B^*_{ij}$, we have $p = \psi^*_{ij}(q)$. From this it follows that $q \in C^*_j \cap B^*_{ji}$ and $p \in C^*_i \cap \psi^*_{ij}(C^*_j \cap B^*_{ji}) = C^*_{ij}$. Hence, $p' = q'$.

We can now build up X^* over the complex space Q^*. For this set $V^*_i = (\pi^*_i)^{-1}(C^*_i)$ and $V^*_{ij} = (\pi^*_i)^{-1}(C^*_{ij})$. The holomorphic $G^{\mathbb{C}}$-space X^* is then the quotient of the disjoint union $\bigsqcup_{i \in I} V^*_i$ with respect to the equivalence relation which is given by the $G^{\mathbb{C}}$-isomorphisms $\phi^*_{ij} : V^*_{ji} \to V^*_{ij}$.

The statements (ii), (iii) and (iv) in the theorem are consequences of the results in section 5 and the remark in section 3.

Now we prove that X^* is a holomorphic $G^{\mathbb{C}}$-complexification of X. Let ϕ be a real analytic G-map from X into a holomorphic $G^{\mathbb{C}}$-space Y. There exist a locally finite and star finite covering $\{U_\alpha\}$ of $\widehat{X}$ consisting of open subsets of X^* which are saturated with respect to the natural map $\pi^* : X^* \to Q^*$ and complexifications $\phi^*_\alpha : U^*_\alpha \to Y$ of the maps $\phi_{|U_\alpha}$. Let $\{V^*_\alpha\}$ be a refinement of $\{U^*_\alpha\}$ consisting of π^*-saturated open subsets such that $\overline{V^*_\alpha} \subset U^*_\alpha$. For $x \in X$ denote by I_x the set of $\alpha \in I$ such that $x \in V^*_\alpha$ and set $J_x = \{\beta \in I \setminus I_x ; \ U^*_\beta \cap \bigcap_{\alpha \in I_x} V^*_\alpha \neq \emptyset\}$. The sets I_x and J_x consist of finitely many elements. Hence, to every $x \in X$ there exists a π^*-saturated open neighborhood T^*_x of x such that :

(a) $T_x^* \subset \cap_{\alpha \in I_x} V_\alpha^*$ and

(b) $T_x^* \cap V_\beta^* = \emptyset$ for all $\beta \in I \setminus I_x$.

After shrinking T_x^*, if necessary, we have $\phi_{\beta|T_x^*}^* = \phi_{\alpha|T_x^*}^*$ for all $\alpha, \beta \in I_x$. If $x, y \in X$ are such that $T_x^* \cap T_y^* \neq \emptyset$, then it follows from (a) that $T_x^* \cap V_\beta^* \neq \emptyset$ for some β with $y \in V_\beta^*$. This implies $\beta \in I_x$. Thus, for $\alpha \in I_x$, $\phi^*|_{T_x^*} = \phi_{\alpha|T_x^*}^*$ defines a holomorphic $G^{\mathbb{C}}$-map ϕ^* from $T^* = \bigcup_{x \in X} T_x^*$ into Y which is a complexification of ϕ. $\square$

REMARK. — The proof shows that X^* can be choosen to be smooth if for every isotropy group K of the G-action on X the kernel of $K \to G^{\mathbb{C}}$ is trivial. This is not always the case.

EXAMPLE. — Let G denote the quotient of $\widetilde{\mathrm{SL}_2(\mathbb{R})} \times S^1$ with respect to a discrete central subgroup Γ of $\widetilde{\mathrm{SL}_2(\mathbb{R})} \times S^1$ which projects onto a dense subgroup of S^1; i.e. the image of $\{1\} \times S^1$ in G is a compact group which is isomorphic to S^1 and $G^{\mathbb{C}} = SL_2(\mathbb{C})$. The $\mathbb{C}^*$-action on $\mathbb{C}^4$ which is defined by $t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4)$ is a complexification of the real S^1-invariant subspace $V = \{(z_1, z_2, \bar{z}_3, \bar{z}_4) ; z_1, z_2 \in \mathbb{C}\}$. Thus $X^* = \mathrm{SL}_2(\mathbb{C}) \times (\mathbb{C}^4 // \mathbb{C}^*)$ is a non smooth complexification of $X = G \times_{S^1} V$ whose set of singular points intersects the image of X in X^* non trivially.

7. Stein extensions

In this section G denotes a Lie group which acts properly on a real analytic manifold X. Let X^* denote a holomorphic $G^{\mathbb{C}}$-complexification of X with the properties which are stated in the complexification theorem. Thus $Q^* \cong X^* // G^{\mathbb{C}}$ is a Stein space (section 3, remark) which is the complexification of the quotient X/G. Note that the fibers and the base of the fibering $\pi^* : X^* \to Q^*$ are Stein spaces. It is natural to ask if X^* is a Stein space.

EXAMPLE. — If X is a real analytic G-principal bundle, then X^* is a holomorphic $G^{\mathbb{C}}$-principal bundle over Q^*. In particular, X^* is a Stein manifold (see [M,M]).

Before we state the main result of this section we give a definition. A Lie group G is said to be *holomorphically reductive* if $G^{\mathbb{C}}$ is the complexification of a compact Lie group; i.e. $G^{\mathbb{C}}$ is a complex reductive group.

Note that a holomorphically reductive group is a Lie group such that a kind of an abstract unitary trick holds.

THEOREM. — *If G is a holomorphically reductive Lie group, then X^* is a Stein space.*

Proof. — There exists a covering $\{Q_\alpha^*\}$ of Q^* consisting of open Stein subsets of Q^* such that $U_\alpha^* = (\pi^*)^{-1}(Q_\alpha^*)$ are open Stein subsets of X^*. Furthermore, since $G^{\mathbb{C}}$ is the complexification of a compact Lie group, we can assume that each U_α^* can be properly and linear equivariantly embedded into some $\mathbb{C}^{n_\alpha}$ (cf. [H2] and section 3).

If $G^{\mathbb{C}}$ acts linearly on $\mathbb{C}^n$, then the sheaves of germs of :

  (a) holomorphic maps,

  (b) holomorphic maps which vanish on X_p^*,

  (c) holomorphic maps which vanish on X_p^* at least of order two,

from X^* into $\mathbb{C}^n$ are coherent analytic $G^{\mathbb{C}}$-sheaves in the sense of ROBERTS (see [R]). Hence the corresponding sheaves of invariants are coherent analytic sheaves over Q^* (see [R]). Since Q^* is a Stein space, an application of THEOREM B shows that to every point $p \in Q^*$ there exists a holomorphic $G^{\mathbb{C}}$-map from X^* into some $\mathbb{C}^{n_p}$ which is an immersion along X_p^* and whose restriction to X_p^* is a closed embedding. The rest of the proof is the same as the proof of the complexification theorem in [H2].   ☐

REMARK. — An analysis of the proof shows that X^* is a holomorphically separable complex space if G is assumed to be a Lie subgroup of a general linear group. In general, there is no known example of an analytic G-space X such that X^* can not be choosen to be a Stein space.

If X is assumed to be only a differentiable G-manifold, then more can be said. Assume that the group G is connected. By Abel's theorem (see [A]) the G-manifold X is diffeomorphic to $G \times_K S$ where K denotes a maximal compact subgroup of G and S is a K-invariant submanifold of X. The manifold S is K-equivariantly diffeomorphic to a real analytic K-manifold which we also denote by S. Here we use an equivariant version of a theorem of Whitney, whose proof can be modified such that it also applies to differentiable K-manifolds. Thus $X \cong G \times_K S$ has a structure of a real analytic manifold so that G acts reall analytically on X. In particular, in the differentiable case there exist some complexification $X^* = (G^{\mathbb{C}} \times S^*)//K$ of X, which is a Stein space.

As a corollary we have the :

EMBEDDING THEOREM. — *Let G be a holomorphically reductive and extendable Lie group which acts properly and real analytically on a real analytic manifold X. Then there exists a linearly equivariant closed embedding of X into some $\mathbb{R}^N$ if and only if X is of finite K-orbit type with respect to some maximal compact subgroup K of G.*

Proof. — If X is of finite K-orbit type, then it is also of finite K-slice type (see [H1]). From the construction of X^* it follows that X^* is a holomorphic Stein $G^{\mathbb{C}}$-manifold of finite $G^{\mathbb{C}}$-slice type. Hence there exists a linearly equivariant closed embedding of X^* into some $\mathbb{C}^M$ (see [H1, Einbettungssatz 1]).

On the other hand it follows from the slice-theorem that every K-subset of $\mathbb{R}^N$ is of finite K-orbit type. $\Box$

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