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UNIVERSAL TOPOLOGICAL STRATIFICATION FOR THE PHAM EXAMPLE

BY

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RÉSUMÉ. — On étudie la stratification de l'espace des modules d'un germe de singularité, par le type topologique de la déformation verselle. On considère une coupe transverse au discriminant de versalité, puis par un procédé inductif, on récupère les informations topologiques en utilisant un certain type de champs de vecteurs stratifiés. Ceci nécessite le calcul explicite du discriminant de versalité, réalisé à l'aide d'un système de calcul formel.

ABSTRACT. — We study the stratification of the moduli space of a germe of singularity by the topological type of the versal deformation. We take a slice to the versality discriminant, then our method becomes an inductive process which recovers topological properties by the use of a special kind of stratified vector fields. This requires the explicit determination of the versality discriminant, performed via a computer algebra system.

Around 1970, F. PHAM [Ph] showed that constant topological type in a family of singularities does not imply constant topological type of the corresponding families of versal deformations. He found an example of a complex curve singularity $f_0(x,y) = y^3 + x^9$ which has a two parameter family of deformations (parametrized by the moduli (s,t))

$$F_1(x,y,s,t) = y^3 + tyx^6 + syx^7 + x^9.$$

This family has constant Milnor number, and hence is topologically trivial. However, the versal deformation of f_0 is not topologically a product

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along the t-axis. In fact, he showed that for $t = 0$ there are fibres arbitrarily close to $f_0^{-1}(0)$ with both E_6 and E_8 singularities in the fibre while this does not happen for $t \neq 0$.

This raised the question for general singularities of how the space of moduli is stratified by the topological type of the versal deformation (such a stratification exists by results of THOM-MATHER [T], [M2], [M3]). For the unimodal hypersurface singularities, the results began with LOOIJENGA [L] and were extended by WIRTHMULLER [W] (and for complete intersections by RONGA [R] and [D1]). This contrasts with the situation for the bimodal singularities on ARNOLD's list [A]. While advances have been made on understanding the topological structure and the adjacencies, e.g. BRIESKORN [B], EBELING [E], EBELING-WALL [E-W], BALKENBORG-BAUER-BILITEWSKI [BBB], the stratification question has remained unanswered.

In this paper we address this question for the Pham example and provide an outline for understanding the topological stratification of versal deformations of bimodal singularities.

The approach initiated by LOOIJENGA demonstrated that the answer to the stratification question for the unimodal singularities follows from the Jacobian algebra being Gorenstein. This can be thought of as first order information about the versal deformation. For the higher modality singularities higher order information must be understood. This involves the consideration of two problems for unfoldings which are partially versal. It requires first a determination of the versality discriminant, which describes where versality fails, and second an understanding of the germ in a neighborhood of the versality discriminant. These two problems are reduced to a single problem for unimodal case.

The role of the Jacobian algebra is replaced by an algebraic criterion for determining the versality discriminant (given in § 2). We are able to geometrically identify a candidate for the versality discriminant for the Pham example; however, to verify that it is correct via the algebraic criterion requires symbolic computations using the system MACSYMA (see § 3).

The theorem we prove uses the results of [D4]. To apply these results, we must determine the structure of the germ in a neighborhood of the versality discriminant and prove that it is stratified topologically trivial (see § 4). By constructing a section to the versality discriminant and using K-action we are able to reduce consideration of the germ in a neighborhood of the versality discriminant to consideration of the multigerm f obtained from this section.

More precisely we determine a local normal form for the multi-germ f, whose initial parts, with respect to certain weights, consists of versal deformations of $\widetilde{E}_8$ and D_4 (the singularities appearing in the special fiber) although the multi-germ itself is not stable. By using algebraic calculations of Looijenga and results from [D3] we are able to prove stratified topological triviality for this multi-germ. Also, a smoothing method is introduced to allow the stratifications to extend outside the neighborhood (PROPOSITION 4.8). This result depends on another piece of second order information, namely, an algebraic linking between the two Jacobian algebras of the germs appearing in the multi-germ (LEMMA 5.6). Such a linking is forced by the finite determinacy of the multi-germ together with the failure of it for the initial terms in the normal form.

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1. Statement of theorem

Consider the following polynomial mapping

$$\boldsymbol{F}(x,y,s,t,\underline{u},\underline{v}) = \big(F(x,y,s,t,\underline{u},\underline{v}), s, t, \underline{u}, \underline{v}\big)$$

where $F(x,y,s,t,\underline{u},\underline{v}) = F_1(x,y,s,t) + \sum_{i=0}^{5} u_{6-i}x^i y + \sum_{i=1}^{7} v_{9-i}x^i.$

The germ at the origin of this polynomial gives the versal deformation of f_0, more generally the germ of $\boldsymbol{F}$ at the point $(0,0,s_0,t_0,0,0)$ is also the versal deformation of the germ $F_1(x,y,s_0,t_0)$ for s_0 and t_0 fixed.

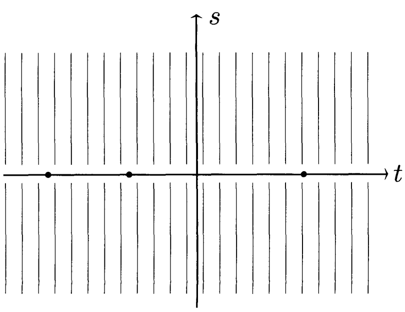


Figure 1.

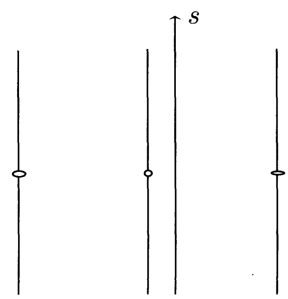


Figure 2.

We either let $K = \mathbb{C}$ and consider holomorphic germs or, $K = \mathbb{R}$ and consider smooth germs.

Since the versal deformation is with respect to $\mathcal{K}$-equivalence, we begin with the $\mathcal{K}$-orbit structure of the (s,t)-subspace given by figure 1. For any given value of t there are only two orbits, the intersection with the t-axis and the complement. The missing points on the t-axis correspond to the values where $4t^3 + 27 = 0$, where $\mathcal{K}$-determinacy fails. This follows from Arnold's classification [A] and the basic results of J. MATHER [M1].

We shall prove in either the smooth case or holomorphic case :

THEOREM. — *The stratification of the* (s,t)*-subspace such that the versal unfolding* (*as a germ of mapping*) *is topologically a product on strata is given by : the s-axis, the punctured lines for* $4t^3 + 27 = 0$, *and the complement* (*see figure* 2).

By a result of WIRTHMULLER [W], the versal deformation is topologically a product along any line parallel to the s-axis with $4t^3 + 27 \neq 0$. Hence, we may assume $s = 0$ and study when the unfolding restricted to this subspace is locally topologically a product along the t-axis.

It was pointed out to us by Terry WALL that if we write the weighted homogeneous part of F_1 as

$$y^3 + ay^2x^3 + byx^6 + cx^9,$$

the modulus is that of the elliptic curve

$$z^2 = y^3 + ay^2 + by + c$$

and that another value ($a = c = 0$, $b \neq 0$), corresponding to $t = \infty$ is also exceptional; indeed the same reasoning as used by PHAM shows that only this case admits two E_7-singularities in the nearby fibre. In fact this other exceptional value was left off of Arnold's list.

Thus, another statement of our theorem takes the j-invariant of the elliptic curve as the modulus parameter. Then $j = 0$ and $j = 1$ are the only values where universal topological triviality fails (the equal roots case $4t^3 + 27 = 0$ disapears at $j = \infty$).

This raises the question of whether this phenomena can be explained by algebraic-geometrical methods.

2. The versality discriminant

We will freely make use of facts and notation concerning $\mathcal{A}$ and $\mathcal{K}$-equivalence, see for example J. MARTINET [Mar], M. GOLUBITSKY et V. GUILLEMIN [G-G], J. MATHER [M1] or [D1].

First we recall the definition :

DEFINITION 2.1. — *Let $g_0 : K^n, 0 \to K^p, 0$ be a finitely $\mathcal{K}$-determined polynomial germ and $g : K^{n+r}, 0 \to K^{p+r}, 0$ be an unfolding of g_0; if $K = \mathbb{R}$ we can construct the complexifications, which we still denote by g_0 and g. As g is finitely $\mathcal{K}$-determined, there is a neighborhood U of 0 such that g has a representive on U (again denoted by g) and a neighborhood W of 0 in $\mathbb{C}^{p+r}$ so that :*

(i) $g_{|\Sigma(g)\cap U} : \Sigma(g) \cap U \to W$ *is proper and finite to one*;

(ii) $g^{-1}(0) \cap \Sigma(g) \cap U = \{0\}$; *where* $\Sigma(g)$ *denotes the critical set of* g.

Then, the *versality discriminant* V of the unfolding g denotes the complement in W of the set $(\underline{z}, \underline{w})$ such that if $g^{-1}(\underline{z}, \underline{w}) \cap \Sigma(g) \cap U = S$, then the multi-germ

$$G(\cdot, \underline{w}) : \mathbb{C}^n, S \to \mathbb{C}^p, \underline{z}$$

is infinitesimally stable.

We call $V_0 = V \cap (\mathbb{C}^p \times \{0\})$, the *versality discriminant* of g_0.

In our case, we denote the restriction of the unfolding $\boldsymbol{F}$ with $s = 0$ by $\boldsymbol{f}$. If we further restrict $t = t_0$ we denote the restricted unfolding by $\boldsymbol{f}_{t_0}$. Then, $\boldsymbol{f}$ viewed as an unfolding of $\boldsymbol{f}_{t_0}$ by the parameter t, is an unfolding of weight 0.

To see that in our case the versality discriminant is more than just a point, we consider the family (where $t \neq 0$)

$$y^3 + t(x - x_0)^4(x + 2x_0)^2 y + (x - x_0)^6(x + 2x_0)^3$$

which we may write in the form :

$$y^3 + tx^6 y + x^9 + \sum_{i=1}^{6} tc_i x_0^i x^{6-i} y + \sum_{i=2}^{9} b_i x_0^i x^{9-i}. \tag{2.2}$$

Near $x = x_0$ with $X = x - x_0$ we have (from the lowest order terms) the germ

$$y^3 + t(3x_0)^2 X^4 y + (3x_0)^3 X^6,$$

which is an $\widetilde{E}_8$-singularity. While near $x = -2x_0$ with $X = x + 2x_0$, we obtain

$$y^3 + t(3x_0)^4 X^2 y + (3x_0)^6 X^3,$$

which is a D_4-singularity.

Thus, along the parametrized curve $\mathcal{C}$ in K^{14} defined by $u_i = tc_i x_0^i$, $v_i = b_i x_0^i$ and $z = b_9 x_0^9$, where z denotes the coordinate for f and x_0 denotes the parameter for the curve $\mathcal{C}$, there are $\widetilde{E}_8$ and D_4 singularities in a fibre. However, for fixed $t \neq 0$, the dimension of the target space is 14 while the codimensions of $\widetilde{E}_8$ and D_4 are 10 and 4 respectively. Thus, if the multi-germ in this fibre were multi-transverse, the set of points where it occurred would be isolated and not along a curve.

Thus, as versality implies multi-transversality [M1V], the curve $\mathcal{C}$ belongs to the versality discriminant of f. In fact, we shall prove :

PROPOSITION 2.3. — *The versality discriminant for f_t, where $t \neq 0$ and $4t^3 + 27 \neq 0$, is exactly the curve $\mathcal{C}$ described above.*

Remark. — As t varies ($t \neq 0$) the curve $\mathcal{C}$ is analytically trivial and a simple change of coordinates makes it constant.

To begin the proof we let V denote the versality discriminant. We recall that V is an analytic set [D1I] which has the following algebraic property, with g as in the above definition :

PROPOSITION 2.4. — *Let $\mathcal{I}'$ be an ideal in $\mathbb{C}_{\underline{z},\underline{w}}$ such that :*

$$\mathcal{I}'\theta(g) \subseteq \mathbb{C}_{\underline{x},\underline{w}}\langle \partial G/\partial \underline{x}_j \rangle + \mathbb{C}_{\underline{z},\underline{w}}\langle \partial/\partial w_i r\rangle \tag{2.5}$$

then on some neighborhood of 0, *V is contained in the analytic set V' defined by $\mathcal{I}'$.*

(Here we have abbreviated the ring of germs at K^{n+r}, 0 by $\mathbb{C}_{\underline{x},\underline{w}}$ and the R-module generated by $h_1, \ldots, h_k$ over a ring R by $R\langle h_1, \ldots, h_k\rangle$ or $R\langle h_i\rangle$ if k is understood.)

For the proof of the proposition, consider the inclusions

$$V' \supseteq V \subseteq \mathcal{C}$$

where V' is defined by an ideal $\mathcal{I}'$. We shall show in § 4 that we can choose $\mathcal{I}'$ such that $\mathcal{C} \supseteq V'$, proving the PROPOSITION 2.3.

In our special case, where $g = f$ and $g_0 = f_{t_0}$, we are able to simplify the inclusion (2.5) :

LEMMA 2.6. — *In order for $\mathcal{I}'$ in $\mathbb{C}_{\underline{z},\underline{u},\underline{v}}$ to satisfy* (2.5) *it is sufficient that :*

$$hyx^6,\ hyx^7 \in \mathbb{C}_{x,y,\underline{u},\underline{v}}\langle \partial f_{t_0} \partial x, \partial f_{t_0}/\partial y\rangle + \mathbb{C}_{z,\underline{u},\underline{v}}\langle 1, \ldots, x^7, y, \ldots, yx^5\rangle \tag{2.7}$$

for a set of generators h of $\mathcal{I}'$ (here $f_{t_0} = z \circ f_{t_0}$).

Proof. — We observe for $F_1(x,y) = y^3 + t_0x^6y + x^9$ that

$$\{1,\dots,x^7,y,\dots,yx^7\}$$

is a basis for $\mathbb{C}_{x,y}/\big(\partial f_1/\partial x, \partial f_1/\partial y\big)$. By the preparation theorem,

$$\mathbb{C}_{x,y,\underline{u},\underline{v}} = \mathbb{C}_{x,y,\underline{u},\underline{v}}\langle f_{t_0}/\partial x, \partial f_{t_0}/\partial y\rangle + \mathbb{C}_{z,\underline{u},\underline{v}}\langle 1,\dots,x^7,y,\dots,yx^7\rangle. \tag{2.8}$$

Multiplying by $\mathcal{I}'$ yields :

$$\mathcal{I}'\theta(f_{t_0}) = \mathcal{I}'\mathbb{C}_{x,y,\underline{u},\underline{v}} \subseteq \mathbb{C}_{x,y,\underline{u},\underline{v}}\big\langle\partial f_{t_0}/\partial x, \partial f_{t_0}/\partial y\big\rangle + \mathbb{C}_{z,\underline{u},\underline{v}}\langle 1,\dots,x^7,y,\dots,yx^5\rangle + \mathcal{I}'\mathbb{C}_{z,\underline{u},\underline{v}}\langle yx^6, yx^7\rangle\cdot \tag{2.9}$$

Since the right hand side of (2.7) is a $\mathbb{C}_{z,\underline{u},\underline{v}}$-module, $\mathcal{I}'\cdot yx^6$ and $\mathcal{I}'\cdot yx^7$ belong to it. This gives (2.5).

Now we will construct an ideal $\mathcal{I}'$ as described above.

Let $\phi_i \in \{1,\dots,x^7,y,\dots,yx^5\}$ and

$$Z = f_{t_0} - \tfrac{1}{9}x\,\partial f_{t_0}/\partial x - \tfrac{1}{3}y\,\partial f_{t_0}/\partial y,$$

then by (2.8) we may write for $k \geq 1$:

$$Z^k\phi_i = h_{k,i}^{(1)}yx^7 + h_{k,i}^{(2)}yx^6 \tag{2.10}$$

modulo the right hand side of (2.7). Observe that :

$$Z^k\phi_i = f_{t_0}^k\phi_i \tag{2.11}$$

modulo the right hand side of (2.7). Besides (2.11), we may also write

$$Zyx^{5+i} = \ell_i^{(1)}yx^7 + \ell_i^{(2)}yx^6 \tag{2.12}$$

modulo the right hand side of (2.7). Form the infinite matrix :

$$H = \begin{bmatrix} h_{1,1}^{(1)} & h_{1,2}^{(1)} & \dots & h_{2,1}^{(1)} & \dots & Z-\ell_1^{(1)} & -\ell_2^{(1)} \\ h_{1,1}^{(2)} & h_{1,2}^{(2)} & \dots & h_{2,1}^{(2)} & \dots & -\ell_1^{(2)} & Z-\ell_2^{(2)} \end{bmatrix}$$

By (2.10)–(2.12) the expressions $h_{k,i}^{(1)}yx^7 + h_{k,i}^{(2)}yx^6$ belong to the right hand side of (2.7). Then, by Cramer's rule the (2×2)-minors of H satisfy (2.7).

Let $\mathcal{I}'$ be the ideal generated by the (2×2)-minors of H; by LEMMA (2.6), $\mathcal{I}'$ satisfies (2.5).

In our case we want to show that the space defined by $\mathcal{J}'$ is the curve $\mathcal{C}$. First, we want to determine the projection of the space defined by $\mathcal{J}'$ onto the (u,v)-subspace by computing $\det(h^{(k)}_{i,j})$ for the possible values of i,j. Second, we will show that on the image of the projection both entries in one of the columns are nonzero. This implies that the $(2\times 2$-determinants using this column and each of the last two columns specifies Z. This forces $V(\mathcal{J}')$ to map bijectively onto the image in (u,v)-space. Thus, it will be enough to show that the image of the projection has the desired form.

3. Symbolic computations

Our goal is to compute sufficiently many generators of $\mathcal{J}'$ (defined at the end of § 2) in order to prove that $\mathcal{C}\supseteq V'$ and hence $V=\mathcal{C}$. A conceptual (versus an effective) way of achieving this goal is the following.

Consider the first derivatives of f with respect to x and y as two polynomials with coefficients in $Q(t)[\underline{u},\underline{v}]$:

$$f_y = 3y^2 + tx^6 + \sum_{0\le i\le 5} u_{6-i}x^i,$$
$$f_x = 6tx^5y + 9x^8 + \sum_{0\le i\le 5} iu_{6-i}x^{i-1}y + \sum_{0\le i\le 7} iv_{9-i}x^{i-1}.$$

To them we add

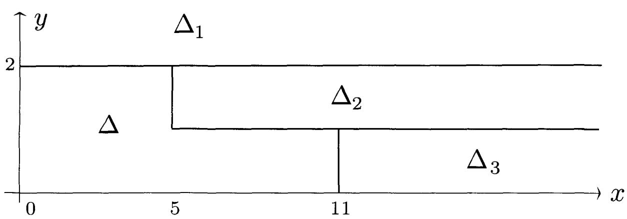
$$\begin{aligned} g &= 4t^2x^5f_y - (2ty-3x^3)f_x \\ &= (4t^3+27)x^{11} + \{\text{terms smaller than } x^{11} \text{ in } x\}, \end{aligned}$$

to form $\{f_x, f_y, g\}$, a standard basis for the jacobian ideal (f_x,f_y) with respect to the following ordering :

$$x^iy^j < x^{i'}y^{j'} \quad \text{if} \quad (i+3j<i'+3j') \text{ or } (i+3j=i+3j \text{ and } j<j')$$

(see [Ga, 3.1]). Then, for any polynomial P in $\mathbb{Q}(t)[\underline{u},\underline{v}][x,y]$ we can apply the generalized Euclidean division algorithm with respect to the following partition of $\mathbb{N}\times\mathbb{N}$:

$$\mathbb{N}\times\mathbb{N} = \Delta\cup\Delta_1\cup\Delta_2\cup\Delta_3$$



(see [G3]), and obtain unique q_1, q_2, q_3, R_1 such that

$$P = q_1 f_y + q_2 f_x + q_3 g + R_1$$

with exponents$(q_1 y^2)$ in Δ_1, exponents$(q_2 x^5 y)$ in Δ_2, exponents$(q_3 x^{11})$ in Δ_3, and exponents(R_1) in Δ.

Very briefly, the algorithm consists of repeatedly replacing

$$\begin{aligned} y^2 &\quad \text{by} \quad \frac{1}{3}(3y^2 - f_y), \\ x^5 y &\quad \text{by} \quad \frac{1}{6t}(6tx^5 y - f_x), \\ x^{11} &\quad \text{by} \quad \frac{1}{4t^3 + 27}\big((4t^3 + 27)x^{11} - g\big) \end{aligned}$$

in such a way that the "generalized degree" $i + 3j$ descreases.

In the remainder R_1, we successively replace x^{10} by $\frac{1}{9}x^2(9x^8 - f_x)$, then x^9 by $\frac{1}{9}x(9x^8 - f_x)$, and then x^8 by $\frac{1}{9}(9x^8 - f_x)$, to obtain the new remainder R.

Thus, we obtain the following decomposition :

$$\mathbb{Q}(t)[\underline{u}, \underline{v}] = \mathbb{Q}(t)[\underline{u}, \underline{v}]\langle 1, \dots, x^7, y, \dots, yx^7\rangle \oplus \mathbb{Q}(t)[\underline{u}, \underline{v}][x, y]\langle f_x, f_y\rangle.$$

This process can, in theory, be programmed on a computer and we would like to carry it out for $P = Z^k x^i y^j$ for $j = 0$, $i = 0, \dots, 7$ or for $j = 1$, $i = 0, \dots, 5$, and e.g. $k = 1, 2$, where we recall

$$Z = f - \tfrac{1}{9}xf_x - \tfrac{1}{3}yf_y.$$

If we collect the coefficients $h^{(1)}_{k,i,j}$ and $h^{(2)}_{k,i,j}$ of $x^7 y$ and $x^6 y$ for the corresponding 28 remainders into a (2×28)-matrix, then the (2×2)-minors provide some of the generators of $\mathcal{J}'$. However, we do not know in advance which k will give us a complete set of generators.

Unfortunately, this "naive" method would lead to symbolic computations for which the number of terms and the size of the coefficients become unmanageable. For example the coefficient $h^{(2)}_{2,7,5}$ is a quasi homogeneous polynomial in $\underline{u}, \underline{v}$ of total weight 22 in 14 variables over $\mathbb{Q}(t)$ and hence may involve several thousand monomials which may as well have very large coefficients.

We avoid this possible computational complexity by using an interative procedure. Specifically, we compute (2×2)-minors as above; however we use the result of each step of the computation to simplify the later steps. This amounts to replacing the ideal $\mathcal{I}'$ by its radical.

Secondly, we change the presentation of f for this computation :

$$\begin{aligned} f = y^3 + x^9 + tx^6y + u_1x^5y - 6u_2x^4y + 4u_3x^3y \\ + 9u_4x^2y - 12u_5xy + 4u_6y - 9v_2x^7 + 6v_3x^6 + 27v_4x^5 \\ - 36v_5x^4 - 15v_6x^3 + 54v_7x^2 - 36v_8x - 8v_9. \end{aligned}$$

When $u_i = tv_i = tx_0^i$, i.e. on the curve $\mathcal{C}$,

$$f = y^3 + t\big[(x-x_0)^2(x+2x_0)^1\big]^2 y + \big[(x-x_0)^2(x+2x_0)^1\big]^3 ;$$

we might then expect the coefficients in our computation to remain "small integers".

Thirdly, we recognize that each division will increase the number of monomials in $\underline{u}$ and $\underline{v}$ appearing as coefficients in the remainders. Consequently we minimize the number of divisions to be performed by considering only polynomials of degree smaller than 2 in y.

Finally, we allowed for the possibility of considering truncated versions of the problem to inductively obtain partial results; but this provision was ultimately not needed.

Now, we will list the principal steps of the calculation (we suppose $t \neq 0$ and $4t^3 + 27 \neq 0$).

Here we let $\equiv$ denote equality modulo (f_x, f_y) :

$$9Zx = u_1x^6y - 12u_2x^5y - 18v_2x^8 + \cdots . \tag{3.1}$$

After division by f_x,

$$9Zx \equiv u_1x^6y - 12(u_2 - tv_2)x^5y + \cdots ,$$

$$9Zx^2 \equiv u_1x^7y - 12(u_2 - tv_2)x^6y + \cdots .$$

Then

$$h^{(2)}_{1,1,0} = h^{(1)}_{1,2,0} = u_1, \quad h^{(4)}_{1,1,0} = 0, \quad h^{(2)}_{1,2,0} = -12(u_2 - tv_2).$$

Thus, the first equation obtained is $(u_1)^2 = 0$. This implies that $u_1 = 0$ which we use to simplify f. Next

$$9Zx^3 \equiv -12(u_2 - tv_2)x^7y + \cdots$$

which gives $u_2 = tv_2$.

After noticing that $h^{(1)}_{1,i,0} = h^{(2)}_{1,i+1,0} = h^{(3)}_{1,i+2,0}$ (where $h^{(3)}$ is the coefficient of x^5y), in the same way we obtain from Zx^i and $i \leq 7$, the relations :

$$u_3 = tv_3, \quad u_4 = 2tv_4 - tv_2^2, \quad u_5 = 2tv_5 - tv_2v_3,$$
$$u_6 = \tfrac{1}{2}t\big(-5v_6 - 2v_3^2 + 27v_2v_4 - 18v_2^3\big).$$

Observe we can't use Zx^8 ; however, v_2Zx^8 will appear as a linear combination of the products $\langle 1, \dots, x^7, y, \dots, yx^5\rangle \cdot \langle Z, Z^2\rangle$. Thus using (3.1),

$$18v_2Zx^8 \equiv Z^2x - 12u_2Zx^5y + \cdots.$$

Provided $v_2 \neq 0$, we can use v_2Zx^8 to obtain :

$$v_7 = 2v_2v_5 + v_3v_4 - 2v_2^2v_3.$$

Also, considering the minor obtained from v_2Zx^8 and

$$2Zy \equiv -\tfrac{1}{27}(4t^3 + 27)(v_2x^7y - v_3x^6y) + \cdots$$

yields :

$$v_8 = -\tfrac{1}{8}\big(10v_2v_6 - 16v_3v_5 + 27v_4^2 - 108v_2^2v_4 + 16v_2v_3^2 + 63v_2^4\big).$$

Since

$$2Zyx \equiv \tfrac{1}{27}(4t^3 + 27)(v_3x^7y) + \cdots$$

we observe that the expression for $A = v_3Zy + v_2Zyx$ has no x^7y term. In fact, a computation shows

$$9A = (4t^3 + 27)\big[v^2v^4 + \tfrac{1}{6}v_3^2 - \tfrac{7}{6}v_2^3\big]x^6y + \cdots.$$

Therefore via a similar procedure the minors formed from Ax^i for $i \leq 3$, yield the relations :

$$v_4 = -\tfrac{1}{6}v_2(v_3^2 - 7v_2^3), \quad v_5 = -\tfrac{1}{10}v_2^2(v_3^2 - 11v_2^3v_3),$$
$$v_6 = \tfrac{1}{10}v_2^3\big(2v_3^4 - 27v_3^2v_2^3 + 35v_2^6\big).$$

To get v_9 and v_3 we need to compute remainders for multiples of Z^2. For that purpose we simplify the expression for Zx^iy^j by evaluating $v_4, v_8, u_2, \dots, u_6$. We write

$$Z^2 \equiv Z(-12u_2x^4y - 18v_2x^7 + \cdots),$$

then we obtain

$$v_9 = \tfrac{1}{8}\big(150v_2v_7 + 30v_3v_6 + 324v_4v_5 - 624v_2^2v_5 - 635v_2v_3v_4 + 16v_3^2 + 732v_2^3v_3\big),$$

$$v_3^2 = v_2^3.$$

Then, everything simplifies to the required relations by parametrizing $v_2 = x_0^2$ and $v_3 = x_0^3$. Now the last case to consider is $v_2 = 0$; the same computations, but considerably simplified yield $u_i = v_i = 0$ for all i.

Lastly, observe that the coefficients for Zy are (up to constant multiples) v_2 and v_3. Therefore, by the comment at the end of § 2, the versality discriminant is exactly the curve we have identified. $\square$

This computation was actually performed with the MACSYMA system [MAC], it could have been done with any other interactive computer algebra system which provides the usual "simplification" routines such that Expand, Substitute or Eval, and where the function "Remainder" can be constructed. The function Remainder(p, m, x) returns the remainder of the multivariate polynomial p divided by the x-monic multivariate polynomial m.

Lastly, to give an idea of the size of the intermediate data, we mention that the listing for the entire session consists of about 40 pages and can be checked with 2 or 3 hours interactive use of a mini-computer.

4. Stratified topological triviality and the structure of the multi-germ

To prove that $\boldsymbol{f}$ is topologically trivial along the t-axis, it is sufficient by theorem 1 of [D4] to prove that $\boldsymbol{f}$ is tratified topologically trivial in a conical neighborhood of the versality discriminant in the sense of [D4] (see below).

We know that the versality discriminant of f_t is a curve $\mathcal{C}$, which on replacing u_i by tu_i, with $t \neq 0$, is defined parametrically by

$$u_i = c_i x_0^i, \quad v_i = b_i x_0^i, \quad z = b_9 x_0^9,$$

for x_0 in K, and for appropriate integers c_i, b_i given in § 3. Then,

$$\boldsymbol{f}_t^{-1}(\mathcal{C}) \cap \Sigma(\boldsymbol{f}_t) = \mathcal{C}''$$

is a curve with two components parametrized by $y = 0$, $x = x_0$ or $x = -2x_0$ and u_i, v_i given above.

We recall that "conical neighborhoods" of $\mathcal{C} \times K$ and $\mathcal{C}' \times K$ are neighborhoods of $(\mathcal{C} \setminus \{0\} \times K$ and $(\mathcal{C}' \setminus \{0\}) \times K$ of the form

$$U = \{(z,t) \in K^{13} \times K : \hat{\rho}(z) < \varepsilon' \rho_0(z)\},$$

$$U' = \{(x,t) \in K^{14} \times K : \hat{\rho}^{(1)}(x) < \varepsilon' \rho_0^{(1)}(z)\},$$

where the various ρ are smooth non-negative "control" functions which vanish on $\{0\} \times K$ for ρ_0 and $\rho_0^{(1)}$ and on $\mathcal{C} \times K$ and $\mathcal{C}' \times K$ for $\hat{\rho}$ and $\hat{\rho}^{(1)}$. Observe that since $\mathcal{C}$ and $\mathcal{C}'$ are invariant under the K^*-action, we may choose the ρ's to be real weighted homogeneous of the same degree and then U and U' are unions of K^*-orbits.

Proposition 4.1. — *There exist conical neighborhoods U and U' such that given a smaller conical neighborhood U_1 of $\mathcal{C} \times K$ such that* $\mathrm{Cl}(U_1) \subset U$ *(where* Cl *denotes closure in $K^{13} \setminus \{0\} \times K$) then there exist K^*-equivariant stratified vector fields ξ, η defined on U' and U respectively (in the sense of* [D4, § 3] *and see below) such that :*

1) *ξ and η project to $\partial/\partial t$,*
2) *ξ and η are smooth on $U' \setminus \boldsymbol{f}^{-1}(\mathrm{Cl}(U_1))$ and $U \setminus \mathrm{Cl}(U_1)$ respectively,*
3) *$\xi(\boldsymbol{f}) = \eta \circ \boldsymbol{f}$.*

Now, if we examine the definition of stratified topological triviality in [D4], we see that the conditions of Proposition 4.1 are conditions 1), 2) and 4) of that definition. Also, the K^*-equivariance implies the remaining condition 3). Hence, we can apply theorem 1 of [D4] to obtain the theorem.

In this section we shall prove this proposition modulo several algebraic lemmas to be established in § 5; the outline of the proof is as follows :

First, slices to $\mathcal{C}$ and $\mathcal{C}'$ are taken, reducing the problem to one about multi-germs. Next the multi-germ is put into a normal form. From this normal form we prove that the multi-germ is stratified topologically trivial. We also prove that this trivialization can be smoothed outside of a small neighborhood. Lastly, the stratified vector fields used for the trivialization are extended by the K^*-action to prove the proposition.

In order to obtain a useful local form, we now take a slice through $\mathcal{C}$ by fixing x_0 (as above) to be small and not zero, and by intersecting $\mathcal{C}$ with the affine hyperplane :

$$v_4 = -(126x_0^4 + 24x_0^2 v_2 + 6x_0 v_3).$$

This is easy seen to be transverse to the curve $\mathcal{C}$ with inverse image in K^{15} defined by the same equation and passing through $\mathcal{C}'$ in two points corresponding to $x = x_0$ and to $x = -2x_0$.

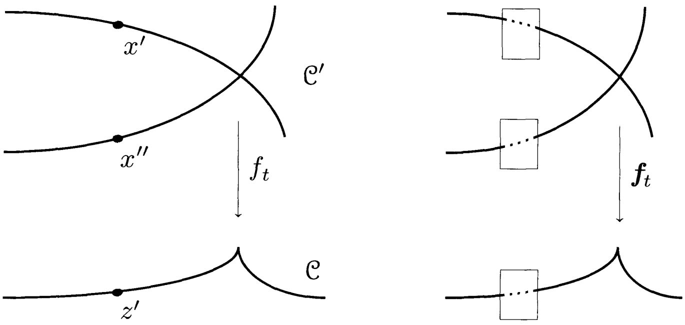
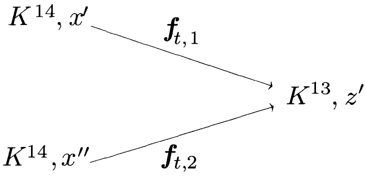


Figure 3.

By restricting f_t to the inverse image of this slice in a neighborhood of $\mathcal{C}''$ we obtain a multi-germ that we continute to denote by $\boldsymbol{f}_t = (\boldsymbol{f}_{t,1}, \boldsymbol{f}_{t,2})$ for a fixed $t \neq 0$; and we will denote by $\boldsymbol{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2)$ the corresponding unfolding along the parameter t.



Next, we place the multi-germ $\boldsymbol{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2)$ into normal form.

PROPOSITION 4.2. — *If $t_0 \neq 0$ and $4t_0^3 + 27 \neq 0$, then by a local change of coordinates near $\boldsymbol{t} = t - t_0 = 0$ and z', x' and x'', we may write the multi-germ in the form*

$$\boldsymbol{f}_1(x', y', w, s, \boldsymbol{t}) = (f_1, w, s, \boldsymbol{t}) \quad \textit{near } x',$$

$$\boldsymbol{f}_2(x'', y'', w, s, \boldsymbol{t}) = (f_2, w, s, \boldsymbol{t}) \quad \textit{near } x'',$$

where $W = (w_1, \dots, w_8)$, $s = (s_1, \dots, s_4)$ and

$$\text{(4.3)} \quad f_1(x', y', w, s, \boldsymbol{t}) = y'^3 + (\boldsymbol{t} + t_0 + cs_4)y'x'^4 + x'^6 + y'\Big(\sum_{i=0}^{3} w_{i+1}x'^i\Big) + \sum_{i=1}^{4} w_{i+4}x'^i + H_1$$

$$(4.4)\qquad f_2(x'',y'',s) = y''^3 + t_0 y'' x''^2 + x''^3 \\ + s_1 + s_2 x'' + y''(s_3 + s_4 x''),$$

where $c \neq 0$.

If we assign weights

$$\mathrm{wt}(x',y',w_1,\ldots,w_8) = (1,2,4,3,2,1,5,4,3,2),$$
$$\mathrm{wt}(x'',y'',s_1,\ldots,s_4) = (2,2,6,4,4,2),$$

then H_1 consists of terms of weight greater than 5 in (x',y') and greater than 6 in all coordinates. Moreover, it is without terms which are products of the form s_4 (terms of weight 6 in (x',y')).

The proof of this proposition will be given at the end of this section.

Next we use this normal form to prove that the multi-germ $\boldsymbol{f}$ is stratified topologically trivial.

PROPOSITION 4.5. — *The multi-germ* $\boldsymbol{f}$ *is stratified topologically trivial, i.e. there exist stratified vector fields* ξ_i *and* η *(in the sense of* [D3, § 2], *see below) which project to* $\partial/\partial t$ *such that*

$$\xi_i(\boldsymbol{f}_i) = \eta \circ \boldsymbol{f}_i.$$

Proof. — We first observe that $\boldsymbol{f}_2$ is a stable germ and that $\partial/\partial w_i$, for $1 \le i \le 8$, and $\zeta_0 = \partial/\partial z + \partial/\partial s_1$ preserve the discriminant of $\boldsymbol{f}_2$ and lift to $\partial/\partial w_i$, respectively $\partial/\partial s_1$. We consider $\mathcal{A}$-equivalence of $\boldsymbol{f}_1$ preserving the discriminant of $\boldsymbol{f}_2$. Let $f_{1,0}$ denote the germ obtained from f_1 by letting $\{s_i = 0\}$ and $\boldsymbol{t} = t - t_0 = 0$. Then, let :

$$\boldsymbol{f}_{1,0}(x',y',w) = \big(f_{1,0}(x',y',w), w\big).$$

Then, $\boldsymbol{f}_1$ is an unfolding of non-decreasing weight of the germ $\boldsymbol{f}_{1,0}$, where the unfolding variables are $(s,\boldsymbol{t})$. We define :

$$\alpha_1'' : \mathbb{C}_{x',y',w}\langle \partial/\partial x', \partial/\partial y'\rangle \oplus \mathbb{C}_{z,w}\langle \zeta_0, \partial/\partial w_1,\ldots,\partial/\partial w_8'\rangle \\ \longrightarrow \mathbb{C}_{x',y',w}\langle \partial/\partial z\rangle$$

by

$$\alpha_1''(\xi,\zeta) = \xi(f_{1,0}) + \zeta'(f_1)_{|(s,\boldsymbol{t})=0} - \eta \circ f_{1,0}$$

where if $\zeta = g_0\zeta_0 + \sum g_i \partial/\partial w_i$ then $\eta = -g_0\partial/\partial z$ and $\zeta' = \zeta + \eta$.

By 1) of LEMMA 5.4, an associated homomorphism α_1' is graded surjective in weight > 0. Since the images only differ in that

$$\alpha_1''(\zeta_0) = \partial/\partial z + \partial f_1/\partial s_1$$

is replaced by $\alpha_1'(\partial/\partial z) = \partial/\partial z$, and $\mathrm{wt}(\partial f_1/\partial s_1) \geq 6$, it follows by the preparation theorem (see e.g. lemma 7.4 of [D3]) that α_1'' is graded surjective in weight > 0.

It then follows by THEOREM 12.5 of [D3] that $\boldsymbol{f}_1$ is stratified topologically trivial along that t-axis. This means that there are vector fields ξ and η projecting to $\partial/\partial t$ such that

$$\xi_1(\boldsymbol{f}_1) = \eta \circ \boldsymbol{f}_1 \tag{4.6}$$

and where η has the form $g_0\zeta_0 + \sum g_i \partial/\partial w_i$ where g_i are continuous and satisfy $|g_i| \leq C\rho^{(a_i/2m)}$ for $C \geq 0$ with

$$\rho = \sum_{1 \leq i \leq 8} |w_i|^{2a_i} + |z|^{2a_0} \quad \text{and} \quad a_i\,\mathrm{wt}(w_i) = a_0\,\mathrm{wt}(z) = 2m.$$

Then, η can be lifted to $\xi_2 = g_0\partial/\partial s_1 + \sum g_i \partial/\partial w_i$ for $\boldsymbol{f}_2$ so that

$$\xi_2(\boldsymbol{f}_2) = \eta \circ \boldsymbol{f}_2. \tag{4.7}$$

It remains to verify that the ξ_i and η are stratified vector fields in the sense of [D3, § 2]. Because this is largely a question of verifying certain technical conditions, we postphone this until we have completed the proof of PROPOSITION 4.1. □

Lastly, to smooth these vector fields outside a neighborhood of z', we need the next proposition. Let $\rho_1 = \sum_{1 \leq i \leq 4} |s_i|^2$.

PROPOSITION 4.8. — *There exist germs of smooth vector fields ξ_1', ξ_2', η' such that :*

$$-\rho_1 \partial f_i/\partial t = \xi_i'(f_i) + \eta' \circ f_i$$

for $i = 1, 2$ and $|\eta'(\rho)| \leq C\rho$ in a neighborhood of 0.

This will be proven in § 5.

Proof of Proposition 4.1. — There exist neighborhoods W of z' and $\widetilde{W}$ of (x', x'') on which the vector fields from PROPOSITIONS 4.5 and 4.8 are defined. If we extend both W's by applying the K^*-action, then we obtain our conical neighborhoods U and U'. Given a smaller conical neighborhood U_1 with $\mathrm{Cl}(U_1) \subset U$, let W_1 denote the intersection of U_1 with the slice through z. Then, $\mathrm{Cl}(W_1) \subset W$.

First consider the vector fields η from PROPOSITION 4.5 and $\partial/\partial t+\rho_1^{-1}\eta'$ from PROPOSITION 4.8. In terms of local coordinates on the slice, η is smooth off of $\{(z,s,w,t) : s = 0\}$ and $\partial/\partial t + \rho_1^{-1}\eta'$ is smooth off of $\{(z,s,w,t) : w = 0, z = 0\}$. Pick a product neighborhood (as shown in figure 4.2) $W' \times W'' \times J \subseteq W_1$ with W' in the w-subspace and W'' in the s-subspace (and J an open interval containing t_0).

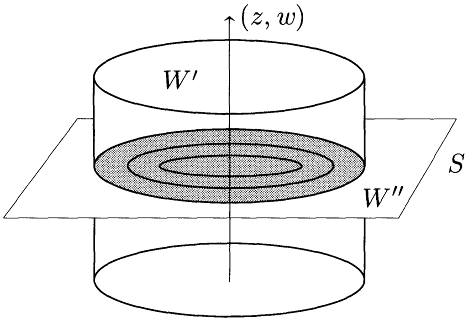


Figure 4.2.

Also, choose neighborhoods of 0, W_i'' and W_1', so that

$$\mathrm{Cl}(W_2'') \subset W_1'' \subset \mathrm{Cl}(W_1'') \subset W'', \quad \mathrm{Cl}(W_1') \subset W'.$$

Let (see figure 4.3) :

$$T_1 = \left[(W'' \times (W' \setminus \mathrm{Cl}(W_1')) \cup W_1'' \times W'\right] \times J,$$

$$T_2 = \left(W'' \setminus \mathrm{Cl}(W_2'')\right) \times W' \times J.$$

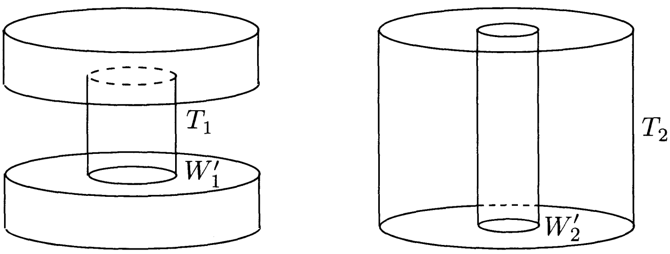


Figure 4.3.

Let $\{\chi_1, \chi_2\}$ be a partition of unity subordinate to $\{T_1, T_2\}$. Then, consider $\eta^{(1)} = \chi_1\eta + \chi_2(\partial/\partial t + \rho_1^{-1}\eta')$. Since $\chi_2 = 0$ on a neighborhood of the subspace where $s = 0$, then $\chi_2\rho_1^{-1}$ is smooth. Hence, $\eta^{(1)}$ is smooth off of $\mathrm{Cl}(W_1'') \times \{0\} \times J$. Hence, it is stratified off of this set relative to the given stratification by proposition 2.5 of [D4]. Lastly, where $\eta^{(1)}$ is not smooth, i.e. on $\mathrm{Cl}(W_1'') \times \{0\} \times J$,

$$\begin{aligned}\big|(\chi_1\eta + \chi_2\big(\partial/\partial t + \rho_1^{-1}\eta')\big)\big| &\leq |\eta(\rho)| + \chi_2\rho_1^{-1}|\eta'(\rho)| \\ &\leq C_1\rho + C_2\rho = C\rho.\end{aligned}$$

That $\eta^{(1)}$ is stratified follows the same arguments to be given for η in the remainder of the proof of PROPOSITION 4.5.

Next, let $\widetilde{\chi}_1, \widetilde{\chi}_2$, and $\tilde{\rho}_1$ denote the composition with $\boldsymbol{f}$. Also, with ξ_i and ξ_i' denoting the vector fields defined in PROPOSITIONS 4.5 and 4.8, we consider, for $i = 1, 2$,

$$\xi_i^{(1)} = \widetilde{\chi}_1\xi_i + \widetilde{\chi}_2\big(\partial/\partial t + \tilde{\rho}_1^{-1}\xi_i'\big).$$

By the same arguments used for $\eta^{(1)}$ we see that $\xi_i^{(1)}$ is smooth off of the inverse image of the subspace where $s = 0$, and stratified relative to the stratification on the slice.

Now extend these vector fields by the K^*-action to give vector fields on the conical neighborhoods U and U'. If we denote these by η and ξ, then by the equivariance they are stratified relative to the stratifications on U and U'. We have already seen that they are smooth where required. Since they are constructed using partitions of unity from vector fields which satisfy 3) (of the proposition), they also satisfy 3). □

Completing the proof of Proposition 4.5. — Actually the vector fields are stratified in the stronger sense of [D4, § 2]; however only the weaker notion of stratified vector field given in [D3, § 2] is needed in the proof.

The stratifications for η and ξ_1 are given by V_1 which in a neighborhood of z' is defined by the $\widetilde{E}_8$-stratum V_1 and its complement and V_1' a stratification of a neighborhood of x' defined by the complement of $\boldsymbol{f}_1^{-1}(V_1)$, $\boldsymbol{f}_1^{-1}(V_1 \setminus V_1')$, and V_1', where V_1' denotes the $\widetilde{E}_8$-stratum in the source.

Secondly, there is a stratification of a neighborhood of z' by strata where multi-germs of a given type occur for $\boldsymbol{f}_2$. Since $\boldsymbol{f}_2$ is a stable D_4-germ, these form a stratification which we denote by V_2. Let V_2' be formed by the inverse images of the strata of V_2 in $\Sigma(\boldsymbol{f}_2)$ and off of $\Sigma(\boldsymbol{f}_2)$. These again form a stratification because they are the pull-backs by the jet extension of $\boldsymbol{f}_2$ of multi-jet orbits. Since V_1 is transverse to $\boldsymbol{f}_2$, V_2 intersects V_1 transversally.

Then, the stratifications for the multi-germ $\boldsymbol{f}$ consist of :

- V formed by the complement of V_1 and the intersections of V_1 with the strata of V_2;
- V' formed by the pull-backs of the stata of V to $\Sigma(\boldsymbol{f}_1)$ and off $\Sigma(\boldsymbol{f}_1)$;
- V'' formed by the pull-backs of the stata of V to $\Sigma(\boldsymbol{f}_2)$ and off $\Sigma(\boldsymbol{f}_2)$.

Again V is a stratification with strata where the multi-germs occur. The strata of V' and V'' are the pull-backs via the multi-jet maps of $\boldsymbol{f}$ of the multi-jet orbits.

It remains to verify that the vector fields are stratified relative to (V, V', V''). We already know that η is stratified relative to V_1, hence η is tangent to V_1. Also, ζ_0 and the $\partial/\partial w_i$ define smooth trivializations of $\boldsymbol{f}_2$ and hence are tangent to the strata of V_2. Thus, any linear combination is still tangent to the strata of V_2. Thus, η is tangent to the strata of V_2 and to V_1 and to their transverse intersections.

Next, the stata of V' and V'' are mapped submersively onto the strata of V by $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. For $\boldsymbol{f}_1$ this follows because $\boldsymbol{f}_1^{-1}(V_1) \cap \Sigma(\boldsymbol{f}_1) = V_1'$ and $\boldsymbol{f}_1$ induces a diffeomorphism of V_1' to V_1. For $\boldsymbol{f}_2$, V_1 is transverse to the strata of V_2; hence $\boldsymbol{f}_2^{-1}(V_1)$ is transverse to the strata of V_2'. The intersection gives the strata of V''. Thus, (4.6) and (4.7) imply that ξ_1 and ξ_2 are tangent to the pull-backs of the strata to $\Sigma(\boldsymbol{f}_i)$ and off of $\Sigma(\boldsymbol{f}_i)$.

Secondly, we must verify the local control conditions : since $\boldsymbol{f}_2$ is trivial in the direction of ζ_0 and the $\partial/\partial w_i$, there is a local control function ρ_z for the stratum V_i of V_2 containing z such that $\zeta_0(\rho_z) = \partial\rho_z/\partial w_i = 0$. Then, $\rho + \rho_z$ is a local control function for $V_i \cap V_1$; and $\eta(\rho_z) = 0$. Thus,

$$\big|\eta(\rho + \rho_z)\big| = \big|\eta(\rho)\big| \leq C\rho \leq C(\rho + \rho_z).$$

The first inequality follows from η being stratified with ρ the control function for V_1.

Also, if ρ' is the control function for V_1', then $\rho' + \boldsymbol{\rho}_z$ is a control function for $\boldsymbol{f}_1^{-1}(V_i \cap V_1) \cap \Sigma(\boldsymbol{f}_1)$ and $(\rho + \rho_z) \circ \boldsymbol{f}_1$ for $\boldsymbol{f}_1^{-1}(V_i \cap V_1) \setminus V_1'$ with $\boldsymbol{\rho}_z = \rho_z \circ \boldsymbol{f}_1$. Since

$$\xi_1(\boldsymbol{\rho}_z) = \mathrm{d}\boldsymbol{f}_1(\xi_1)(\rho_z) = \eta(\rho_z) \circ \boldsymbol{f}_1 = 0,$$

the local control condition is satisfied because it is for ξ_1 using ρ' and $\rho \circ \boldsymbol{f}_1$.

Lastly, for ξ_2 let ρ_x be a local control function for the stratum V_1' of V_2' containing x such that $\xi_i'(\rho_x) = 0$. Hence, the stratum $V_i' \cap \boldsymbol{f}_2^{-1}(V_1)$ has local control function $\rho_x + \rho \circ \boldsymbol{f}_2$. Again, by (4.7)

$$\begin{aligned}\left|\xi_2(\rho_x + \rho \circ \boldsymbol{f}_2)\right| = \left|\xi_2(\rho \circ \boldsymbol{f}_2)\right| = \left|\mathrm{d}\boldsymbol{f}_2(\xi_2)(\rho)\right| = \left|\eta(\rho \circ \boldsymbol{f}_2)\right| \\ \leq C\rho \circ \boldsymbol{f}_2 \leq C(\rho_x + \rho \circ \boldsymbol{f}_2).\end{aligned}$$

This completes the verification that the vector fields are stratified. $\square$

Proof of Proposition 4.2. — We begin with a change of coordinates $x' = x - x_0$ so that $x = x_0 + x'$. Upon substitution into f_1 we obtain :

$$z = y^3 + tyx'^4 + ty\Big(\sum_{0 \leq i \leq 5} u'_i x'^{6-i}\Big) + x'^9 + 9x_0 x'^8 + \sum_{2 \leq i \leq 8} v'_i x'^{9-i}$$

where $v'_4 = 0$ by the choice of the slice.

Each u'_i, v'_i, z is an affine function of the u_i's, respectively v_i's, respectively v_i's and z. Since we have an inverse transformation obtained by resubstituting $x' = x - x_0$, we conclude that this transformation is invertible.

Secondly, we consider the multi-germ at $x = -2x_0$. We let

$$\boldsymbol{x} = x + 2x_0 = x' + 3x_0, \quad \text{or} \quad x' = \boldsymbol{x} - 3x_0.$$

Upon substituing, we obtain :

$$\begin{aligned}z = y^3 + (-3x_0)^4 ty\boldsymbol{x}^2 + (-3x_0)^6 \boldsymbol{x}^3 + ty\big(u''_2 + u''_1 \boldsymbol{x}\big) \\ + \big(v''_3 + v''_2 \boldsymbol{x} + v''_1 \boldsymbol{x}^2\big) + H\end{aligned}$$

where H contains terms of weights greater than 5 in $(\boldsymbol{x}, y)$ and than 6 in all variables.

Furthermore, remembering that $v'_4 = 0$, we see by direct calculation that modulo $(u'_3, \ldots, u'_6, v'_5, \ldots, v'_8)$ the u''_i, respectively v''_i, are affine functions of (u'_1, u'_2), respectively (v'_2, v'_3), with linear parts given by, respectively :

$$5(-3x_0)u'_1 + 4u'_2, \quad (-3x_0)u'_1 + u'_2,$$

$$7(-3x_0)v'_2 + 6v'_3, \quad (-3x_0)v'_2 + v'_3.$$

These are easily seen to be linearly independent.

Next, we absorb $v''_1 \boldsymbol{x}^2$ by a substitution $x'' = \boldsymbol{x} + \frac{1}{3}(-3x_0)^{-6} v''_1$. Because each term of H is at least cubic (and at least quadratic in x'') the coefficients of 1 and x'' will still differ from v''_3 and v''_2 by higher order terms in the v'_i. Also, the linear terms of the coefficients of y and yx''

will differ from those of u_2'' and u_1'' by at most terms in v_1'' whose linear terms involve v_i'. Thus, the linear terms of the coefficients are still linearly independent, modulo $(u_3', \dots, u_6', v_5', \dots, v_8')$.

Hence, the coefficients together with $u_3', \dots, u_6', v_5', \dots, v_8'$ and either (x', y) or (x'', y) or z form systems of local coordinates (giving $w_1, \dots, s_1 \dots$) vanishing at x', x'', z'.

Next, we incorportate the powers of $(-3x_0)^2$ into x'' and (since $t \neq 0$) replace tw_i by w_i for $1 \leq i \leq 4$, and ts_i by s_i for $i = 3, 4$. To verify that the coefficient c is non-zero we note that u_2' is replaced by a linear combination of u_1'' and u_2'' modulo $(u_3', \dots, u_6', \{v_i'\})$; by direct calculation we see that the coefficient of u_2'' is non-zero. The preceding step will also leave the coefficient of u_2'' non-zero.

Now everything has the desired form except that f_2 differs from the desired form by the terms $\boldsymbol{t}y''x''^2 + H$, with $\boldsymbol{t} = t - t_0$. These terms give a deformation of non-decreasing weight of the germ in (4.3), which is the versal unfolding of a D_4 singularity. Moreover, it is versal in a graded sense by 2) of LEMMA 5.4; hence the vector fields trivializing f_2 viewed as an unfolding of (4.3) with unfolding parameters $(w, \boldsymbol{t})$ have weight ≥ 0. Thus, the germs of diffeomorphisms trivializing f_2 are of nondecreasing weight. This introduces a change of coordinates for s, depending on $(w, \boldsymbol{t})$ which is the identity when $(w, \boldsymbol{t}) = 0$. Thus, f_1 is only changed by terms of weight > 6 in (x', y') or terms which are products of the form $s_i s_j \phi$, $\boldsymbol{t}s_4\phi$, or $w_i s_j \phi$ with ϕ a term of weight 6 in (x', y'). Hence, f_1 still has the desired form. $\square$

5. The algebraic lemmas

In this section we prove PROPOSITION 4.8 and give the algebraic justifications needed in PROPOSITION 4.2 and 4.5. First, we view $\boldsymbol{f}_i$ as unfoldings of germs $\boldsymbol{f}_{i,0}$. Let $f_{1,0}$ denote the germ obtained from f_1 by letting $\{s_i = 0\}$ and $t = t_0$ i.e. $\boldsymbol{t} = t - t_0 = 0$. Then,

$$\boldsymbol{f}_{1,0}(x', y', w) = \big(f_{1,0}(x', y', w), w\big).$$

Also, $\boldsymbol{f}_2$ is a constant unfolding of the germ $\boldsymbol{f}_{2,0}$ obtained from it by setting $w = 0$, $\boldsymbol{t} = 0$. Here

$$(5.1) \qquad \begin{cases} f_{1,0} = y'^3 + t_0 y' x'^4 + x'^6 + y'\Big(\displaystyle\sum_{i=0}^{3} w_{i+1} x'^i\Big) \\ \qquad\qquad\qquad\qquad + \displaystyle\sum_{i=1}^{4} w_{i+4} x'^i + H_1', \\ f_{2,0} = y''^3 + t_0 y'' x''^2 + x''^3 + s_1 + s_2 x'' + y''(s_3 + s_4 x''). \end{cases}$$

Again to simplify the description of modules of vector fields that follow, we denote $R\langle h_1,\dots,h_k\rangle$ by $R\langle h_i\rangle$ when the index k is clear from the context. Weight filtrations on modules of vector fields are defined by $\mathrm{wt}(g\partial/\partial\lambda)=\mathrm{wt}(g)-\mathrm{wt}(\lambda)$ for weighted homogeneous g and $\lambda=x',y',z,s_i$ or w_i. For an arbitrary vector field ξ we define $\mathrm{wt}(\xi)\geq k$ if the terms of ξ each have weight $\geq k$. This defines weight filtrations on :

$$\begin{aligned}&\theta(f_{1,0})=\mathbb{C}_{x',y',w}\langle\partial/\partial z\rangle,\\ &\mathbb{C}_{x',y',w}\langle\partial/\partial x',\partial/\partial y'\rangle,\\ &\mathbb{C}_{z,w}\langle\partial/\partial z,\partial/\partial w_i\rangle,\end{aligned}$$

for $f_{1,0}$, as well the corresponding modules for $f_{2,0}$ obtained by replacing (x',y',w) by (x'',y'',s) and $\mathbb{C}_{z,w}\langle\partial/\partial z,\partial/\partial w_i\rangle$ by $\mathbb{C}_{z,s}\langle\partial/\partial s_i\rangle$. Given such a module M with a weight filtration, we denote the submodule of vector fields of weight $\geq k$ (respectively $>k$) by $M_{(\geq k)}$ (respectively $M_{(>k)}$).

Next, for $f_{i,0}$ we define the maps α'_i which are essentially the infinitesimal orbit maps. For $f_{1,0}$, we define :

$$\alpha'_1:\mathbb{C}_{x',y',w}\langle\partial/\partial x',\partial/\partial y'\rangle\oplus\mathbb{C}_{z,w}\langle\partial/\partial w_i\rangle\oplus\mathbb{C}_{z,w}\langle\partial/\partial z\rangle \longrightarrow \mathbb{C}_{x',y',w}\langle\partial/\partial z\rangle$$

by $\alpha'_1(\xi,\zeta,\eta)=\xi(f_{1,0})+\zeta(f_{1,0})-\eta\circ f_{1,0}$.

We denote that on the first summand α'_1 is a $\mathbb{C}_{x',y',w}$-module homomorphism, while on the second and third summand α'_1 is a module homomorphism over $f^*_{1,0}:\mathbb{C}_{z,w}\to\mathbb{C}_{x',y',w}$. For $f_{2,0}$, we define :

$$\alpha'_2:\mathbb{C}_{x'',y'',s}\langle\partial/\partial x'',\partial/\partial y''\rangle\oplus\mathbb{C}_{s}\langle\partial/\partial s_i\rangle \longrightarrow \mathbb{C}_{x'',y'',s}\langle\partial/\partial z\rangle$$

by $\alpha'_2(\xi,\zeta)=\xi(f_{2,0})+\zeta(f_{2,0})$.

We note that the α'_i preserve the filtrations, i.e.

$$\mathrm{wt}(\xi)\geq k\quad\text{implies}\quad\mathrm{wt}(\alpha'_i(\xi))\geq k.$$

A homomorphism of filtered vector spaces $\alpha:M\to N$ is said to be *graded surjective in weight* $\geq L$ (respectively $>L$) if for any $k\geq L$ (respectively $k>L$)

$$N_{(\geq k)}=\alpha\big(M_{(\geq k)}\big)+N_{(\geq k+1)}. \tag{5.3}$$

We say that α is *graded surjective* if (5.3) holds for all k.

LEMMA 5.4.

1) α'_1 *is graded surjective in weight* > 0;

2) α'_2 *is graded surjective*;

3) *if* V *denotes the subspace spanned by* $y'x'^4\partial/\partial z$ *and*

$$i' : V \to \mathbb{C}_{x',y',w}\langle\partial/\partial z\rangle$$

denotes the inclusion, then $\alpha'_1 + i'$ *is graded surjective in weight* ≥ 0.

Proof. — First consider α'_1. It is enough to replace $f_{1,0}$ by the germ $f'_{1,0}$ defined by the terms of lowest weight (i.e.= 6) in $f_{1,0}$. This amounts to discarding $\boldsymbol{t}y'x'^4$ and H'_1. Then, $\boldsymbol{f}'_{1,0}$ is exactly the unfolding of an $\widetilde{E}_8$-germ, which is versal except for the modulus term. By the algebraic calculation of Looijenga [L, prop. 2.1], the corresponding map α'_1 for $\boldsymbol{f}'_{1,0}$ is graded surjective in weight > 0 and its image in weight 0 has codimension 1 spanned by $y'x'^4$. This implies that we can solve (5.3) for α'_1 of $f_{1,0}$ for $k > 0$, or $k = 0$ adding i'.

For α'_2, the germ $f_{2,0}$ is weight homogeneous and is the versal unfolding of D_4 with respect to right equivalence. Hence, the corresponding map α'_2 for $f'_{2,0}$ is surjective in all weights, which implies that (5.3) for α'_2 can be solved for all k. $\square$

We can draw several consequences of this lemma for each $\boldsymbol{f}_i$ viewed as an unfolding of $\boldsymbol{f}_{i,0}$. For this, we extend the filtrations in such a way that $(s, \boldsymbol{t})$, respectively $(w, \boldsymbol{t})$ are treated as unfolding parameters for $\boldsymbol{f}_1$, respectively $\boldsymbol{f}_2$. To simplify the notations, let :

$$R_1 = \mathbb{C}_{x',y',w,s,\boldsymbol{t}}, \quad R_2 = \mathbb{C}_{x",y",w,s,\boldsymbol{t}}, \quad S = \mathbb{C}_{z,w,s,\boldsymbol{t}}.$$

We then define induced filtrations on $R_i\langle\partial/\partial z\rangle$, $R_1\langle\partial/\partial x', \partial/\partial y'\rangle$, etc., by taking the submodules of vector fields of weight $\geq k$ to be those generated by vector fields in $\mathbb{C}_{x',y',w}\langle\partial/\partial z\rangle$, etc., of weight $\geq k$. Moreover, we shall replace $\partial/\partial z$ by $\zeta_0 = (\partial/\partial z + \partial/\partial s_1, -\partial/\partial s_1)$ and denote $\partial/\partial w_i$ by ζ_i. We let :

$$T_1 = R_1\langle\partial/\partial x', \partial/\partial y'\rangle \oplus S\langle\zeta_i\rangle,$$

$$T_2 = R_2\langle\partial/\partial x', \partial/\partial y'\rangle \oplus S\langle\partial/\partial s_i\rangle.$$

Then, we define homomorphisms $\alpha_i : T_i \to R_i\langle\partial/\partial z\rangle = \theta(f_i)$ by

$$\alpha_i(\xi, \zeta) = \begin{cases} \xi(f_1) + \zeta'(f_1) - \zeta'' \circ \boldsymbol{f}_1 & \text{if } i = 1, \\ \xi(f_2) + \zeta(f_2) & \text{if } i = 2, \end{cases}$$

where for $i = 1$, if we write $\zeta = \sum g_i\zeta_i$, then $\zeta'' = -g_0\partial/\partial z$ and $\zeta' = \zeta - \zeta''$. Again these homomorphisms preserve filtrations. Then, the α_i also satisfy surjectivity results.

LEMMA 5.5.

1) α_1 *is graded surjective in weight* > 0;

2) α_2 *is graded surjective*;

3) *if* $i : \mathbb{C}_{s,t}\langle y'x'^4\partial/\partial z\rangle \to R_1\langle\partial/\partial z\rangle$ *is the inclusion, then* $\alpha_1 + i$ *is graded surjective in weight* ≥ 0.

Proof. — By LEMMA 5.4, α'_1 is graded surjective in weight > 0, $\alpha'_1 + i'$ is graded surjective. By LEMMA 7.4 of [D3], α_1, $\alpha_1 + i$, and α_2 are graded surjective as claimed. □

To understand the interaction between the multi-germs we define :

$$\alpha : T_1 \oplus T_2 \longrightarrow \theta(f_1) \oplus \theta(f_2)$$

so that

$$\begin{aligned}
&\alpha_{|R_1\langle\partial/\partial x',\partial/\partial y'\rangle} = (\alpha_1, 0), \quad \alpha_{|R_2\langle\partial/\partial x'',\partial/\partial y''\rangle} = (0, \alpha_2),\\
&\alpha(\partial/\partial z) = \big(\partial/\partial z, \partial/\partial z\big), \quad \alpha(\partial/\partial w_i) = \big(\partial f_1/\partial w_i, 0\big),\\
&\alpha(\partial/\partial s_i) = \big(\partial f_1/\partial s_i, \partial f_2/\partial s_i\big)
\end{aligned}$$

and α extends S-linearly over $S\langle\zeta_i\rangle \oplus S\langle\partial/\partial s_i\rangle$.

The surjectivity of the α_i no longer implies the surjectivity of α. Nevertheless, the algebraic linking that we referred to earlier can be established by using jumps in filtration that occur for certain vector fields. Define :

$$M_1 = T_{1(>0)}, \qquad M_2 = T_{1(\geq 0)},$$

$$W_1 = \theta(f_1)_{(>0)}, \quad W_2 = \theta(f_1)_{(\geq 0)}.$$

LEMMA 5.6. — *For each* i, *there are vector fields* $\beta_i \in T_{2(\geq 0)}$ *so that* :

$$\alpha(\beta_i) = \big(s_i y'x'^4\partial/\partial z, 0\big) \quad \textit{modulo} \quad (m_S^2 W_2 + W_1).$$

Proof. — To prove this lemma, we need one more lemma about $f_{2,0}$.

LEMMA 5.7. — *The restriction of* α'_2 :

$$\begin{aligned}
\mathbb{C}_{x'',y'',s}&\langle\partial/\partial x'', \partial/\partial y''\rangle\\
&+ \mathbb{C}_s\langle\partial/\partial s_1, \ldots, \partial/\partial s_3, z\partial/\partial s_1, \ldots, z\partial/\partial s_4\rangle\\
&\qquad\qquad\qquad\longrightarrow \theta(f''_{2,0})
\end{aligned}$$

is surjective in weight $\neq -2$, *and in particular* $m_S\,\theta\,(F_{2,0})$ *is contained in the image of* α'_2.

Proof. — If we had $\partial/\partial s_4$ in place of the $z\partial/\partial s_i$, the mapping would be surjective by the preparation theorem and the stability of D_4. The argument in Looijenga [L, prop. 2.1], which was given for simple elliptic singularities, implies that $\partial/\partial s_4$ may be replaced by $\{z\partial/\partial s_i\}$ and the homomorphism remains surjective in weight $\neq -2$ with complement spanned by $y''x''\partial/\partial z$ (alternately see e.g. [D1 II, thm 6.5]). □

Proof of lemma 6.6 *continued.* — Let :

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (1, x'', y'', x'', y''),$$
$$\gamma_i = 6z\partial/\partial s_i - 2\phi_i x''\partial/\partial x'' - 2\phi_y y''\partial/\partial y''.$$

Then, by the Euler relation,

$$\gamma_i(f_{2,0}) = \Big(\sum_{1\leq i\leq 4} \mathrm{wt}(s_j)s_j\phi_i\phi_j\Big)\partial/\partial z. \tag{5.8}$$

By LEMMA 5.7, if $\mathrm{wt}(\phi_i) + \mathrm{wt}(\phi_j) \neq 4$, then there exists $\xi_{i,j}$ of weight $\mathrm{wt}(\phi_i) + \mathrm{wt}(\phi_j) - 6$ such that

$$\xi_{i,j}(f_{2,0}) = \phi_i\phi_j \tag{5.9}$$

and for which the coefficient of $\partial/\partial s_4$ vanishes when $z = 0$.

As the Jacobian algebra of D_4 is Gorenstein, if $\mathrm{wt}(\phi_i\phi_j) = 4$ then there exists a constant $c_{i,j}$ so that :

$$\phi_i\phi_j = c_{i,j}x''y'' \quad \text{modulo } \mathbb{C}_{x'',y'',s}\langle\partial f_{2,0}/\partial x'', \partial f_{2,0}/\partial y''\rangle + m_s\mathbb{C}_{x'',y'',s}. \tag{5.10}$$

In addition by LEMMA 5.7 we may set $c_{i,j} = 0$ when $\mathrm{wt}(\phi_i)+\mathrm{wt}(\phi_j) \neq 4$.

The fact that the Jacobian algebra is Gorenstein further implies that $(c_{i,j})$ is non singular. Then, (5.10) together with LEMMA 5.7 implies that there is a $\xi_{i,j}$ such that :

$$\alpha_2(\xi_{i,j}) = c_{i,j}y''x''. \tag{5.11}$$

Let $\beta_i'' = \gamma_i - \sum_{1\leq i\leq 4} \mathrm{wt}(s_j)s_j\xi_{i,j}$. Then, by (5.9) to (5.11) :

$$\alpha_2(\beta_i'') = \Big(\sum c_{i,j}\,\mathrm{wt}(s_j)s_j\Big)y''x''\partial/\partial z.$$

Hence, if we let

$$\beta'_i = \beta''_i - \Big(\sum c_{i,j}\,\mathrm{wt}(s_j)s_j\Big)\partial/\partial s_4$$

then $\alpha_2(\beta'_i) = 0$. Next we determine $\alpha(\beta'_i)$. We observe :

$$\gamma_i(f_1) = 6z\partial f_1/\partial s_i \in W_1.$$

• First consider $i = 4$; by weight considerations $c_{4,j} = 0$ if $j \neq 1$. Also, $\mathrm{wt}(\gamma_4) = 4$ and $\mathrm{wt}(s_i) \leq 4$ if $i > 1$. Thus, $\mathrm{wt}(\xi_{4,j}) \geq 0$ for $j > 1$. Thus,

$$s_j\xi_{4,j}(f_1) \in m_s^2W_2 + W_1 \quad \text{for} \quad j > 1.$$

Hence :

$$\alpha(\beta'_1) = \big(5c'c_{4,1}s_1y'x'^4\partial/\partial z, 0\big) \text{ modulo } \big(m_s^2W_2 + W_1\big). \tag{5.12}$$

We let $\beta_1 = (5c'c_{4,1})^{-1}\beta'_1$.

• Next, for $i = 2, 3$, $c_{2,j} = 0$ and $c_{3,j} = 0$ if $j = 1, 4$. Also, $\mathrm{wt}(\gamma_2) = \mathrm{wt}(\gamma_3) = 2$. Then, $\mathrm{wt}(\xi_{2,4}), \mathrm{wt}(\xi_{3,4}) \geq 0$ and again

$$s_4\xi_{2,4}(f_1),\ s_4\xi_{3,4}(f_1) \in m_s^2W_2 + W_1.$$

Thus, for $i = 2, 3$,

$$\alpha(\beta'_i) = \big(4c(c_{i,2}s_2 + c_{i,3}s_3)y'x'^4\partial/\partial z, 0\big) \quad \text{modulo } \big(s_1W_2 + m_s^2W_2 + W_1\big).$$

Since $(c_{i,2}, c_{i,3})$ are linearly independent for $i = 2, 3$, there are linear combinations of β'_2 and β'_3 minus multiples of β_1 which yield β_2 and β_3.

• Finally, for $i = 1$, we note by LEMMA 6.7 that in $\xi_{4,4}$ the coefficient of $\partial/\partial s_4$ vanishes when $z = 0$; and $c_{i,4} = 0$ if $i \neq 1$ by weight considerations. Thus :

$$\alpha(\beta'_4) = \big(2c'c_{1,4}s_4y'x'^4\partial/\partial z, 0\big) \text{ modulo } \big(s_1, s_2, s_3\big)W_2 + m_s^2W_2 + W_1.$$

We let $\beta_4 = \big(2c'c_{1,4}\big)^{-1}\beta'_4$ minus an appropriate linear combination of $\beta_1, \beta_2, \beta_3$. □

Now we are ready to turn to the proof of PROPOSITION 4.8 :

Proof. — Multiplying 3) of LEMMA 5.5 by m_s and applying LEMMA 5.6 yields :

$$\alpha\big(m_s M_2 + \mathbb{C}_{s,t}\langle\beta_i\rangle\big) + m_s^2 W_2 + W_1 = m_s W_2 + W_1. \tag{5.9}$$

From 1) of LEMMA 5.5, $\alpha(M_1) = W_1$. Hence, for $W = m_s W_2 + W_1$ (5.9) becomes :

$$\alpha\big(M_1 + m_s M_2 + \mathbb{C}_{s,t}\langle\beta_i\rangle\big) + m_s W = W. \tag{5.10}$$

Thus, by the preparation theorem :

$$\alpha\big(M_1 + m_s M_2 + \mathbb{C}_{s,t}\langle\beta_i\rangle\big) = W.$$

Hence, we may solve

$$-s_i \partial f_j/\partial t = \xi_j^{(i)}(f_i) + \eta^{(i)} \circ f_j$$

where $\eta^{(i)}$ has weight ≥ 0 with respect to (z,w) in the $\partial/\partial w_i$ and $\partial/\partial z$ terms. Multiplying by $-\mathrm{conjugate}(s_i)$ and summing over i yields :

$$-\rho_1 \partial f_j/\partial t = \xi_j'(f_j) + \eta' \circ f_j.$$

Lastly, if

$$\eta' = g_0 \partial/\partial z + \sum_{1\leq i\leq 8} g_i \partial/\partial w_i + \sum_{1\leq i\leq 4} g_i' \partial/\partial s_i$$

then $\mathrm{wt}(g_i) \geq \mathrm{wt}(w_i)$ or ($\mathrm{wt}(z)$ if $i=0$) with respect to (z,w); thus,

$$|g_i| \leq C\rho^{\mathrm{wt}(w_i)/2m}$$

in a neighborhood of 0. Then,

$$\begin{aligned}\big|\eta'(\rho)\big| &\leq |g_0|\cdot|\partial\rho/\partial z| + \sum_{1\leq i\leq 8} |g_i|\cdot|\partial\rho/\partial w_i| \\ &\leq C_0 \rho^{(\mathrm{wt}(z)/2m)}\rho^{1-(\mathrm{wt}(z)/2m)} \\ &\qquad\qquad + \sum C_i \rho^{(\mathrm{wt}(w_i)/2m)}\rho^{1-(\mathrm{wt}(w_i)/2m)} \\ &\leq C\rho. \quad \square\end{aligned}$$

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