BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 121, nº 1 (1993), p. 91-107
<http://www.numdam.org/item?id=BSMF_1993__121_1_91_0>

Bull. Soc. math. France,
121, 1993, p. 91–107.

MELLIN TRANSFORMS OF WHITTAKER FUNCTIONS

BY

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RÉSUMÉ. — Soit G un groupe algébrique réductif connexe défini et quasi-déployé sur $\mathbb{R}$. Dans cet article, on étudie la transformée de Mellin de la fonction de Whittaker associée à G. On montre que cette transformée de Mellin se prolonge en une fonction méromorphe et satisfait à certaines équations de différences. On obtient un algorithme pour ces équations de différences, et on l'illustre dans le cas $G = \mathrm{GL}(n)$ pour n petit.

ABSTRACT. — Let G be a connected reductive algebraic group defined and quasi-split over $\mathbb{R}$. In this paper the Mellin transform of the Whittaker function associated to G is studied. It is shown that this Mellin transform has a meromorphic continuation and satisfies certain explicit difference equations. An effective algorithm for obtaining these difference equations is presented, and is illustrated in low rank cases for $G = \mathrm{GL}(n)$.

1. Introduction

Let G be a simply connected Chevalley group defined over a local field k, and let A be a maximal k-split torus of G. In his thesis, JACQUET [J] showed how to define a Whittaker function W associated to the unramified principal series of $G(k)$. In this paper we shall study the Mellin transform of Jacquet's Whittaker function when k is archimedean. This Mellin transform, which is by definition the integral of W against a character of A, converges absolutely in a subspace which is, after identifications, a product of right half-planes. We show that it has a meromorphic continuation

Texte reçu le 8 avril 1991, révisé le 30 janvier 1992.
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Research supported in part by NSF grant n° 9003907.
Classification AMS : 22E30, 11F70, 11F72, 22G20, 22E45.

to the full space of characters and satisfies certain explicit difference equations. Our method provides an effective algorithm for obtaining these difference equations which has been implemented by O. McGuinness in some low rank cases. We thank him for allowing us to incorporate his calculations in section 5.

We shall present our argument in detail when $G = \mathrm{GL}(n+1,\mathbb{R})$, for simplicity, and then (in section 6) indicate the modifications to be made for a general connected reductive algebraic group G defined and quasi-split over $\mathbb{R}$. Archimedean Whittaker functions in this generality were first studied by Schiffmann [Sc]. It is our pleasure to thank H. Jacquet for many helpful and enlightening conversations. The first named author would also like to express his thanks to the Columbia University mathematics department for its hospitality during the 1990–1991 academic year.

2. Notation and statement of results

Let $G = \mathrm{GL}(n+1,\mathbb{R})$, with $n \geq 1$. Let $N \leq G$ be the subgroup of upper-triangular matrices with diagonal entries equal to 1. Set $K = \mathrm{O}(n+1,\mathbb{R})$, the standard maximal compact subgroup of G. The Iwasawa decomposition identifies the symmetric space $H = G/K\,\mathbb{R}^{\times}$ of G with the set of matrices of the form $z = xy$ with $x \in N$ and $y = \mathrm{diag}(y_1 \cdots y_n, y_2 \cdots y_n, \ldots, y_n, 1)$ with $y_i > 0$ for all $1 \leq i \leq n$.

Let us recall the definition of Jacquet's Whittaker function [J]. Let $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$. Define a function $Y_\nu : H \to \mathbb{R}^+$ by

$$Y_\nu(z) = \prod_{i=1}^{n} \prod_{j=1}^{n} y_i^{b_{i,j}\nu_j}$$

with

$$b_{i,j} = \begin{cases} ij & \text{if } i+j \leq n+1, \\ (n+1-i)(n+1-j) & \text{if } i+j \geq n+1. \end{cases}$$

Let $\mathcal{D}$ denote the algebra of G-invariant differential operators on H, which is known to be commutative and isomorphic to a polynomial ring in n variables. Then Y_ν is an eigenfunction of $\mathcal{D}$. Let λ_ν denote the resulting character of $\mathcal{D}$ (the Harish-Chandra homomorphism) :

$$\delta Y_\nu = \lambda_\nu(\delta) Y_\nu \qquad (\delta \in \mathcal{D}).$$

Fix $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, with $m_1 \cdots m_n \neq 0$. Define a character ψ of N by

$$\psi(x) = \exp\bigl(2\pi i (m_1 x_{1,2} + \cdots + m_n x_{n,n+1})\bigr)$$

for $x = (x_{i,j}) \in N$. Then Jacquet's Whittaker function

$$W_\nu : H \to \mathbb{C}$$

is characterized (up to scalars) by the properties

$$\begin{cases} \delta W_\nu = \lambda_\nu(\delta) W_\nu & \text{for all } \delta \in \mathcal{D}; \\ W_\nu(xz) = \psi(x) W_\nu(z) & \text{for all } x \in N; \\ W_\nu(z) \text{ decays exponentially as } y_i \to \infty \text{ for each } 1 \le i \le n. \end{cases}$$

For $c \in \mathbb{R}$, define the half-plane $\mathrm{Re}(s) > c$ to be the set of $s \in \mathbb{C}^n$ such that $\mathrm{Re}(s_i) > c$ for all $i = 1, \ldots, n$. Consider an arbitrary continuous function $F : H \to \mathbb{C}$. For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ we formally define the Mellin transform of F,

$$\widetilde{F}(s) = \int_{y_1=0}^{\infty} \cdots \int_{y_n=0}^{\infty} y_1^{s_1} \cdots y_n^{s_n} \, F(y) \prod_{i=1}^{n} \frac{dy_i}{y_i}.$$

If F has sufficient decay as each $y_i \to \infty$ and at worst slow growth as each $y_i \to 0$, the above integral converges absolutely in a half-plane $\mathrm{Re}(s) > c$ for some fixed constant c.

The primary purpose of this paper is to obtain the meromorphic continuation and growth properties of the Mellin transform of the Whittaker function W_ν, regarded as a function of s and ν. It is known that there exists a real number $N(\nu)$ such that the integral $\widetilde{W_\nu}(s)$ converges absolutely for $\mathrm{Re}(s) > N(\nu)$. We shall prove :

THEOREM 1. — *Let $n \ge 1$. The Mellin transform $\widetilde{W_\nu}(s)$ has a meromorphic continuation to all $s, \nu \in \mathbb{C}^n$.*

Let $s = (s_1, \ldots, s_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ be complex variables. For brevity let $\mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_n]$ and $\mathbb{C}[s, \nu] = \mathbb{C}[s][\nu_1, \ldots, \nu_n]$. If $f(s, \nu) \in \mathbb{C}[s, \nu]$, then $f(s, \nu)$ may be written as a sum $f(s, \nu) = \sum_\mu f_\mu(\nu)$, where the f_μ are homogeneous polynomials of order μ in the variables $\nu_1, \ldots, \nu_n$ with coefficients in $\mathbb{C}[s]$. We define $\deg_\nu(f(s, \nu))$ to be the largest integer μ such that f_μ is not identically zero. Also, we write $\mathbb{Z}^+$ for the set of positive integers.

THEOREM 2. — *Let $n \ge 1$ and let $\delta_1, \ldots, \delta_n$ be a set of generators for $\mathcal{D}$ over $\mathbb{C}$. Then for all sufficiently large positive integers M, there exist*

(1) *a finite set of shift vectors $\Delta_M \subset (2\mathbb{Z}^+)^n$;*

(2) *polynomials $p_\sigma(s) \in \mathbb{C}[s]$ for $\sigma \in \Delta_M$;*

(3) *a polynomial* $q(s,\nu) \in \mathbb{C}[s][\lambda_\nu(\delta_1),\ldots,\lambda_\nu(\delta_n)]$ *satisfying*

$$\deg_\nu(q(s,\nu)) \geq M,$$

and $q(s,\nu)=0$ *when* $s_i=\sum_j b_{i,j}\nu_j$ *for all* i, $1\leq i\leq n$ *such that* :

$$\widetilde{W}_\nu(s) = \frac{1}{q(s,\nu)} \sum_{\sigma\in\Delta_M} p_\sigma(s)\widetilde{W}_\nu(s+\sigma) \quad \textit{for all } s,\nu\in\mathbb{C}^n. \tag{2.1}$$

REMARKS. — When $n\leq 2$ it is known (see [B, chap. X] and [BF1] for the case $n=2$) that $\widetilde{W}_\nu(s)$ is a ratio of products of Gamma functions. Similarly, in the non-archimedean case for $n\leq 2$ the p-adic Mellin transform of the p-adic Whittaker function is a ratio of products of Gauss sums. In the non-archimedean case, this phenomenon breaks down when $n\geq 3$. By analogy, one does *not* expect $\widetilde{W}_\nu(s)$ to be a ratio of products of Gamma functions when $n\geq 3$. Motivated by their integral representation of the exterior square L-function, BUMP and FRIEDBERG [BF2] have conjectured that for $n\geq 3$, $\widetilde{W}_\nu(s)$ should restrict to a ratio of products of Gamma functions on a certain subspace of $s\in\mathbb{C}^n$ of dimension two. This conjecture was recently verified by STADE [Sta] in the case $n=3$.

The continuation of $\widetilde{W}_\nu(s)$ for fixed ν was first proved by JACQUET and SHALIKA [JS1] using difference equations. The proof of THEOREM 1 given here, while also utilizing difference equations, differs in that it is based on an elementary counting argument involving integer partitions. This allows us to obtain the quantitative form of THEOREM 2 which appears to be new. It follows from (2.1) that for any fixed s, $\widetilde{W}_\nu(s)$ has polynomial decay in ν. Our methods are completely constructive, and hence, for any $n\geq 1$, M sufficiently large, the set of shift vectors Δ_M and the polynomials p_σ (for $\sigma\in\Delta_M$), $q(s,\nu)$ can be effectively determined. Some examples are given in section 5. Our method may also be used to give analogous results for the partial Mellin transforms of W_ν.

Since $q(s,\nu)$ vanishes when $s_i=\sum_j b_{i,j}\nu_j$ for $1\leq i\leq n$, THEOREM 2 supports the conjecture [G] that $\widetilde{W}_\nu(s)$ has poles when $s_i=\sum_j b_{i,j}\nu_j$. It may also be shown that the polynomial $q(s,\nu)$ has zeros at

$$s+\sigma=\sum_j b_{i,j}\nu_j$$

for σ in a certain finite set of shift vectors. See the proof of THEOREM 2 for details.

Let $z = xy \in H$ be as above, and let $f(y)$ be a rapidly decreasing function of $y_1, \cdots, y_n$. Extend the character ψ to a function on H by the formula $\psi(z) = \psi(x)f(y)$. Also let $N(\mathbb{Z}) = N \cap \mathrm{SL}(n+1, \mathbb{Z})$. Recall that a meromorphic function $G(s)$ in a tube domain $\mathrm{Re}(s) > c$ has polynomial growth if there exist two polynomials $p(s)$, $q(s)$, such that $G(s)q(s)/p(s)$ is holomorphic and bounded in the tube domain. As an application of THEOREM 2, we have :

COROLLARY 1.—*Let $s \in \mathbb{C}^n$. Then the Poincaré series* [BFG, F, G, Ste]

$$P(z; s, \psi) = \sum_{\gamma \in N(\mathbb{Z}) \backslash \mathrm{SL}(n+1,\mathbb{Z})} Y_s(\gamma z)\psi(\gamma z)$$

has a meromorphic continuation to all $s \in \mathbb{C}^n$, and has at most polynomial growth in s in any tube domain $\mathrm{Re}(s) > c$.

Here the condition that $f(y)$ be rapidly decreasing in y guarantees that $P(z; s, \psi) \in L^2(\mathrm{SL}(n+1, \mathbb{Z}) \setminus H)$. COROLLARY 1 is proved by computing the spectral expansion of $P(z; s, \psi)$. We briefly indicate the continuation of the cuspidal projection; the arguments for the continuous and residual projections are similar. Let $\phi(z)$ be a cuspidal automorphic form in $L^2(\mathrm{SL}(n+1, \mathbb{Z}) \setminus H)$, and let $\nu \in \mathbb{C}^n$ be chosen such that $\delta\phi = \lambda_\nu(\delta)\phi$ for all $\delta \in \mathcal{D}$. Following [BFG], the Petersson inner product $\langle P, \phi \rangle$ is the Whittaker-Fourier coefficient of ϕ corresponding to $\psi(x)$ times a sum of shifts of the Mellin transform $\widetilde{W_\nu}$ (the shifts arise from doing a Taylor expansion of $f(y)$). The continuation and polynomial growth of the cuspidal projection follows from this and from (2.1) after summing over a basis of cusp forms. Here we strongly use the fact that $\widetilde{W_\nu}(s)$ has polynomial decay in the variable ν and at most polynomial growth in the variable s.

3. Proof of theorem 1

Let $s = (s_1, \ldots, s_n)$ be variables. In order to prove THEOREM 1, we shall study the action of $\mathcal{D}$ on the function

$$I_s(z) = \psi(x)\, y_1^{s_1} \cdots y_n^{s_n} \qquad (z = xy)$$

as s ranges over $\mathbb{C}^n$. First, we must develop some basic information on the differential operators in $\mathcal{D}$. We shall refine this to give an explicit construction of a basis for $\mathcal{D}$ in section 5 below. However, this is not needed for the proofs of the Theorems.

Write B for the standard Borel subgroup of G of upper triangular matrices. Denote the Lie algebras of G, $\mathrm{SL}(n+1, \mathbb{R})$, B, and K by $\mathtt{G}$, $\mathtt{G}_1$, $\mathtt{B}$, and $\mathtt{K}$, respectively. Given an arbitrary Lie algebra $\mathcal{L}$, we shall denote its universal enveloping algebra by $U(\mathcal{L})$, and its subalgebra of right K-invariant elements by $U(\mathcal{L})^K$. Then the G-invariant differential operators on H are naturally identified with the quotient space

$$U(\mathfrak{g}_1)^K/(U(\mathfrak{g}_1)^K \cap U(\mathfrak{g}_1)\mathfrak{k})$$

(cf. [H, chap. II, thm 4.6]). Elements of $U(\mathfrak{g}_1)^K$ operate on $C^\infty(H)$ in the standard manner by successive application of the formula

$$Xf(z) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(z\exp(tX)\big)_{|t=0} \qquad (X \in \mathfrak{g}). \tag{3.1}$$

Let us extend (3.1) to an action of $U(\mathfrak{g})$ on $C^\infty(H)$ (even though the action is no longer compatible with the passage from group to symmetric space). Then since a function on H is right K-invariant, the left ideal $U(\mathfrak{g})\mathfrak{k}$ acts trivially.

LEMMA 1. — *Suppose $u \in U(\mathfrak{g})$.*

(1) *There exists an element of $U(\mathfrak{b})$ whose action on $C^\infty(H)$ is the same as that of u.*

(2) *The action of u on $C^\infty(H)$ is given by a (non-commutative) polynomial in the operators*

$$y_{n+1-i}\frac{\partial}{\partial y_{n+1-i}} - y_{n-i+2}\frac{\partial}{\partial y_{n-i+2}}$$

for $1 \le i \le n+1$ (with $y_0 = y_{n+1} = 0$), and

$$\sum_{k=1}^{i} y_{n-j+2}\cdots y_{n+1-i}\, x_{k,i}\frac{\partial}{\partial x_{k,j}}$$

for $1 \le i < j \le n+1$ (with $x_{i,i} = 1$).

Proof. — By the Iwasawa decomposition,

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{k}.$$

Consequently

$$U(\mathfrak{g}) = U(\mathfrak{b}) \otimes U(\mathfrak{k}).$$

From this it follows that the coset $u + U(\mathfrak{g})\mathfrak{k}$ in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}$ has a coset representative u' in $U(\mathfrak{b})$. Since every function in $C^\infty(H)$ is annihilated by the left ideal $U(\mathfrak{G})\mathfrak{K}$, the action of u is the same as that of u'. Part (1) is then established.

As for part (2), let $X \in \mathfrak{B}$. Since $\exp(tX) \in B$, we may explicitly compute the action of X on $C^\infty(H)$ by applying the chain rule to compute the derivative (3.1). Let $E_{i,j}$ denote the elementary matrix with 1 at the (i,j)-th position and 0 elsewhere. Then a basis for $\mathfrak{B}$ is given by the elementary matrices $E_{i,j}$ with $1 \le i \le j \le n+1$. The action of these basis elements is computed as follows. For $1 \le i \le n+1$, one has

$$\exp(tE_{i,i}) = \mathrm{diag}(1,\dots,1,\mathrm{e}^t,1,\dots,1)$$

with the e^t in the i-th position, while for $1 \le i < j \le n+1$,

$$\exp(tE_{i,j}) = I + tE_{i,j}.$$

If $f(z) \in C^\infty(H)$ and $z = xy$ in Iwasawa coordinates, let us write $f(z) = f(x;y) = f(x_{i,j};y_i)$. Then for $1 \le i \le n+1$,

$$f\big(z\exp(tE_{i,i})\big) = f\big(x;y_1,\dots,y_{n-i},y_{n-i+1}\,\mathrm{e}^t,y_{n-i+2}\,\mathrm{e}^{-t},y_{n-i+3},\dots,y_n\big).$$

Similarly, for $1 \le i < j \le n+1$, one has $f(z\exp(tE_{i,j})) = f(x';y)$ with

$$\begin{cases} x'_{k,j} = x_{k,j} + ty_{n-j+2}\cdots y_{n-i+1}x_{k,i}, & k = 1,\dots,i; \\ x'_{\ell,m} = x_{\ell,m} \qquad m \ne j \quad \text{or} \quad \mathrm{m} = \mathrm{j},\ \ell > \mathrm{i}. \end{cases}$$

The proof of part (2) immediately follows.

LEMMA 2. — *Suppose $\delta \in \mathcal{D}$. Then*

$$\delta I_s(z) = \sum_{\sigma \in S(\delta)} c(s;\sigma,\delta)\, I_{s+\sigma}(z),$$

where the sum is over a finite set $S(\delta)$ of shift vectors $\sigma = (\sigma_1,\dots,\sigma_n)$ with each σ_i a non-negative integer, and $c(s;\sigma,\delta) \in \mathbb{C}[s_1,\dots,s_n]$.

Proof. — Since the action of δ is given by an element of $U(\mathfrak{G})$, this follows immediately from LEMMA 1.

REMARKS :

(1) LEMMA 2 defines the set $S(\delta)$ for $\delta \in \mathcal{D}$.

(2) Using the explicit description for $\mathcal{D}$ to be presented in section 5 below, one may sharpen LEMMA 2 by showing that if $\sigma \in S(\delta)$, then $\sigma_i = 0$ or 2.

(3) Let N_1 be the subgroup of N of matrices $(x_{i,j})$ with $x_{i,i+1} = 0$ for $1 \le i \le n$, and N_1 denote its Lie algebra. Since the function $I_s(z)$ is constant on N_1, one finds that giving a precise description of the shift vectors $S(\delta)$ for all $\delta \in \mathcal{D}$ is equivalent to giving a description of

$$U(\mathrm{G}_1)^K / \big(U(\mathrm{G}_1)^K \cap U(\mathrm{G}_1)\mathrm{K}\big)\big(U(\mathrm{G}_1)^K \cap U(\mathrm{G}_1)\mathrm{N}_1\big).$$

Let $L = \mathbb{C}(s_1, \ldots, s_n)$. Define $\mathcal{S}$ to be the infinite dimensional vector space over L freely generated by the set of all shift vectors $\sigma \in S(\delta)$ with $\delta \in \mathcal{D}$. By LEMMA 2 there is a natural map

$$M : \mathcal{D} \to \mathcal{S}$$

which assigns to each $\delta \in \mathcal{D}$ the sum of shift vectors $\bigoplus_{\sigma \in S(\delta)} c(s; \sigma, \delta) \cdot \sigma$. This map extends to a map, again written M, from $\mathcal{D} \otimes L$ to $\mathcal{S}$.

LEMMA 3. — *The map $M : \mathcal{D} \otimes L \to \mathcal{S}$ is injective.*

Proof. — If f, g are $N(\mathbb{Z})$-invariant functions on H such that $f(z)\,\overline{g(z)} \in L^1(N(\mathbb{Z}) \setminus H)$, let us write $\langle f, g \rangle$ for the integral

$$\langle f, g \rangle = \int_{N(\mathbb{Z})\setminus H} f(z)\,\overline{g(z)}\,\mathrm{d}^{\times} z,$$

where $\mathrm{d}^{\times} z$ denotes the (unique up to a constant) smooth G-invariant measure on H. If $\delta \in \mathcal{D}$, let δ^* be the adjoint of δ with respect to $\langle\ ,\ \rangle$. Extend the adjoint map anti-linearly to a map $^* : \mathcal{D} \otimes L \to \mathcal{D} \otimes \overline{L}$, where $\overline{L} = \mathbb{C}(\overline{s_1}, \ldots, \overline{s_n})$.

If LEMMA 3 is false, then there exists $\delta \in \mathcal{D} \otimes L$ such that $\delta(I_s) = 0$ for all $s \in \mathbb{C}^n$. Choose a basis $\{e_1, e_2, e_3, \ldots\}$ for $\mathcal{D}$ as a complex vector space, and set $\delta^* = \sum p_j(s) e_j$ with $p_j(s) \in \overline{L}$. It follows that for $\mathrm{Re}(s)$ sufficiently large,

$$\begin{aligned} 0 = \langle W_\nu, \delta I_s \rangle &= \langle \delta^* W_\nu, I_s \rangle \\ &= \Big(\sum p_j(s) \lambda_\nu(e_j)\Big) \langle W_\nu, I_s \rangle. \end{aligned}$$

Since for $\mathrm{Re}(s)$ sufficiently large $\langle W_\nu, I_s \rangle$ converges absolutely to a function which is not identically zero, this implies that

$$\sum p_j(s) \lambda_\nu(e_j) = 0.$$

But for $j \ge 1$ the functions $\lambda_\nu(e_j) \in \mathbb{C}[\nu_1, \ldots, \nu_n]$ are linearly independent over $\mathbb{C}$. Indeed, if not, one would have a non-zero $\delta \in \mathcal{D}$ such that

$\delta(Y_\nu) = 0$ for all ν. This would contradict the injectivity of the HARISH-CHANDRA homomorphism [HC]. Consequently, the $p_j(s)$ all vanish identically and $\delta^* = \delta = 0$.

The key to the proof of THEOREM 1 is that there are more differential operators than shift vectors which do not shift some particular coordinate. To state this precisely, given a shift vector $\sigma = (\sigma_1, \dots, \sigma_n)$, define its *weight* to be $w(\sigma) = \sum_{i=1}^{n} \sigma_i$. For a positive integer k, let $\mathcal{S}_k$ denote the finite dimensional subspace of $\mathcal{S}$ generated by all shift vectors $S(\delta)$, $\delta \in \mathcal{D}$, satisfying $w(S(\delta)) \leq k$. For $1 \leq j \leq n$, let $S(j)$ denote the subspace of S spanned by all shift vectors $\sigma = (\sigma_1, \dots, \sigma_n)$ with $\sigma_j = 0$. Also, recall that $\mathcal{D}$ is isomorphic to a polynomial ring in n variables. Choose a set of generators $\delta_1, \dots, \delta_n$ for $\mathcal{D}$ as an algebra over $\mathbb{C}$. Define $\mathcal{D}_k$ to be the subspace of $\mathcal{D}$ spanned by the monomials $\delta_1^{k_1} \cdots \delta_n^{k_n}$ whose total degree $\sum k_i$ is less than or equal to k.

LEMMA 4. — *Let $m = \max w(\sigma)$, where σ ranges over $\bigcup_{1 \leq i \leq n} S(\delta_i)$. Then for $k \gg n(n!) \cdot m^{n-1}$,*

$$\dim_{\mathbb{C}}(\mathcal{D}_k) > \dim_{\mathbb{C}}\big(\mathcal{S}_{mk} \cap \mathcal{S}(j)\big)$$

for each $j = 1, 2, \dots n$.

Proof. — If $n = 1$ the Lemma is trivial, so suppose that $n > 1$. Let $P(a,b)$ denote the number of partitions of a non-negative integer a into at most b parts, with the convention that $P(0,b) = 1$. Then

$$\dim_{\mathbb{C}}(\mathcal{D}_k) \geq \sum_{a=0}^{k} P(a,n).$$

On the other hand, each vector σ of weight a with $\sigma_j = 0$ gives a partition of a into $(n-1)$ pieces. Thus

$$\dim_{\mathbb{C}}\big(\mathcal{S}_{mk} \cap \mathcal{S}(j)\big) \leq (n-1)! \sum_{a=0}^{mk} P(a, n-1).$$

To compare these sums, recall the asymptotic formula of ERDÖS and LEHNER [EL] for the partition function

$$P(a,n) \sim \frac{a^{n-1}}{n!\,(n-1)!} \qquad (a \to \infty, n \text{ fixed}).$$

Combining this with the trivial bound $P(a,n) = \mathrm{O}(a^{n-1})$, we see that

$$\dim_{\mathbb{C}}(\mathcal{D}_k) \gtrsim \frac{k^n}{(n\,!)^2} \qquad \text{as} \quad k \to \infty,$$

while

$$\dim_{\mathbb{C}}\big(\mathcal{S}_{mk} \cap \mathcal{S}(j)\big) \lesssim \frac{(mk)^{n-1}}{(n-1)\,!} \qquad \text{as} \quad k \to \infty.$$

The result follows.

We now prove THEOREM 1. Observe first that $M(\mathcal{D}_k) \subset \mathcal{S}_{mk}$. It then follows from LEMMAS 3 and 4 and elementary linear algebra that for k sufficiently large there are differential operators $d_j \in \mathcal{D}_k \otimes L$, $1 \leq j \leq n$, such that $M(d_j)$ has no components in $S(j)$. In other words, for $1 \leq j \leq n$,

$$d_j(I_s) = \sum_{\sigma \in S(d_j)} c(s;\sigma,d_j) I_{s+\sigma}$$

where $c(s;\sigma,d_j) \in L^{\times}$ and all shift vectors $\sigma \in S(d_j)$ have $\sigma_j > 0$. Then for $s \in \mathbb{C}^n$ with $\mathrm{Re}(s) > N(\nu)$ and fixed j, $1 \leq j \leq n$,

$$\begin{aligned} \overline{\lambda}_\nu(d_j^*)\langle I_s, W_\nu\rangle &= \langle I_s, d_j^* W_\nu\rangle \\ &= \langle d_j I_s, W_\nu\rangle \\ &= \sum_{\sigma \in S(d_j)} c(s;\sigma,d_j)\langle I_{s+\sigma}, W_\nu\rangle. \end{aligned}$$

(In the above equation, λ_ν has been extended $\overline{L}$-linearly to a map on $\mathcal{D}\otimes\overline{L}$. Since d_j^* is a function of $\overline{s}$, $\overline{\lambda}_\nu(d_j^*)$ is thus a function of s.) But the right hand side is a meromorphic function in the larger region

$$\Big\{(s,\nu) \in \mathbb{C}^n \times \mathbb{C}^n \mid \mathrm{Re}(s_j) > N(\nu) - c_j, \mathrm{Re}(s_i) > N(\nu) \\ \text{for } 1 \leq \mathrm{i} \neq \mathrm{j} \leq \mathrm{n}\Big\},$$

where $c_j \geq 2$ is the coefficient of the lowest shift in s_j which occurs. Iterating, this gives the meromorphic continuation of $\widetilde{W_\nu}(s)$ to $\mathbb{C}^n \times \mathbb{C}^n$.

4. Proof of theorem 2

Let $s \mapsto s'$ denote the linear transformation of $\mathbb{C}^n$ such that $y_1^{s_1} \cdots y_n^{s_n} = Y_{s'}$. Suppose that the algebra $\mathcal{D}$ is generated by $\delta_1, \ldots, \delta_n$. Then the algebra $\mathcal{D} \otimes L$ is generated over L by

$$\delta_1 - \lambda_{s'}(\delta_1), \delta_2 - \lambda_{s'}(\delta_2), \ldots, \delta_n - \lambda_{s'}(\delta_n).$$

The set of shifts associated to $\delta_i - \lambda_{s'}(\delta_i)$ acting on I_s does not contain the trivial shift $(0, \ldots, 0)$. (Note that the set of shifts associated to $\delta_i - \lambda_{s'}(\delta_i)$ acting on I_w for some other $w \in \mathbb{C}^n$ may contain the trivial shift, but by LEMMA 2 will not contain any negative shifts.) The proof of THEOREM 1 thus shows that there are differential operators $d_i = d_i(s)$, given by polynomials in these generators with no degree zero term, such that for all $s \in \mathbb{C}^n$ with $\mathrm{Re}(s) > N(\bar{\nu})$

$$\overline{\lambda}_{\bar{\nu}}(d_i^*)\langle I_s, W_{\bar{\nu}}\rangle = \sum_{\sigma \in S(d_i)} c(s; \sigma, d_i)\langle I_{s+\sigma}, W_{\bar{\nu}}\rangle$$

where $\sigma \in S(d_i)$ implies $\sigma_i > 0$.

To prove THEOREM 2, one must apply successively the differential operators $d_i(s + \sigma^{(i)})$, $i = 1, \ldots, n$, with $\sigma^{(i)}$ ranging over a set of relevant shift vectors. First apply d_1. Then, inductively, we may assume that after j steps one obtains

$$q_j(s, \nu)\langle I_s, W_{\bar{\nu}}\rangle = \sum_{\sigma \in S^{(j)}} p_{j,\sigma}(s)\langle I_{s+\sigma}, W_{\bar{\nu}}\rangle \tag{4.1}$$

where the $S^{(j)}$ are finite sets of shift vectors such that $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $S^{(j)}$ implies that $\sigma_i > 0$ for $1 \leq i \leq j$; $p_{j,\sigma} \in \mathbb{C}[s]$; and $q_j(s, \nu)$ is a polynomial in $\mathbb{C}[s][\lambda_\nu(\delta_1), \ldots, \lambda_\nu(\delta_n)]$ with $q_j(s, s') = 0$. To this expression we apply the operator

$$\prod_{\sigma^{(j)} \in S^{(j)}} d_{j+1}(s + \sigma^{(j)}),$$

thereby obtaining an expression similar to (4.1) but with j replaced by $j + 1$. The set $S^{(j+1)}$ is the set of all shifts obtained by adding shifts in $S^{(j)}$ and $|S^{(j)}|$ shifts arising from the action of d_{j+1}. Continuing inductively, one obtains the desired equation (2.1), with a particular M. But then iterating (2.1), we see that we may make the degree in ν of $q(s, \nu)$ as large as desired. Since $\nu = s'$ if and only if $s_i = \sum_j b_{i,j}\nu_j$ for all i, $1 \leq i \leq n$, THEOREM 2 is proved.

Observe that there are additional zeroes of the polynomial $q(s, \nu)$ at $\nu = (s + \sigma^{(j)})'$ for $\sigma^{(j)} \in S^{(j)}$, $1 \leq j \leq n - 1$.

5. Explicit computations

In this section we shall illustrate THEOREMS 1 and 2 with some explicit examples when n is small. These examples are based on computer calculations done with the help of Oisin McGUINNESS. The calculations are based on an explicit set of generators for the algebra $\mathcal{D}$ of G-invariant differential operators on H, which we now describe.

Recall that the algebra $\mathcal{D}$ may be naturally identified with the quotient space

$$U(\mathrm{G}_1)^K/\big(U(\mathrm{G}_1)^K \cap U(\mathrm{G}_1)\mathrm{K}\big).$$

In the case at hand, the natural map to this space from the center $\mathcal{Z}$ of the universal enveloping algebra of G is surjective (see [H, chap. II]). The algebra $\mathcal{Z}$ may be described as follows.

Let m be a positive integer, and write S_m for the symmetric group on m elements. For a monomial $M = X_1 \cdots X_m \in U(\mathrm{G})$, with the $X_i \in \mathrm{G}$, let $s(M)$ denote its symmetrization $s(M) = 1/m!\sum_{\tau\in S_m} X_{\tau(1)}\cdots X_{\tau(m)}$. For $g \in G$, $0 \le r \le n$, put $\omega_r(g) = \operatorname{trace}\bigwedge^{\mathrm{r}+1}\mathrm{g}$, the sum of the principal $(r+1)\times(r+1)$ minors of g. Define $z_r \in \mathcal{Z}$ as $z_r = s(\omega_r(E_{i,j}))$; that is, apply ω_r to a formal matrix $g = (g_{i,j})$, substitute for each entry $g_{i,j}$ the elementary matrix $E_{i,j} \in \mathrm{G}$, and symmetrize. Then (see for example [JS2]) $\mathcal{Z} \cong \mathbb{C}[z_0,\dots,z_n]$. We shall write the differential operators on H associated to the z_r as δ_r. The action of δ_0 is easily seen to be trivial; we have

$$\mathcal{D} \cong \mathbb{C}\big[\delta_1,\dots,\delta_n\big].$$

The δ_r may be obtained as explicit differential operators as follows. Find the $z_r \in \mathcal{Z}$ as above; each z_r is a sum of products (in the universal enveloping algebra) of elementary matrices. To compute the action of z_r, observe first that if $u \in U(\mathrm{G})$, $1 \le i,j \le n+1$, then

$$uE_{i,j} \equiv uE_{j,i} \pmod{U(\mathrm{G})\mathrm{K}}; \tag{5.1}$$

this holds since $E_{i,j} - E_{j,i} \in \mathrm{K}$. Hence for any i and j, $uE_{i,j}$ and $uE_{j,i}$ give the same action on $C^\infty(H)$. Applying repeatedly the commutation relations

$$E_{p,q}E_{s,t} - E_{s,t}E_{p,q} = \delta_{q,s}E_{p,t} - \delta_{p,t}E_{s,q} \qquad \text{(Kronecker } \delta)$$

and using repeatedly (5.1) when $i > j$, one may obtain an element $b_r \in U(\mathrm{B})$ whose action on $C^\infty(H)$ is the same as z_r. (This is an alternative, algorithmic, proof of LEMMA 1 part (1).) Then writing b_r in terms of the basis $\{E_{i,j}, 1 \le i \le j \le n+1\}$ of B, and using the formulas given in LEMMA 1 part (2), one obtains the differential operators δ_r.

This procedure may be carried out in practice to obtain explicitly the ingredients used in the proofs of THEOREMS 1 and 2. For example, the shift vectors occurring in $S(\delta_r)$ are given by :

$$n=1: \quad S(\delta_1)=\{(0),(2)\}\cdot$$

$$n=2: \quad S(\delta_1)=S(\delta_2)=\{(0,0),(2,0),(0,2)\}\cdot$$

$$\begin{aligned} n=3: \quad & S(\delta_1)=S(\delta_2)=\{(0,0,0),(2,0,0),(0,2,0),(0,0,2)\}\cdot \\ & S(\delta_3)=\{(0,0,0),(2,0,0),(0,2,0),(0,0,2),(2,0,2)\}\cdot \end{aligned}$$

$$\begin{aligned} n=4: \quad S(\delta_1)=S(\delta_2)=\{&(0,0,0,0),(2,0,0,0),(0,2,0,0),\\ &(0,0,2,0),(0,0,0,2)\}\cdot \\ S(\delta_3)=S(\delta_4)=\{&(0,0,0,0),(2,0,0,0),(0,2,0,0),\\ &(0,0,2,0),(0,0,0,2),(2,0,0,2),\\ &(0,2,0,2),(2,0,2,0)\}\cdot \end{aligned}$$

One may also give the shift coefficients. For economy, we shall take all $m_i = 1$ in the definition of $\psi(x)$, list the coefficents each divided by $(2\pi i)^2$, and omit the formulae for the coefficients $c(s;(0),\delta_i)=\lambda_{s'}(\delta_i)$ of the (0) shift vector.

$$n=1: \quad c\big(s_1;(2),\delta_1\big)=-1.$$

$$\begin{aligned} n=2: \quad & c\big(s_1,s_2;(2,0),\delta_1\big)=c\big(s_1,s_2;(0,2),\delta_1\big)=-1. \\ & c\big(s_1,s_2;(2,0),\delta_2\big)=1-s_2, \\ & c\big(s_1,s_2;(0,2),\delta_2\big)=s_1-1. \end{aligned}$$

$$\begin{aligned} n=3: \quad & c\big(s_1,s_2,s_3;(2,0,0),\delta_1\big)=c\big(s_1,s_2,s_3;(0,2,0),\delta_1\big) \\ & \qquad\qquad\qquad\qquad\qquad\;\; =c\big(s_1,s_2,s_3;(0,0,2),\delta_1\big)=-1, \\ & c\big(s_1,s_2,s_3;(2,0,0),\delta_2\big)=2-s_2, \\ & c\big(s_1,s_2,s_3;(0,2,0),\delta_2\big)=s_1-s_3, \\ & c\big(s_1,s_2,s_3;(0,0,2),\delta_2\big)=s_2-2, \\ & c\big(s_1,s_2,s_3;(2,0,0),\delta_3\big)=-\tfrac{7}{6}+s_3(s_3-1)+\tfrac{3}{2}s_2-s_2s_3, \\ & c\big(s_1,s_2,s_3;(0,2,0),\delta_3\big)=\tfrac{11}{6}+s_1s_3-\tfrac{3}{2}(s_1+s_3), \\ & c\big(s_1,s_2,s_3;(0,0,2),\delta_3\big)=-\tfrac{7}{6}+s_1(s_1-1)+\tfrac{3}{2}s_2-s_2s_1, \\ & c\big(s_1,s_2,s_3;(2,0,2),\delta_3\big)=(2\pi i)^2. \end{aligned}$$

For $n=1$, the difference equation (2.1) is given by

$$\widetilde{W_\nu}(s_1)=\frac{4\pi^2}{\nu(1-\nu)-s_1(1-s_1)}\,\widetilde{W_\nu}(s_1+2),$$

while for $n = 2$, it takes the form

$$\widetilde{W_\nu}(s_1, s_2) = \frac{(4\pi^2)^2}{q(s,\nu)}(2 - s_1 - s_2)(s_1 + s_2) \cdot \widetilde{W_\nu}(s_1 + 2,\ s_2 + 2),$$

where

$$\begin{aligned} q(s,\nu) = \Big[&(s_1 - 1)\big(\lambda_\nu(\delta_1) - \lambda_{((2s_2-s_1)/3,(2s_1-s_2)/3)}(\delta_1)\big) \\ &\qquad + \big(\lambda_\nu(\delta_2) - \lambda_{((2s_2-s_1)/3,(2s_1-s_2)/3)}(\delta_2)\big)\Big] \\ \times \Big[&(1 - s_2)\big(\lambda_\nu(\delta_1) - \lambda_{((2s_2-s_1-2)/3,(2s_1-s_2+4)/3)}(\delta_1)\big) \\ &\qquad + \big(\lambda_\nu(\delta_2) - \lambda_{((2s_2-s_1-2)/3,(2s_1-s_2+4)/3)}(\delta_2)\big)\Big]. \end{aligned}$$

Here

$$\begin{aligned} \lambda_\nu(\delta_1) &= \nu_1 + \nu_2 + \nu_1\nu_2 - \nu_1^2 - \nu_2^2, \\ \lambda_\nu(\delta_2) &= (\nu_2 - \nu_1)\big(1 - \nu_1 - \nu_2 + \nu_1\nu_2\big). \end{aligned}$$

6. Generalization to other groups

In this section we shall indicate the generalizations of THEOREMS 1 and 2 to a connected reductive algebraic group G defined and quasi-split over $\mathbb{R}$.

Let $B = AN$ be a Borel subgroup defined over $\mathbb{R}$, with unipotent radical N and Levi factor A. Let T be the maximal $\mathbb{R}$-split torus contained in A, and let Z denote the center of G. Write $X(T)$ for the $\mathbb{Z}$-module of rational characters of T which are trivial on $Z \cap T$, and let $X(T)_\mathbb{C} = X(T) \otimes_\mathbb{Z} \mathbb{C}$. If the semisimple real rank of G is n, then $X(T)_\mathbb{C}$ has a canonical structure as a complex analytic manifold of dimension n.

Let $\overline{N}$ be the group opposed to N, and ψ be a linear character of $N(\mathbb{R})$ which is nondegenerate, i.e. such that ψ is nontrivial on $N(\mathbb{R}) \cap s^{-1}\bar{N}(\mathbb{R})s$ for each s in the normalizer of $A(\mathbb{R})$ in $G(\mathbb{R})$ but not in $A(\mathbb{R})$. Let K denote a maximal compact subgroup of $G(\mathbb{R})$. For each $\nu \in X(T)_\mathbb{C}$, there is a right K-invariant Whittaker function $W_\nu \in \mathrm{Ind}_\mathrm{N}^\mathrm{G}(\psi)$ with trivial central character (see [Sc]). Let ${}^0T$ denote the topologically-connected component of the identity of $T(\mathbb{R})$. Then the Mellin transform of W_ν is the function on $X(T)_\mathbb{C}$ defined by

$$\widetilde{W_\nu}(\alpha) = \int_{{}^0T/{}^0T \cap Z(\mathbb{R})} \alpha(a)\, W_\nu(a)\, \mathrm{d}^\times a, \tag{6.1}$$

where $\mathrm{d}^\times a$ denotes a choice of Haar measure on ${}^0T/{}^0T \cap Z(\mathbb{R})$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots of G with respect to B. Then one may show that for each $\nu \in X(T)_{\mathbb{C}}$ there exists a real number $N(\nu)$ such that the integral (6.1) converges if $\alpha = \sum_{i=1}^{n} \alpha_i \otimes s_i$ with $\mathrm{Re}(s_i) > N(\nu)$ for each i. We have :

THEOREM 3. — *Suppose that G is a connected reductive algebraic group defined and quasi-split over $\mathbb{R}$. Then the Mellin transform $\widetilde{W_\nu}(\alpha)$ has a meromorphic continuation to all $\alpha, \nu \in X(T)_{\mathbb{C}}$.*

Let $H = G(\mathbb{R})/KZ(\mathbb{R})$ be the symmetric space for $G(\mathbb{R})$. Let $\mathcal{D}$ denote the algebra of $G(\mathbb{R})$-invariant differential operators on H. This algebra is again isomorphic to a polynomial algebra in n variables. Let λ_ν be the associated Harish-Chandra homomorphism. Then regarding ν as a function on H which is one on the cosets of $N(\mathbb{R})$, $\delta(\nu) = \lambda_\nu(\delta) \cdot \nu$ for all $\delta \in \mathcal{D}$. As a function of ν, the map $\lambda_\nu(\delta)$ is a Weyl-invariant morphism from $X(T)_{\mathbb{C}}$ to $\mathbb{C}$. Identify $X(T)_{\mathbb{C}}$ with $\mathbb{C}^n$ via the map $\alpha = \sum_{i=1}^{n} \alpha_i \otimes s_i \mapsto s = (s_1, \ldots, s_n)$. We may then regard the $\lambda_\nu(\delta)$ as polynomials in n complex variables. Also, define a linear transformation $\nu \mapsto \nu'$ of $\mathbb{C}^n$ by the condition $\delta W_{\nu'} = \delta(\prod \alpha_i^{\nu_i})$ for all $\delta \in \mathcal{D}$.

THEOREM 4. — *Suppose that the hypotheses of Theorem 3 hold. Let $\delta_1, \ldots, \delta_n$ be a set of generators for $\mathcal{D}$ over $\mathbb{C}$. Then for all sufficiently large positive integers M, there exist*

(1) *a finite set of shift vectors $\Delta_M \subset (\mathbb{Z}^+)^n$,*
(2) *polynomials $p_\sigma(s) \in \mathbb{C}[s]$ for $\sigma \in \Delta_M$,*
(3) *a polynomial $q(s, \nu) \in \mathbb{C}[s][\lambda_\nu(\delta_1), \ldots, \lambda_\nu(\delta_n)]$ satisfying*

$$\deg_\nu(q(s,\nu)) \geq M, \quad q(s,s') = 0$$

such that

$$\widetilde{W_\nu}(s) = \frac{1}{q(s,\nu)} \sum_{\sigma \in \Delta_M} p_\sigma(s) \widetilde{W_\nu}(s+\sigma)$$

for all $s, \nu \in \mathbb{C}^n \cong X(T)_{\mathbb{C}}$.

The proofs of these results may be obtained by the same method as that used above for $\mathrm{GL}(n+1, \mathbb{R})$. Namely, by the Iwasawa decomposition, elements z of H may again be represented as $z = xy$, with $x \in N(\mathbb{R})$, $y \in {}^0T$. Then one studies the action of $\mathcal{D}$ on the function

$$I_s(z) = \psi(x)\, \alpha_1^{s_1}(y) \cdots \alpha_n^{s_n}(y).$$

Write the Lie algebras associated to $G(\mathbb{R})$, $B(\mathbb{R})$, $T(\mathbb{R})$, $N(\mathbb{R})$ and K as G, B, T, N and K respectively. Then LEMMA 1, part (1) holds if one replaces $U(\mathrm{B})$ by $U(\mathrm{T}) \otimes U(\mathrm{N})$, with a similar proof (note that

$G(\mathbb{R}) = KT(\mathbb{R})N(\mathbb{R})$ even though in general $B(\mathbb{R}) \neq T(\mathbb{R})N(\mathbb{R})$). One may deduce from this that LEMMA 2 holds in general.

The proof of LEMMA 3 in the present case is similar to that given for $\mathrm{GL}(n+1, \mathbb{R})$. One introduces an inner product on functions on $\Gamma \setminus H$, where Γ is a discrete subgroup of $N(\mathbb{R})$ such that ψ is trivial on Γ, and then follows the proof above. Key use is made here of Harish-Chandra's theorem [HC], which asserts that the characters $\lambda_\nu(d)$ defined above are linearly independent as d ranges over a basis of $\mathcal{D}$. Since the algebra of G-invariant differential operators on H is again isomorphic to a polynomial algebra in n variables, one may employ the counting argument above to establish LEMMA 4 in this generality. With the lemmas in place, the proofs of THEOREMS 3 and 4 then follow by the same arguments used to prove THEOREMS 1 and 2.

If ψ is a degenerate but nontrivial linear character, then a similar method may be used to establish the meromorphic continuation of a suitable Mellin transform of the Whittaker function associated to ψ, and to give finite difference equations for such a Mellin transform.

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