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CONTINUATION OF ANALYTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS UP TO CONVEX CONICAL SINGULARITIES

BY

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RÉSUMÉ. — On étudie le prolongement analytique des solutions d'équations aux dérivées partielles linéaires jusqu'au sommet d'un cône convexe fermé. On démontre en particulier un théorème général sur les singularités apparentes des solutions analytiques réelles d'équations aux dérivées partielles à coefficients variables.

ABSTRACT. — We study analytic continuation of the solutions of linear partial differential equations up to the vertex of a closed convex cone. In particular, we prove a general theorem on removable singularities of the real analytic solutions of partial differential equation with variable coefficients.

0. Introduction

In this paper, we study problems of local continuation of real analytic solutions of differential equations in a unified manner and, in particular, prove the following theorems :

THEOREM 0.1. — *Let K be a C^1-convex closed subset of a real analytic manifold M, having a conical singularity at x (cf. section 1.1). Let $P = P(x, D)$ be a second order differential operator with analytic coefficients defined in a neighborhood of x. Assume that P is of real principal type and is not elliptic. Then any real analytic solution to the equation $Pu = 0$ defined outside K is analytically continued up to x.*

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THEOREM 0.2. — *Let (K, x) be as in the above theorem. Let $\mathcal{M}$ be an elliptic system of linear differential equations in a neighborhood of x. Assume that the characteristic variety of $\mathcal{M}$ has codimension ≥ 2 in the fibre of x. Then any (real analytic) solution of $\mathcal{M}$ defined outside K is analytically continued up to x.*

We also give a similar theorem to 0.2 for a class of overdetermined systems of linear differential equations including systems which are not elliptic. In section 6, we prove a theorem on removable isolated singularities of the real analytic solutions of higher order partial differential equations with analytic coefficients, without any growth condition (cf. COROLLARY 6.2). This theorem extends the result of [G] for C^∞-solutions of differential equations with constant coefficients to those with variable (analytic) coefficients.

As for global continuation of solutions of linear differential equations with constant coefficients, there is much literature : [E], [M], [P], [Ko], [G], [Kn5], and the references cited there. In the case of overdetermined systems of differential equations with analytic coefficients, KAWAI [Kw] has given general results on local continuation of hyperfunction/analytic function solutions. On the other hand, in the case of single differential equations with variable coefficients, KANEKO (cf. survey report [Kn1], [Kn2]) proved, by a different method, a theorem on local continuation of real analytic solutions with thin singularity; however, his argument holds good only when the singularity of an analytic solution is contained in a hypersurface. The purpose of this paper is to deal with the local continuation problem of solutions of differential equations, with sharp-pointed bulky singularity. It should be noted that our argument is closely related to that of [Kn2, part 3].

1. Main Results

1.1. Notations. — Let M be a real analytic manifold of dimension $n \geq 2$, X a complex neighborhood of M, $\pi : T^*X \to X$ the cotangent bundle of X. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, $\mathcal{A}_M$ the sheaf of real analytic functions on M (i.e., $\mathcal{A}_M = \mathcal{O}_{X\,|\,M}$).

Let $P = P(x, D)$ be a differential operator with analytic coefficients on M. We denote by ${\mathcal{A}_M}^P$ the sheaf of real analytic solutions to the equation $Pu = 0$; i.e., ${\mathcal{A}_M}^P$ denotes the kernel of the sheaf homomorphism $P : \mathcal{A}_M \to \mathcal{A}_M$.

Let K be a closed subset of M. K is said to be C^α*-convex at* $x \in M$ $(1 \leq \alpha \leq \omega)$ if there exist a neighborhood U of x and an open C^α-

immersion $\phi : U \to \mathbb{R}^n$ such that $\phi(U \cap K)$ is convex in $\mathbb{R}^n$. For $x \in K$, the tangent cone $C_x(K)$ of K at x is defined in a system of local C^1-coordinates by

$$C_x(K) = \big\{v \in T_xM \mid \text{there are sequences } \{x_\nu\} \subset K \text{ and } \{a_\nu\} \subset \mathbb{R}_+ \\ \text{such that} \quad x_\nu \to x,\ a_\nu(x_\nu - x) \to v\big\}$$

(cf. e.g. [KS2]). K is said to have a *conical singularity at* x if $x \in K$ and $C_x(K)$ is a closed proper cone of T_xM or equivalently if, for a choice of local C^1-coordinates, there is a closed convex proper cone Γ of $\mathbb{R}^n$ with vertex at x containing K locally.

1.2. — Let $x \in M$. Let $P = P(x, D)$ be a second order differential operator with analytic coefficients defined in a neighborhood of x and let $f = f(z, \zeta)$ denote the principal symbol of P. We assume the following conditions :

(a.1) P is of real principal type :

$$\operatorname{Im} f_{|T^*_M X} = 0 \text{ and } \mathrm{d}_\zeta f \neq 0 \text{ on } f^{-1}(0) \cap \pi^{-1}(x) \setminus T^*_X X,$$

where d_ζ denotes the differential along the fibres of $\pi : T^*X \to X$.

(a.2) P is not elliptic at x :

$$f^{-1}(0) \cap \pi^{-1}(x) \cap T^*_M X \not\subset T^*_X X.$$

THEOREM 1. — *Let K be a closed C^1-convex subset of M, having a conical singularity at x. Let $P = P(x, D)$ be a second order differential operator with analytic coefficients, satisfying conditions* (a.1) *and* (a.2) *at x. Then*

$$\mathcal{A}_M{}^P \longrightarrow j_* j^{-1} \mathcal{A}_M{}^P \tag{1.1}$$

is an isomorphism at x, where j denotes the open embedding $M \backslash K \hookrightarrow M$; i.e., any real analytic solution to $Pu = 0$ defined on $M \backslash K$ can be continued up to x as a real analytic solution.

REMARK 1. — KANEKO [Kn4] conjectured the result of THEOREM 1 in the case of the wave equation $P = D_1^2 + \cdots + D_{n-1}^2 - D_n^2$ for $K = \big\{(x_1, x') \in \mathbb{R}^n \mid x_1 \leq -|x'|\big\}$ through the observations of loci of singularities of its solutions (cf. also [Kn1], [Kn2], and [Kn3]).

REMARK 2. — A generalization of THEOREM 1 to higher order differential equations in the case $K = \{x\}$ is given later in section 6.

1.3. — Let $\mathcal{D}_X$ denote the sheaf of rings of differential operators on X. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called a *system of linear differential equations*, and $\mathrm{Char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. In order to state a result for overdetermined systems, we first recall the notion of a virtual bicharacteristic manifold of $\mathcal{M}$.

Let $V = \mathrm{Char}(\mathcal{M})$; V^c denotes the complex conjugate of V with respect to T_M^*X. Let $p \in V \cap (T_M^*X \setminus M)$. Assume that V satisfies the following conditions at p :

(b.1) V is nonsingular at p.

(b.2) V and V^c intersect cleanly at p; i.e., $V \cap V^c$ is a smooth manifold and :

$$T_pV \cap T_pV^c = T_p(V \cap V^c).$$

(b.3) $V \cap V^c$ is regular; i.e., $\omega|_{V\cap V^c} \neq 0$, with ω being the fundamental 1-form on T^*X.

(b.4) The generalized Levi form of V has constant rank in a neighborhood of p.

Then one can find $\mathbb{C}^\times$-conic complex involutive manifolds V_1, V_2, V_3 so that :

(c.1) $V = V_1 \cap V_2 \cap V_3$;

(c.2) V_1 is regular and $V_1 = V_1^c$;

(c.3) V_2 and V_2^c intersect transversally and their intersection is regular and involutive;

(c.4) the generalized Levi form of V_3 is non degenerate

(cf. [SKK, chap. III, sect. 2.4]). The virtual bicharacteristic manifold Λ_p of $\mathcal{M}$ is by definition the real bicharacteristic manifold of $V_1 \cap V_2 \cap T_M^*X$ passing through p (cf. [SKK, chap. III, def. 2.2.7]; cf. also [Kw, p. 222]). In the theorem below, we assume :

(b.5) $\mathrm{d}\pi(T_p\Lambda_p) \neq \{0\}$,

where $T_p\Lambda_p$ denotes the tangent space of Λ_p at p, and

$$\mathrm{d}\pi : T_pT_M^*X \to T_{\pi(p)}M.$$

THEOREM 2. — *Let $x \in M$ and let K be a closed C^1-convex subset of M having a conical singularity at x. Let $\mathcal{M}$ be a system of differential equations. Assume that* $\mathrm{Char}(\mathcal{M}) \cap \pi^{-1}(x)$ *has codimension* ≥ 2 *in* $\pi^{-1}(x)$ *and that* $V = \mathrm{Char}(\mathcal{M})$ *satisfies conditions* (b.1)–(b.5) *at each point* p

of $V \cap (T^*_M X \setminus M) \cap \pi^{-1}(x)$. *Then*

$$(1.2) \qquad \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \longrightarrow j_* j^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)$$

is an isomorphism at x, *where* $j : M \setminus K \hookrightarrow M$.

COROLLARY. — *Let* (K, x) *be as in theorem 2. Let* $\mathcal{M}$ *be an elliptic system of differential equations and assume that* $\mathrm{Char}(\mathcal{M}) \cap \pi^{-1}(x)$ *has codimension* ≥ 2 *in* $\pi^{-1}(x)$. *Then any solution* u *of* $\mathcal{M}$ *defined outside* K *can be analytically continued up to* x.

REMARK 1. — For a system $\mathcal{M}$ of differential equations with constant coefficients, conditions (b.3), (b.4), (b.5) are fulfilled on conditions (b.1), (b.2) if $\mathcal{M} \neq {\mathcal{D}_X}^r$. In this case COROLLARY is a special case of theorem 4 of [P, chap. VIII, sect. 14].

REMARK 2. — Let $\mathcal{M}$ be an overdetermined system of differential equations with constant coefficients on $\mathbb{R}^n$. Then any (generalized function) solutions of $\mathcal{M}$ defined outside a compact convex set can be continued on the whole $\mathbb{R}^n$ (cf. [E], [Ko], [M], [P] and the references cited there). Apparently THEOREM 2 and COROLLARY are regarded as corresponding local results in the case of variable coefficients.

REMARK 3. — See [Kw], theorems 4 and 5, for general results on analytic continuation of the solutions of overdetermined systems of differential equations. Let $\mathcal{M}$ be a system of differential equations. Let K be a closed subset of M. Let $\{S_t; t \in \mathbb{R}\}$ be a family of C^ω-hypersurfaces of M, given by $S_t : \varphi(x) = t$, with $\mathrm{d}\varphi \neq 0$, such that :

- every S_t is noncharacteristic for $\mathcal{M}$ $(t \geq 0)$;
- $S_t \cap K$ is compact for every t and is empty for $t < 0$.

Let $\mathcal{M}_t$ denote the induced system of $\mathcal{M}$ on S_t. KAWAI [Kw] remarked that, if $\mathcal{M}_t$ $(t \geq 0)$ satisfies microlocal-geometrical conditions (5), (6), (7), and either (13′) or (14′) of [Kw], we have an isomorphism

$$\Gamma\big(M, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)\big) \xrightarrow{\sim} \Gamma\big(M \setminus K, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)\big).$$

In this result, imposing good microlocal conditions on $\mathcal{M}_t$ $(t \geq 0)$ is required in the argument. Note in this respect that, only the sharp-pointed singularity K being treated in THEOREM 2 and in COROLLARY, no conditions are imposed on the induced systems $\mathcal{M}_t$ of $\mathcal{M}$ for any sweeping family of hypersurfaces.

2. Preliminaries

2.0. General notations. — Let X be a topological space, $\mathcal{R}$ a sheaf of rings on X. We denote by $D^b(\mathcal{R})$ the derived category of the category of bounded complexes of left $\mathcal{R}$-modules; in particular, we set $D^b(X) = D^b(\mathbb{C}_X)$.

2.1. — Let M be a real analytic manifold, X a complex neighborhood of M.

Let Ω be an open subset of M, and let $T^*_\Omega X$ denote the microsupport of the sheaf $\mathbb{C}_\Omega$ (cf. [KS2], [KS3]). Let $\mathcal{C}_{\Omega|X} = \mu\mathrm{hom}(\mathbb{C}_\Omega, \mathcal{O}_X) \otimes or_M[n]$ (cf. [S1], [S2], [S3]); $\mathcal{C}_{\Omega|X}$ is an object of the derived category $D^b(\pi^{-1}\mathcal{D}_X)$ of $\pi^{-1}\mathcal{D}_X$-modules on T^*X. Recall that $\mathcal{C}_{\Omega|X|\pi^{-1}(\Omega)} \cong \mathcal{C}_{M|\pi^{-1}(\Omega)}$, where $\mathcal{C}_M$ denotes the sheaf of Sato's microfunctions on M.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $\mathcal{B}_M$ be the sheaf of Sato's hyperfunctions on M. Then we have the spectral map

$$(2.1) \qquad \alpha : \pi^{-1}j_*j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow H^0\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}),$$

where $j : \Omega \hookrightarrow M$. Following Schapira [S1], [S2], [S3], for a hyperfunction solution u of $\mathcal{M}$ on Ω, we set $\mathrm{SS}^{\mathcal{M}}_\Omega(u) = \mathrm{supp}(\alpha(u))$; this is a closed conic subset of $T^*_\Omega X \cap \mathrm{Char}(\mathcal{M})$. We simply denote $\mathrm{SS}^P_\Omega(u) = \mathrm{SS}^{\mathcal{D}_X/\mathcal{D}_X P}_\Omega(u)$ for a differential operator P. For $u \in \mathcal{B}_M(\Omega)$, we set $\mathrm{SS}_\Omega(u) = \mathrm{SS}^{\mathcal{D}_X}_\Omega(u)$; this set is contained in $T^*_M X$ and called the boundary analytic wavefront set of u. For $u \in \mathcal{B}_M(\Omega)$, with $\mathrm{SS}(u)$ denoting the singular spectrum of u over Ω, the equality $\mathrm{SS}(u) = \mathrm{SS}_\Omega(u) \cap \pi^{-1}(\Omega)$ holds.

Let $P = P(x, D)$ be an $r_1 \times r_0$ matrix of differential operators, and set $\mathcal{M} = \mathcal{D}^{r_0}_X / \mathcal{D}^{r_1}_X P$.

Proposition 2.1 [S1]. — *Assume that $M \setminus \Omega$ is C^ω-convex. Let $u = (u_i)_{i=1,\cdots,r_0}$ be a hyperfunction solution of $Pu = 0$ on Ω. Then*

$$\mathrm{SS}^{\mathcal{M}}_\Omega(u) \cap T^*_M X = \bigcup_{i=1}^{r_0} \mathrm{SS}_\Omega(u_i).$$

Let Ω' be an open subset of M contained in Ω. Then we have the morphism $\mathcal{C}_{\Omega|X} \to \mathcal{C}_{\Omega'|X}$. In particular, we have

$$\mathrm{SS}_{\Omega'}(u_{|\Omega'}) \subset \mathrm{SS}_\Omega(u)$$

for all $u \in \mathcal{B}_M(\Omega)$.

Refer to [S1], [S2], [S3] for the details of the theory of $\mathcal{C}_{\Omega|X}$ and boundary value problems. See also [U1] for further study.

2.2. — Let $x \in M$. Let $\varphi_1, \cdots, \varphi_r$ be real-valued C^ω-functions on M with $\varphi_1(x) = \cdots = \varphi_r(x) = 0$ and $\mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_r(x) \neq 0$. Set

$$\Omega_\alpha^\pm = \{\pm\varphi_\alpha > 0\} \quad \text{for} \quad 1 \leq \alpha \leq r \quad \text{and} \quad \Omega^+ = \Omega_1^+ \cup \cdots \cup \Omega_r^+.$$

PROPOSITION 2.2.

(1) *Let u be a hyperfunction defined on Ω^+. Then we have*

$$\mathrm{SS}_{\Omega^+}(u) \cap \pi^{-1}(x) = \bigcup_{\alpha=1}^{r} \mathrm{SS}_{\Omega_\alpha^+}(u) \cap \pi^{-1}(x).$$

In particular, we have the inequality

$$\mathrm{SS}_{\Omega'}(u) \cap \pi^{-1}(x) \subset \bigcup_{\alpha=1}^{r} \mathrm{SS}_{\Omega_\alpha^+}(u) \cap \pi^{-1}(x)$$

for any open subset $\Omega' \subset \Omega^+ = \Omega_1^+$.

(2) *Set $Z = \{\varphi_1 = \cdots = \varphi_r = 0\}$. Then the equality*

$$\mathrm{SS}_{M\setminus Z}(u) \cap \pi^{-1}(x) = \bigcup_{1\leq\alpha\leq r,\pm} \mathrm{SS}_{\Omega_\alpha^\pm}(u) \cap \pi^{-1}(x)$$

holds for any hyperfunction u defined on $M \setminus Z$.

Sketch of proof. — Let $Z_\alpha^- = \{\varphi_\alpha \leq 0\}$ for $1 \leq \alpha \leq r$ and $Z^- = Z_1^- \cap \cdots \cap Z_r^-$. Then claim (1) is equivalent to the equality

$$\mathcal{C}_{Z^-|X,p} = \bigcap_{1\leq\alpha\leq r} \mathcal{C}_{Z_\alpha^-|X,p}$$

in $\mathcal{C}_{M,p}$, for $p \in T_M^*X$ (cf. section 3.1 for the definition of the sheaves $\mathcal{C}_{Z_\alpha^-|X}$ and $\mathcal{C}_{Z^-|X}$). By using a quantized complex contact transformation (cf. the proof of [S1, prop. 3.1], this equality is reduced to the following lemma on holomorphic continuation. Refer to the proof of [U1, prop. 5.1].

In the same way, claim (2) is also reduced to part (2) of the lemma :

LEMMA. — *Let U be an open neighborhood of $0 \in \mathbb{C}^n$. Let*

$$D_\alpha^\pm = \Big\{z \in \mathbb{C}^n \mid y_n > (\pm y_\alpha)_+^2 + \sum_{\substack{k\neq\alpha \\ 1\leq k<n}} y_k^2\Big\}, \qquad 1 \leq \alpha \leq r,$$

$$\widetilde{D}^+ = \Big\{z \in \mathbb{C}^n \mid y_n > \sum_{1\leq j\leq r} (y_j)_+^2 + \sum_{r<k<n} y_k^2\Big\},$$

where $y = \operatorname{Im} z$ and $(y_\alpha)_+ = \max\{y_\alpha, 0\}$.

(1) *There is a neighborhood U' of $0 \in \mathbb{C}^n$ such that every holomorphic function defined on $U \cap (D_1^+ \cup \cdots \cup D_r^+)$ is analytically continued on $U' \cap \widetilde{D}^+$.*

(2) *Set $\widetilde{D} = \Big\{z \in \mathbb{C}^n \mid y_n > \sum_{r<k<n} y_k^2\Big\}$. Then there is a neighborhood U' of $0 \in \mathbb{C}^n$ such that every holomorphic function defined on*

$$U \cap (D_1^+ \cup \cdots \cup D_r^+ \cup D_1^- \cup \cdots \cup D_r^-)$$

is analytically continued on $U' \cap \widetilde{D}$.

The first part of the lemma follows from a local version of the Bochner's tube theorem (cf. [KKK, prop. 3.8.6]). The second claim follows immediately from the first one, because $\widetilde{D} = \cup_\varepsilon \widetilde{D}^\varepsilon$, where

$$\widetilde{D}^\varepsilon = \Big\{z \in \mathbb{C}^n \mid y_n > \sum_{1\le i\le r} (\varepsilon_i y_i)_+^2 + \sum_{r<k<n} y_k^2\Big\}$$

for $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_r) \in \{+1, -1\}^r$. $\square$

2.3. — Let $\Omega = \{\varphi > 0\}$ be an open subset of M with C^ω-boundary N, with $d\varphi \neq 0$, Y the complexification of N in X. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and assume that Y is non-characteristic for $\mathcal{M}$ (i.e., $T_Y^*X \cap \operatorname{Char}(\mathcal{M}) \subset T_X^*X$). The induced system of $\mathcal{M}$ on Y is denoted by $\mathcal{M}_Y$. Then we have the isomorphism of Cauchy-Kowalewski-Kashiwara :

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)_{|N} \cong \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{A}_N). \tag{2.2}$$

Let Ω, $\mathcal{M}$ be as above. We have the boundary value map (cf. [S1], [S3])

$$\gamma : j_* j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_{|N} \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N), \tag{2.3}$$

where $j : \Omega \hookrightarrow M$. Let ρ and ϖ be the natural maps associated with $Y \hookrightarrow X$:

$$T^*Y \xleftarrow{\rho} Y \times_X T^*X \xrightarrow{\varpi} T^*X.$$

Proposition 2.3 [S1, prop. 4.1]. — *Let u be a hyperfunction solution of $\mathcal{M}$ defined on Ω. Let $\operatorname{SS}(\gamma(u))$ denote the singular spectrum of the trace $\gamma(u)$ on N. Then :*

$$\operatorname{SS}\big(\gamma(u)\big) = \rho\varpi^{-1}\operatorname{SS}_\Omega^{\mathcal{M}}(u).$$

2.4. — Let Ω, $\mathcal{M}$ be as in 2.3. Let $p \in N \times_M (T_M^*X \setminus M)$. Assume that $V = \operatorname{Char}(\mathcal{M})$ satisfies conditions (b.1)–(b.4) of 1.3 at p. Let Λ_p denote the virtual bicharacteristic manifold of $\mathcal{M}$ passing through p, and set $\Lambda_p^+ = \Lambda_p \cap \pi^{-1}(\Omega)$. Then we have :

PROPOSITION 2.4. — *Suppose that Λ_p is not tangent to the boundary : $T_p\Lambda_p \not\subset \mathrm{d}\pi^{-1}T_{\pi(p)}N$, with $\mathrm{d}\pi : T_pT_M^*X \to T_{\pi(p)}M$. Let u be a hyperfunction solution of $\mathcal{M}$ defined on Ω. If Λ_p^+ is not contained in $\mathrm{SS}_\Omega^{\mathcal{M}}(u)$ in a neighborhood of p, then $p \notin \mathrm{SS}_\Omega^{\mathcal{M}}(u)$.*

Proof. — We can find complex manifolds V_1, V_2, V_3 which satisfy conditions (c.1), (c.2), (c.3), (c.4) of 1.3. Let $\mathcal{E}_X$ (resp. $\mathcal{E}_X^\infty$) denote the sheaf of rings of microdifferential operators of finite order (resp. of infinite order). For $i = 1,2,3$, let $\mathcal{E}_X/\mathcal{J}_i$ be a coherent $\mathcal{E}_X$-module that has simple characteristics along V_i; set $\mathcal{M}_0 = \mathcal{E}_X/(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3)$. Then $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}$ is microlocally isomorphic to a direct summand of the direct sum of a finite number of copies of $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_0$ (cf. [SKK, chap. III, thm 2.4.1]). Thus we have an injective sheaf-homomorphism (cf. (3.4))

$$H^0\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) \longrightarrow \sum H^0\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}_0, \mathcal{C}_{\Omega|X}),$$

where $\sum$ denotes the direct sum of a finite number of copies. For this reason, we may assume from the beginning that $\mathcal{M} = \mathcal{M}_0$.

Let $P \in \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$. Consider the sheaf-homomorphism induced from $\mathcal{E}_X/\mathcal{E}_XP \to \mathcal{M}_0$:

$$H^0\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}_0, \mathcal{C}_{\Omega|X}) \longrightarrow H^0\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_XP, \mathcal{C}_{\Omega|X}).$$

Since $H^i(\mathcal{C}_{\Omega|X})_{|T_M^*X} = 0$ for $i < 0$, this homomorphism is injective on T_M^*X ; therefore $p \in \mathrm{SS}_\Omega^{\mathcal{M}_0}(u)$ implies $p \in \mathrm{SS}_\Omega^P(u)$.

The virtual bicharacteristic manifold Λ of $\mathcal{M}$ passing through p is nothing but the real bicharacteristic manifold of $V_1 \cap V_2 \cap T_M^*X$. For $i = 1,2$, let Λ_i denote the real bicharacteristic manifold of $V_i \cap T_M^*X$ passing through p. Then $\Lambda_1 \cup \Lambda_2 \subset \Lambda$ and $T_p\Lambda_1 + T_p\Lambda_2 = T_p\Lambda$.

Case 1. — $T_p\Lambda_1 \not\subset \mathrm{d}\pi^{-1}T_{\pi(p)}N$. We can find a microdifferential operator $P \in \mathcal{J}_1$ of which the principal symbol f satisfies the following :

$$\operatorname{Im} f_{|T_M^*X} \equiv 0, \quad \mathrm{d}\varphi\big(\pi(p)\big) \cdot \mathrm{d}\pi\big(H_f(p)\big) > 0,$$

where H_f denotes the Hamiltonian vector field of f on T_M^*X. Let b_f^+ be the positive half integral curve of H_f issued from p. Then $b_f^+ \subset \Lambda_1 \cap \{\varphi > 0\}$, and it follows from [S2, thm 3.1] that, if $p \in \mathrm{SS}_\Omega^P(u)$, then $b_f^+ \subset \mathrm{SS}_\Omega^P(u)$.

Case 2. — $T_p\Lambda_2 \not\subset \mathrm{d}\pi^{-1}T_{\pi(p)}N$. Then we can find $P \in \mathcal{J}_2$ and an involutive conic C^ω-manifold W of T_M^*X with $p \in W \subset V_2$ so that

$d(\varphi \circ \pi)(p)$ is non-microcharacteristic for P on the complexification $W^{\mathbb{C}}$ of W (cf. [S4, sect. 5]). Let b_W be the real bicharacteristic manifold of W passing through p; then $b_W \subset \Lambda_2$. Set $b_W^+ = b_W \cap \{\varphi > 0\}$. It follows from [S4, thm 5.1] (cf. also [U2, sect. 1]) that, if $p \in \mathrm{SS}_\Omega^P(u)$, then $b_W^+ \subset \mathrm{SS}_\Omega^P(u)$ in a neighborhood of p.

Since $\Lambda_p^+ \cap \mathrm{SS}(u)$ is open and closed in Λ_p^+ [SSK, chap. III, thm 2.2.99], Λ_p^+ is contained in $\mathrm{SS}(u)$ in a neighborhood of p if $p \in \mathrm{SS}_\Omega^{\mathcal{M}}(u)$. This completes the proof. $\square$

3. A Basic Lemma

3.1. — Let M, X be as in section 1.1. Let K be a closed C^ω-convex subset of M, and let T_K^*X denote the microsupport of the sheaf $\mathbb{C}_K$. Let $\mathcal{C}_{K|X}$ be the sheaf of microfunctions along T_K^*X; i.e. $\mathcal{C}_{K|X} = \mathcal{H}^n(\mu\mathrm{hom}(\mathbb{C}_K, \mathcal{O}_X)) \otimes or_M$, with or_M being the orientation sheaf of M (cf. [S1], [S3]).

The following fact is a part of Proposition 11.4.4 of [KS3]. Let $\mathcal{E}_X$ denote the sheaf of rings of microdifferential operators on X. Then we have :

LEMMA 3.1. — *$\mathcal{C}_{K|X}$ is an $\mathcal{E}_X$-module.*

Proof. — Let F be an object of $D^b(X)$; then we have a canonical morphism [KS3, cor. 4.4.10]

$$\mu\mathrm{hom}(F, \mathcal{O}_X) \otimes_{\mathbb{C}} \mu\mathrm{hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mu\mathrm{hom}(F, \mathcal{O}_X).$$

This induces the morphism :

$$\mu\mathrm{hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mathbb{R}\mathcal{H}om_{\mathbb{C}}\big(\mu\mathrm{hom}(F, \mathcal{O}_X), \mu\mathrm{hom}(F, \mathcal{O}_X)\big).$$

Note that this morphism ic compatible with

$$\mu\mathrm{hom}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mu\mathrm{hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mu\mathrm{hom}(\mathcal{O}_X, \mathcal{O}_X)$$

because the diagram composed of

$$\begin{aligned} &\mu\mathrm{hom}(F_1, F_2) \otimes \mu\mathrm{hom}(F_2, F_3) \otimes \mu\mathrm{hom}(F_3, F_4) \\ &\qquad \to \mu\mathrm{hom}(F_1, F_3) \otimes \mu\mathrm{hom}(F_3, F_4) \\ &\qquad\qquad \to \mu\mathrm{hom}(F_1, F_4) \end{aligned}$$

and

$$\begin{aligned}\mu\text{hom}(F_1,F_2)\otimes\mu\text{hom}(F_2,F_3)\otimes\mu\text{hom}(F_3,F_4)\\ \to\mu\text{hom}(F_1,F_2)\otimes\mu\text{hom}(F_2,F_4)\\ \to\mu\text{hom}(F_1,F_4)\end{aligned}$$

with $F_1=F$, $F_2=F_3=F_4=\mathcal{O}_X$, is commutative. Hence

$$H^0\big(\mu\text{hom}(\mathcal{O}_X,\mathcal{O}_X)\big)$$

endowed with a natural ring structure acts on the object $\mu\text{hom}(\mathcal{O}_X,\mathcal{O}_X)$ in $D^b(T^*X)$.

On the other hand, we have a morphism $\mathcal{E}_X\longrightarrow\mu\text{hom}(\mathcal{O}_X,\mathcal{O}_X)$ (cf. [KS2, sect. 10.6]). The composition yields a ring homomorphism

$$\begin{aligned}(3.1)\qquad \mathcal{E}_X(U)&\to H^0\mathbb{R}\text{Hom}_{\mathbb{C}}\big(\mu\text{hom}(F,\mathcal{O}_X)_{|U},\mu\text{hom}(F,\mathcal{O}_X)_{|U}\big)\\ &=\text{Hom}_{D^b(\mathbb{C}_U)}\big(\mu\text{hom}(F,\mathcal{O}_X)_{|U},\mu\text{hom}(F,\mathcal{O}_X)_{|U}\big)\end{aligned}$$

on every open subset U of T^*X. Thus $\mathcal{E}_X$ acts on the sheaf

$$H^i\big(\mu\text{hom}(F,\mathcal{O}_X)\big)$$

for every $i\in\mathbb{Z}$. $\square$

3.2. — Suppose that K has a conical singularity at x. Let C^* be the polar set of $C_x(K)$ in T^*_xX; then $C^*=T^*_KX\cap\pi^{-1}(x)$. Let p be an interior point of the cone C^* in $\pi^{-1}(x)$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined in a neighborhood of p. Set $V'=\text{supp}(\mathcal{M})\cap\pi^{-1}(x)$. Then we have the basic lemma :

LEMMA 3.2. — *Assume $V'\neq\pi^{-1}(x)$ at p and let $V'=V_1\cup\cdots\cup V_r$ be the irreducible decomposition of (V',p).*

1) *Let u be a section of $\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{K|X})$ defined in a neighborhood of p; then* supp(u) *is the union of some of the hypersurface components V_i (i.e.* $\dim_{\mathbb{C}}(V_i,p)=n-1$*) in a neighborhood of p. In particular, we have :*

2) *Suppose that (V',p) is an irreducible hypersurface of $(\pi^{-1}(x),p)$. Let u be a section of $\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{K|X})$ in a neighborhood of p; then* supp$(u)=V'$ *if $u\not\equiv 0$.*

3) *If* $\dim_{\mathbb{C}}(V',p)\leq n-2$, *then $\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{K|X})\equiv 0$ in a neighborhood of p.*

Proof. — Since K is convex and contained in a closed proper cone Γ with vertex at x for a choice of local coordinates, by the definition of microsupports (cf. [KS2, sect. 3.1]), we have $T^*_K X = T^*_{\{x\}} X$ in a neighborhood of p. Hence the morphism $\mathbb{C}_K \to \mathbb{C}_{\{x\}}$ is an isomorphism in $D^b(X;p)$ (cf. the proof of [KS2, prop. 6.2.1]) ; therefore $\mathcal{C}_{K|X}$ and $\mathcal{C}_{\{x\}|X}$ are isomorphic as $\mathcal{E}_X$-modules in a neighborhood of p. Thus we may assume from the beginning that $K = \{x\}$. Let $z = (z_1, \cdots, z_n)$ be a local coordinate system of (X, x) with $x = (0, \cdots, 0)$, $(z; \zeta)$ the associated coordinates of T^*X.

First we prove claim (2) in the case where V' is a smooth hypersurface defined by $f(\zeta) = 0$ for a homogeneous holomorphic function f. We may assume that $\partial_{\zeta_1} f(p) \neq 0$. Set $Y = \{z_1 = 0\} \subset X$ and $\rho : T^*_x X \to T^*_x Y$, $(\zeta_1, \zeta') \mapsto \zeta'$. In this case $\rho_{|V'}$ is a holomorphic isomorphism from (V', p) to $(T^*_x Y, \rho(p))$. Since Y is non characteristic for $\mathcal{M}$ in a neighborhood Δ_p $(\subset \pi^{-1}(x))$ of p, the division theorem of microdifferential operators for microfunctions with holomorphic parameters (cf. [KK], [KS1]) gives the isomorphism

$$(3.2)\quad \alpha_p : (\rho_{|\Delta_p})_* \big(\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\{x\}|X})_{|\Delta_p}\big) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{E}_Y}\big(\mathcal{M}_Y^{\Delta_p}, \mathcal{C}_{\{x\}_{|Y}}\big),$$

where $\mathcal{M}_Y^{\Delta_p}$ denotes the induced system of microdifferential equations of $\mathcal{M}_{|\Delta_p}$ on Y, defined on $\rho(\Delta_p)$. Thus one can identify

$$\big(V' \cap \Delta_p, \mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\{x\}|X})_{|V' \cap \Delta_p}\big) \cong \big(\rho(\Delta_p), \mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y^{\Delta_p}, \mathcal{C}_{\{x\}_{|Y}})\big)$$

through (ρ, α_p). Since $\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y^{\Delta_p}, \mathcal{C}_{\{x\}_{|Y}})$ is a subsheaf of the direct sum of finite number of copies of $\mathcal{C}_{\{x\}_{|Y}}$ in a neighborhood of $\rho(p)$, the conclusion follows from the fact that every section of $\mathcal{C}_{\{x\}_{|Y}}$ has the variables $\zeta' = (\zeta_2, \cdots, \zeta_n)$ as holomorphic parameters (cf. [SKK], [KK] and [KS1]).

Next consider the general case and let $V' = V_1 \cup \cdots \cup V_r$ be the irreducible decomposition of (V', p). Let W_i be an irreducible hypersurface containing V_i for $1 \leq i \leq r$ and set $W = W_1 \cup \cdots \cup W_r$. We may assume that Y, $z_1 = 0$, is non characteristic for W in a neighborhood of p; i.e., $\rho^{-1}\rho(p) \cap W = \{p\}$ in a neighborhood of p, with $\rho : T^*_x X \to T^*_x Y$, $(\zeta_1, \zeta') \mapsto \zeta'$. Then there is a hypersurface Σ of $(T^*_x Y, \rho(p))$ such that $W_i \setminus \rho^{-1}(\Sigma)$ is smooth, connected, and dense in W_i for $1 \leq i \leq r$. We may also supose that $W \setminus \rho^{-1}(\Sigma)$ is smooth.

Now let u be a section of $\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\{x\}|X})$ defined in a neighborhood of p; then $\mathrm{supp}(u) \subset W$. Suppose that $W_1 \not\subset \mathrm{supp}(u)$ (as germs at p), and we shall prove that $\mathrm{supp}(u) \subset W_2 \cup \cdots \cup W_r$. Since there is a point

$q \in W_1 \setminus \rho^{-1}(\Sigma)$ where $u = 0$, it follows from the result of the smooth hypersurface case that $u_{|W_1 \setminus \rho^{-1}(\Sigma)} \equiv 0$. Let

$$p_1 \in W_1 \cap \rho^{-1}(\Sigma) \setminus (W_2 \cup \cdots \cup W_r)\,;$$

then, $Y \hookrightarrow X$ being non characteristic for $\mathcal{M}$ at p_1, the division theorem of microdifferential operators again gives the isomorphism α_{p_1} (cf. (3.2)) for a small neighborhood Δ_{p_1} of p_1. Since $\alpha_{p_1}(u_{|\Delta_{p_1}}) \equiv 0$ outside Σ, it follows from uniqueness of holomorphic continuation of sections of $\mathcal{C}_{\{x\}_{|Y}}$ that $\alpha_{p_1}(u_{|\Delta_{p_1}}) \equiv 0$ on the whole of $\rho(\Delta_{p_1})$; therefore $u_{|\Delta_{p_1}} \equiv 0$, i.e., $p_1 \notin \operatorname{supp}(u)$. Since p_1 is taken arbitrarily, we have $\operatorname{supp}(u) \subset W_2 \cup \cdots \cup W_r$. Repeating this argument, we find that $\operatorname{supp}(u)$ is the union of some W_i's. This proves claim (1). $\square$

3.3. — Let $\Omega = M \setminus K$. Then $\mathcal{C}_{\Omega|X}$ is an object of $D^b(\mathcal{E}_X)$, and there is a canonical morphism (cf. [S1], [S3])

$$\delta : \mathcal{C}_{\Omega|X} \longrightarrow \mathcal{C}_{K|X}[1] \tag{3.3}$$

in $D^b(\mathcal{E}_X)$. Outside $T^*_M X$, the morphism δ is an isomorphism.

Remark. — By the proof of Lemma 3.1, $\mathcal{C}_{K|X}$ is in fact an $\mathcal{E}^\infty_X$-module. Hence $\mathcal{C}_{\Omega|X}$, with $\Omega = M \setminus K$, is an object of $D^b(\mathcal{E}^\infty_X)$. By this fact, for any $\mathcal{E}_X$-module $\mathcal{L}$, we have

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{L}, \mathcal{C}_{\Omega|X}) \cong \mathbb{R}\mathcal{H}om_{\mathcal{E}^\infty_X}(\mathcal{E}^\infty_X \otimes_{\mathcal{E}_X} \mathcal{L}, \mathcal{C}_{\Omega|X}), \tag{3.4}$$

since $\mathcal{E}^\infty_X$ is flat over $\mathcal{E}_X$.

4. Proof of theorem 1

4.1. — Since, for a choice of C^1-local coordinates, K is convex and contained in a convex proper cone Γ with vertex at x, the injectivity of (1.1) at x follows from uniqueness of analytic continuation.

We shall prove the surjectivity of (1.1) at x. In virtue of uniqueness of continuation, we may suppose

$$M = \mathbb{R}^n, \quad K = \{(x_1, x') \in M \mid x_1 \leq -\epsilon_0 |x'|\}, \quad x = (0, \cdots, 0),$$

with $\epsilon_0 > 0$, where $|x'|$ denotes the Euclidean norm. Set $\Omega = M \setminus K$, $M_+ = \{x_1 > 0\}$, $N = \{x_1 = 0\}$, $x_0 = (0, \ldots, 0)$. We may assume that N is non characteristic for P, because P is non degenerate at x.

Let $K_\epsilon = \{(x_1, x') \in M \mid x_1 \le -\epsilon|x'|\}$, $\Omega_\epsilon = M \setminus K_\epsilon$ $(0 < \epsilon < \epsilon_0)$.

Let $z = (z_1, z')$ be the complexification of the coordinates (x_1, x') of $\mathbb{R}^n$ and let (z, ζ) be the associated coordinates of $T^*\mathbb{C}^n$.

4.2. — Let $P = P(x, D)$ be a second order differential operator satisfying conditions (a.1) and (a.2). Denote by $f = f(z; \zeta)$ the principal symbol of P.

Let u be a real analytic solution to the equation $Pu = 0$ defined on Ω. We shall first prove

$$\mathrm{SS}^P_{\Omega_\epsilon}(u_{|\Omega_\epsilon}) \cap T^*_M X \cap \pi^{-1}(x_0) \subset T^*_X X$$

for some ϵ $(0 < \epsilon < \epsilon_0)$.

Let $p \in T^*_M X \cap \pi^{-1}(x_0) \setminus T^*_X X$ with $f(p) = 0$. Let $b(p)$ denote the bicharacteristic curve of P (i.e., the integral curve of the Hamiltonian vector field H_f on $T^*_M X$) passing through p.

Let us take n linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of $\mathbb{R}^n$ satisfying the following conditions for $0 < \epsilon \ll \epsilon_0$:

- $\mathbf{a}_k = (a_{k1}, a'_k)$ with $a_{k1} > |a'_k|/\epsilon_0$, for $1 \le k \le n$;
- $\{(\xi_1, \xi') \in \mathbb{R}^n \mid \xi_1 > |\xi'|/\epsilon\} \subset \mathbb{R}_+\mathbf{a}_1 + \cdots + \mathbb{R}_+\mathbf{a}_n$;
- $\langle \mathbf{a}_k, \mathrm{d}_\zeta f(p) \rangle \ne 0$ for $1 \le k \le n$;

where, for $\mathbf{a} \in \mathbb{R}^n$, $\langle \mathbf{a}, x \rangle$ denotes the linear function defined by the inner product. Set

$$\Omega^+_k = \{x \in M \mid \langle \mathbf{a}_k, x \rangle > 0\}, \qquad 1 \le k \le n.$$

Then $\Omega_\epsilon \subset \bigcup_{1 \le k \le n} \Omega^+_k \subset \Omega$ and $b(p)$ is transversal to each boundary of Ω^+_k. Since the micro-analyticity of u propagates from Ω^+_k up to p along the curve $b(p)$, we have $p \notin \mathrm{SS}^P_{\Omega^+_k}(u)$ for $1 \le k \le n$. Hence it follows from PROPOSITIONS 2.1 and 2.2 that $p \notin \mathrm{SS}^P_{\Omega_\epsilon}(u_{|\Omega_\varepsilon})$.

4.3. — Letting

$$\begin{aligned} L_+ &= f^{-1}(0) \cap T^*_{M_+} X \cap \pi^{-1}(x_0) \\ &= \{(\zeta_1, \zeta') \in \mathbb{C}^n \mid f(x_0\,;\ \zeta_1, \zeta') = 0,\ \operatorname{Re}\zeta_1 \le 0,\ \operatorname{Re}\zeta' = 0\}, \end{aligned}$$

we prove that the set $\mathrm{SS}^P_{\Omega_\epsilon}(u_{|\Omega_\epsilon})$, for $\epsilon > 0$ as small as in 4.2, does not contain any point of $L_+ \setminus T^*_X X$.

We may assume $L_+ \not\subset T^*_M X$. Since $f(x_0;\zeta_1,\zeta')$ is a polynomial in ζ_1 of degree 2 with real coefficients, it follows from (a.2) that the closure of any connected component L'_+ of $L_+ \setminus T^*_M X$ has nonvoid intersection with $T^*_M X$. Let $p \in \overline{L'_+} \cap T^*_M X$; then, by 4.2, we have $\Delta_p \cap \mathrm{SS}^P_{\Omega_\epsilon}(u) = \emptyset$ for some neighborhood Δ_p of p in $\pi^{-1}(x_0)$. Consider the sheaf homomorphisms (cf. (2.1) and (3.3))

$$\alpha_\epsilon : \pi^{-1} j_{\epsilon *} j_\epsilon^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P,\, \mathcal{B}_M) \longrightarrow H^0 \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P,\, \mathcal{C}_{\Omega_\epsilon|X}),$$

$$\delta : H^0 \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P,\, \mathcal{C}_{\Omega_\epsilon|X}) \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \mathcal{C}_{K_\epsilon|X}),$$

where $j_\epsilon : \Omega_\epsilon \hookrightarrow M$; then we have $\delta \circ \alpha_\epsilon(u_{|\Omega_\epsilon})_{|\Delta_p} \equiv 0$. Noting that any point $q \in L_+ \setminus T^*_M X$ belongs to the interior of $T^*_{K_\epsilon} X \cap \pi^{-1}(x_0)$ and $\mathrm{d}_\zeta f(q) \neq 0$, we have by LEMMA 3.2 (2) :

$$\delta \circ \alpha_\epsilon(u_{|\Omega_\epsilon})_{|L'_+} \equiv 0$$

for any L'_+. The morphism δ being an isomorphism outside $T^*_M X$, this implies that $\alpha_\epsilon(u_{|\Omega_\epsilon}) = 0$ on $L_+ \setminus T^*_M X$. Hence, noting 4.2, we get

$$L_+ \cap \mathrm{SS}^P_{\Omega_\epsilon}(u_{|\Omega_\epsilon}) \subset T^*_X X.$$

4.4. — In summary, for any real analytic solution u to the equation $Pu = 0$ defined outside K, we have

$$\mathrm{SS}^P_{\Omega_\epsilon}(u_{|\Omega_\epsilon}) \cap T^*_{M_+} X \cap \pi^{-1}(x_0) \subset T^*_X X$$

for $0 < \epsilon \ll \epsilon_0$; in particular,

$$\mathrm{SS}^P_{M_+}(u_{|M_+}) \cap \pi^{-1}(x_0) \subset T^*_X X.$$

Let $\gamma(u)$ denote the boundary values on N of $u_{|M_+}$:

$$\gamma(u) = \big(u_{|x_1 \to +0}, D_1 u_{|x_1 \to +0}\big).$$

Then $\gamma(u)$ is real analytic in a neighborhood of x_0 in N (PROPOSITION 2.3). Hence u can be continued analytically in a neighborhood of the origin by solving the Cauchy problem for P with the analytic data $\gamma(u)$ by Cauchy-Kowalewski Theorem. This completes the proof of the surjectivity of (1.1).

5. Proof of theorem 2

5.1. — It is enough to prove the surjectivity of (1.2) at x. We may suppose $M = \mathbb{R}^n$, $K = \{(x_1, x') \in M \mid x_1 \leq -\epsilon_0 |x'|\}$ (with $\epsilon_0 > 0$), $x = (0, \ldots, 0)$. Set $\Omega = M \setminus K$, $M_+ = \{x_1 > 0\}$, $N = \{x_1 = 0\}$, $x_0 = (0, \ldots, 0)$, and $Y = \{(z_1, z_1') \in \mathbb{C}^n \mid z_1 = 0\}$; we may assume in addition that Y is non characteristic for $\mathcal{M}$.

5.2. — Since $T^*_{M_+}X \cap \pi^{-1}(x_0) \setminus T^*_M X$ is contained in the interior of $T^*_K X \cap \pi^{-1}(x_0)$ and $\mathrm{Char}(\mathcal{M}) \cap \pi^{-1}(x_0)$ has by assumption codimension ≥ 2 in $\pi^{-1}(x_0)$, it follows from LEMMA 3.2(3) that

$$H^0 \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) \cong \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{K|X}) = 0$$

on $T^*_{M_+}X \cap \pi^{-1}(x_0) \setminus T^*_M X$. In particular, we have

$$\mathrm{SS}^{\mathcal{M}}_{M_+}(u_{|M_+}) \cap \pi^{-1}(x_0) \subset T^*_M X$$

for any hyperfunction solution u of $\mathcal{M}$ on Ω.

5.3. — Let u be an analytic solution of $\mathcal{M}$ defined on Ω. Let

$$p \in \mathrm{Char}(\mathcal{M}) \cap (T^*_M X \setminus M) \cap \pi^{-1}(x_0).$$

Let Λ_p be the virtual bicharacteristic manifold of $\mathcal{M}$ passing through p. By condition (b.5), we can find $a' \in \mathbb{R}^{n-1}$ with $0 < |a'| < \epsilon_0$ so that each $\mathrm{d}x_1 \pm \langle a', \mathrm{d}x' \rangle$ is not orthogonal to $\mathrm{d}\pi(T_p \Lambda_p)$. Let

$$M'_\pm = \{x \in M \mid x_1 \pm \langle a', x' \rangle > 0\};$$

then $M'_\pm \subset \Omega$ (because $|a'| < \epsilon_0$) and, by the choice of a', $\pi(\Lambda_p)$ is not tangent to each boundary of $M'_\pm$. Hence, by propagation of the micro-analyticity of u along Λ_p (PROPOSITION 2.4), we have $p \notin \mathrm{SS}^{\mathcal{M}}_{M'_\pm}(u)$. Since $M_+ \subset M'_+ \cup M'_-$, it follows from PROPOSITIONS 2.1 and 2.2 that $p \notin \mathrm{SS}^{\mathcal{M}}_{M_+}(u)$.

5.4. — In summary, we have

$$\mathrm{SS}^{\mathcal{M}}_{M_+}(u) \cap \pi^{-1}(x_0) \subset T^*_X X$$

for any germ u of the sheaf $j_* j^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)$, with j being the embedding $M \backslash K \hookrightarrow M$. Let $\mathcal{M}_Y$ denote the induced system of differential equations of $\mathcal{M}$ on Y and let $\gamma(u) \in \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$ be the boundary value of $u_{|M_+}$ to N as hyperfunction. Then $\gamma(u)$ is real analytic in a neighborhood of x_0 in N (PROPOSITION 2.3). Hence u can be continued up to x_0 by solving the Cauchy problem for $\mathcal{M}$ with the analytic data $\gamma(u)$ (cf. (2.2)). This completes the proof of THEOREM 2.

6. Removable isolated singularities of real analytic solutions

6.1. — Let M be a real analytic manifold of dimension $n \geq 2$, and let $x_0 \in M$. Let $P = P(x, D)$ be a differential operator with analytic coefficients defined on M, $f = f(z, \zeta)$ the principal symbol of P. Set $V = \{q \in T^*X \mid f(q) = 0\}$. For $p \in V$, we say that p is a simple point of V if the germ (V, p) is given by $g = 0$ for a holomorphic function g on T^*X with $d_\zeta g \neq 0$, where d_ζ is the differential along the fibres of $\pi : T^*X \to X$. Let $V' = V \cap \pi^{-1}(x_0)$; then V' is an algebraic variety of $\mathbb{C}^n_\zeta$. We impose the following conditions on P :

(d.1) P has real principal symbol : $\operatorname{Im} f_{|T^*_M X} \equiv 0$.

(d.2) Let $V' = V_1 \cup \cdots \cup V_r$ be the irreducible decomposition of V'. Then every V_j has a real simple point p_j of V; i.e., there is a point p_j of $V_j \cap T^*_M X$ which is a simple point of V.

THEOREM 6.1. — *Assume that $P = P(x, D)$ satisfies conditions* (d.1) *and* (d.2) *at x_0. Let u be a real analytic solution to the equation $Pu = 0$ defined on $M \setminus \{x_0\}$. Then there is a unique hyperfunction $\tilde{u}$ defined on M such that :*

$$\tilde{u}_{|M\setminus\{x_0\}} = u \quad \textit{and} \quad P\tilde{u} = 0 \quad \textit{on } M.$$

COROLLARY 6.2. — *Let $P = P(x, D)$ be a differential operator of real principal type. Assume that the polynomial $f(x_0; \zeta)$ in ζ has no elliptic factors. Then any real analytic solution to the equation $Pu = 0$ defined in a neighborhood of x_0 except x_0 can be analytically continued on the whole of a neighborhood of x_0.*

REMARK 1. — COROLLARY 6.2 is a generalization of THEOREM 1 to higher order differential equations in the case $K = \{x_0\}$.

REMARK 2. — Cf. [Kn2, cor. 22] for a similar result in the case where V' is irreducible, where a few more conditions are imposed on V' in addition to (d.2). Note that Theorem 17 of [Kn2] is also a special case of COROLLARY 6.2. See also [G] and [Kn5] for similar results for differential equations with constant coefficients.

6.2. *Proof.* — Set $M^* = M \setminus \{x_0\}$. Let P be a differential operator satisfying conditions (d.1) and (d.2). Let u be a real analytic solution to the equation $Pu = 0$ on M^*.

6.2.1. — Let $p \in V' \cap T^*_M X$ with $p \neq 0$. Assume that p is a simple point of V. Let $b(p)$ denote the real bicharacteristic curve of $V \cap T^*_M X$ passing through p; then $\pi(b(p))$ is a smooth curve of M passing through x_0. Let us take n real-valued C^ω-functions $\varphi_1, \dots, \varphi_n$ on M with

$$\varphi_1(x_0) = \cdots = \varphi_n(x_0) = 0, \quad \mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_n \neq 0$$

so that $\pi(b(p))$ is transversal to every $\varphi_\alpha = 0$. Set $\Omega_\alpha^\pm = \{\pm\varphi_\alpha > 0\}$ for $1 \leq \alpha \leq n$; then, by propagation theorem of regularity up to the boundary (cf. 2.4), we have $p \notin \mathrm{SS}^P_{\Omega_i^\pm}(u)$. Hence it follows from PROPOSITIONS 2.1 and 2.2 that $p \notin \mathrm{SS}^P_{\mathcal{M}^*}(u)$.

6.2.2. — Let us denote for short by $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ the following sheaves (valued in microfunctions) on T^*X

$$\begin{aligned}
\mathcal{S}_0 &= \mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P, \mathcal{C}_{M|X}),\\
\mathcal{S}_1 &= H^0\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P, \mathcal{C}_{\mathcal{M}^*|X}),\\
\mathcal{S}_2 &= \mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P, \mathcal{C}_{\{x_0\}|X}).
\end{aligned}$$

Then we have a natural sheaf-homomorphisms (cf. section 2.1 and 3.3) $\rho : \mathcal{S}_0 \to \mathcal{S}_1$ and $\delta : \mathcal{S}_1 \to \mathcal{S}_2$, and the sequence

$$0 \to \mathcal{S}_0 \xrightarrow{\rho} \mathcal{S}_1 \xrightarrow{\delta} \mathcal{S}_2 \tag{6.2}$$

is exact (by the definition of $\mathcal{C}_{M^*|X}$).

6.2.3. — Let u_1 be the section of $\mathcal{S}_1$ on T^*X defined by u (cf. (2.1)). From 6.2.1 and condition (d.2), every V_j has a point p_j ($\notin T^*_X X$) where $u_1 = 0$. Let $\delta(u_1)$ be the global section of $\mathcal{S}_2$ defined by $\delta : \mathcal{S}_1 \to \mathcal{S}_2$ from u_1; then $\delta(u_1) = 0$ at p_j. By LEMMA 3.2 (2) for $K = \{x_0\}$, we have $\delta(u_1) = 0$ on $V_j \setminus T$, where T denotes the singular locus of V'. Therefore the support of $\delta(u_1)$ is contained in T. Since $\dim_{\mathbb{C}}(T, q) < n-1$ at any point $q \in T$, by LEMMA 3.2 (1), we have $\delta(u_1) = 0$ at q. Thus $\delta(u_1) = 0$ on T^*X.

Let q_0 denote the point $(x_0; 0)$ of T^*X. By exact sequence (6.1), there is a germ u_0 of $\mathcal{S}_0$ at q_0 such that $\rho(u_0) = u_1$. Since $\rho_{q_0} : \mathcal{S}_{0,q_0} \to \mathcal{S}_{1,q_0}$ coincides with the restriction map

$$\Gamma(U, \mathcal{B}_M{}^P) \longrightarrow \Gamma(U \setminus \{x_0\}, \mathcal{B}_M{}^P) \tag{6.2}$$

with U being a neighborhood of x_0, where $\mathcal{B}_M{}^P$ denote the sheaf of hyperfunction solutions of $Pu = 0$ on M, u_0 gives an extention of u on a neighborhood of x_0 as a hyperfunction solution of $Pu = 0$.

Since P is nondegenerate at x_0, by the Holmgren theorem, (6.2) is injective; i.e. extension of u is unique.

This completes the proof of THEOREM 6.1.

6.2.4. — Assume moreover P to be of real principal type. For a point q of $V' \cap T^*_M X \setminus M$, let $b(q)$ denote the real bicharacteristic curve of P passing through q; then $b(q) \setminus \{q\} \subset \pi^{-1}(M \setminus \{x_0\})$. Since the regularity of $\tilde{u}$ propagates along $b(q)$, $\tilde{u}$ is microlocally analytic at q. This completes the proof of COROLLARY 6.2.

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