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ON IDEALS GENERATED BY BOUNDED ANALYTIC FUNCTIONS IN THE BIDISC

BY

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RÉSUMÉ. — Nous considérons une certaine généralisation du problème de la couronne dans le bidisque.

ABSTRACT. — We consider a certain generalization of the corona problem in the bidisc.

Introduction

Let D be the unit disc in $\mathbb{C}$ with boundary $\partial D = T$. We denote by $H(D\times D)$ the analytic functions on $D\times D$ and by $H^p(D\times D)$ the functions in $H(D\times D)$ with boundary values in $L^p(T\times T)$; the measure on T is the normalized Lebesgue measure $\mathrm{d}\sigma$ and the measure on $T\times T$ is $\mathrm{d}\sigma\otimes\mathrm{d}\sigma$.

In this note we prove that a certain generalized corona problem (for two generators) always has a solution that can be estimated by the Cauchy transform of a bounded function plus a bounded function.

Related results have been obtained by CHANG [2], LIN [3] and VAROPOULOS [4].

1. The $\bar\partial$-problem

Suppose $H = H_1\,\mathrm{d}\bar z_1 + H_2\,\mathrm{d}\bar z_2$ is a $\bar\partial$-closed form on $D\times D$ with coefficients continuous on $\overline{D\times D}$. Define :

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$$(1)\qquad U(z_1,z_2)=\frac{1}{2\pi i}\int_D \frac{H_1(\xi_1,z_2)}{\xi_1-z_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1 + \frac{1}{(2\pi i)^2}\int_{T\times D}\frac{H_2(\xi_1,\xi_2)\,\mathrm{d}\xi_1}{\xi_1-z_1}\,\frac{\mathrm{d}\xi_2\wedge \mathrm{d}\bar{\xi}_2}{\xi_2-z_2}.$$

Then $\bar{\partial}U = H$ for $\partial U/\partial \bar{z}_1 = H_1$ and since $\partial H_1/\partial \bar{z}_2 = \partial H_2/\partial \bar{z}_1$

$$\frac{\partial U}{\partial \bar{z}_2}=\frac{1}{2\pi i}\left\{\int_D \frac{\partial H_2/\partial \bar{z}_1(\xi_1,z_2)}{\xi_1-z_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\xi_1+\int_T\frac{H_2(\xi_1,z_2)\,\mathrm{d}\xi_1}{\xi_1-z_1}\right\}=H_2$$

by Cauchy's general integral formula.

We now want to rewrite (1) and therefore first consider the formula

$$(2)\qquad 0=4\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar{\xi}_1 z_1}\Big|^2\frac{\partial H_1}{\partial z_1}(\xi_1,z_2)\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1 + \int_D \frac{H_1(\xi_1,z_2)}{\xi_1-z_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1+\bar{z}_1\int_D\frac{H_1(\xi_1,z_2)}{1-\xi_1\bar{z}_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1$$

which follows from the fact that the right hand side is harmonic (in z_1) and has boundary values zero.

Hence, (1) is now

$$(3)\quad U(z_1,z_2)=-\frac{2}{\pi i}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar{\xi}_1 z_1}\Big|^2\frac{\partial H_1}{\partial z_1}(\xi_1,z_2)\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1 - \frac{1}{2\pi i}\bar{z}_1\int_D\frac{H_1(\xi_1,z_2)}{1-\xi_1\bar{z}_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar{\xi}_1 - \frac{2}{(2\pi i)^2}\int_{T\times D}\log\Big|\frac{z_2-\xi_2}{1-\bar{\xi}_2 z_2}\Big|^2\frac{\partial H_2(\xi_1,\xi_2)}{\partial z_2}\,\mathrm{d}\xi_2\wedge \mathrm{d}\bar{\xi}_2\,\frac{\mathrm{d}\xi_1}{\xi_1-z_1} - \frac{1}{(2\pi i)^2}\int_{T\times D}\frac{\bar{z}_2 H_2(\xi_1,\xi_2)}{1-\xi_2\bar{z}_2}\,\mathrm{d}\xi_2\wedge \mathrm{d}\bar{\xi}_2\,\frac{\mathrm{d}\xi_1}{\xi_1-z_1}.$$

2. A corona problem for two generators

In this section, we prove the following theorem :

THEOREM. — *Assume that g, f_1 and $f_2 \in H^\infty(D\times D)$ with*

$$|g|\le \left(|f_1|^2+|f_2|^2\right)^{1+\varepsilon}$$

for some $\varepsilon>0$. Then there is a function

$$a(z_1,z_2)=\int_T\frac{K(\xi_1,z_2)}{\xi_1-z_1}\,\mathrm{d}\sigma(\xi_1)+L(z_1,z_2)$$

where $K \in L^\infty(T \times D)$ *and* $L \in L^\infty(D \times D)$ *so that*

$$\frac{g\bar{f}_1}{|f_1|^2+|f_2|^2} - f_2 a = g_1 \in H(D \times D)$$
$$\frac{g\bar{f}_2}{|f_1|^2+|f_2|^2} + f_1 a = g_2 \in H(D \times D).$$

In particular, $f_1 g_1 + f_2 g_2 = g$.

Proof. — Consider

$$g_1 = \frac{g\bar{f}_1}{|f_1|^2+|f_2|^2} - f_2 a, \quad g_2 = \frac{g\bar{f}_2}{|f_1|^2+|f_2|^2} + f_1 a.$$

If we could find $a \in C^1(D \times D)$ so that $\bar{\partial} g_1 = 0$, then

$$0 = \bar{\partial} g = \bar{\partial}(f_1 g_1 + f_2 g_2) = f_2 \bar{\partial} g_2,$$

which means that $\bar{\partial}(g_2) = 0$ outside an analytic set, hence $\bar{\partial} g_2 = 0$ on $D \times D$ which proves that g_2 is also analytic.

Set

$$\Phi_i = f_2 \frac{\partial f_1}{\partial z_i} - f_1 \frac{\partial f_2}{\partial z_i} \cdot$$

That $\bar{\partial} g_1 = 0$ means that :

$$\begin{aligned} H = \bar{\partial} a &= \frac{g\,\overline{\Phi}_1}{\left(|f_1|^2+|f_2|^2\right)^2}\,\mathrm{d}\bar{z}_1 + \frac{g\,\overline{\Phi}_2}{\left(|f_1|^2+|f_2|^2\right)^2}\,\mathrm{d}\bar{z}_2 \\ &= H_1\,\mathrm{d}\bar{z}_1 + H_2\,\mathrm{d}\bar{z}_2. \end{aligned}$$

To avoid regularity problems, we can dilatate g, f_1 and f_2, prove uniform estimates and use weak convergence and normal family arguments. Moreover, we can assume that :

$$\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty} \le 1.$$

We now claim that there exist two functions $K_1 \in L^\infty(T \times D)$ and $K_2 \in L^\infty(T \times T)$ such that $\|K_1\|_{L^\infty} + \|K_2\|_{L^\infty} \le C$

$$\begin{aligned} &\int_{T \times D} A(\xi_1,\xi_2) K_1(\xi_1,\xi_2)\,\mathrm{d}\sigma(\xi_1)\,\mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2 \\ &\qquad = \int_{D \times D} A(\xi_1,\xi_2) H_1(\xi_1,\xi_2)\,\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_1\,\mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2 \end{aligned}$$

and

$$\int_{T\times T} B(\xi_1,\xi_2)K_2(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\sigma(\xi_2) = \frac{1}{(2\pi i)}\int_{T\times D} B(\xi_1,\xi_2)H_2(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar{\xi}_2$$

for every $A\in L^1(T\times D)$, analytically extendable in the first variable and every $B\in L^1(T\times T)$, analytic extendable in the second variable.

By Stokes' formula (cf. [1, Thm 2]), we have for almost all $\xi_2\in D$:

$$\left|\int_D \frac{Ag\,\overline{\Phi}_1}{2\xi_1\big(|f_1|^2+|f_2|^2\big)^2}\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar{\xi}_1\right| \le C\int_T \big|A(\xi_1,\xi_2)\big|\,\mathrm{d}\sigma(\xi_1).$$

Applying this to $\xi_1 A$

$$\left|\int_{D\times D} A(\xi_1,\xi_2)H_1(\xi_1,\xi_2)\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar{\xi}_1\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar{\xi}_2\right| \le C\left|\int_{T\times D}\big|A(\xi_1,\xi_2)\big|\,\mathrm{d}\sigma(\xi_1)\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar{\xi}_2\right|,$$

so the linear functional

$$A\mapsto\int_{D\times D} AH_1$$

can, by the Hahn-Banach theorem, be extended to $L^1(T\times D)$ without increase the norm. Since the dual is $L^\infty(T\times D)$ we have proved that K_1 exist.

We also have from [1] that

$$\left|\int_D \frac{Bg\,\overline{\Phi}_2}{2\xi_2\big(|f_1|^2+|f_1|^2\big)^2}\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar{\xi}_2\right| \le C\int_T \big|B(z_1,\xi_2)\big|\,\mathrm{d}\sigma(\xi_2)$$

for almost all $z_1\in T$, so therefore

$$\left|\int_{T\times D} \frac{Bg\,\overline{\Phi}_2}{2\xi_2\big(|f_1|^2+|f_2|^2\big)^2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar{\xi}_2\right| \le C\int_{T\times T}\big|B(\xi_1,\xi_2)\big|\,\mathrm{d}\sigma(\xi_1)\,\mathrm{d}\sigma(\xi_2)$$

so the linear functional

$$B\mapsto\int_{T\times D} BH_2$$

can be extended to $L^1(T\times T)$ without increase the norm and since the dual of $L^1(T\times T)$ is $L^\infty(T\times T)$ the claim is proved.

Note that if f is analytic in a neighborhood of $\overline{D}$, then

$$\int_T f(\xi_1)K_1(\xi_1,\xi_2)\,\mathrm{d}\sigma(\xi_1)=\int_D f(\xi_1)H_1(\xi_1,\xi_2)\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1,\qquad \xi_2\in D;$$

$$\int_T f(\xi_2)K_2(\xi_1,\xi_2)\,\mathrm{d}\sigma(\xi_2)=\int_D f(\xi_2)H_2(\xi_1,\xi_2)\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2,\qquad \xi_1\in T.$$

We now use formula (3) and get

$$\begin{aligned}
U(z_1,z_2)&=-\frac{2}{\pi i}\int_D\log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1z_1}\Big|^2\frac{\partial H_1}{\partial z_1}(\xi_1,z_2)\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1\\
&\quad-\frac{1}{2\pi i}\bar z_1\int_D\frac{H_1(\xi_1,z_2)}{1-\xi_1\bar z_1}\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1\\
&\quad-\frac{2}{(2\pi i)^2}\int_{T\times D}\log\Big|\frac{z_2-\xi_2}{1-\bar\xi_2z_2}\Big|^2\frac{\partial H_2}{\partial z_2}(\xi_1,\xi_2)\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&\quad-\frac{1}{(2\pi i)^2}\bar z_2\int_{T\times D}\frac{H_2(\xi_1,\xi_2)}{1-\xi_2\bar z_2}\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&=-\frac{-2}{\pi i}\int_D\log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1z_1}\Big|^2\frac{\partial H_1}{\partial z_1}(\xi_1,z_2)\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1\\
&\quad-\frac{\bar z_1}{2\pi i}\int_T\frac{K_1(\xi_1,z_2)}{1-\xi_1\bar z_1}\,\mathrm{d}\sigma(\xi_1)\\
&\quad-\frac{2}{(2\pi i)^2}\int_T\Big[\int_D\log\Big|\frac{z_2-\xi_2}{1-z_2\bar\xi_2}\Big|^2\frac{\partial H_2}{\partial z_2}(\xi_1,\xi_2)\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2\Big]\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&\quad-\frac{1}{2\pi i}\int_T\Big[\bar z_2\int_T\frac{K_2(\xi_1,\xi_2)}{1-\xi_2\bar z_2}\,\mathrm{d}\sigma(\xi_2)\Big]\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&=-\frac{-2}{\pi i}\int_D\log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1z_1}\Big|^2\frac{\partial H_1}{\partial z_1}(\xi_1,z_2)\,\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1\\
&\quad-\frac{1}{2\pi i}\int_T\frac{1-|z_1|^2}{\xi_1|\xi_1-z_1|^2}K_1(\xi_1,z_2)\,\mathrm{d}\sigma(\xi_1)\\
&\quad+\frac{1}{2\pi i}\int_T\frac{K_1(\xi_1,z_2)}{\xi_1-z_1}\,\mathrm{d}\sigma(\xi_1)\\
&\quad-\frac{2}{(2\pi i)^2}\int_T\Big[\int_D\log\Big|\frac{z_2-\xi_2}{1-\bar\xi_2z_2}\Big|^2\frac{\partial H_2}{\partial z_2}(\xi_1,\xi_2)\,\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2\Big]\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&\quad-\frac{1}{(2\pi i)}\int_T\Big[\int_T\frac{1-|z_2|^2}{\xi_2|\xi_2-z_2|^2}K_2(\xi_1,\xi_2)\,\mathrm{d}\sigma(\xi_2)\Big]\frac{\mathrm{d}\xi_1}{\xi_1-z_1}\\
&\quad+\frac{1}{2\pi i}\int_{T\times T}\frac{K_2(\xi_1,\xi_2)}{(\xi_2-z_2)(\xi_1-z_1)}\,\mathrm{d}\xi_1\,\mathrm{d}\sigma(\xi_2).
\end{aligned}$$

• We estimate the first term as follows. Since

$$H_1 = \frac{g\,\bar\Phi_1}{\big(|f_1|^2+|f_2|^2\big)^2}$$

we get :

$$\frac{\partial H_1}{\partial z_1} = \frac{\partial g}{\partial z_1}\,\frac{\bar\Phi_1}{\big(|f_1|^2+|f_2|^2\big)^2} - 2g\,\frac{\big(\partial f_1/\partial z_1 \bar f_1 + \partial f_2/\partial z_1 \bar f_2\big)\bar\Phi_1}{\big(|f_1|^2+|f_2|^2\big)^3}\,.$$

Choose δ so that $\dfrac{1-\delta/2}{1+\varepsilon/2}(1+\varepsilon) = 1$. Then

$$\Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{\partial H_1}{\partial z_1}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1\Big|$$

$$\le \Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{|\partial g/\partial z_1|}{|g|^{1-\delta/2}}\,\frac{|g|^{1-\delta/2}}{\big(|f_1|^2+|f_2|^2\big)^{1+\varepsilon/2}}$$

$$\times \frac{\big(|\partial f_1/\partial z_1|^2+|\partial f_2/\partial z_1|^2\big)^{1/2}}{\big(|f_1|^2+|f_2|^2\big)^{1/2-\varepsilon/2}}\Big|\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1$$

$$+ \Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{|g|}{\big(|f_1|^2+|f_2|^2\big)^{1+\varepsilon}}$$

$$\times \frac{|\partial f_1/\partial z_1|^2+|\partial f_2/\partial z_1|^2}{\big(|f_1|^2+|f_2|^2\big)^{1-\varepsilon}}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1\Big|$$

$$\le \Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{|\partial g/\partial z_1|^2}{|g|^{2-\delta}}\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1\Big|^{1/2}$$

$$\times \Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \Big[\frac{|\partial f_1/\partial z_1|^2}{|f_1|^{2-2\varepsilon}} + \frac{|\partial f_2/\partial z_1|^2}{|f_2|^{2-2\varepsilon}}\Big]\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1\Big|^{1/2}$$

$$+ \Big|\frac{1}{2\pi}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \Big[\frac{|\partial f_1/\partial z_1|^2}{|f_1|^{2-2\varepsilon}} + |\partial f_2/\partial z_1|^2 |f_2|^{2-2\varepsilon}\Big]\,\mathrm{d}\xi_1\wedge \mathrm{d}\bar\xi_1\Big|$$

$$= \frac{1}{2\pi}\Big|\frac{1}{\delta^2}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \delta|g|^{\delta}\Big|^{1/2}$$

$$\times \Big|\frac{1}{4\varepsilon^2}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \big(\Delta|f_1|^{2\varepsilon}+\Delta|f_2|^{2\varepsilon}\big)\Big|^{1/2}$$

$$+ \frac{1}{2\pi}\Big|\frac{1}{4\varepsilon^2}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \big(\Delta|f_1|^{2\varepsilon}+\Delta|f_2|^{2\varepsilon}\big)\Big|$$

$$
\begin{aligned}
&= \frac{1}{2\delta\varepsilon}\left|\int_T \frac{1-|z_1|^2}{|\xi_1 - z_1|^2}\left|g(\xi_1 z_2)\right|^\delta \mathrm{d}\sigma(\xi_1) - \left|g(z_1,z_2)^\delta\right|\right|^{1/2}\\
&\times\left|\int_T \frac{1-|z_1|^2}{|\xi_1 - z_1|^2}\left(|f_1|^{2\varepsilon} + |f_2|^{2\varepsilon}\right)\mathrm{d}\sigma(\xi_1) - \left|f_1(z_1,z_2)\right|^{2\varepsilon} - \left|f_2(z_1,z_2)\right|^{2\varepsilon}\right|^{1/2}\\
&+ \frac{1}{4\varepsilon^2}\left[\int_T \frac{1-|z_1)^2}{|z_1 - \xi_1|^2}\left(|f_1|^{2\varepsilon} + |f_2|^{2\varepsilon}\right)\mathrm{d}\sigma(\xi_1) - \left|f_1(z_1,z_2)\right|^{2\varepsilon} - \left|f_2(z_1,z_2)\right|^{2\varepsilon}\right]\\
&\le \frac{1}{\delta\varepsilon} + \frac{1}{\varepsilon^2}
\end{aligned}
$$

by Riesz representation theorem.

- The second term is bounded by $\|K_1\|_{L^\infty}$ and as above, we can prove that

$$\left|\frac{1}{2\pi}\int_D \log\left|\frac{z_2 - \xi_2}{1-\bar{\xi}_2 z_2}\right|^2 \frac{\partial H_2}{\partial z_2}(z_1,\xi_2)\,\mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2\right| \le \frac{1}{\delta\varepsilon} + \frac{1}{\varepsilon^2}.$$

Since the last term is analytic it can be removed and therefore the proof of the theorem is complete.

Remark

There exist $K \in L^\infty(T)$ such that there is no $f \in H(D\times D)$ with

$$\bar{z}_2 \int_T \frac{K(\xi)}{\xi - z_1}\,\mathrm{d}\xi + f \in L^\infty(D\times D).$$

For if $K \in L^\infty(T)$ but $\displaystyle\int_T \frac{K(\xi)}{\xi - z}\,\mathrm{d}\xi \notin L^\infty(D)$, there exists for every m a harmonic polynomial P_m with

$$\int \left|P_m(z)\right|\mathrm{d}\sigma(z) = 1 \qquad \text{but} \qquad \left|\int_T\int_T \frac{K(\xi)}{\xi - z}\,\mathrm{d}\xi P(z)\,\mathrm{d}\sigma(z)\right| \ge m$$

so if f is any function in $H^1(D\times D)$ then

$$
\begin{aligned}
&\left|\int_T\int_T\left\{\bar{z}_2\int_T \frac{K(\xi)}{\xi - z_1}\,\mathrm{d}\xi + f(z_1,z_2)\right\} z_2 P(z_1)\,\mathrm{d}\sigma(z_1)\,\mathrm{d}\sigma(z_2)\right|\\
&\qquad = \left|\int_T\int_T \frac{K(\xi)}{\xi - z_1}\,\mathrm{d}\xi P(z_1)\,\mathrm{d}\sigma(\xi)\right| \ge m,
\end{aligned}
$$

so $\displaystyle \bar{z}_2\int_T \frac{K(\xi)}{\xi - z_1} + f(z_1,z_2) \notin L^\infty(D\times D).$

REMARK

It follows from the proof of the theorem that the equation

$$\frac{\partial V}{\partial \bar z_1} = H_1$$

has a bounded solution V. (In particular, $\dfrac{\partial^2 V}{\partial \bar z_1 \partial \bar z_2} = \dfrac{\partial H_1}{\partial \bar z_2}$, cf. CHANG [2]). Take V to be :

$$V(z_1, z_2) = -\frac{-2}{\pi i}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{\partial H_1}{\partial z_1}\,\mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1 - \int_T \frac{1-|z_1|^2}{\xi_1|\xi_1 - z_1|^2} K_1(\xi_1, z_2)\,\mathrm{d}\sigma(\xi_1).$$

Then, $V \in L^\infty(D\times D)$ and

$$\begin{aligned}\frac{\partial V}{\partial z_1} &= \frac{\partial}{\partial \bar z_1}\Big[-\frac{-2}{\pi i}\int_D \log\Big|\frac{z_1-\xi_1}{1-\bar\xi_1 z_1}\Big|^2 \frac{\partial H_1}{\partial z_1}(\xi_1, z_2)\,\mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1 \\ &\qquad - \frac{1}{2\pi i}\int_D \frac{\bar z_1}{1-\xi_1 \bar z_1} H_1(\xi_1, z_1)\,\mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1\Big] \\ &= \frac{\partial}{\partial \bar z_1}\Big[\frac{1}{2\pi i}\int_D \frac{H_1(\xi_1, z_1)}{\xi_1 - z_1}\,\mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1\Big] = H_1.\end{aligned}$$

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