ANDRÉA SOLOTAR

Bivariant cohomology and $S^1$-spaces

Bulletin de la S. M. F., tome 120, n° 4 (1992), p. 397-412

<http://www.numdam.org/item?id=BSMF_1992__120_4_397_0>
BIVARIANT COHOMOLOGY AND $S^1$-SPACES

BY

ANDRÉA SOLOTAR (*)

0. Introduction

The bivariant version of cyclic cohomology was introduced by Jones and Kassel in [J-K]. In the other hand, there is a topological definition for $S^1$-spaces $X$ and $Y$ of the bivariant $S^1$-equivariant cohomology, denoted $H^*_S(X,Y)$ which can be found in [C].

In the following work we prove the bivariant version of the theorem of Jones [J], Goodwillie [G] and Burghelea-Fiedorowicz [B-F], which
says that if $X$ is an $S^1$-space, then its equivariant cohomology is isomorphic to the cyclic cohomology of the singular complex module of $X$:

$$H^*_S(X) = HC^*(S.(X)).$$

One of the main results is the following:

**Theorem.** — Let $X$ and $Y$ have the homotopy type of CW-complexes equipped with a pointed $S^1$-action, such that $Y$ has the homotopy type of a finite complex. Then there exists a natural isomorphism:

$$H^*_S(X,Y) \cong HC^n(S.X,S.Y)$$

where $HC^n(S.X,S.Y)$ is the reduced bivariant cyclic cohomology of $S.X$ and $S.Y$.

This isomorphism sends Connes’ long exact sequence in bivariant cyclic cohomology to a Gysin long exact sequence of $X$ and $Y$ in the topological context.

We also prove that in certain cases the bivariant cyclic cohomology of two cyclic modules $M._{et}N.$ can be computed as the cyclic cohomology of the mixed complex $M. \otimes DN.$, where $DN.$ is the dual of $N.$, considered as a chain complex.

The paper is organized as follows:

In sections 1–3 we recall the definitions and some properties of bivariant cohomology, bivariant cyclic cohomology and the stable homotopy category $Stab$, respectively. This category, studied principally in [D-P], is used to provide an intermediate result during the proof.

In section 4 we extend the theorem of [J], [G] and [B-F] to the $Stab$ category (PROPOSITION 4.1).

Section 5 gives the following preliminary result which is used in the proof of the main theorem:

**Proposition.** — Under the hypotheses of the theorem, there is a natural isomorphism between $HC^n(S.X,S.Y)$ and $HC^n(S.(X \wedge DY))$, where $DY$ is the Spanier-Whitehead dual of $Y$ in the $Stab$ category and $S.$ denotes the reduced singular complex module.

This result is proved in section 6.

Finally, in section 7, we show that the diagram which relates the reduced Gysin long exact sequence of $ES^1 \times_{S^1}(X \wedge DY)$ and Connes’ long exact sequence of the reduced bivariant cyclic cohomology of $S.X$ and $S.Y$ is commutative.
All the spaces that we are going to consider have the homotopy type of a CW-complex, are connected and base pointed.

1. Bivariant cohomology

Given two CW-complexes $X$ and $Y$, their bivariant cohomology with integral coefficients is defined, using maps of spectra as $[\Sigma^\infty X, \Sigma^\infty Y \wedge H]$, where $H$ is the Eilenberg-Mac Lane spectrum, $H_i = K(\mathbb{Z}, i)$, $\Sigma^\infty X$ is the spectrum defined by $(\Sigma^\infty X)^n = S^n \wedge X$ and $[ , ]$ denotes homotopy classes of morphisms that fix the base point [C, p. 3].

As $(\Sigma^\infty Y \wedge H)_n = Y \wedge H_n$ [S, Cor. 13.39], we may define:

$$H^i(X, Y) = \lim_{j \to \infty} [\Sigma^j X, Y \wedge K(\mathbb{Z}, j + i)] \quad (i \in \mathbb{Z}).$$

There are other definitions of the same object which are equivalent, such as: $H^i(X, Y)$ is the group of chain homotopy classes of chain maps of degree $i$ from the reduced singular chain complex of $X$ to the reduced singular chain complex of $Y$ [C-S, p. 398]), and one has a split short exact sequence

$$0 \to \text{Ext}^{n+1}(\overline{H}_*(X), \overline{H}_*(Y)) \to H^n(X, Y) \to \text{Hom}_{-n}(\overline{H}_*(X), \overline{H}_*(Y)) \to 0,$$

where $\overline{H}$ denotes reduced homology: $\overline{H}(X) = H(X)/H(\ast)$.

As a consequence, $H^n(X, Y)$ is $H_{-n}(\text{Hom}(S.X, S.Y))$.

2. Bivariant cyclic cohomology

For the definition and properties of cyclic $k$-modules (where $k$ is a commutative ring with unit), we refer to [Co], [L1] and [L2].

We recall that a mixed complex $(M, b, B)$ is a nonnegatively graded $k$-module $(M_n)_{n \in \mathbb{N}}$ endowed with a degree $-1$ morphism $b$ and a degree $+1$ morphism $B$, such that $b^2 = B^2 = [B, b] = 0$.

The cyclic homology of a cyclic $k$-module is defined in [Co] and [L2], and the cyclic homology of a mixed complex is defined in [B] and [K1].

Given cyclic $k$-modules $M$ and $N$, Kasel [K2] has defined the bivariant Hochschild cohomology of $M$ and $N$.
Definition. — $HH^n(M, N) = H_{-n}(\text{Hom}(M, N), d)$ $(n \in \mathbb{Z})$, where $d(f) = b_N \cdot f - (-1)^{\deg(f)}f \cdot b_M$ and $\text{Hom}(M, N)$ is the complex such that $(\text{Hom}(M, N))_j = \Pi_p \text{Hom}(M_p, N_{p-j})$.

Given augmented cyclic $k$-modules $M.$ and $N.$, their reduced bivariant Hochschild cohomology $\overline{HH}^n(M, N.)$ is defined as $HH^n(M., N.)$, where $M. = k \oplus M., N. = k \oplus N..$ It verifies that:

$$HH^n(M, N.) = \overline{HH}^n(M, N.) \oplus HH^n(k) \oplus \overline{HH}^n(M.) \oplus HH^n(k., N.).$$

So we observe that $H^n(X, Y) \cong \overline{HH}^n(S.X, S.Y)$.

Jones and Kassel [J-K] have also defined the bivariant cyclic cohomology of $M.$ and $N.$ in the following way: to the mixed complex $(M, b, B)$ is associated the complex

$\beta(M) = k[u] \otimes M,$

(\text{where deg}(u) = 2), with differential

$$d(u^n \otimes m) = \begin{cases} u^n \otimes bm + u^{n-1} \otimes Bm & \text{if } n > 0, \\ u^n \otimes bm & \text{if } n = 0. \end{cases}$$

The natural projection $S: \beta(M) \rightarrow \beta(M)[2]$ is given by:

$$S(u^n \otimes m) = \begin{cases} u^{n-1} \otimes m & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

which is a morphism of complexes.

The module $\beta(M)$ is then called an $S$-module.

We consider $\text{Hom}_S(\beta(M), \beta(N))$, the submodule of $\text{Hom}(\beta(M), \beta(N))$ consisting of elements which commute with $S$.

Definition. — $HC^n(M, N) = H_{-n}(\text{Hom}_S(\beta(M), \beta(N)))$ $(n \in \mathbb{Z})$.

Remarks:
1) $HC^n$ is a contravariant functor in $M$ and a covariant functor in $N$.
2) If $N = k$, $HC^i(M, k) = HC^i(M)$.
3) If $M = k$, $HC^i(k, N) = HC^i_-(N)$.

(For a definition of $HC^i_-$, see [J].)

Following the ideas of [L-Q, § 4], the following definition of the reduced bivariant cyclic cohomology is given in [K3, 8.2].
Definition. — If $M.$ and $N.$ are augmented cyclic $k$-modules, the reduced bivariant cyclic cohomology of $M.$ and $N.$ is \( \overline{HC}^n(M., N.) = HC^n(M., N.). \) where $M. = k. \oplus M.$ and $N. = k. \oplus N.$.

Remark :

\[
HC^n(M., N.) = \overline{HC}^n(M., N.) \oplus \overline{HC}^n(M.) \oplus \overline{HC}^{-n}(N.) \oplus HC^n(k.).
\]

Examples. — If $X$ and $Y$ are $S^1$-spaces, then their reduced singular complex $k$-modules, denoted $\mathcal{S}.X$ and $\mathcal{S}.Y$ are generated by their reduced singular complexes $\mathcal{S}.X$ and $\mathcal{S}.Y$. These $k$-modules are not only simplicial $k$-modules but also cyclic $k$-modules, with the cyclic action defined by (see [G]) :

\[
C_n \times \mathcal{S}_n(X) \longrightarrow \mathcal{S}_n(X) \quad \quad \quad \quad (t_n, s) \quad \quad \quad \quad \quad t_n \cdot \sigma,
\]

where $t_n \cdot \sigma(u_0, ..., u_n) = e^{2\pi i u_0} \cdot \sigma(u_1, ..., u_n, u_0)$.

In this case $\mathcal{S}.X = k \oplus \mathcal{S}.X$ and the same for $Y$, so the $S^1$-spaces $X$ and $Y$ give rise to the bivariant cyclic cohomology groups $HC^n(\mathcal{S}.X, \mathcal{S}.Y)$ $(n \in \mathbb{Z})$, which by definition are the reduced bivariant cyclic cohomology groups of $\mathcal{S}.X$ and $\mathcal{S}.Y$, $\overline{HC}^n(\mathcal{S}.X, \mathcal{S}.Y)$.

3. The stable homotopy category

From now on all the spaces considered are base pointed, compactly generated $CW$-complexes.

3.1. — We recall from [D-P] that the stable homotopy category $\text{Stab}$ is the category whose objects are pairs $(X, n)$, where $X$ is a space, $n \in \mathbb{Z}$, and whose maps are :

\[
\text{Stab}((X, n); (Y, m)) = \lim_k [\Sigma^{n+k}X, \Sigma^{m+k}Y].
\]

The product $(X, n) \otimes (Y, m) = (X \wedge Y, n + m)$ makes $\text{Stab}$ a monoidal category.

We shall make use of the following objects :

Definitions :

(1) Given $(X, n)$, if $(X', n')$ is an object in $\text{Stab}$ such that

\[
\text{Stab}((X, n) \otimes (Z, k), (S^0, 0))
\]
is isomorphic to \( \text{Stab}((Z, k); (X', n') \otimes (S^0, 0)) \), for every object \((Z, k)\) in Stab, then \((X', n')\) is called (up to canonic isomorphism) the \textit{weak dual} of \((X, n)\) and denoted \(D(X, n)\).

(2) If \(\text{Stab}((X, n) \otimes ((Z, k), (W, \ell)))\) is isomorphic to

\[
\text{Stab}((Z, k), (X', n') \otimes (W, \ell))
\]

for every pair of objects \((Z, k)\) and \((W, \ell)\) in Stab, then \((X', n')\) is called (up to canonic isomorphism) the \textit{strong dual} of \((X, n)\) and also denoted \(D(X, n)\).

The uniqueness of a (weak) dual object is assured by [S, cor. 14.25] and the existence of a strong dual for a finite CW-complex, by [S, th.14.34].

Then \(D : \text{Stab}^* \to \text{Stab}\) is a contravariant functor, where \(\text{Stab}^*\) is the full subcategory whose objects are those \((X, n)\) such that \(X\) is a finite CW-complex.

According to the definition of the homology of spectra [S], it is clear that:

\[
H_n((X, h)) = H_{n-h}(X).
\]

Similarly, for cohomology we have that \(H^n((X, h)) = H^{n-h}(X)\).

4. \(S^1\)-spaces

Let \(Z\) be an \(S^1\)-space, such that the action of \(S^1\) over \(Z\) is pointed, that is, it preserves the base point (ex. : \(Z = \text{Map}(S^1, Z')\)), and consider, for \(n \in \mathbb{Z}\), the \(S^1\)-space \(\Sigma^n Z = S^n \wedge Z\) with the trivial action of \(S^1\) on \(S^n\).

In this situation we want to define \(ES^1 \times_{S^1} (Z, n)\), for \((Z, n) \in \text{Stab}\).

We observe that if \(n = 0\), then

\[
ES^1 \times_{S^1} (Z, 0) = ES^1 \times_{S^1} Z = (ES^1 \times_{S^1} Z, 0).
\]

If \(n > 0\), we identify \(ES^1\) with \((ES^1, 0)\), then

\[
ES^1 \times_{S^1} (Z, n) = ES^1 \times_{S^1} (\Sigma^n Z, 0) = (ES^1 \times_{S^1} \Sigma^n Z, 0).
\]

As the space \(ES^1 \times_{S^1} \Sigma^n Z\) is homotopy equivalent to \(\Sigma^n(ES^1 \times_{S^1} Z)\), we define \(ES^1 \times_{S^1} (Z, n) = (ES^1 \times_{S^1} Z, n)\) and we want to show now that, as in the case of \(S^1\)-spaces, we have:

**Proposition 4.1.** — \(H_*((ES^1 \times_{S^1} Z, n)) = HC_*(S.(Z, n))\).

**Proof.** — We first observe that \(S.(Z, n)\) though not a cyclic module is a mixed complex.
We have already seen that:

\[ H_*(ES^1 \times S^1 Z, n) = H_{*-n}(ES^1 \times S^1 Z) \]

and by [J], [G] and [B-F], this last term equals \( HC_{*-n}(S.Z) \).

In order to calculate \( HC_*(S.(Z, n)) \), we use the \( \beta \)-complex of [L-Q], whose total complex is such that \((\text{Tot} \beta)i = (\text{Tot} \beta')i_{-n}\) (where \( \beta' \) is the double complex of \( S.Z \)). We obtain that \( HC_*(S.(Z, n)) \) is isomorphic to \( HC_{*-n}(S.Z) \).

**Corollary 4.2.** — \( HC^*(S.(Z, n)) \) is isomorphic to \( H^*_S((Z, n)) \), where \( H^*_S \) is the \( BS^1 \) reduced cohomology (i.e. \( H^n_{S^1}(pt) = H^n(BS^1) = 0 \) for all \( n \)).

### 5. Duality, bivariant Hochschild cohomology and bivariant cyclic cohomology

Let us consider now the category \( \text{Ho}(\partial\text{-mod}_k) \), whose objects and morphisms are respectively chain \( k \)-complexes and homotopy classes of chain maps.

The tensor product of complexes makes \( \text{Ho}(\partial\text{-mod}_k) \) a monoidal category, with neutral object \( I = (I_q)_{q \in \mathbb{Z}} \) (where \( I_q = k \) if \( q = 0 \) and 0 if not).

Every chain complex \( A \) has a weak dual \( DA \) defined by \( (DA)_q = \text{Hom}_k(A_{-q}, k) \) and, by [D-P], a chain complex \( A \) is strongly dualizable in \( \text{Ho}(\partial\text{-mod}_k) \) if and only if it has the homotopy type of a finitely generated and projective chain complex.

DOLD and PUPPE give an extension \( S' \) to the category \( \text{Stab} \) of the functor which associates to a pointed space \((X, x_0)\), its singular complex module \( S.X \), such that

\[ H_n(S'(X, k)) = \begin{cases} H_{n-k}(S.X) & \text{if } n - k \geq 0, \\ 0 & \text{if } n - k < 0. \end{cases} \]

**Remark.** — Singular reduced homology with coefficients in \( k \) in the \( \text{Stab} \) category coincides with the definition given in this category by means of spectra, and similarly for cohomology.

In this context we have the following result:

**Proposition 5.1.** — *If \( M. \) and \( N. \) are simplicial modules, there is an isomorphism:*

\[ HH^*(M., DN) \cong HH^*(M \otimes N.). \]
Proof.

\[ HH^*(M, DN.) = H_{-\epsilon}(\text{Hom}(M, DN.)) \]
\[ = H_{-\epsilon}(\text{Hom}(M \otimes N., k)) \]
\[ = HH^*(M \otimes N.). \]

The second equality is obtained by definition of the weak dual in the category.

Similarly, we can prove the following results:

Corollary 5.2.

(i) If \( N. \) is such that \( DDN. \) is homotopically equivalent to \( N. \) (for example, is \( N. \) is finitely generated and projective), then:

\[ HH^*(M., N.) \text{ is isomorphic to } HH^*(M. \otimes DN.). \]

(ii) If \( M. \) is such that \( DDM. \) is homotopically equivalent to \( M. \) (for example, is \( M. \) is finitely generated and projective), then:

\[ HH^*(M., N.) \text{ is isomorphic to } HH^*(DM. \otimes N.). \]

We have already seen that \( \overline{H}^*(X \wedge DY) = HC_*(S.(X \wedge DY)) \) and the same for cohomology.

Now we want to show that \( HC^n(S.(X \wedge DY)) \) is isomorphic to \( \overline{H}C^n(S.X.S.Y) \). We shall consider a more general framework.

Let \( C \) be the subcategory of \( \text{Ho}(\partial\text{-mod}_k) \) whose objects are the chain complexes which have a degree \(-2\) action \( S \) and whose morphisms are those ones of \( \text{Ho}(\partial\text{-mod}_k) \) that commute with \( S \).

The subcategory \( C \) consists, then, of the complexes which are \( k[u]\)-comodules (\( \text{dg}(u) = 2 \)), where \( k[u] \) is a coalgebra with coproduct

\[ \triangle(u^n) = \sum_{i+j=n} u^i \otimes u^j \]

and counit \( \kappa(u^i) = 1 \) if \( i = 0 \) and 0 if not.

The cotensor product \( \boxtimes_k[u] \) makes \( C \) a monoidal category.

Proposition 5.4. — Let \( (M, b, B) \) be a mixed complex and let \( DM. \) denote the weak dual of \( M. \) in \( \text{Ho}(\partial\text{-mod}_k) \), and \( BM = k[u] \otimes M. \) the associated total complex, with differential

\[ \partial(u^i \otimes m) = u^i \otimes bm + u^{i-1} \otimes Bm. \]

Then \( DM. \) is also a mixed complex and \( BM. \) is the weak dual of \( BM \) in \( C \). Moreover, if \( M. \) has the homotopy type of a projective finitely generated chain complex (and so \( DM \) is the strong dual of \( M. \) in \( \text{Ho}(\partial\text{-mod}_k) \)), then \( BM. \) is the strong dual of \( BM \) in \( C \).
Proof. — We have the mixed complex \((M, b, B)\). Then \(DM\) is the complex defined by \((DM)_j = \text{Hom}_k(M_{-j}, k)\) (\(DM\) is zero in positive degrees).

We define \(b : DM_j \to DM_{j-1}\) and \(B : DM_j \to DM_{j+1}\) from \(b\) and \(B\) by composition. Then \((DM, b, B)\) is also a mixed complex.

The fact that the evaluation \(\varepsilon : DM \otimes M. \to k\) is a morphism of mixed complexes implies that \(BDM\) is the dual of \(BM\) in \(C\).

If \(DM\) is the strong dual of \(M\) in \(\text{Ho}(\partial\text{-mod}_k)\), we consider the morphisms of \(k[u]\)-comodules:

\[
\varepsilon' : BM \boxtimes_{k[u]} BDM \to k[u] \quad \text{(evaluation) and}
\]
\[
\nu' : k[u] \to BM \boxtimes_{k[u]} BDM \quad \text{(coevaluation)}.
\]

The last one is defined by \(\nu'(u^j) = \sum_i (u^j \otimes a_i) \otimes f_i\) (where \(\{a_1,...,a_n\}\) is a basis of \(\bigcup_{q \in \mathbb{Z}} M_q\) and \(\{f_1,...,f_n\}\) is the dual basis in \(\bigcup_{q \in \mathbb{Z}} (DM)_q\)).

They are such that:

(i) \((\text{id} \otimes \varepsilon') \circ (\nu' \otimes \text{id}) = \text{id} ;
(ii) \((\varepsilon' \otimes \text{id}) \circ (\text{id} \otimes \nu') = \text{id}.

So the proof is finished.

Remark. — \(\text{Hom}_C(BM, BN) = \text{Hom}_S(BM, BN)\).

We have then the following result:

**Theorem 5.5.** — If \(M\) and \(N\) are mixed complexes and \(N\) is such that \(DDN\) is homotopically equivalent to \(N\), there is an isomorphism:

\[
HC^n(M, N.) \cong HC^n(M. \otimes DN.).
\]

Proof. — By the previous Proposition, \(DN\) is also a mixed complex and \(BDN\), being the strong dual of \(BN\) verifies:

\[
HC^n(M, N.) = H_{-n}(\text{Hom}_S(k[u] \otimes M., k[u] \otimes N.)
= H_{-n}(\text{Hom}_S((k[u] \otimes M.) \boxtimes_{k[u]} (k[u] \otimes DN.), k[u])
= H_{-n}(\text{Hom}_S(k[u] \otimes (M. \otimes DN.), k[u]),
\]

and this, by definition, is \(HC^n(M. \otimes DN.)\).

The last isomorphism is due to Eilenberg-Moore [E-M].

**Corollary 5.6.** — If \(M\) has the homotopy type of a projective finitely generated chain complex, there is an isomorphism:

\[
HC^n(M, N.) \cong HC_{-n}(DM. \otimes N.)
\]
COROLLARY 5.7. — If $X$ and $Y$ are pointed paces provided of a pointed action of $S^1$, and $Y$ is homotopically equivalent to $DDY$, then $\bar{HC}^n(S.X, S.Y)$ is isomorphic to $\bar{H}^n_{S^1}(X \wedge DY)$.

Proof. — In this conditions, $S.(DY)$ is the dual of $S.(Y)$ in $\text{Ho}(\partial\text{-mod}_k)$ and so :

$$\bar{HC}^n(S.X, S.Y) \cong HC^n(S.X, S.Y) \cong HC^n(S.X \otimes S.DY)$$

$$\cong HC^n(S.(X \wedge DY)) \cong \bar{H}^n_{S^1}(X \wedge DY).$$

6. Bivariant $S^1$-equivariant cohomology and bivariant cyclic cohomology

In [C], CRABB defines the bivariant $S^1$-equivariant cohomology of two CW-complexes equipped with an $S^1$-action, $X$ and $Y$, as :

$$H^i_{S^1}(X, Y) = H^i_B(E S^1 \times_1 X, E S^1 \times_1 Y)$$

and this last object is defined as

$$\lim_k \left[ (B \times S^k) \wedge_B (E S^1 \times_1 X), (E S^1 \times_1 Y) \wedge_B (B \times \mathbb{H})_{k+i} \right]_B,$$

(the homotopy classes of morphisms that commute with the projections)

where $B = BS^1$ and given two fibrations $Z \to BS^1$ and $Z' \to BS^1$ with sections $s$ and $s'$, $Z \wedge_B Z'$ is the push-out of the diagram :

$$\begin{array}{ccc}
BS^1 & \longrightarrow & Z \\
\downarrow s' & & \downarrow \\
Z' & \longrightarrow & Z \wedge_B Z'.
\end{array}$$

From now on, if $X$ is an $S^1$ space, we denote $X = ES^1 \times_1 X$.

If $D_{BS^1}(Y)$ is the $B$-dual [D-P, 6] of $Y$, then the last expression in the definition of $H^i_{S^1}(X, Y)$ is :

$$\lim_k \left[ S^k \wedge (X \wedge_B D_B(Y)/B), \mathbb{H}_{k+i} \right] = \bar{H}^i((X \wedge_B D_B(Y)/B).$$

But, if $DY$ is the dual of $(Y, 0)$ in Stab, following ([B-G, 4]), then $E S^1 \times_1 DY$ is canonically isomorphic to the $B$-dual of $Y$, that will be denoted $D Y$, so :

$$H^i_{S^1}(X, Y) = \bar{H}^i((X \wedge_B D Y)/B).$$

This is a reduced cohomology, in the sense that if $X = Y = \text{pt}$, then $H^i_{S^1}(X, Y) = 0$, for every $i.$
Proposition 6.1. — \( \overline{H}^n(X \wedge_B \mathbb{D}Y) = \overline{H}^n_{S^1}(X \wedge DY) \) and we have exact sequences (for \( k \in \mathbb{Z} \))

\[
0 \rightarrow H^{2k}((X \wedge_B \mathbb{D}Y)/B) \rightarrow H^{2k}(X \wedge_B \mathbb{D}Y) \xrightarrow{i'} H^{2k}(B) \\
\rightarrow H^{2k+1}((X \wedge_B \mathbb{D}Y)/B) \rightarrow H^{2k+1}(X \wedge_B \mathbb{D}Y) \rightarrow 0,
\]

where \( i' \) splits by means of the section \( s : B \rightarrow X \wedge_B \mathbb{D}Y \).

Proof. — We observe that \( X \wedge_B \mathbb{D}Y \rightarrow B \) is a fibration of fibre \( X \wedge DY \), so we have the long Gysin exact sequence for this fibration.

There is another fibration \( ES^1 \times_{S^1} (X \wedge DY) \rightarrow B \) also of fibre \( X \wedge DY \), and a morphism \( F : ES^1 \times_{S^1} (X \wedge DY) \rightarrow X \wedge \mathbb{D}Y \) given by \( F((e, x, y')) = ((e, x), (e, y')) \) which induces the identity on the fibre (where \( (e, x) \) is the class of \( (e, x) \)). Then \( H^n(X \wedge_B \mathbb{D}Y) = H^n_{S^1}(X \wedge DY) \) for \( n \in \mathbb{N} \) and we also have the corresponding isomorphisms for the reduced cohomology theory.

Next, if we regard (using the section \( s : B \rightarrow X \wedge_B \mathbb{D}Y \)), \( B \) as a subspace of \( X \wedge_B \mathbb{D}Y \), whose quotient space is \( (X \wedge_B \mathbb{D}Y)/B \) we get the desired exact sequences.

From now on we shall take \( k = \mathbb{Z} \).

Theorem 6.2. — Under the conditions of Theorem 5.5, there is an isomorphism :

\[
\overline{H}^n(S, X, S, Y) \cong H^n_{S^1}(X, Y).
\]

Proof. — By Corollary 6.7, \( \overline{H}^n(S, X, S, Y) = \overline{H}^n(S, X, S, Y) \) is isomorphic to \( H^n_{S^1}(X \wedge DY) \), and by the first part of Proposition 6.1, the latter is \( \overline{H}^n(X \wedge_B \mathbb{D}Y) \).

Then, using the exact sequences of this proposition, we obtain that :

\[
H^n(X \wedge_B \mathbb{D}Y) = \begin{cases} 
H^n((X \wedge_B \mathbb{D}Y)/B) & \text{if } n = 2k + 1, \\
H^n((X \wedge_B \mathbb{D}Y)/B) \oplus \mathbb{Z} & \text{if } n = 2k.
\end{cases}
\]

Therefore, \( \overline{H}^n(X \wedge_B \mathbb{D}Y) \) is isomorphic to \( H^n((X \wedge_B \mathbb{D}Y)/B) \), which is, by definition, \( H^n_{S^1}(X, Y) \).

Example. — If the action of \( S^1 \) on \( Y \) is trivial, we observe that \( Y = BS^1 \times Y \) and \( H^n(X, Y) \) is \( H^n_B(X, Y) \) because :

\[
H^n(X, Y) = \lim_{k} [\Sigma^k_X, Y \wedge H_{k+n}],
\]

BULLETIN DE LA SOCIETE MATHEMATIQUE DE FRANCE
while
\[
H^n_B(X, Y) = \lim_{k} \left[ \Sigma^k_B X, (B \times Y) \wedge_B (B \times \mathbb{H})_{k+n} \right]_B
\]
\[
= \lim_{k} \left[ \Sigma^k_B X, B \times (Y \wedge \mathbb{H})_{k+n} \right]_B
\]
\[
= \lim_{k} \left[ \Sigma^k_B X, Y \wedge \mathbb{H}_{k+n} \right]
\]
(see [C-S]).

If we consider now the structure of \(S.Y\) as a cyclic module, we find that it is trivial, and it is known then that \(HC^*(S.Y) = k[u] \otimes H^*(S.Y)\), with trivial \(S\) coaction on \(H^*(S.Y)\).

In this case, \(HC^n(S.X, S.Y) = \text{Hom}(HC^*(S.X), H_*(S.Y))_{-n}\) and this is \(\text{Hom}(H_*(X), H_*(S.Y))_{-n}\) (cf. [J-K, §7]).

Taking the reduced bivariant cyclic cohomology, we get the following identification :
\[
\bar{H}C^n(S.X, S.Y) = \text{Hom}(\bar{H}_*(X), \bar{H}_*(S.Y))_{-n}.
\]

By the isomorphism of the last theorem, \(\bar{H}C^n(S.X, S.Y) = H^n(X, Y)\), which is also \(\text{Hom}(\bar{H}_*(X), \bar{H}_*(Y))_{-n}\) by using the short exact sequence of (1) if we suppose, as in [J-K, §4], that \(HC_*(S.X) = H_*(X)\) is \(k\)-projective.

### 7. Connes’ long exact sequence

Kassel has shown [K2, I.2.3] that there is a long exact sequence (l.e.s.) for bivariant cyclic cohomology :

If \(M.\) and \(N.\) are cyclic \(k\)-modules and \(M\) is \(k\)-projective, there is a long exact sequence :

\[
\cdots \rightarrow HC^{n-2}(M., N.) \xrightarrow{S} HC^n(M., N.) \xrightarrow{I} HH^n(M., N.) \xrightarrow{B} HC^{n-1}(M., N.) \rightarrow \cdots
\]

and he has also described the morphisms \(S, B\) and \(I\).

There is also a long (Gysin) exact sequence :

\[
\cdots \rightarrow H^{n-2}_{S^1}(X \wedge DY) \xrightarrow{S'} H^n_{S^1}(X \wedge DY) \xrightarrow{I'} H^n(X \wedge DY) \xrightarrow{B'} H^{n-1}_{S^1}(X \wedge DY) \rightarrow \cdots
\]

and we want to show that, taking \(M. = S.X\) and \(N. = S.Y\), if we relate the reduced versions of both sequences by the isomorphism of the above paragraphs, then the diagram is commutative:
**Proposition 7.1.** — If $X$ and $Y$ are $S^1$-spaces satisfying the conditions of Theorem 5.5, then the following diagram is commutative:

\[
\cdots \to \overline{H}C^{n-2}(S.X, S.Y) \xrightarrow{S} \overline{H}C^n(S.X, S.Y) \to \cdots
\]

\[
\begin{array}{ccc}
\phi_{n-2} & & \phi_n \\
\downarrow & & \downarrow \\
\cdots & \longleftarrow & \cdots
\end{array}
\]

\[
\cdots \to H^{n-2}_{S^1}(X, Y) \longrightarrow H^n_{S^1}(X, Y)
\]

\[
\begin{array}{ccc}
\phi'^n & & \phi_{n-1} \\
\downarrow & & \downarrow \\
H^n(X, Y) & \longrightarrow & H^{n-1}_{S^1}(X, Y) \longrightarrow \cdots
\end{array}
\]

Proof. — We can introduce an additional row in the middle and consider the following diagram:

\[
\cdots \to \overline{H}\overline{H}^n(S.X, S.Y) \longrightarrow \overline{H}C^{n-1}(S.X, S.Y) \longrightarrow \cdots
\]

\[
\begin{array}{ccc}
\downarrow & & \\
h^n(S.(X \wedge DY)) & \longrightarrow & \overline{H}C^{n-1}(S.(X \wedge DY)) \\
\downarrow & & \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

\[
\begin{array}{ccc}
\overline{H}C^{n+1}(S.X, S.Y) & \longrightarrow & \overline{H}\overline{H}^{n+1}(S.X, S.Y) \longrightarrow \cdots \\
\downarrow & & \\
\overline{H}C^{n+1}(S.(X \wedge DY)) & \longrightarrow & \overline{H}^{n+1}(S.(X \wedge DY)) \longrightarrow \cdots \\
\downarrow & & \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \\
H^{n+1}_{S^1}(X, Y) & \longrightarrow & H^{n+1}(X, Y) \longrightarrow \cdots
\end{array}
\]

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE
As the lower part commutes [J, thm 3.3]), we have to show that the upper part also commutes.

The first exact sequence is a consequence of [K2, I, prop. 2.1], while the second one follows from the short exact sequence:

\[ 0 \to \ker(\text{AdS})_{-n+2} \to \ker(\text{AdS})_{-n} \to \text{Hom}_{-n}(\Sigma(X \wedge DY), k) \to 0, \]

where

\[ \text{AdS} : \text{Hom}_{-n+2}(k[u] \otimes \Sigma(X \wedge DY), k[u]) \to \text{Hom}_{-n}(k[u] \otimes \Sigma(X \wedge DY), k[u]) \]

is defined by \((\text{AdS})(f) = S \circ f - f \circ S\).

So, the proof reduces to verify the commutativity of the following square, as the maps \(\phi_i, \phi_i'\) are defined between the complexes before taking homology:

\[
\begin{array}{ccc}
\ker(\text{AdS})_{-n+2} & \longrightarrow & \ker(\text{AdS})_{-n} \\
\phi'_{n-2} \downarrow & & \downarrow \phi'_n \\
\ker(\text{AdS})_{-n+2} & \longrightarrow & \ker(\text{AdS})_{-n}.
\end{array}
\]

We observe that if \(f\) is an element of \(\text{Hom}_{-n+2}(k[u] \otimes \Sigma X, k[u] \otimes \Sigma Y)\), then \(\phi'_{n-2}(f) = (\text{id} \otimes \varepsilon) \circ (f \square_{k[u]} \text{id}_{\Sigma(DY)})\).

So, if \(f : (k[u] \otimes \Sigma X)_j \to (k[u] \otimes \Sigma Y)_{j-n+2}\) and

\(f \in \ker(\text{AdS}), \quad \phi'_{n-2}(f) \in \ker(\text{AdS}),\)

then we have that:

\[
\begin{array}{ccc}
f & \longrightarrow & (Sf) \\
\downarrow & & \downarrow \\
(\text{id} \otimes \varepsilon) \circ (f \square_{k[u]} \text{id}) & \longrightarrow & S\phi'_{n-2}(f) = \phi'_n(Sf)
\end{array}
\]

is commutative.
Aknowledgements. — I wish to thank Jean-Louis Loday, who proposed me this problem and helped me very much during the realization of this work.
I also thank Ch. Kassel for his commentaries, and the Université de Strasbourg where this paper was written.

BIBLIOGRAPHY


