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Y.-J. CHIANG
A. RATTO

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HARMONIC MAPS ON SPACES WITH CONICAL SINGULARITIES

BY

Y.-J. CHIANG and A. RATTO (*)

Dedicated to Prof. J. EELLS and Prof. J.H. SAMPSON.

RÉSUMÉ. — Nous considérons un espace métrique $X = X^{m+1}$ avec des singularités coniques Σ : le sous ensemble $X - \Sigma$ est une variété riemannienne $(m+1)$-dimensionnelle ouverte et dense dans X. Soit N une variété riemannienne compacte : nous disons qu'une application $f : N \to X$ est *harmonique* si f est continue et si sa restriction à $X - \Sigma$ est harmonique dans les sens usuels. Nous appliquons la méthode du flot de la chaleur pour démontrer un résultat d'existence pour variétés N à courbure non positive. En dimension 2, nos résultats sont valables pour les singularités conformément coniques et nous conduisent à un théorème d'existence pour les courbes algébriques dans les espaces projectifs complexes.

ABSTRACT. — Let $X = X^{m+1}$ be a compact metric space with conical singularities Σ : then $X - \Sigma$ is an open, $(m+1)$-dimensional Riemannian manifold dense in X. Let N be a compact Riemannian manifold; we say that a map $f : X \to N$ is *harmonic* if f is continuous and its restriction to $X - \Sigma$ is harmonic in the usual sense. We apply heat flow techniques to prove an existence result for nonpositively curved N. In dimension 2, our results hold for conformally conical singularities and lead us to an existence theorem for complex projective algebraic curves.

1. Introduction

Harmonic maps $f : M \to N$ between Riemannian manifolds are the smooth critical points of the energy functional

$$E(f) = \frac{1}{2} \int_M |\mathrm{d}f|^2 ;$$

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Y-.J. CHIANG, Department of Mathematics, Mary Washington College, Fredericksburg Virginia 22401-5358, USA.
A. RATTO, Department of Mathematics, University of Warwick, Coventry CV 47AL, Angleterre.

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i.e. maps whose tension field $\tau(f) = \operatorname{Trace}\nabla \mathrm{d}f$ vanishes identically. Their study provides a rich source of interplay between elliptic analysis, geometry and topology (see the surveys [9], [10]); for instance, geodesics, minimal immersions, harmonic functions and holomorphic maps between Kähler manifolds are all special cases of harmonic maps.

In this paper we extend the notion of harmonic map to the case that the domain of f is a space with conical singularities (see [1], [2], [3]). Let M be an m-dimensional Riemannian manifold with metric g_M; the metric cone $C(M)$ on M is the space $M \times (0, +\infty)$, equipped with the metric

$$r^2 g_M + \mathrm{d}r^2. \tag{1.1}$$

Set

$$C_{u,v}(M) = \{(x, r) \in C(M) \mid u < r < v\}; \tag{1.2}$$

and

$$C^*(M) = C(M) \cup p, \tag{1.3}$$

the completion of $C(M)$ by adding the vertex p. We shall consider the following class of spaces

DEFINITION 1.4. — A compact metric space $X^{m+1} = X$, $m \geq 1$, is a space with *conical singularities* if there exist points $p_j \in X$, $j = 1, \dots, k$, such that $X - \bigcup_{j=1,\dots,k} p_j$ is an open $(m+1)$-dimensional Riemannian manifold and each p_j has a neighbourhood U_j such that $U_j - p_j$ is isometric to $C_{0,v_j}(M_j)$, for some v_j and compact (not necessarily connected) manifolds M_j. We denote $\Sigma = \bigcup_{j=1,\dots,k} p_j$.

The standard De Rham-Hodge theory admits an extension to spaces with conical singularities, provided that the usual cohomology and homology are replaced by L^2-cohomology and intersection homology respectively (see the survey [4]). A related type of L^2 analysis concerns the spectral theory of Δ : that leads to a detailed study of wave and heat operators on X, as carried out in [1], [2], [3]. In particular, a smooth, symmetric kernel is associated with the heat operator $\mathrm{e}^{-\Delta t}$, a fact which will be the starting point of our analysis.

We shall study maps $f : X \to N$, where N is a compact Riemannian manifold. We introduce the following :

DEFINITION 1.5. — A map $f : X \to N$ is said to be *harmonic* if it is continuous and its restriction to $X - \Sigma$ is harmonic in the usual sense.

Our main result is :

THEOREM 1.6. — *Assume N compact and* $\operatorname{Riem} N \leq 0$. *Then any continuous map $f_0 : X \to N$ is homotopic to a harmonic map.*

The proof is an adaptation of Eells-Sampson's method (see [11]) to this context.

The paper is organized as follows : in section 2 we recall some properties of the operators Δ and $\mathrm{e}^{-\Delta t}$ on $C(M)$. In section 3 we prove THEOREM 1.6 above. In section 4 we show that the singularities of complex projective algebraic curves C are *conformally conical*; from that we deduce an existence result for maps $f : C \to N$.

2. Functional analysis on $C(M)$

We recall here some basic facts from [1], [2], [3]. Set :

$$
\begin{aligned}
\operatorname{dom} \mathrm{d} &= \{g \in C^\infty \cap L^2 \mid \mathrm{d}g \in L^2\}, \\
\operatorname{dom} \delta &= \{\omega \in \Lambda^1 \cap L^2 \mid \delta\omega \in L^2\}, \\
\operatorname{dom} \Delta &= \{g \in C^\infty \cap L^2 \mid \mathrm{d}g \in \operatorname{dom} \delta\}.
\end{aligned}
\tag{2.1}
$$

The Laplacian Δ on functions, on $C(M)$, is given by

$$
\Delta = \delta\,\mathrm{d} = -\frac{\partial^2}{\partial r^2} - \frac{m}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^M,
\tag{2.2}
$$

where Δ^M denotes the Laplacian on the cross-section M, which we assume to be compact. The operators d and δ have well defined strong closures $\bar{\mathrm{d}}$, $\bar{\delta}$ which satisfy Stokes' theorem

$$
\langle \bar{\mathrm{d}}g, \omega\rangle = \langle g, \bar{\delta}\omega\rangle
\tag{2.3}
$$

for all $g \in \operatorname{dom} \mathrm{d}$, $\omega \in \operatorname{dom} \delta$. It follows that the strong closure of Δ, still denoted by Δ, is a *self-adjoint* operator; in particular, we shall be able to form functions $f(\Delta)$ via spectral theory.

Let $\{\varphi_i\}$ be an orthonormal basis of eigenfunctions of Δ^M, with eigenvalues μ_i. If $g = g(r,x) \in L^2(C(M))$, we can write

$$
g = \sum_i g_i(r)\varphi_i(x)
\tag{2.4}
$$

where the sum converges in the L^2 norm. If we set

$$
\Delta_\mu h(r) = \Big\{-\frac{\partial^2}{\partial r^2} - \frac{m}{r}\frac{\partial}{\partial r} + \frac{\mu}{r^2}\Big\}h(r),
$$

then :

$$\Delta g = \sum_i \big[\Delta_{\mu_i} g_i(r)\big]\varphi_i(x). \tag{2.5}$$

Setting

$$\alpha = -\tfrac{1}{2}(m-1), \quad \nu_i = (\mu_i + \alpha^2)^{1/2}, \tag{2.6}$$

the eigenfunctions of Δ on $C_{0,a}(M)$, for $a > 0$, are given by

$$r^\alpha J_{\nu_i}(\lambda r)\varphi_i, \tag{2.7}$$

where J_ν are Bessel functions (see [15]). Hence the associate eigenvalue is λ^2.

Remark. — Here and below, if $m = 1$, r^α is to be replaced by a suitable logarithmic expression.

Using polar coordinates (r, x) makes it natural replacing Fourier transform with Hankel transform; if $h(r)$ is a smooth function with compact support in $(0, +\infty)$, its Hankel transform is :

$$H_\nu(h)(\lambda) = \int_0^\infty h(r) J_\nu(\lambda r)\,\mathrm{d}r. \tag{2.8}$$

Hankel inversion formula and Plancherel formula are

$$h(r) = H_\nu\big(H_\nu(h)\big)(r) \tag{2.9}$$

and

$$\int_0^\infty \big|h(r)\big|^2 r\,\mathrm{d}r = \int_0^\infty \big|H_\nu(h)(\lambda)\big|^2 \lambda\,\mathrm{d}\lambda \tag{2.10}$$

respectively. Thus H_ν can be extended to an isometry of $L^2((0,+\infty), r\,\mathrm{d}r)$; from that, together with (2.4) and (2.9), it is easy to deduce that the map $\mathcal{H}_\nu$ defined by

$$\mathcal{H}_\nu\big[g(r,x)\big] = \big[H_{\nu_0}(r^{-\alpha}g_0), H_{\nu_1}(r^{-\alpha}g_1), \ldots\big] \tag{2.11}$$

provides an isometry of $L^2(C(M))$ with the Hilbert space of measurable functions $k : (0, +\infty) \to \ell^2$ such that $\int_0^\infty \|k(\lambda)\|^2 \lambda\,\mathrm{d}\lambda < +\infty$. Moreover, using (2.7), it is not difficult to show that $\mathcal{H}_\nu[\Delta g] = \lambda^2 \mathcal{H}_\nu[g]$: therefore (2.11) provides the spectral representation of $\sqrt{\Delta}$.

From standard spectral theory (see [8]) we obtain the following formal expression for the kernel of the operator $f(\Delta)$:

$$k_f(r_1,x_1,r_2,x_2) = \sum_i k_f(r_1,r_2,\nu_i)\varphi_i(x_1)\varphi_i(x_2) \tag{2.12}$$

where

$$k_f(r_1,r_2,\nu_i) = (r_1r_2)^\alpha \int_0^\infty f(\lambda^2)J_{\nu_i}(\lambda r_1)J_{\nu_i}(\lambda r_2)\lambda\,\mathrm{d}\lambda. \tag{2.13}$$

In many interesting cases the above series converges uniformly; in particular, we have :

Proposition 2.14. — *If $f(\Delta) = \mathrm{e}^{-\Delta t}$, for all $a,b,c > 0$, (2.12) converges uniformly on $C_{0,a}(M)\times C_{0,a}(M)\times[b,c]$. The associated kernel, denoted by $H(r_1,x_1,r_2,x_2\,;\,t)$, is smooth, bounded and symmetric. Similarly for the iterated Green's function G^k on $C_{0,a}(M)\times C_{0,a}(M)$ (corresponding to $f(\Delta)=\Delta^{-k}$, $k\geq 1$).*

The distance function on $C(M)$ is given by

$$\begin{aligned}\rho^2&\big((r_1,x_1),(r_2,x_2)\big)\\ &= \begin{cases} r_1^2+r_2^2-2r_1r_2\cos\big(\theta(x_1,x_2)\big), & \text{if } \theta(x_1,x_2)\leq\pi\\ (r_1+r_2)^2, & \text{if } \theta(x_1,x_2)>\pi,\end{cases}\end{aligned} \tag{2.15}$$

where θ denotes distance between points in M. Writting v,z for two arbitrary points in $C(M)$, the parametrix for the heat equations is :

$$P(v,z,t) = \frac{1}{(2\sqrt{\pi})^{m+1}t^{(m+1)/2}}\exp\{-\rho^2(v,z)/4t\}. \tag{2.16}$$

Let $K = K(v,z,t)$ be the fundamental solution of the heat equation associated with P. The following estimates follow by simple modification of the arguments of the nonsingular case (see [13], [16]).

Lemma 2.17. — *For each $0<\mu<1$,*

$$\begin{gathered} K(v,z,t)\leq Bt^{-\mu}\big[\rho(v,z)\big]^{2\mu-m-1}\\ (\partial K/\partial v_i)(v,z,t)\leq Bt^{-\mu}\big[\rho(v,z)\big]^{2\mu-m-2}\\ \big(\partial^2K/\partial v_i\partial v_j\big)(v,z,t)\leq Bt^{-\mu}\big[\rho(v,z)\big]^{2\mu-m-3}\end{gathered}$$

for some $B>0$.

3. Proof of Theorem 1.6

Step 1. — Given $f_0 : X \to N$, we consider on $X - \Sigma$ the parabolic system

$$\frac{\partial f_t}{\partial t} = \tau(f_t). \tag{3.1}$$

We can assume that f_0 is of class C^1 on $X - \Sigma$ and that its energy density $e(f_0)$ is bounded. Following EELLS-SAMPSON [11], we obtain solutions f_t, for $t \geq 0$, which converge uniformly to a continuous map $f_\infty : X \to N$ whose restriction to $X - \Sigma$ is harmonic. Since the general case is not harder, we can assume

$$X = C^*_{0,1}(M) \cup Y, \tag{3.2}$$

where Y is a compact Riemannian manifold with boundary $\partial Y = M$, and the union is along the boundary; i.e., Σ is just one point, the vertex of $C_{0,1}(M)$.

We can blend together (see [1], [2]) the fundamental solution K on $C_{0,1}(M)$ (see (2.16)) with a corresponding fundamental solution on Y to obtain a global fundamental solution K on $X - \Sigma$. Similarly, on $X - \Sigma$ we have a heat kernel H and iterated Green's functions G^k which behave qualitatively, near the singular point, as those of PROPOSITION 2.14.

If $M = S^m$, then the vertex of $C_{0,1}(S^m)$ is actually a regular point and X is an ordinary compact manifold : in this case, THEOREM 1.6 is Eells-Sampson's theorem. If M is not a unit sphere, Σ is a singularity; however, the compactness of M insures that, as $r \to 0$, the kernels H, G^k (expressed in polar coordinates near Σ) behave qualitatively as in the nonsingular case : that is because, after separation of variables, the various functions of r which occur (see (2.7), (2.13) for example) are always solutions of an O.D.E. which qualitatively does not depend on the cross-section M (M compact). Thus the proof of THEOREM 1.6 follows essentially by the arguments of Eells-Sampson for compact manifolds. We indicate the basic steps.

Step 2. — For technical reasons we can assume that N is isometrically embedded in some Euclidean space $\mathbb{R}^q$ by a map $i : N \to \mathbb{R}^q$ whose second fundamental form is denoted by $A = (A^\gamma_{ij})$.

Let $L = -\Delta - (\partial/\partial t)$ be the heat operator, and set $W_t = i \circ f_t$. Then (3.1) becomes :

$$L(W_t) = A(W_t)\big(\mathrm{d}W_t, \mathrm{d}W_t\big)\,; \tag{3.3}$$

in charts,

$$(3.4)\qquad \big(L(W_t)\big)^\gamma = A^\gamma_{\beta\delta}(W_t^\beta)_i(W_t^\delta)_j\, g^{ij}, \quad 1 \le \gamma \le q,$$

where $(W_t^\beta)_i = \partial(W_t^\beta)/\partial x_i$ and $g = (g_{ij})$ is the metric on $X - \Sigma$.

Step 3. — Let $u : X \to \mathbb{R}$ be a continuous function. The integral operator associated to the heat kernel is

$$(3.5)\qquad \mathcal{U}(z,t) = \int_{X-\Sigma} H(z,v\,;\ t)u(v)\,\mathrm{d}v$$

for $z \in X - \Sigma$, $t \ge 0$. From now on we always write $\int$ for $\int_{X-\Sigma}$.

The operator $\mathcal{U}$ is well-defined because $u \in L^2(X - \Sigma)$. Moreover

$$(3.6)\qquad L\mathcal{U}(z,t) = 0 \qquad \text{for all } z \in X - \Sigma,\ t > 0,$$

$$(3.7)\qquad \lim_{t\to 0}\mathcal{U}(z,t) = u(z) \qquad \text{for all } z \in X - \Sigma,$$

and $\mathcal{U}(z,t)$ is continuous for $t \ge 0$.

Step 4. — Because Stokes' theorem (2.3) holds, we can compare the heat kernel H with the fundamental solution K (see [6], [11]) :

$$(3.8)\quad H(z,v\,;\ t) - K(z,v\,;\ t) = q(z,v\,;\ t) \quad \text{for all } z,v \in X - \Sigma,\ t > 0,$$

where q, together with its derivatives of all orders, are smooth, bounded functions which tend uniformly to 0 as $t \to 0$. Moreover, at least for small t and for a suitable $B > 0$, the estimates of LEMMA 2.17 apply to H.

Step 5. — We have the following a priori estimates :

LEMMA 3.9. — *Let $u : X \to \mathbb{R}$ be a continuous function with continuous, bounded first derivatives on $X - \Sigma$. Writing a subscrit i for $\partial/\partial z_i$, we have*

$$(3.10)\qquad \big|\mathcal{U}(z,t)\big| \le \sup_X |u|$$

$$(3.11)\qquad \big|\mathcal{U}_i(z,t)\big| \le C \sup_{X-\Sigma} |u_i|, \quad 0 < t < \delta,$$

for some C, $\delta > 0$.

Proof. — Inequality (3.10) is an application of the maximum principle for the heat equation. As for (3.11), we proceed as follows :

$$(3.12)\qquad \begin{aligned}\big|\mathcal{U}_i(z,t)\big| &= \Big|\int H_i(z,v\,;\ t)u(v)\,\mathrm{d}v\Big| \\ &\le \tilde{e}\Big|\int K_i(z,v\,;\ t)u(v)\,\mathrm{d}v\Big|, \quad 0 < t < \delta,\end{aligned}$$

by Step 4, for some $\tilde{e}$, $\delta > 0$.

Integrating by parts and using Stokes' theorem (2.3), from (3.12) we deduce :

$$\begin{aligned}(3.13)\qquad \big|\mathcal{U}_i(z,t)\big| &\le \tilde{e}\Big|\int K(z,v\,;\ t)u_i(v)\,\mathrm{d}v\Big| \\ &\le \tilde{e}\sup_{X-\Sigma}|u_i| \times \Big|\int K(z,v\,;\ t)\,\mathrm{d}v\Big|.\end{aligned}$$

Now (3.11) follows from (3.13), LEMMA 2.17 and $\mathrm{d}v = r^m\,\mathrm{d}v_M \wedge \mathrm{d}r$ around Σ. ∎

By similar arguments, we also have :

LEMMA 3.14. — *If $u = u(z,t)$ is a continuous function on $X \times [0,\delta]$, then*

$$(3.15)\qquad \Big|\int_0^t \mathrm{d}\tau \int H(z,v\,;\ t-\tau)u(v,\tau)\,\mathrm{d}v\Big| \le D \sup_{X\times[0,t]}|u|$$

for some $D > 0$. If furthermore u_i is continuous and bounded on $(X-\Sigma)\times[0,t]$,

$$(3.16)\qquad \Big|\int_0^t d\tau \int H_i(z,v\,;\ t-\tau)u(v,\tau)\,\mathrm{d}v\Big| \le E \sup_{(X-\Sigma)\times[0,t]}|u_i|$$

for some $E > 0$.

Step 6. — If f_t is a solution of (3.1), then its energy density $e(f_t)$ satisfies :

$$\begin{aligned}(3.17)\quad L\,e(f_t) &= \big|\nabla\,\mathrm{d}f_t\big|^2 \\ &\quad - \big\langle \mathrm{Ricci}(X-\Sigma)\,\mathrm{d}f_t,\,\mathrm{d}f_t\big\rangle - \big\langle \mathrm{Riem}^N(\mathrm{d}f_t,\,\mathrm{d}f_t)\,\mathrm{d}f_t,\,\mathrm{d}f_t\big\rangle.\end{aligned}$$

Clearly the metric structure of X near Σ guarantees that $\mathrm{Ricci}(X-\Sigma)$ is bounded; therefore, as in [11], we can use (3.17), together with $\mathrm{Riem}^N \le 0$, to obtain :

LEMMA 3.18. — *Let $f_t \in C^2((X-\Sigma),N)$ be a solution of* (3.1) *for $0 < t < t_1$, with $t_1 > 1$; assume that $e(f_0)$ is bounded on $X-\Sigma$ and $e(f_t)(v)$ is continuous at $t = 0$. Then*

$$(3.19)\qquad e(f_t)(v) \le c_1 \int e(f_0)\,\mathrm{d}v, \quad 1 \le t < t_1,$$

and

$$e(f_t)(v) \le c_2 \sup_{X-\Sigma}\{e(f_0)\}, \quad 0 \le t \le 1, \tag{3.20}$$

for some $c_1, c_2 > 0$ *independent of* f_t.

Step 7. — Let $W_0 = i \circ f_0$, as in step 2. For integers $\nu \ge 0$, $1 \le \gamma \le q$, we define

$$W^{0,\gamma}(z,t) = \int H(z,v\,;\ t) W_0^\gamma(v)\,\mathrm{d}v \tag{3.21}$$

and

$$W^{\nu,\gamma}(z,t) = -\int_0^t \mathrm{d}\tau \int H\big(z,v\,;\ t-\tau\big) F^{\nu-1,\gamma}(v,\tau)\,\mathrm{d}v + W^{0,\gamma}(z,t) \tag{3.22}$$

where

$$F^{\nu,\gamma}(v,\tau) = A^\gamma_{\alpha\beta}(W^{\nu,\alpha})_i (W^{\nu,\beta})_j\, g^{ij}.$$

Using steps 5 and 6, we can repeat the arguments of section 10 of [11] (see also [6]) to show that, as $\nu \to +\infty$, the successive approximations (3.22) converge to the required solution W_t of (3.3). Moreover, each W_t extends continuously across Σ, because of (3.7) and the fact that W_0 does. $\square$

Remark. — As in [11], THEOREM 1.6 extends to the case that N is complete and satisfies some growth conditions at ∞. On the other hand, *a* straightforward modification of [7] gives examples of initial data $f_0 : C^*(M) \to S^n$ such that the associated solution of the heat equation blows up in finite time (see also [5]).

4. An application of Theorem 1.6

(4.1). — Let C be a (possibly singular) complex projective algebraic curve : i.e., C is the locus of common zeros of a certain finite set of homogeneous polynomials over $\mathbb{C}^{n+1}$. To avoid trivialities, we can assume that C has only isolated singular points, whose union is denoted by Σ. The inclusion $C \to \mathbb{C}P^n$ induces a Kähler metric ω on $C - \Sigma$.

LEMMA 4.2. — *Around each point of* Σ, ω *is conformally conical* (*i.e., conformally equivalent to a metric of type* (1.1)).

Proof. — Let p be a point of C. In terms of local Euclidean coordinates, each branch of C through p can be expressed by

$$F(z) = \big(F_1(z), \dots, F_n(z)\big),$$

for some analytic functions F_j, with $F(z_0) = p$.

Let $\beta(p)$ be the order of vanishing of the Jacobian $(\partial F_j/\partial z)$; i.e.,

$$\beta(p) = \min\big(\mathrm{ord}_{z_0}(\partial F_j/\partial z)\big).$$

And let Ω be the associated $(1,1)$-form of the Fubini-Study metric on $\mathbb{C}P^n$. We show that

$$F^*\Omega = \tfrac{1}{2}\sqrt{-1}\,|z - z_0|^{2\beta(p)} h(z)\,\mathrm{d}z \wedge \mathrm{d}\bar z \tag{4.3}$$

with $h(z)$ smooth and nonzero at z_0.

Indeed, let $v(z)$ be any lifting of F to $\mathbb{C}^{n+1}$, near z_0. Then (see [14])

$$F^*\Omega = \tfrac{1}{2}\sqrt{-1}\,\partial\bar\partial \log\big\|v(z)\big\|^2,$$

where $\|\ \|$ denotes Euclidean norm on $\mathbb{C}^{n+1}$.

In particular, we may take the lifting :

$$v(z) = \big[1, F_1(z), \dots, F_n(z)\big].$$

Then a computation shows

$$F^*\Omega = \tfrac{1}{2}\sqrt{-1}\,\|v\|^{-4}\,\mathrm{d}z \wedge \mathrm{d}\bar z \left\{ \sum_j \Big|\frac{\partial F_j}{\partial z}\Big|^2 + \sum_{j\neq k} \Big|\frac{\partial F_j}{\partial z}\,F_k - \frac{\partial F_k}{\partial z}\,F_j\Big|^2 \right\},$$

from which (4.3) follows easily. Now from (4.3) :

$$\omega = |z - z_0|^{2\beta(p)} h(z)\,\mathrm{d}z \otimes \mathrm{d}\bar z. \tag{4.4}$$

But in polar coordinates

$$\mathrm{d}z \otimes \mathrm{d}\bar z = r^2\,\mathrm{d}\theta + \mathrm{d}r^2,$$

and so the proof of the Lemma is complete. □

Remark 4.5. — Expression (4.4) tells us that $p \in C - \Sigma$ iff C has only one branch through p and $\beta(p) = 0$. If $p \in \Sigma$, then, around p, C is conformally equivalent to $C_{0,1}(M)$, where M is a disjoint union of as many copies of S^1 as there are branches of C through p.

(4.6). — For two dimensional domains, harmonicity is invariant under conformal changes of metric; this fact, together with THEOREM 1.6 and LEMMA 4.2, gives :

THEOREM 4.7. — *In the notation of* (4.1), *let* $f_0 : C \to N$ *be a continuous map into a compact manifold* N *with* $\mathrm{Riem}^N \leq 0$. *Then* f_0 *is homotopic to a continuous map* $f : C \to N$ *whose restriction* $f : (C - \Sigma, \omega) \to N$ *is harmonic.*

(4.8). — Let W, Y be compact Riemann surfaces. Suppose that there exists a non-constant $\pm$ holomorphic map $\varphi : W \to C$ (i.e., $\varphi : W \to \mathbb{C}P^n$ is $\pm$ holomorphic and $\varphi(W) \subset C$) of degree d_φ. Let $f : C \to Y$ be a non-constant map of degree d_f which is harmonic on $C - \Sigma$. Applying the main THEOREM of [12] to $f \circ \varphi$ and using the characterization

$$\pm\text{holomorphic} = \text{weakly conformal},$$

it is not difficult to deduce the following :

If $e(W) + |\mathrm{d}_f \mathrm{d}_\varphi e(Y)| > 0$, *then* f *is* $\pm$ *holomorphic on* $C - \Sigma$.

(Here $e(W)$, $e(Y)$ denote Euler characteristics.)

(4.9). — If we replace (1.1) by metrics of type

$$f^2(r) g_M + \mathrm{d}r^2, \quad \lim_{r \to 0} f(r) = 0, \tag{4.10}$$

the separation of variables methods of this paper do not immediately generalize : the main difficulty is the more complicated type of singularity (at $r = 0$) of the associated O.D.E. We note the following construction : consider the metric

$$g = \sin^2 r \, g_{S^{n-1}} + \sin^{2(n-1)} r \, \mathrm{d}r^2$$

on $S^n = S^{n-1} \times [0, \pi]$, with $n \geq 2$. Then g (after reparametrization) has two singularities of type (4.10) at the two poles. The map $\Phi : (S^n, g) \to S^1$ defined by $\Phi(x, r) = 2r$ is continuous on S^n and harmonic on S^n minus the two poles (Note that any harmonic map $S^n \to S^1$, $n \geq 2$, is constant.)

BIBLIOGRAPHY

[1] Cheeger (J.). — On the spectral geometry of spaces with conical singularities, *Proc. Nat. Acad. Sci. U.S.A.*, t. **76**, 1979, p. 2103–2106.

[2] Cheeger (J.). — Singular Riemannian spaces, *J. Differential Geom.*, t. **18**, 1983, p. 575–657.

[3] Cheeger (J.) and Taylor (M.). — On the diffraction of waves by conical singularities I, II, *Comm. Pure Appl. Math.*, t. **35**, 1982, p. 275–331 and p. 487–529.

[4] Cheeger (J.), Goresky (M.) and MacPherson (R.). — *L^2-cohomology and intersection homology of algebraic varieties.* — Annals Math. Studies 102, Princeton, 1982.

[5] Chen (Y.) and Ding (W.-Y.). — Blow-up and global existence for heat flows of harmonic maps, *Invent. Math.*, t. **99**, 1990, p. 567–578.

[6] Chiang (Y.-J.). — Harmonic maps of V-manifolds, *Ann. Global Anal. Geom.*, t. **8**, 1990, p. 315–344.

[7] Coron (J.-M.) et Ghidaglia (J.-M.). — Explosion en temps fini pour le flot des applications harmoniques, *C.R. Acad. Sci. Paris*, t. **308**, 1989, p. 339–344.

[8] Dunford (N.) and Schwartz (J.-T.). — *Linear operators I, II.* — Wiley-Interscience, 1963.

[9] Eells (J.) and Lemaire (L.). — A report on harmonic maps, *Bull. London Math. Soc.*, t. **10**, 1978, p. 1–68.

[10] Eells (J.) and Lemaire (L.). — Another report on harmonic maps, *Bull. London Math. Soc.*, t. **20**, 1988, p. 385–524.

[11] Eells (J.) and Sampson (J.-H.). — Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, t. **86**, 1964, p. 109–164.

[12] Eells (J.) and Wood (J.-C.). — Restrictions on harmonic maps of surfaces, *Topology*, t. **15**, 1976, p. 263–266.

[13] Friedmann (A.). — *Partial differential equations of parabolic type.* — Prentice-Hall Inc., 1964.

[14] Griffiths (P.) and Harris (J.). — *Principles of algebraic geometry.* — Wiley-Interscience, 1978.

[15] Lebedey (N.). — *Special functions and their applications.* — Dover, New York, 1972.

[16] Pogorzelsky (W.). — Propriétés des intégrales de l'équation parabolique normale, *Ann. Math. Polinici*, t. **4**, 1957, p. 61–92.