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PERIOD MAPPING VIA BRIESKORN MODULES

BY

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RÉSUMÉ. — Pour une déformation μ-constante de fonctions holomorphes à singularités isolées, on définit une application de période qui associe le module de Brieskorn à chaque point, où on utilise le système de Gauss-Manin pour définir la translation parallèle. On démontre qu'elle induit un morphisme fini de la strate μ-constante d'une déformation miniverselle. Cela signfie que le module local de fonction est déterminé par le module de Brieskorn à une ambiguité finie près.

ABSTRACT. — For a μ-constant deformation of holomorphic functions with isolated singularities, we define a period mapping by associating the Brieskorn module to each point, where we use the Gauss-Manin system to define the parallel translation. We show that it induces a finite morphism of the μ-constant stratum of a miniversal deformation. This means that the local moduli of function is determined by Brieskorn module up to finite ambiguity.

Introduction

Let $f : (\mathbb{C}^n, 0) \times (S, 0) \to (\mathbb{C}, 0)$ be a μ-constant deformation of holomorphic function with isolated singularity at $0 \in \mathbb{C}^n$, parametrized by a complex analytic space $(S, 0)$. Let f_s be the restriction of f to $(\mathbb{C}^n, 0) \times \{s\}$ for $s \in S$. By [26] we have a canonical mixed Hodge structure on the cohomology of the Milnor fiber of f_s. Then we can define a period mapping, assuming S contractible (by shrinking S if necessary). The weight filtration of the mixed Hodge structure is determined by the monodromy (so called the monodromy filtration) and remains constant by μ-constant deformation, and only the Hodge filtration varies.

Here we use Deligne's vanishing cycle sheaf [6] along f, which enables us to avoid a delicate problem about the topological triviality of μ-constant deformation. This is a locally constant sheaf of vanishing cohomologies (up to a shift) on $\{0\} \times S$, and induces the parallel translation of the Hodge

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filtration which is needed to define the period mapping. We can show also that the mapping is holomorphic (i.e., the Hodge filtration determines holomorphic vector subbundles of the vector bundle corresponding to the local system of vanishing cohomologies), even if S is singular. Note that the notion of variation of (mixed) Hodge structure is not yet defined on singular spaces.

Unfortunately this period mapping does not provide enough information, because the fibers of the mapping have positive dimensions in general (e.g., $f_0 = x^5 + y^4$), even when S is the μ-constant stratum of the base space of a miniversal deformation of f_0, cf. also (3.4). So we consider a refinement of the period mapping using *Brieskorn module.* For $s \in S$, Brieskorn module of f_s is defined by :

$$\mathcal{H}''_s = \Omega^n_{\mathbf{C}^n,0}/\mathrm{d}f_s \wedge \mathrm{d}\Omega^{n-2}_{\mathbf{C}^n,0}$$

[3] which has the structure of $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-module [13], [17], i.e. $\mathbb{C}\{t\}$-module with (regular) singular connection ∇ such that the inverse of $\partial_t = \nabla_{\partial/\partial_t}$ is well-defined, cf. [3], where t is the coordinate of $\mathbb{C}$. Let $\mathcal{G}_s$ be the localization of $\mathcal{H}''_s$ by the action of ∂_t^{-1}. It is a regular holonomic $\mathcal{D}_{\mathbb{C},0}$-module, called the *Gauss-Manin system* of f_s [17], and $\mathcal{H}''_s$ is a $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodule of $\mathcal{G}_s$. By VARCHENKO [27] (see also [17], [20], [21], [24]), the mixed Hodge structure on the cohomology of Milnor fiber can be obtained by taking Gr_V of $\mathcal{H}''_s$, cf. (2.6.1), where V is the filtration of $\mathcal{G}_s$ by eigenvalues of the action of $\partial_t t$.

So the Brieskorn module gives finer information than the mixed Hodge structure. Using Deligne's vanishing cycle sheaf again, we show that the $\mathcal{G}_s$ ($s \in S$) form a locally constant sheaf of regular holonomic $\mathcal{D}_{\mathbb{C},0}$-modules on S, and get the parallel translation of the elements of $\mathcal{G}_s$, cf. (2.9). Assume S contractible so that $\mathcal{G}_s$ for $s \in S$ is identified with each other by parallel translation, and denote $\mathcal{G}_s$ by $\mathcal{G}$. Then the $\mathcal{H}''_s$ are $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodules of $\mathcal{G}$ parametrized holomorphically by S, i.e., $\mathcal{H}''_s$ ($s \in S$) determines a locally free subsheaf of the holomorphic scalar extension of the above locally constant sheaf (cf. (2.7-8) for a precise statement). So we get a refined period mapping :

(0.1) $$\Psi : S \to \mathbf{L}(\mathcal{G})$$

by associating Brieskorn module $\mathcal{H}''_s$ to $s \in S$, where $\mathbf{L}(\mathcal{G})$ is a set of $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodules of $\mathcal{G}$ satisfying some conditions, cf. (2.9). Here the locally constant sheaf of regular holonomic $\mathcal{D}_{\mathbb{C},0}$-modules plays the role of the local system of vanishing cycles in the period mapping via the Hodge filtration on the vanishing cohomology. The problem is whether (0.1) is injective when S is the μ-constant stratum of the base space of a miniversal deformation of f_0, i.e. :

(0.2) *Problem : Is the local moduli of μ-constant deformation determined completely by Brieskorn modules ?*

See Supplement of [20]. In [22, 2.10] we proved :

THEOREM 0.3. — (0.1) *is (locally) injective on the smooth points of the μ-constant stratum*, cf. (3.2).

In this paper we show in general :

THEOREM 0.4. — (0.1) *is finite to one by shrinking the μ-constant stratum S if necessary*, cf. (3.3).

The proof is not difficult, once the period mapping is defined. In fact, since Ψ is analytic, (0.4) is reduced to $\dim \Psi^{-1}\Psi(0) = 0$, and follows from (0.3), restricting to the smooth points of $\Psi^{-1}\Psi(0)$ if $\dim \Psi^{-1}\Psi(0) > 0$. So the local moduli is determined up to finite ambiguity. For the moment, I don't have enough evidence to conjecture the injectivity or non injectivity of Ψ.

As a corollary of (0.4), we can get some information about the failure of the injectivity of the period mapping via mixed Hodge structures as above, cf. (3.4), using the structure of Brieskorn module [22], because the Hodge filtration of the vanishing cohomology is obtained by the graduation of the Brieskorn module by the filtration V, cf. (2.6.1), where the information lost by the graduation is expressed by the linear mappings $c_{\beta,\alpha}$ in [*loc. cit.*]. But it is not easy to relate this directly with the geometry of the discriminant as in [8].

Note that the mixed Hodge structure and the associated period mapping in [*loc. cit.*] are not well-defined because he considers deformation of hypersurface instead of function. Although there is an embedding of the base space of the miniversal deformation of hypersurface into the product of an open disc with the base space of the miniversal deformation of function, it is not unique and the period mapping of the μ-constant stratum obtained by composition is not well-defined. It is not clear whether we can get an interesting result using a not well-defined mapping. Note also that the theory of logarithmic vector fields and primitive forms are not so useful for the study of the Torelli problem as in [*loc. cit.*], because the

Euler vector field E is everywhere non zero (i.e. $\mathrm{Fix}(E) = \emptyset$) and the variation of mixed Hodge structure is not constant on the logarithmic strata contained in the μ-constant stratum in general, cf. (3.5).

We also give a counter example to [8, (5.3)], cf. (3.9). It is possible to give a correct (but rather transcendental) version of [*loc. cit.*] using MALGRANGE's extension of a good basis [31], cf. (3.10).

In § 1, we review some elementary facts from the theory of regular holonomic $\mathcal{D}$-module of one variable. Using this, we construct the period mapping Ψ of a not necessary smooth base space in § 2. Then the main theorems are proved in § 3.

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1. Regular holonomic $\mathcal{D}$-modules of one variable

We review some facts from the elementary theory of regular holonomic $\mathcal{D}$-module of one variable which will be needed in the next section, see also [1], [2], [7], [22, § 1], etc.

1.1. — We denote by Δ an open disc $\{t \in \mathbb{C} : |t| < \delta\}$. Put :

$$\mathcal{O} := \mathcal{O}_{\Delta,0} = \mathbb{C}\{t\}, \quad \mathcal{D} := \mathcal{D}_{\Delta,0} = \mathbb{C}\{t\}[\partial_t].$$

Let $M_{\mathrm{rh}}(\mathcal{D})$ be the category of regular holonomic $\mathcal{D}$-modules, i.e. $\mathcal{D}$-modules M of finite type such that $M[t^{-1}]$ (localization by t) are $\mathbb{C}\{t\}[t^{-1}]$-modules of finite type with regular singular connection in the classical sense, cf. [7]. Let $M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$ be the category of regular holonomic $\mathcal{D}_\Delta$-Modules whose characteristic varieties are contained in $T_0^*\Delta$ (i.e. their stalks at 0 belong to $M_{\mathrm{rh}}(\mathcal{D})$ and their restrictions to the punctured disc Δ^* are locally free finite $\mathcal{O}_{\Delta^*}$-Modules with integrable connection). Then $M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$ is independent of Δ by restriction morphisms, because the locally free finite $\mathcal{O}_{\Delta^*}$-Module with integrable connection can be uniquely extended to a larger punctured disc. So we have an equivalence of categories :

$$M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0 \xrightarrow{\sim} M_{\mathrm{rh}}(\mathcal{D}) \tag{1.1.1}$$

using coherent extension of $\mathcal{D}$-modules. For $M \in M_{\mathrm{rh}}(\mathcal{D})$ and $\alpha \in \mathbb{C}$, let :

$$M^\alpha = \bigcup_{i \geq 0} \mathrm{Ker}\big((\partial_t t - \alpha)^i : M \to M\big). \tag{1.1.2}$$

Then M is generated by M^α ($\alpha \in \mathbb{C}$) over $\mathcal{O} = \mathbb{C}\{t\}$, and we get a natural inclusions :

$$\bigoplus_{\alpha\in\mathbb{C}} M^\alpha \longrightarrow M \longrightarrow \prod_{\alpha\in\mathbb{C}} M^\alpha \tag{1.1.3}$$

which induces an infinite decomposition of M. This gives the asymptotic expansion of an element of M, cf. [22, 1.5]. This is inspired by VARCHENKO's theory of asymptotic Hodge filtration [27], and gives a version of Varchenko's theory used in [23], cf. (2.6) below. Note that (1.1.3) is classical if M is meromorphic type as in (1.3.2), and the general case is reduced to this case using the localization morphism $M \to M[t^{-1}]$. Since the action of $(\partial_t t - \alpha)$ is nilpotent on M^α, we have bijections :

$$t : M^\alpha \xrightarrow{\sim} M^{\alpha+1}, \quad \partial_t : M^{\alpha+1} \xrightarrow{\sim} M^\alpha \quad \text{for} \quad \alpha \neq 0. \tag{1.1.4}$$

Let Λ be a subset of $\mathbb{C}$ such that $\Lambda \to \mathbb{C}/\mathbb{Z}$ is bijective and $1 \in \Lambda$. We define :

$$\psi_t M = \bigoplus_{\alpha\in\Lambda} M^\alpha. \tag{1.1.5}$$

Then $\psi_t M$ depends only on $M[t^{-1}]$, i.e. the localization morphism $M \to M[t^{-1}]$ induces an isomorphism :

$$\psi_t M \xrightarrow{\sim} \psi_t M[t^{-1}]. \tag{1.1.6}$$

In fact, we can show that ψ_t is an exact functor, and the kernel and cokernel of $M \to M[t^{-1}]$ are supported in $\{0\}$ (i.e. the supports of their coherent extensions are contained in $\{0\}$), where

$$M^\alpha = 0 \quad \text{for} \quad -\alpha \notin \mathbb{N} \quad \text{if} \quad \operatorname{supp} M = \{0\} \tag{1.1.7}$$

by (1.1.4).

1.2. — We say that $M \in M_{\mathrm{rh}}(\mathcal{D})$ is *quasi-unipotent* if $M^\alpha = 0$ for $\alpha \notin \mathbb{Q}$. In this case, we choose Λ in (1.1.5) so that $\Lambda \cap \mathbb{Q} = (0,1] \cap \mathbb{Q}$. If M is quasi-unipotent, we define the filtration V by :

$$V^\alpha M = M \cap \prod_{\beta\geq\alpha} M^\beta \tag{1.2.1}$$

using (1.1.3). (See also [10], [14].) By definition, we have a canonical isomorphism :

$$(1.2.2) \qquad M^{\alpha} \xrightarrow{\sim} \mathrm{Gr}_V^{\alpha} M.$$

We can check :

$$(1.2.3) \qquad \begin{array}{l} V^{\alpha}M \text{ for } \alpha > 0 \text{ is finite free} \\ \text{over } \mathbb{C}\{t\} \text{ and also over } \mathbb{C}\{\{\partial_t^{-1}\}\}, \end{array}$$

where $\mathbb{C}\{\{\partial_t^{-1}\}\}$ is a discrete valuation ring defined by

$$\Big\{ \sum_{i \geq 0} a_i \partial_t^{-i} : \sum_{i \geq 0} a_i r^i / i! < \infty \text{ for some } r > 0 \Big\}.$$

Let $M \in M_{\mathrm{rh}}(\mathcal{D}_{\Delta})_0$. We say that M is *quasi-unipotent*, if so is its stalk at 0, or equivalently, so is the monodromy of its corresponding local system on Δ^*, cf. (1.4.5). In this case, M has the filtration V defined by (1.2.1) at 0 and by $M_{|\Delta^*}$ outside 0, cf. [10], [14]. Note that $V^{\alpha}M$ are coherent $\mathcal{O}_{\Delta}$-sub-Modules of M, because the localization of $V^{\alpha}M_0 \to M_0$ by t is an isomorphism by (1.1.7) and the restriction to Δ^* of a coherent extension of $V^{\alpha}M_0 \to M_0$ is an isomorphism.

1.3. — Let $M_{\mathrm{rh}}(\mathcal{D}, *)$ (resp. $M_{\mathrm{rh}}(\mathcal{D}, !)$) be the full subcategory of $M_{\mathrm{rh}}(\mathcal{D})$ such that the action of t (resp. ∂_t) on its objects is bijective. This condition is equivalent to :

$$(1.3.1) \qquad t : M^0 \xrightarrow{\sim} M^1 \quad (\text{resp. } \partial_t : M^1 \xrightarrow{\sim} M^0).$$

Combined with (1.1.3–4), $M \in M_{\mathrm{rh}}(\mathcal{D}, *)$ (resp. $M_{\mathrm{rh}}(\mathcal{D}, !)$) is uniquely determined by $\psi_t M = \bigoplus_{\alpha \in \Lambda} M^{\alpha}$ with the action of $N := -(\partial_t t - \alpha)$. We can easily check :

$$(1.3.2) \qquad \begin{array}{l} M \text{ is finite free over } \mathbb{C}\{t\}[t^{-1}] \quad (\text{resp. } \mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]) \\ \text{if } M \in M_{\mathrm{rh}}(\mathcal{D}, *) \quad (\text{resp. } M_{\mathrm{rh}}(\mathcal{D}, !)). \end{array}$$

We say that $M \in M_{\mathrm{rh}}(\mathcal{D}, *)$ (resp. $M_{\mathrm{rh}}(\mathcal{D}, !)$) is *meromorphic* (resp. *microlocal*) *type*. Let $M_{\mathrm{rh}}(\mathcal{D}_{\Delta}, *)_0$ (resp. $M_{\mathrm{rh}}(\mathcal{D}_{\Delta}, !)_0$) be the full subcategory of $M_{\mathrm{rh}}(\mathcal{D}_{\Delta})_0$ defined by the condition :

$$(1.3.3) \qquad M \xrightarrow{\sim} M[t^{-1}] \quad \big(\text{resp. } j_! j^{-1} \mathrm{DR}_{\Delta}(M) \xrightarrow{\sim} \mathrm{DR}_{\Delta}(M)\big)$$

where DR_Δ is the de Rham functor defined by :

$$\mathrm{DR}_\Delta(M) = C(\partial_t : M \to M),$$

and $j : \Delta^* \to \Delta$ is the natural inclusion. Note that the condition $M \xrightarrow{\sim} M[t^{-1}]$ is equivalent to $\mathrm{DR}_\Delta(M) \xrightarrow{\sim} \mathbf{R}j_*j^{-1}\,\mathrm{DR}_\Delta(M)$ by the Riemann-Hilbert correspondence [11], [15]. We have equivalences of categories :

$$(1.3.4) \qquad M_{\mathrm{rh}}(\mathcal{D}_\Delta, *)_0 \xrightarrow{\sim} M_{\mathrm{rh}}(\mathcal{D}, *), \quad M_{\mathrm{rh}}(\mathcal{D}_\Delta, !)_0 \xrightarrow{\sim} M_{\mathrm{rh}}(\mathcal{D}, !)$$

induced by (1.1.1). In fact, the assertion for $*$ is clear, and the assertion for ! follows from :

$$(1.3.5) \qquad \mathrm{DR}_\Delta(M)_{|\{0\}} = C(\partial_t : (M_0)^1 \to (M_0)^0)$$

for $M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$, cf. (1.1.4), where M_0 is the stalk of M at 0.

1.4. — Let $M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$. Then $\mathrm{DR}_\Delta(M)[-1]_{|\Delta^*}$ is a $\mathbb{C}$-local system. Let $L(\Delta^*, \mathbb{C})$ be the category of $\mathbb{C}$-local systems on Δ^*. Using Riemann-Hilbert correspondence [11], [15], $M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta, *)_0$ or $M_{\mathrm{rh}}(\mathcal{D}_\Delta, !)_0$ is uniquely determined by its restriction to Δ^*, because it holds for $\mathrm{DR}_\Delta(M)$ by definition (1.3.3). So we get equivalences of categories :

$$(1.4.1) \qquad M_{\mathrm{rh}}(\mathcal{D}_\Delta, *)_0 \xrightarrow{\sim} L(\Delta^*, \mathbb{C}), \quad M_{\mathrm{rh}}(\mathcal{D}_\Delta, !)_0 \xrightarrow{\sim} L(\Delta^*, \mathbb{C}),$$

which can be also checked using the local classification of regular holonomic $\mathcal{D}$-modules of one variable, cf. [1], [2], [22, 1.4.2]. For $M_{\mathrm{rh}}(\mathcal{D}_\Delta, *)_0$, the inverse functor is constructed explicitly by DELIGNE [7] :

Let $L \in L(\Delta^*, \mathbb{C})$, and $\{u_1, \dots, u_n\}$ be a basis of the multivalued sections of L. Then the corresponding $M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta, *)_0$ is a $\mathcal{O}_\Delta[t^{-1}]$-sub-Module of $j_*(\mathcal{O}_{\Delta_*} \otimes_{\mathbb{C}} L)$ with a basis $\{v_1, \dots, v_n\}$ over $\mathcal{O}_\Delta[t^{-1}]$ defined by :

$$(1.4.2) \qquad v_j = \exp\bigl(-\log t(\log T)/2\pi i\bigr)u_j \in \Gamma\bigl(\Delta, j_*(\mathcal{O}_{\Delta^*} \otimes_{\mathbb{C}} L)\bigr).$$

Moreover, $\{v_1, \dots, v_n\}$ is a basis of $V^\alpha M$ (resp. $V^{>\alpha}M$) over $\mathcal{O}_\Delta$, if L has quasi-unipotent monodromy (so that M_0 is quasi-unipotent) and the eigenvalues of $-(\log T)/2\pi i$ are contained in $[\alpha - 1, \alpha[$ (resp. $]\alpha - 1, \alpha]$). In particular, $V^\alpha M$ (resp. $V^{>\alpha}M$) coincides with DELIGNE's extension [7].

Let $L \in L(\Delta^*, \mathbb{C})$. Then Deligne's nearby cycles $\psi_t L$ [6] is isomorphic to the multivalued sections of L, and is endowed with the action of monodromy T. Let $V(\mathbb{C}, T)$ be the category of $\mathbb{C}$-vector spaces with automorphism T. Then we have an equivalence of categories :

$$(1.4.3) \qquad \psi_t : L(\Delta^*, \mathbb{C}) \xrightarrow{\sim} V(\mathbb{C}, T)$$

as well-known. So we get equivalences of categories :

$$(1.4.4) \qquad M_{\mathrm{rh}}(\mathcal{D},*) \xrightarrow{\sim} V(\mathbb{C},T), \quad M_{\mathrm{rh}}(\mathcal{D},!) \xrightarrow{\sim} V(\mathbb{C},T)$$

by (1.3.4), (1.4.1), (1.4.3). We can also get this using the functor ψ_t in (1.1.5), because we have the compatibility of ψ_t with the de Rham functor :

$$(1.4.5) \qquad \psi_t M = \psi_t(\mathrm{DR}_\Delta(M)_{|\Delta^*})[-1] \quad \text{for} \quad M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0,$$

where the action of T on the right hand side corresponds to $\exp(-2\pi i\partial_t t)$ on $\psi_t M = \bigoplus_{\alpha\in\Lambda} M^\alpha$, cf. (1.1.5). This compatibility is a special case of [10], [14]. It is reduced to the meromorphic case (i.e. $M \in M_{\mathrm{rh}}(\mathcal{D}_\Delta,*)_0$) by (1.1.6), and follows from [7] using the morphism $\exp(-\log t(\log T)/2\pi i)$ in (1.4.2). As a corollary, the inverse functor of the equivalence of categories (1.4.4) is given as follows :

Let $L \in V(\mathbb{C},T)$, and $L = \bigoplus_\lambda L_\lambda$ the decomposition by eigenvalues of T. Then the corresponding M is defined by :

$$(1.4.6) \qquad M^\alpha = L_\lambda \quad \text{for} \quad \lambda = \exp(-2\pi i\alpha),$$

using (1.1.3–4), (1.3.1), where T on L_λ corresponds to $\exp(-2\pi i\partial_t t)$ on M^α.

2. Gauss-Manin systems and Brieskorn modules

We construct the period mapping via Brieskorn modules for a μ-constant deformation of function. The underlying idea is to imitate the construction of the variation of Hodge structure associated with a smooth projective family, cf. remark after (2.8). Since the base space may be singular, we cannot apply directly the theory of mixed Hodge Modules, because it produces an object different from what we need to construct the holomorphic period mapping (i.e., the holomorphic vector bundle corresponding to a local system) in the singular case. So we take a rather classical approach.

2.1. — Let X be an open set of $\mathbb{C}^n$ containing 0, S a reduced complex analytic space, and $f : X \times S \to \mathbb{C}$ a holomorphic function. In this paper we assume $n > 1$, because the case $n = 1$ is trivial. Put $f_s = f_{|X\times\{s\}}$. Assume $f_s(0) = 0$, and $\mathrm{Sing}\, f_s = \{0\}$. Then it is well-known that the Milnor number $\mu(f_s)$ of f_s is constant (using the discriminant, for example), and it is denoted by μ. So f is called a μ-constant deformation (which is preferable to μ-constant unfolding). We define :

$$(2.1.1) \qquad \mathcal{H}''_f := \Omega^n_{X\times S/S}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-2}_{X\times S/S}{}_{|\{0\}\times S}.$$

By the same argument as [3], $\mathcal{H}''_f$ has a structure of $\mathcal{O}_S\{t\}$-Module with action of ∂_t^{-1} such that ∂_t^{-1} commutes with the action of $\mathcal{O}_S$, where $\mathcal{O}_S\{t\} = \mathcal{O}_{\mathbb{C}\times S\,|\,\{0\}\times S}$ with t the coordinate of $\mathbb{C}$. We define a subcomplex $(K^\bullet, \mathrm{d})$ of $(\Omega^\bullet_{X\times S/S}, \mathrm{d})$ by :

$$K^i = \mathrm{Ker}\big(\mathrm{d}f\wedge : \Omega^i_{X\times S/S} \longrightarrow \Omega^{i+1}_{X\times S/S}\big). \tag{2.1.2}$$

Since $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ is a regular sequence, the complex :

$$(\Omega^\bullet_{X\times S/S}, \mathrm{d}f\wedge)$$

is acyclic except for the degree n, and :

$$\mathrm{d}f\wedge \mathrm{d}\Omega^{n-2}_{X\times S/S} = \mathrm{d}\big(\mathrm{d}f\wedge\Omega^{n-2}_{X\times S/S}\big) = \mathrm{d}(K^{n-1}).$$

So we get :

$$\mathcal{H}''_f = \mathcal{H}^n(K)_{\,|\,\{0\}\times S}. \tag{2.1.3}$$

Let $Z := \{(x,s) \in \mathbb{C}^n \times S : |x| < \epsilon,\ |f(x)| < \delta\}$ for $0 < \delta \ll \epsilon \ll 1$, and $g : Z \to Y := \Delta \times S$ the morphism induced by $f \times \mathrm{pr}_2$, where Δ is the open disc of radius δ, and we embed $(S, 0)$ in $(\mathbb{C}^m, 0)$ and replace S by its intersection with a ball of radius δ' for δ' sufficiently small. Let $\Delta^* = \Delta \setminus \{0\}$. By the same argument as [16], g induces a Milnor fibration, i.e. :

(2.1.4) The restriction of g over $Y^* = \Delta^* \times S$ is a topological fibration for $0 < \epsilon \ll 1$, $0 < \delta < \delta(\epsilon)$, $0 < \delta' < \delta'(\epsilon)$, and the topological type of the fiber (denoted by F_0) is independent of ϵ,

where $\delta(\epsilon), \delta'(\epsilon)$ are constants depending on ϵ such that $g^{-1}(t,s)$ is transversal to the sphere of radius ϵ for $0 < \epsilon \ll 1$, $|t| < \delta(\epsilon)$, $|s| < \delta'(\epsilon)$. Using an extension of f to $X \times \mathbb{C}^m$, this can be reduced to the S smooth case (but g becomes a topological fibration over a Zariski open subset of $\Delta \times S$). Note that $\dim_{\mathbb{Q}} \tilde{H}^{n-1}(F_0, \mathbb{Q}) = \mu$ the Milnor number of f_0.

Let $s \in S$ with S as in (2.1.4). Replacing 0 in (2.1.4) with s, we have a Milnor fibration defined over a neighborhood of s. But its range of ϵ may be smaller than the range of ϵ of the Milnor fibration in (2.1.4), and it is not trivial whether the inclusion of the Milnor fibers $F_s \to F_0$ induces an isomorphism of cohomologies even though they have the same dimension. In fact, this is related with the delicate problem about the topological triviality of Milnor fibration, cf. [30]. But the cohomological assertion with rational coefficients can be easily checked analytically, cf. remark after (2.3).

Since g is a Stein morphism, we have :

$$R^n g_* K = g_* \Omega^n_{Z/S} / \mathrm{d}(g_* K^{n-1}), \tag{2.1.5}$$

and

(2.1.6) $R^n g_* K$ is compatible with base change by a closed embedding of S.

The following is well-known to specialists in the S smooth case :

(2.1.7) $R^n g_* K$ is a free $\mathcal{O}_Y$-Module of rank μ for ϵ, δ, δ' as in (2.1.4), and the restriction morphisms (for different ϵ) induce isomorphisms.

In fact, the invariance by ϵ is checked as in [3], and the coherence follows from a standard argument (using for example [12]). Then the remaining assertion of (2.1.7) is reduced to the case $S = \mathrm{pt}$ by (2.1.6) and follows from well-known results of BRIESKORN [3] and SEBASTIANI [25] (which follows also from the natural inclusion $\mathcal{H}''_f \to V^{>0}\mathcal{G}_f$ in (2.6.2), cf. also [13], [22, 2.6].) Applying the same argument to (S, s) for $s \in S$ and taking the limit for $\epsilon \to 0$, we get :

(2.1.8) $(\mathcal{H}''_f)_s$ is a free $\mathcal{O}_{\mathbb{C}\times S,(0,s)}$-module of rank μ for $s \in S$.

LEMMA 2.2. — *Let $\pi : S' \to S$ be a morphisms of analytic spaces. Then we have a canonical isomorphism :*

$$\pi^* \mathcal{H}''_f \xrightarrow{\sim} \mathcal{H}''_{\pi^* f} \tag{2.2.1}$$

compatible with the action of ∂_t^{-1}, where $\pi^*\mathcal{H}''_f$ is defined by :

$$\mathcal{O}_{S'}\{t\} \otimes_{\pi^{-1}\mathcal{O}_S\{t\}} \pi^{-1}\mathcal{H}''_f$$

and $\pi^* f$ is the abbreviation of $(\mathrm{id} \times \pi)^* f$. Moreover, (2.2.1) is compatible with the composition of morphisms of the base spaces S.

Proof. — We have the canonical morphism (2.2.1) compatible with the composition of the morphisms of base spaces, by the right exactness of tensor product (because $\mathrm{d} : \Omega^i_{X\times S/S} \to \Omega^{i+1}_{X\times S/S}$ is $\mathcal{O}_S$-linear), and this implies also the isomorphism (2.2.1) when π is a closed embedding. By the compatibility with the composition of $\{s'\} \to S'$ and π, the restriction of (2.2.1) to each point is an isomorphism, and the assertion follows from (2.1.8).

PROPOSITION 2.3. — *$\mathcal{H}''_f$ is locally free of rank μ over $\mathcal{O}_S\{t\}$.*

Proof. — The assertion follows from (2.1.8) if $S = \text{pt}$. For $g : Z \to Y$ as in (2.1.4), we have a natural morphism :

$$(R^n g_* K)_{|\{0\}\times S} \to \mathcal{H}''_f \tag{2.3.1}$$

by (2.1.3), and the assertion is reduced to the bijectivity of (2.3.1) by (2.1.7). But it is enough to show its surjectivity by (2.1.7–8). Taking the pull-back by $\{s\} \to S$, the assertion is reduced to the case $S = \text{pt}$ by (2.1.6) (but ϵ might be bigger than the range of Milnor fibration around s, cf. the remark after (2.1.4)). Then $\mathcal{H}''_f$ is a finite $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module [13], [17], and is generated over the discrete valuation ring $\mathbb{C}\{\{\partial_t^{-1}\}\}$ by generators of a finite dimensional vector space $\mathcal{H}''_f/\partial_t^{-1}\mathcal{H}''_f = \Omega^n_{X,0}/\mathrm{d}f\wedge\Omega^{n-1}_{X,0}$ (cf. [3]). So the assertion is clear by (2.1.5).

Remark. — By (2.3.1), the restriction morphism $H^{n-1}(F_0, \mathbb{Q}) \to H^{n-1}(F_s, \mathbb{Q})$ is an isomorphism in the notation of the remark after (2.1.4), cf. also [30]. The above argument is a quite simple version of the argument used in the proof of Scherk's surjectivity of the restriction morphism of the cohomology of a good compactification of Milnor fiber [36].

2.4. — With the notation and assumption of (2.1), let $\varphi_f\mathbb{C}_{X\times S}$ be Deligne's vanishing cycle sheaf [6]. Then $\operatorname{supp}\varphi_f\,\mathbb{C}_{X\times S} \subset \{0\}\times S$, because (2.1.4) holds also for points outside $\{0\}\times S$, where the Milnor number is zero. Put :

$$L_f := \varphi_f\mathbb{C}_{X\times S}[n-1]_{|\{0\}\times S}. \tag{2.4.1}$$

By the above remark, L_f is a local system on S with the action of monodromy T, and

$$(L_f)_s \simeq \tilde{H}^{n-1}(Z\cap g^{-1}(t,s), \mathbb{C}) \quad \text{for} \quad 0 < |t| < \delta,\ s \in S \tag{2.4.2}$$

by (2.1.4) (choosing a lifting of t to the universal covering of Δ^*), where Z is as in (2.1.4) and $Z \cap g^{-1}(t,s)$ is the Milnor fiber. We can also check that L_f is compatible with base change of S, i.e.

$$\pi^* L_f = L_{\pi^* f} \tag{2.4.3}$$

for $\pi : S' \to S$ as in (2.2). By equivalence of categories (1.4.4), we have a local system of $\mathcal{D}$-modules $M_f(*)$ (resp. $M_f(!)$) on S whose stalks belong to $M_{\mathrm{rh}}(\mathcal{D},*)$ (resp. $M_{\mathrm{rh}}(\mathcal{D},!)$) and correspond to those of L_f by (1.4.4). More explicitly, we have natural inclusions :

$$(2.4.4) \qquad \bigoplus_{\alpha\in\mathbb{C}} M_f(*)^\alpha \longrightarrow M_f(*) \longrightarrow \prod_{\alpha\in\mathbb{C}} M_f(*)^\alpha$$

(same for $M_f(!)$) inducing (1.1.3) at each stalk (in particular, $M_f(*)$ is generated by $M_f(*)^\alpha$ over $\mathbb{C}\{t\}$ or $\mathbb{C}\{\{\partial_t^{-1}\}\}$), where $M_f(*)^\alpha$ are local systems on S such that :

$$(2.4.5) \qquad M_f(*)^\alpha = L_{f,\lambda} \quad \text{for} \quad \lambda = \exp(-2\pi i\alpha)$$

(same for $M_f(!)$) by (1.4.6). Here $L_f = \bigoplus_\lambda L_{f,\lambda}$ is the decomposition by the eigenvalue of the monodromy T. Since T is quasi-unipotent, the stalks of $M_f(*), M_f(!)$ are quasi-unipotent, and the decompositions (2.4.4) are indexed by $\mathbb{Q}$. So $M_f(*), M_f(!)$ have the filtration V indexed by $\mathbb{Q}$ such that $V^\alpha M_f(*), V^\alpha M_f(!)$ are local systems of $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-modules whose stalks are finite over $\mathbb{C}\{t\}$ and also over $\mathbb{C}\{\{\partial_t^{-1}\}\}$, cf. (1.2.1). We have a natural morphism of local systems of $\mathcal{D}$-modules :

$$(2.4.6) \qquad M_f(!) \longrightarrow M_f(*),$$

because $M_f(*) = M_f(!)[t^{-1}]$. By (1.1.6) it induces an isomorphism :

$$(2.4.7) \qquad V^\alpha M_f(!) \to V^\alpha M_f(*) \quad \text{for} \quad \alpha > 0.$$

Let $\mathcal{O}_S\{t\}[\partial_t]$ be the ring of relative differential operators (i.e. $\mathcal{D}_{\Delta\times S/S\,|\,\{0\}\times S}$). We define :

$$(2.4.8) \qquad \tilde{M}_f(*) = \mathcal{O}_S\{t\}[\partial_t] \otimes_{\mathbb{C}\{t\}[\partial_t]} M_f(*) = \mathcal{O}_S\{t\} \otimes_{\mathbb{C}\{t\}} M_f(*)$$

(same for $\tilde{M}_f(!)$). We have a filtration V of $\tilde{M}_f(*)$, $\tilde{M}_f(!)$ by :

$$(2.4.9) \qquad V^\alpha \tilde{M}_f(*) = \mathcal{O}_S\{t\} \otimes_{\mathbb{C}\{t\}} V^\alpha M_f(*)$$

(same for $V^\alpha \tilde{M}_f(!)$). Put $V^{>\alpha}\tilde{M}_f(*) = \bigcup_{\beta>\alpha} V^\beta \tilde{M}_f(*)$ (same for $V^{>\alpha}\tilde{M}_f(!)$). By definition, we have :

(2.4.10) $V^\alpha \tilde{M}_f(*)$ and $V^{>\alpha}\tilde{M}_f(*)$ have locally a free basis over $\mathcal{O}_S\{t\}$ induced by a basis in (1.4.2).

2.5. — With the above notation and assumption, assume further S smooth. Let $\mathcal{G}_f$ be the localization of $\mathcal{H}''_f$ by the action of ∂_t^{-1}. We define a filtration of $\mathcal{G}_f$ by :

$$F_p\mathcal{G}_f = \partial_t^{p+n-1}\mathcal{H}''_f \quad \text{for} \quad p \in \mathbb{Z}. \tag{2.5.1}$$

Then $\mathcal{G}_f$ has a natural structure of regular holonomic $\mathcal{D}_{\mathbb{C}\times S\,|\,\{0\}\times S}$-Module (i.e. the restriction to $\{0\} \times S$ of a regular holonomic $\mathcal{D}_{\mathbb{C}\times S}$-Module) such that the characteristic variety $Ch(\mathcal{G}_f)$ is contained in $T^*_{\{0\}\times S}\mathbb{C} \times S$ the conormal bundle of $\{0\} \times S$ in $\mathbb{C} \times S$. This can be checked using the Gauss-Manin system associated with the morphism $g : Z \to Y$ in (2.1.4) together with the isomorphism (2.3.1) (or we can use also the microlocal Gauss-Manin system), cf. [11], [17], [22, § 2], etc. In fact, the assertion is local and we may shrink S so that (2.1.4) holds. The Gauss-Manin system can be defined by :

$$\int_g \mathcal{O}_Z = \mathbf{R}g_*\big(\Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]\big)[\dim X], \tag{2.5.2}$$

where the differential of the complex $\Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is given by :

$$\omega \otimes \partial_t^i \longmapsto \mathrm{d}\omega \otimes \partial_t^i - \mathrm{d}f \wedge w \otimes \partial_t^{i+1}$$

(cf. [22, § 2]). Here $\Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is the abbreviation of :

$$\Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]\,\delta(f-t)$$

with $\delta(f-t)$ the delta function supported on $f = t$ such that :

$$\partial_{x_i}\delta(f-t) = -(\partial_{x_i}f)\,\partial_t\delta(f-t).$$

In particular, K is a subcomplex of $\Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$, and we get a natural morphism :

$$R^n g_* K \to \int_g^0 \mathcal{O}_Z := \mathcal{H}^0\Big(\int_g \mathcal{O}_Z\Big). \tag{2.5.3}$$

Since the restrictions of $K, \Omega^\bullet_{Z/S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ to $Z^* = g^{-1}(Y^*)$ are both quasi-isomorphic to $g^{-1}(\mathcal{O}_Y)[-1]_{\,|\,Z^*}$, we have :

$$\big(R^{n-1}g_*\mathbb{C}_Z\big)_{\,|\,Y^*} \otimes_{\mathbb{C}} \mathcal{O}_{Y^*} \xrightarrow{\sim} (R^n g_* K)_{\,|\,Y^*} \xrightarrow{\sim} \Big(\int_g^0 \mathcal{O}_Z\Big)_{\,|\,Y^*}. \tag{2.5.4}$$

So (2.5.3) is injective by (2.1.7). We define the Hodge filtration F on $\int_g^0 O_Z$ by :

$$F_p \int_g^0 O_Z = \partial_t^{p+n-1} R^n g_* K \quad \text{for} \quad p \in \mathbb{Z}. \tag{2.5.5}$$

We can also check that $(\int_g^0 O_Z)_{|\{0\}\times S}$ is independent of ϵ in (2.1.4). Passing to the limit, it is generated by $\mathcal{H}_f''$ over $\mathbb{C}[\partial_t]$. So we get an isomorphism of $\mathcal{O}_s\{t\}[\partial_t]$-Modules :

$$\mathcal{G}_f \xrightarrow{\sim} \Big(\int_g^0 O_Z\Big)_{|\{0\}\times S}, \tag{2.5.6}$$

which gives the structure of $\mathcal{D}_{\mathbb{C}\times S\,|\,\{0\}\times S}$-Module on $\mathcal{G}_f$. Using $\int_g^0 O_Z$ we can also check the compatibility of $\mathcal{G}_f$ with base change of S as in (2.2), i.e., for a morphism of complex manifolds $\pi : S' \to S$, we have a canonical isomorphism of $\mathcal{D}$-Modules :

$$\pi^* \mathcal{G}_f \xrightarrow{\sim} \mathcal{G}_{\pi^* f}, \tag{2.5.7}$$

where $\pi^*\mathcal{G}_f$ is the pull-back of $\mathcal{D}$-Modules and is defined by :

$$\mathcal{O}_{S'}\{t\} \otimes_{\pi^{-1}\mathcal{O}_S\{t\}} \pi^{-1}\mathcal{G}_f.$$

Since $\mathrm{Ch}(\mathcal{G}_f) \subset T^*_{\{0\}\times S}\mathbb{C}\times S$, $\mathcal{G}_f$ is locally trivial along $\{0\}\times S$, i.e., $\mathcal{G}_f$ is locally isomorphic to the pull-back of a regular holonomic $\mathcal{D}$-module M of one variable by the projection $p : \mathbb{C}\times S \to \mathbb{C}$. So $\mathcal{G}_f$ has a filtration V induced by V on M in (1.2.1), which coincides with the filtration V of KASHIWARA [10] and MALGRANGE [14] along $\{0\}\times S$ indexed by $\mathbb{Q}$. Note that $\mathrm{Gr}_V^\alpha \mathcal{G}_f$ is a smooth holonomic $\mathcal{D}_S$-Module, i.e., a locally free $\mathcal{O}_S$-Module with integrable connection. By definition (2.4.8), $\tilde{M}_f(*)$, $\tilde{M}_f(!)$ have a structure of $\mathcal{D}_{\mathbb{C}\times S\,|\,\{0\}\times S}$-Modules, and they are regular holonomic, because they are locally isomorphic to the pull-back of regular holonomic $\mathcal{D}$-modules by the smooth projection p as above. By (2.5.4), (2.5.6), we get a canonical isomorphism :

$$\mathcal{G}_f \xrightarrow{\sim} \tilde{M}_f(!), \tag{2.5.8}$$

because $\int_g^0 O_Z$ is the unique extension of $(\int_g^0 O_Z)_{|Y^*}$ as regular holonomic $\mathcal{D}_Y$-Module such that the action of ∂_t on $(\int_g^0 O_Z)_{|\{0\}\times S}$ is bijective (this is a relative version of (1.4.1), and follows, for example, from the Riemann-Hilbert correspondence [11], [15]). By (2.5.3) together with (2.3.1), (2.5.6), (2.5.8), we get a canonical morphism of $\mathcal{O}_s\{t\}$-Modules :

$$\mathcal{H}''_f \to \tilde{M}_f(!). \tag{2.5.9}$$

compatible with base change via (2.2.1), (2.4.3). By (2.3), (2.5.9) is uniquely determined by its restriction to the points of S.

2.6. — With the above notation and assumption, assume further $S = \text{pt}$. Then the filtration F on $\mathcal{G}_f$ in (2.1.2) induces the Hodge filtration F of the mixed Hodge structure [26] on the vanishing cohomology $\tilde{H}^{n-1}(Z \cap f^{-1}(t), \mathbb{C})$ $(0 < |t| \ll 1)$ via the isomorphisms :

$$\mathrm{Gr}_V^{\alpha}\mathcal{H}''_f = F^p\tilde{H}^{n-1}\big(Z \cap f^{-1}(t), \mathbb{C}\big)_{\lambda} \tag{2.6.1}$$

for $\lambda = \exp(-2\pi i\alpha)$, $n-1-p < \alpha \le n-p$ (choosing a lifting of t to the universal covering of Δ^*), which is induced by (1.2.2), (1.4.5), (2.4.2), (2.5.8) and the action of ∂_t. Here $\tilde{H}^{n-1}(Z \cap f^{-1}(t), \mathbb{C})_{\lambda}$ denotes the λ-eigenspace by the action of the monodromy. This fact is essentially due to VARCHENKO [27], see also [18], [20], [21], [24], etc. Note that the paper [24] quoted in [18], [20], [21] is its first version, where no $\mathcal{D}$-Modules were used (except for meromorphic type).

We have :

$$V^{>0}\mathcal{G}_f \supset \mathcal{H}''_f \supset V^{n-1}\mathcal{G}_f, \tag{2.6.2}$$

where the first inclusion follows from the positivity of the minimal exponent [9], [13], and the second from the symmetry of the exponents (or spectrum) [26] (using [27]). In fact, the maximal exponent is less than n, and

$$\dim_{\mathbb{C}} \mathrm{Gr}_V^{\alpha}\mathcal{H}''_f = \dim_{\mathbb{C}} \mathrm{Gr}_V^{\alpha+1}\mathcal{H}''_f \ \ (= \dim_{\mathbb{C}} \mathrm{Gr}_V^{\alpha}\mathcal{G}_f) \quad \text{for } \alpha \ge n-1,$$

where $\mathcal{H}''_f \supset V^{\alpha}\mathcal{G}_f$ for α large enough follows from $\bigcup_i \partial_t^i\mathcal{H}''_f = \mathcal{G}_f$. Note that :

$$\begin{array}{l}\dim_{\mathbb{C}} \mathrm{Gr}_V^{\alpha}\mathcal{H}''_f \text{ remains constant under}\\ \mu\text{-constant deformation of } f,\end{array} \tag{2.6.3}$$

because the exponents and the Hodge numbers are constant under μ-constant deformation [28].

Remark. — We loose a lot of information by passing from the Brieskorn module $\mathcal{H}''_f$ to the Hodge filtration on vanishing cohomology taking the graduation, cf. (2.6.1). The lost information is expressed by the linear mapping $c_{\beta,\alpha}$ in [22]. But we don't loose information at all in the quasihomogeneous case, where $c_{\beta,\alpha} = 0$. This is because $\mathcal{H}''_f$ is stable by the action of $\partial_t t$, and the infinite decomposition (1.1.3) is compatible with $\mathcal{H}''_f$, i.e., $\mathcal{H}''_f$ is generated by $\mathbb{C}\{t\}$ or $\mathbb{C}\{\{\partial_t^{-1}\}\}$ over $\mathcal{G}^\alpha \cap \mathcal{H}''_f$ so that $\mathcal{G}^\alpha \cap \mathcal{H}''_f = \mathrm{Gr}_V^\alpha \mathcal{H}''_f$.

PROPOSITION 2.7. — *With the notation and assumption of* (2.1), *we have a canonical morphism of $\mathcal{O}_S\{t\}$-Modules :*

$$\mathcal{H}''_f \longrightarrow V^{>0}\tilde{M}_f(!) = V^{>0}\tilde{M}_f(*). \tag{2.7.1}$$

compatible with base change via (2.2.1), (2.4.3). *Moreover, it coincides with* (2.5.9), *if S is smooth.*

Proof. — By the compatibility with base change to each point of S, (2.7.1) is unique. So the assertion is local on S. Take :

$$g : Z \to Y = \Delta \times S$$

as in (2.1.4) by restricting S. Let $\omega \in g_*\Omega^n_{Z/S}$. Since the restriction of g over $Y^* = \Delta^* \times S$ is smooth, $\omega/\mathrm{d}f$ determines a section $s(\omega/\mathrm{d}f)$ of $j_*(R^{n-1}g_*\mathbb{C}_Z\,|_{Y^*} \otimes_{\mathbb{C}} \mathcal{O}_{Y^*})\,|_{\{0\}\times S}$, where $j : Y^* \to Y$ is the natural inclusion. This can be checked by extending the function f to $X \times S''$, where S'' is a smooth space containing S. Let M' be a coherent extension of $V^\alpha\tilde{M}_f(*)$, i.e. a coherent $\mathcal{O}_U$-Module whose restriction to $\{0\} \times S$ is $V^\alpha\tilde{M}_f(*)$, where U is an open neighborhood of $\{0\} \times S$ in Y. We define $\tilde{M}_f(\infty) = j_*j^*M'\,|_{\{0\}\times S}$, which is independent of α. Then we get a morphism :

$$\mathcal{H}''_f \to \tilde{M}_f(\infty) \tag{2.7.2}$$

compatible with base change of S. Using a local basis $\{v_1, \dots, v_\mu\}$ of $V^{>0}\tilde{M}_f(*)$ in (2.4.10), the image of $\omega \in \mathcal{H}''_f$ in $\tilde{M}_f(\infty)$ is expressed by $\sum_i a_i v_i$ with $a_i \in (j_*j^*\mathcal{O}_Y)\,|_{\{0\}\times S}$, and it is enough to show $a_i \in \mathcal{O}_s\{t\}$. If S is smooth, (2.7.2) coincides with the morphism induced by (2.5.3) via (2.3.1), (2.5.6), (2.5.8), and it induces (2.7.1) by the positivity of the minimal exponent [9], [13], cf. (2.6.2). Since the basis in (2.4.10) (and hence the coefficients a_i) is compatible with base change of S, the assertion is reduced to the smooth case if S is normal, using a desingularization of S.

In general, let $\pi : S' \to S$ be the normalization, and $f' = \pi^* f$, cf. (2.2). Then we have $\mathcal{H}''_f \to \pi_* V^{>0}\tilde{M}_{f'}(*)$, and it is enough to show the vanishing of the composition :

$$\text{(2.7.3)} \qquad \mathcal{H}''_f \longrightarrow \pi_* V^{>0}\tilde{M}_{f'}(*) \longrightarrow \pi_* V^{>0}\tilde{M}_{f'}(*)/V^{>0}\tilde{M}_f(*).$$

The image is a coherent subsheaf, and annihilated by a power of t, because $a_i \in (j_* j^* \mathcal{O}_Y)_{|\{0\}\times S}$. So we get the assertion, because t is a non-zero divisor of $\pi_*\mathcal{O}_{S'}\{t\}/\mathcal{O}_S\{t\}$ and $\pi_* V^{>0}\tilde{M}_{f'}(*)/V^{>0}\tilde{M}_f(*)$ is a direct sum of $\pi_*\mathcal{O}_{S'}\{t\}/\mathcal{O}_S\{t\}$.

COROLLARY 2.8. — *The morphism* (2.7.1) *is injective and its cokernel is a free $\mathcal{O}_S$-Module of finite rank. Moreover the natural inclusion $V^{n-1}\tilde{M}_f(!) \to V^{>0}\tilde{M}_f(!)$ is factorized by* (2.7.1).

Proof. — This follows from (2.6.2–3) and the compatibility with base change.

Remark. — There is some similarity between the above situation and the usual variation of (mixed) Hodge structure. In fact, the local system of $\mathcal{D}$-modules $M_f(!)$ corresponds to the underlying $\mathbb{C}$-local system of the variation, $\tilde{M}_f(!)$ to the underlying holomorphic vector bundle which is the holomorphic scalar extension of the local system, and $\mathcal{H}''_f$ to the Hodge bundles. Moreover, $\mathcal{H}''_f/V^{n-1}\tilde{M}_f(!)$ is a subbundle of a free $\mathcal{O}_S$-Module $V^{>0}\tilde{M}_f(!)/V^{n-1}\tilde{M}_f(!)$, and the analogy becomes closer if we restrict $\mathcal{H}''_f$ to $V^{>0}\tilde{M}_f(!)/V^{n-1}\tilde{M}_f(!)$. Note that, restricting $\mathcal{H}''_f$ to $\mathrm{Gr}_V\,\tilde{M}_f(!)$, we get the Hodge filtration of the variation of mixed Hodge structures on the vanishing cohomologies, cf. (2.6.1).

2.9. — With the notation and assumption of (2.1), we define $\mathcal{G}_s := \mathcal{G}_{f_s}$ by applying (2.5) to f_s. Then $\mathcal{G}_s$ is a regular holonomic $\mathcal{D}$-module as in (1.1), and we have a canonical isomorphism :

$$\text{(2.9.1)} \qquad \mathcal{G}_s = M_f(!)_s,$$

because $M_f(!)$ is compatible with base change of S. So we get a locally constant sheaf on S, denoted by $\mathcal{G}_S$, such that its stalk at $s \in S$ is $\mathcal{G}_s$ and $\mathcal{G}_S = M_f(!)$. This means that we have the parallel translation of the elements of $\mathcal{G}_s$. Assume S contractible by shrinking S if necessary. Then the locally constant sheaf is a constant sheaf, and we denote its stalk by $\mathcal{G}$ so that we have a canonical isomorphism $\mathcal{G}_s = \mathcal{G}$ for any $s \in S$. Let $\mathcal{H}''_s = \mathcal{H}''_{f_s}$ and $m_\alpha = \dim_{\mathbb{C}} \mathrm{Gr}_V^\alpha \mathcal{H}''_{f_s}$ for $s \in S$. By (2.6.3),

m_α is independent of s. Let $\mathbf{L}(\mathcal{G})'$ be the set of $\mathbb{C}$-vector subspaces M of $V^{>0}\mathcal{G}$ such that $V^{>0}\mathcal{G} \supset M \supset V^{n-1}\mathcal{G}$, and $\mathbf{L}(\mathcal{G})$ the subset of $\mathbf{L}(\mathcal{G})'$ consisting of M such that M is a $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodules of $V^{>0}\mathcal{G}$ and $\dim_{\mathbb{C}} \mathrm{Gr}_V^\alpha M = m_\alpha$ for $\alpha \in \mathbb{Q}$. Then $\mathbf{L}(\mathcal{G})'$ has naturally a structure of complex manifold, and $\mathbf{L}(\mathcal{G})$ is a locally closed analytic subspace of $\mathbf{L}(\mathcal{G})'$. By (2.6.3), (2.7) and (2.8), we get :

THEOREM 2.10. — *We have a period mapping :*

$$\Psi : S \to \mathbf{L}(\mathcal{G}) \tag{2.10.1}$$

associating $\mathcal{H}''_s \in \mathbf{L}(\mathcal{G})$ to $s \in S$, and it is holomorphic.

Since the Hodge filtration on the vanishing cohomology is obtained by taking the graduation of $\mathcal{H}''_s$ by V, cf. (2.6.1), this implies the following (which is not trivial in the case S non-normal) :

COROLLARY 2.11. — *We have a period mapping $\overline{\Psi}$ associating the Hodge filtration of mixed Hodge structure on the cohomology of Milnor fiber, and it is holomorphic.*

Proof. — For the target $\overline{\mathbf{L}}(\mathcal{G})$ of $\overline{\Psi}$, we take the product of flag varieties of $(L_{f,\lambda})_0$, cf. (2.4.2), (2.6.1), where $L_{f,\lambda}$ denotes the λ-eigenvalue part of L_f so that $(L_{f,\lambda})_0 = \tilde{H}^{n-1}(Z \cap f^{-1}(t,0), \mathbb{C})_\lambda$. Then the natural morphism $\mathbf{L}(\mathcal{G}) \to \overline{\mathbf{L}}(\mathcal{G})$ is holomorphic.

3. Period mapping of the μ-constant stratum

3.1. — Let $f' : (X,0) \times (S',0) \to (\mathbb{C},0)$ be a miniversal deformation of a holomorphic function f'_0 with isolated singularity, where X is an open neighborhood of 0 in $\mathbb{C}^n$, and S' is a complex manifold such that :

$$\Big\{1, \frac{\partial f'}{\partial s_1}\Big|_{s=0}, \dots, \frac{\partial f'}{\partial s_{\mu-1}}\Big|_{s=0}\Big\}$$

is a basis of $\mathcal{O}_{X,0}/(\partial f'_0)$, where $(x_1,\dots,x_n)$, $(s_1,\dots,s_{\mu-1})$ are coordinate systems of X and S' respectively, and $(\partial f'_0) = \sum_i \mathcal{O}_{X,0}\partial f'_0/\partial x_i$. Here μ is the Milnor number of f'_0. We may assume $\mathrm{Sing} f_s = \{0\}$ and $f_s(0) = 0$ if s belongs to the μ-constant stratum, by replacing f' if necessary. Put :

$$\mathcal{O}_{C'} = \mathcal{O}_{X\times S'} \Big/ \sum_i \mathcal{O}_{X\times S'}\, \partial f'/\partial x_i.$$

Let $\pi' : X \times S' \to S'$. Shrinking S' if necessary, $\pi'_* \mathcal{O}_{C'}$ is a free $\mathcal{O}_{S'}$-Module of rank μ, and we have a canonical inclusion :

$$\eta' : \Theta_{S'} \to \pi'_* , \mathcal{O}_{C'} \tag{3.1.1}$$

whose cokernel is free of rank 1, by associating $vf' \in \pi'_* \mathcal{O}_{C'}$ to a vector field $v \in \Theta_{S'}$, where $vf' \in \pi'_* \mathcal{O}_{C'}$ is defined by lifting v to $X \times S'$ and is independent of the choice of lifting by definition of $\mathcal{O}_{C'}$. We can also define the Gauss-Manin system $\int^0_{g'} \mathcal{O}_{Z'}$ with Hodge filtration F as in (2.5.2), (2.5.5), where $g' : Z' \to \Delta \times S'$ is defined as in (2.1.4), cf. [17], [22, § 2]. Let $p' : \Delta \times S' \to S'$ be the natural projection. We can check that the Hodge filtration F on $\int^0_{g'} \mathcal{O}_{Z'}$ is exhaustive (cf. [*loc. cit.*]), and :

$$\begin{array}{l} p'_* \mathrm{Gr}^F_p \int^0_{g'} \mathcal{O}_{Z'} \text{ is a free } \pi'_* \mathcal{O}_{C'}\text{-Module} \\ \text{generated by } \partial_t^{p+n-1} \delta(f'-t) \text{ for } p \in \mathbb{Z}, \end{array} \tag{3.1.2}$$

where $\delta(f'-t)$ is the delta function as in (2.5), and the coordinates of $\mathbb{C}^n$ are used to trivialize $\Omega^n_{Z'/S'}$. This is essentially due to BRIESKORN [30] (cf. also [19, (1.4.3)]) admitting the injectivity of (2.5.3) in this case. Here we can also use the theory of microlocal filtered Gauss-Manin system [22, § 2]. Then :

$$\begin{array}{l} \partial_t^{-1} v \in \mathrm{End}(p'_* \mathrm{Gr}^F_p \int^0_{g'} \mathcal{O}_{Z'}) \text{ for } v \in \Theta_{S'} \text{ is identified} \\ \text{via (3.1.2) with the multiplication by } -\eta'(v) \text{ on } \pi'_* \mathcal{O}_{C'}, \end{array} \tag{3.1.3}$$

because $v\delta(f'-t) = -(vf')\,\partial_t \delta(f'-t)$. (Compare [19].) Let S be a closed analytic subspace of the μ-constant stratum $\{s \in S' : \mu(f'_s) = \mu(f'_0)\}$, and S_{reg} the smooth points of S. Let f be the restriction of f' over S, and define $\mathcal{O}_C$ and π by replacing S', f' with S, f. The Gauss-Manin system is compatible with base change so that (3.1.2–3) holds with Z', S', f', etc. replaced by Z, S, f, etc., and (3.1.1) induces an injective morphism :

$$\eta : \Theta_{S_{\mathrm{reg}}} \to \pi_* \mathcal{O}_{C \,|\, S_{\mathrm{reg}}} = i^* \pi'_* \mathcal{O}_{C' \,|\, S_{\mathrm{reg}}}, \tag{3.1.4}$$

whose cokernel is also locally free (because $(i^* \Theta_{S' \,|\, S_{\mathrm{reg}}}) / \Theta_{S_{\mathrm{reg}}}$ is locally free), where $i : S \to S'$ is the natural inclusion.

THEOREM 3.2. — *For S as above, the restriction of Ψ in* (2.10.1) *to S_{reg} is locally a closed embedding.*

Proof. — This follows from the same argument as in [22, 2.10] using (3.1.3–4). Let $H = V^{>0}\mathcal{G}/V^{n+1}\mathcal{G}$, and $F^p_s = \partial_t^{-p} \mathcal{H}''_s / V^{n+1}\mathcal{G} \subset H$

for $p = 0,1,2$. Then $m_p := \dim F_s^p$ is independent of $s \in S$. Let $\mathrm{Flag}(H; m_0, m_1, m_2)$ be the flag variety of three subspaces with dimension m_0, m_1, m_2. We have a holomorphic mapping :

$$(3.2.1) \qquad \Phi : S_{\mathrm{reg}} \longrightarrow \mathrm{Flag}(H; m_0, m_1, m_2)$$

by associating (F_s^0, F_s^1, F_s^2) to $s \in S_{\mathrm{reg}}$. By definition, this is factorized by $\Psi|_{S_{\mathrm{reg}}}$, and it is enough to show the injectivity of $\mathrm{d}\Phi$. Let $\mathcal{H}$ be a trivial holomorphic vector bundle on S_{reg} with fiber H. Let $\mathcal{F}^p$ be the subbundle of $\mathcal{H}$ whose fiber at s is F_s^p for $p = 0,1,2$, and $\mathcal{F}^{-1} = \mathcal{H}$, $\mathcal{F}^3 = 0$. Since $v\mathcal{F}^p \subset \mathcal{F}^{p-1}$ for a vector field v, the image of $\mathrm{d}\Phi$ is contained in the horizontal tangent bundle whose corresponding locally free sheaf is $\bigoplus_{0\le p\le 2} \mathcal{H}om_{\mathcal{O}}(\mathrm{Gr}_{\mathcal{F}}^p, \mathrm{Gr}_{\mathcal{F}}^{p-1})$. So it is enough to show that the morphism :

$$(3.2.2) \qquad \Theta_{S_{\mathrm{reg}}} \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathrm{Gr}_{\mathcal{F}}^1, \mathrm{Gr}_{\mathcal{F}}^0)$$

is injective and its cokernel is locally free. But this follows from (3.1.3-4).

Remark. — This proof was inspired by a discussion with J. STEVENS about VARCHENKO's work [29] at Leiden in 1984.

THEOREM 3.3. — *Let S be the μ-constant stratum. Then Ψ in* (2.10.1) *is finite to one by shrinking S if necessary.*

Proof. — It is enough to show the dimension of $Z = \Psi^{-1}(\Psi(0))$ at $0 \in S$ is 0. (Take sufficiently small relatively compact neighborhoods U, U' of 0 in S such that $\overline{U} \subset U'$, and replace S by $U \setminus \Psi^{-1}(\Psi(\overline{U}' \setminus U))$. Then Ψ becomes proper over the image, and has finite fibers.) Assume $\dim Z > 0$. Then we get the local injectivity of the restriction of Ψ to Z_{reg} by (3.2). This is contradiction.

3.4 Example : quasihomogeneous case. — Assume f_0 is a quasihomogeneous polynomial of weight $w_1, \ldots, w_n$, i.e., f_0 is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ whose degree is one, where :

$$\deg x_1^{m_1} \cdots x_n^{m_n} = m_1 w_1 + \cdots + m_n w_n .$$

Let $g_1, \ldots, g_{\mu-1}$ be monomials such that $\{1, g_1, \ldots, g_{\mu-1}\}$ is a basis of $\mathcal{O}_{X,0}/(\partial f_0)$. Then the miniversal deformation is given by :

$$f' = f_0 + \sum g_i s_i ,$$

where $(s_1, \dots, s_{\mu-1})$ is the coordinate system of $\mathbb{C}^{\mu-1}$, and S' is an open subset of $\mathbb{C}^{\mu-1}$. Each s_i has the degree $\nu_i := 1 - \deg g_i$. Let $\bar{S} = \mathbb{C} \times \mathbb{C}^{\mu-1}$, and $s_0 = t$ the coordinate of $\mathbb{C}$ with $\nu_0 = \deg s_0 = 1$. Then $\bar{S}$ has a $\mathbb{C}^*$-action associated with the degree $(\nu_0, \dots, \nu_{\mu-1})$, i.e., $\alpha(s_0, \dots, s_{\mu-1}) = (\alpha^{\nu_0} s_0, \dots, \alpha^{\nu_{\mu-1}} s_{\mu-1})$ for $\alpha \in \mathbb{C}^*$. The associated vector field E is given by $\sum \nu_i s_i (\partial/\partial s_i)$, and coincides with the Euler vector field in [19] by definition. We have a decomposition :

$$\overline{S} = \overline{S}^+ \times \overline{S}^0 \times \overline{S}^-,$$

where $\overline{S}^+, \overline{S}^0, \overline{S}^-$ denotes respectively the positive, zero, and negative degree part of $\overline{S}$ by the $\mathbb{C}^*$-action. Then the μ-constant stratum S is $(\{0\} \times S') \cap (\overline{S}^0 \times \overline{S}^-)$ by [29], where we may assume $S = S^0 \times S^-$ with :

$$S^0 = S \cap \overline{S}^0, \quad S^- = S \cap \overline{S}^-.$$

So S is smooth and the period mapping Ψ is injective by (3.2) (by shrinking S' if necessary). But $\overline{\Psi}$ in (2.11) is not injective in general. In fact, let $\pi : S \to S^0$ denote the natural projection which is also induced by the $\mathbb{C}^*$-action. By the same argument as [22, 4.2], [23], the Hodge filtration of vanishing cohomology is invariant under a deformation of f_0 obtained by adding monomials of higher degree, and $\overline{\Psi}$ is factorized by π. The image S^0 of π is the parameter space of deformation of f_0 as quasihomogeneous polynomials. By remark after (2.6), the restriction of $\overline{\Psi}$ to S^0 is equivalent to the restriction of Ψ, and is injective. So we get :

(3.4.1) There is a projection π of the μ-constant stratum S to its closed subspace, on which the local Torelli holds and π is the identity, and $\overline{\Psi}$ is constant on the fibers of π.

In particular, THEOREMS (3.2), (3.3) do not hold for $\overline{\Psi}$. Note that, if f_0 is homogeneous, deformation of homogeneous polynomials is essentially equivalent to deformation of projective hypersurfaces, and the latter was studied by CARLSON and GRIFFITHS [4], where the local Torelli was proved in a different way.

3.5 Remark. — In [8] KARPISHPAN tried to extend (3.4.1) to the non-quasihomogeneous case using the theory of primitive forms and logarithmic vector fields [19]. But his arguments contain many gaps, and his idea seems too optimistic. The situation seems much more complicated than is described in [8]. Although the linear mappings $c_{\beta,\alpha}$ in [22, § 3] express the information which is lost by passing to the graduation of $\mathcal{H}''_s$

by V, it is not easy to relate this with the geometry of the discriminant in the non-quasihomogeneous case. For example, the Euler vector field E cannot be used to construct the projection π, because :

(3.5.1) The Euler vector field E is everywhere non-zero, i.e. $\mathrm{Fix}(E) = \emptyset$, in the non-quasihomogeneous case,

(3.5.2) The variation of mixed Hodge structure is not constant on the logarithmic strata contained in the μ-constant stratum, especially on the integral curves of E, if the Milnor monodromy of f_0 is not semisimple,

cf. (3.6). Note also :

(3.5.3) The logarithmic vector fields are defined on $\Delta \times S'$ and not on S'.

This makes the arguments more complicated, because we have to use a vector field on $\Delta \times S$ coming form S (more generally, the coefficient of ∂_t is divisible by t^2) to get the correct action on the variation of mixed Hodge structure which is obtained by graduation of the Gauss-Manin system by the filtration V along $\{0\} \times S$, where S is a complex manifold parametrizing a μ-constant deformation.

In fact, if we extend a vector field on $\{0\} \times S$ to a vector field on $\Delta \times S$ in a bad way (i.e., the coefficient of ∂_t is not divisible by t^2), the action on the variation may be different from that of the Gauss-Manin connection on the variation. (Here the coordinate t on $\Delta \times S'$ is fixed, because we consider deformation of function.) This kind of problem occurs, because the inclusion $\{0\} \times S' \to \Delta \times S'$ is characteristic and we have to use the filtration V. We don't have such a problem in a non-characteristic case, e.g., $\Delta \times D \to \Delta \times S'$ for a locally closed submanifold D of S', because the restriction is defined simply by tensor of the structure sheaf.

3.6 Remark. — Let $\mathbf{v} = {}^t(v_1, \dots, v_\mu)$ be a good basis of the Brieskorn module $\mathcal{H}''_{f_0}$ in [22]. Let f' be a miniversal deformation of f_0 as in (3.1), and S' its base space. Let $(s_1, \dots, s_{\mu-1})$ be a coordinate system of S', and $\partial_i = \partial/\partial s_i$. By MALGRANGE [31], the good basis is uniquely extended to a basis (also denoted by $\mathbf{v} = {}^t(v_1, \dots, v_\mu)$) of the Brieskorn module $\mathcal{H}''_{f'}$ associated with f' so that :

(3.6.1) $$t\partial_t \mathbf{v} = A_0 \partial_t \mathbf{v} + (A_1 - 1)\mathbf{v},$$

(3.6.2) $$\partial_i \mathbf{v} = B_i \partial_t \mathbf{v} \quad (1 \le i < \mu),$$

where A_0, B_i are matrices with coefficients in holomorphic functions on S', and A_1 is a semisimple matrix with constant coefficients such that v_i is an eigenvector of A_1 with eigenvalue α_i (where α_i are the exponents of f_0), cf. [22, (4.3.4)]. By (3.1.2), A_1 and B_i express the action of f' and $\partial_i f'$ respectively on $\pi'_*\mathcal{O}_{C'}$ in the notation of (3.1). Let $S \subset S'$ denote the μ-constant stratum. We have :

(3.6.3) The restriction of $\mathbf{v}$ to $s \in S$ sufficiently near 0 is also a good basis.

(This is essentially equivalent to that the restriction of v_i to $s \in S$ sufficiently near 0 belongs to V^{α_i}, cf. [22, (3.6)].) It follows from the uniqueness of MALGRANGE's extension [31]. In fact, a good basis at the origin is uniquely extended to a good basis at $s \in S$ by fixing an opposite filtration in [22] and using the parallel translation of the splitting of the Hodge filtration in [*loc. cit.*] at $0 \in S$. Here we take the extension v_j so that the projection of v_j to the parallel translation of G^α at 0 is constant. This extension coincides with MALGRANGE's extension by its uniqueness, because we can take a one-parameter family and check Malgrange's condition similar to (3.6.1–2) for the basis obtained above using [22, (3.4.1)] and the constantness of the opposite filtration.

Let E be the Euler vector field. By definition [19], we have :

(3.6.4) $$t\partial_t = E' + E \quad \text{with} \quad E' = g(s)\partial_t + \sum_{1\le i<\mu} g_i(s)\partial_i,$$

where $g(s), g_i(s)$ are holomorphic functions on S'. In particular,

(3.6.5) $$E'\mathbf{v} = g(s)\partial_t\mathbf{v} + \sum_{1\le i<\mu} g_i(s)B_i\partial_t\mathbf{v}.$$

Since E is a logarithmic vector field, we have $E(\mathcal{H}''_{f'}) \subset \mathcal{H}''_{f'}$ and :

(3.6.6) $$E'\mathbf{v} = A_0\partial_t\mathbf{v}, \quad E\mathbf{v} = (A_1 - 1)\mathbf{v},$$

(3.6.7) $$A_0 = g(s) + \sum_{1\le i<\mu} g_i(s)B_i$$

by (3.6.1), because $\mathbf{v}$ is a basis of $\mathcal{H}''_{f'}$ over $\mathcal{O}_{S',0}\{\{\partial_t^{-1}\}\}$, cf. [22]. (This argument is same as [22, 4.3] where we proved that the primitive form is an eigenvector of E.) In particular, if f_0 is not quasihomogeneous, we have $A_0(0) \neq 0$ (using [32]), and E' is not zero at $0 \in \Delta \times S'$ by (3.1.3), (3.6.6).

So we get (3.5.1). (The assertion (3.5.1) is also proved by KARPISHPAN in his recent manuscript.)

Let $D \subset S'$ be an integral curve of E contained in the μ-constant stratum. Replacing f' if necessary, we may assume $(\partial_i f')|_{\{0\}\times D} = 0$. This implies $g(s) = 0$ for $s \in D$ by (3.6.7) (using the identification with the action on $\pi'_* \mathcal{O}_{C'}$), because $f'|_{\{0\}\times D} = 0$. So we get :

$$E'|_{\Delta\times D} = \Big(\sum_{1\le i<\mu} g_i(s)\partial_i\Big)\Big|_{\Delta\times D}, \tag{3.6.8}$$

i.e., the coefficient of ∂_t is zero, cf. the remark after (3.5.3). Here $\Delta\times D \to \Delta \times S'$ is non characteristic, cf. the last remark in (3.5). Let f'' denote the restriction of f' over D. By (3.6.3), $\{\partial_t^p v_i\}_{i\in I(\alpha,p)}$ is a free basis of $\mathrm{Gr}^F_{p-n+1}\mathrm{Gr}^\alpha_V \mathcal{G}_{f''}$ over $\mathcal{O}_D$, where $I(\alpha,p) = \{i : \alpha_i - \alpha = p\}$. By (3.6.6),

$$\mathrm{Gr}^F \mathrm{Gr}_V E'|_{\Delta\times D} : \mathrm{Gr}^F_p \mathrm{Gr}^\alpha_V \mathcal{G}_{f''} \longrightarrow \mathrm{Gr}^F_{p+1}\mathrm{Gr}^\alpha_V \mathcal{G}_{f''}$$

coincides with $\mathrm{Gr}^F \mathrm{Gr}_V \partial_t t$ which is identified with $-\mathrm{Gr}^F N$ by the isomorphism (2.6.1), where $N = \log T_u/2\pi i$ with T_u the unipotent part of the Milnor monodromy T. If the Milnor monodromy is not semisimple, $\mathrm{Gr}^F N \neq 0$ by the strict compatibility of N with the Hodge filtration F on the vanishing cohomology, and the Hodge filtration F is not locally constant along D, because it is not stable by the action of E'. (Note that the restriction of E to D coincides with that of $-E'$, and we have to use $-E'$ to get the correct action, cf. the remark after (3.5.3).) So we get (3.5.2).

We can also verify directly (3.5.2) in the example $f_0 = x^p + y^q + z^r + xyz$ with $1/p + 1/q + 1/r < 1$. In fact, we have a $\mathbb{C}^*$-action on $\mathbb{C}\times\mathbb{C}^{\mu-2}\times\mathbb{C}^*$ which contains $\Delta\times S'$ by taking :

$$f = x^p + y^q + z^r + xyzs_{\mu-1} + \sum_{1\le i\le\mu-2} g_i s_i,$$

where $\{1, xyz, g_1, \ldots, g_{\mu-2}\}$ is a monomial basis of $\mathcal{O}_{X,0}/(\partial f_0)$. The vector field corresponding to the $\mathbb{C}^*$-action is the Euler vector, but it is nowhere zero. Moreover, we can check that the mixed Hodge structure on the vanishing cohomology varies really along the μ-constant stratum $\{s_i = 0 \ (i \neq \mu - 1)\}$.

3.7 Remark. — The mixed Hodge structure on the cohomology of Milnor fiber depends on the choice of the coordinate t of the open disc Δ, because the isomorphism (1.4.5) depends on t, where the Hodge filtration is defined on the left hand side, and the rational structure on the right. So the period map in [8] is not well-defined, because he considers the map of the parameter space of a deformation of hypersurface instead of function. To make the period map well-defined, we have to take a quotient space of the classifying space of mixed Hodge structure. But it is not clear what kind of quotient space we should take, because the ambiguity of defining equation is given by a function on $(\mathbb{C}^n, 0)$ (not only by a function on Δ).

To solve this problem, there may be two possibilities. One is to choose an embedding of the base space of the miniversal deformation of hypersurface into the product of an open disc Δ with the base space S' of the miniversal deformation of function, cf. also remark below. But we cannot get a well-defined period mapping $\overline{\Psi}$ of the μ-constant stratum in the base space of the deformation of hypersurface, taking the composition with the embedding. The second is to give up the local Torelli for hypersurfaces, and consider only the moduli of functions. In any case, the arguments in [8] do not seem to be useful by (3.5.1–2).

3.8 Remark. — Let S be the μ-constant stratum of the base space of the miniversal deformation of a function. We have a stratification of S by :

$$\tau(s) = \tau(f_s) := \dim_{\mathbb{C}} \mathcal{O}_{X,0}/(f_s, \partial f_s) \quad \text{for } s \in S,$$

with the notation and assumption of (2.1). Let $S_\tau = \{s \in S : \tau(s) = \tau\}$. It is a union of logarithmic strata (like foliation). Note that the logarithmic stratification in [19] is not locally finite. Let D be a logarithmic stratum, and T_s a submanifold of $\Delta \times S'$ transversal to D at $s \in D$, where $\Delta \times S'$ is as in (3.1). According to KARPISHPAN, we have :

$$\text{(3.8.1)} \qquad \dim D = \mu - \tau(s) \quad \text{for} \quad s \in D,$$

(cf. p. 290 in [33]), and $\dim T_s = \tau(s)$. Since the fibers are analytically trivial along D, T_s should be identified with the base space of the miniversal deformation of the hypersurface $f_s^{-1}(0)$ by Kas-Schlessinger. So we get an embedding of the base space into $\Delta \times S'$. But, of course, the restriction of $\overline{\Psi}$ to the base space depends on the choice of T_s. Note that a miniversal deformation of f_0 induces that of f_s for $s \in S$ sufficiently near 0. One might expect that the local Torelli for hypersurfaces holds on $T_s \cap S_{\tau(s)}$, i.e.,

(3.8.2) The restriction of $\overline{\Psi}$ to $T_s \cap S_{\tau(s)}$ is injective on a neighborhood of s.

If s is a smooth point of $S_{\tau(s)}$, then $T_s \cap S_{\tau(s)}$ is smooth, and (3.8.2) is equivalent to the infinitesimal Torelli (i.e. the injectivity of the differential). But (3.8.2) is not true as is seen in the following :

Example. — With the notation of (3.4), assume $S^0 = \{0\}$ so that $\overline{\Psi}$ is constant. If (3.8.2) is true, we have $T_s \cap S_{\tau(s)} = \{s\}$ for any $s \in S = S^-$, or equivalently :

(3.8.3) $$\dim S_\tau = \mu - \tau \quad \text{for any} \quad \tau$$

using (3.8.1). But this is not true in general. For example, let $f_0 = x^6 + y^5$. Then $\dim S = 3$, and the μ-constant deformation is given by :

$$f = x^6 + y^5 + ax^4y^3 + bx^4y^2 + cx^3y^3,$$

where (a, b, c) is the coordinate system of S. By definition,

$$\mu - \tau = \dim \operatorname{Im}\big(f_s : \mathcal{O}_{X,0}/(\partial f_s) \to \mathcal{O}_{X,0}/(\partial f_s)\big),$$

and we can check $\mu - \tau = 0$ if $(a, b, c) = (0, 0, 0)$, 1 if $a \neq 0, (b, c) = (0, 0)$, and 2 otherwise, because $x^5y^2, x^3y^4 \in (\partial f_s)$. (In fact, in the case $b \neq 0$, we can check $x^3y^4, x^6y^2, x^7y, x^5y^3, x^2y^5, x^5y^2 \in (\partial f_s)$ inductively.)

3.9 Remark. — Another major problem in [8] is that the filtration $''V$ depends not only on the choice of a good basis, but also on the representative of the basis. We can find easily a counter example to (5.3) in [*loc. cit.*]. It gives also a counter example to Proposition and Corollary in (5.8) of [*loc. cit.*]. The error comes form a misinterpretation of the determinant theorem in [*loc. cit.*]; there is no relation between $\omega_1, \ldots, \omega_\mu$ associated with f and those associated with f_r as long as the properties (ii) and (iii) are concerned. Here KARPISHPAN uses the decomposition $X \times S'$ of the total space to extend a form on X to the total space. But the extension depends heavily on the choice of the decomposition (associated with the choice of the miniversal deformation). If one uses relative differential forms, there is a problem about the ambiguity of the extension as relative forms, and it is not easy to get something well-defined without loosing information, see remark (3.10) below.

Example. — Let $f = x^4 + y^4 \in \mathbb{C}\{x, y\}$. We have :

$$\mathrm{d}f \wedge \mathrm{d}(x^iy^j) = 4(jx^{i+4}y^j - ix^iy^{j+4})\mathrm{d}x/x \wedge \mathrm{d}y/y,$$

and $\mathcal{H}''_f = \Omega^2_{\mathbb{C}^2,0}/\mathrm{d}f \wedge \mathrm{d}\mathcal{O}_{\mathbb{C}^2,0}$ has a basis $x^i y^j \mathrm{d}x\mathrm{d}y$ $(0 \le i,j \le 2)$ over $\mathbb{C}\{t\}$ or $\mathbb{C}\{\{\partial_t^{-1}\}\}$, where $x^i y^j \mathrm{d}x\mathrm{d}y = 0$ in $\mathcal{H}''_f$ if $(i-3)$ or $(j-3)$ is divisible by 4. With the notation of [8, § 5], we take $x^i y^j \mathrm{d}x\mathrm{d}y$ $(0 \le i,j \le 2;$ $(i,j) \ne (2,2))$ and $(x^2y^2 + x^3)\mathrm{d}x\mathrm{d}y$ for the representative of a good basis $\{\omega_1, \ldots, \omega_\mu\}$. Then we can calculate the filtration $''V$ on $\mathbb{C}\{x,y\}$ explicitly, and $x^i y^j$ $(0 \le i,j \le 2)$ can be taken for $\varphi_1, \ldots, \varphi_\mu$. In particular, x^2y^2 is a part of a basis of $\mathrm{Gr}_{''V} M^f$, and $f_r = f + \lambda(r)x^2y^2$ is a μ-constant deformation where $\lambda(r)$ is a non-zero holomorphic function of r. But x^2y^2 does not belong to $''V^1\mathcal{O}$, and consequently the corresponding curve does not belong to $\mathcal{K}$. So $\mathcal{K}$ is smaller than the μ-constant stratum D_μ.

For COROLLARY in (5.8) of [*loc. cit.*], we replace f by $f + z^2$ to get a function of three variables. (This may be done also for (5.3).) Then the exponents are added by $\frac{1}{2}$ because $\mathrm{d}z$ has degree $\frac{1}{2}$. Take $\omega = (x^2y^2 + x^3)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ and $f_r = f + rx^2y^2$. Then $x^3\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is not zero in Ω_{f_r} for $r \ne 0$, and hence $\alpha(s_\omega(r)) = \frac{7}{4}$ for $r \ne 0$ because $\deg x^3\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{7}{4}$. But $\alpha(\omega) = \frac{8}{4}$ at $r = 0$, because $x^3\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$ in $\mathcal{H}''$ at $r = 0$. This gives also a counter example to PROPOSITION in (5.8), because ω is a part of a basis satisfying the three conditions in the determinant theorem applied to f.

3.10 Remark. — Using MALGRANGE's extension [31] of a good basis in [22] (cf. also (3.6)), it is possible to get a correct version of THEOREM in [8, (5.3)] by (3.6.3). Let $\iota : (C,0) \to (S',0)$ be a holomorphic mapping of an open disc C with coordinate s. Let $\{\iota^*v_1, \ldots, \iota^*v_\mu\}$ denote the pull-back of the good basis in (3.6) to $\iota^*\mathcal{H}''_{f'}$. Then :

(3.10.1) $\iota(C)$ is contained in the μ-constant stratum if and only if $\alpha\big(\partial_s^k(\iota^*v_j)_{\,|\,s=0}\big) \ge \alpha_j$ for any $k > 0$, j,

where $\alpha(w) = \max\{\alpha : w \in V^\alpha\mathcal{G}_{f_0}\}$ for $w \in \mathcal{G}_{f_0}$, and α_j is as in (3.6) so that $\alpha_j = \alpha(\iota^*v_{j\,|\,s=0})$. In fact, if $\iota(C)$ is contained in the μ-constant stratum, we have $\alpha(\partial_s^k(\iota^*v_j)_{\,|\,s=0}) \ge \alpha_j$ by (3.6.3).

The proof of the converse is essentially same as in [8], where the argument holds for any basis. Assume $\iota(C)$ is not contained in the μ-constant stratum. Let $D \subset \Delta \times S'$ denote the discriminant. Let $\{\gamma_1, \ldots, \gamma_\mu\}$ be a basis of the multivalued section of the local system on the complement of D, whose stalk is the homology of the Milnor fiber. We have a multivalued holomorphic function $I_{i,j}$ defined by the integration of $v_j/\mathrm{d}f'$ along the stalk of γ_i at each point of $\Delta \times S' \setminus D$. It is well known that $h^{1-n/2}\det(I_{i,j})$ is a unit, where h is a reduced equation of D. (This follows immediately from the calculation of the Brieskorn module in the ordinary double point case.) Here we use the fact that $\{v_1, \ldots, v_\mu\}$ is a basis of $\mathcal{H}''_{f'}$ over $\mathcal{O}_{\Delta\times S'}$, and :

(3.10.2) The degree of the projection $D \to S'$ is μ.

It is also well known that :

(3.10.3) The μ-constant stratum coincides with $\{P \in \Delta \times S' : \mathrm{mult}_P D = \mu\}$.

If $n > 2$, the assumption and (3.10.2–3) imply that $\iota^* h$ is not divisible by t^μ, i.e., $\det(\iota^* I_{i,j})^2$ is not divisible by $t^{(n-2)\mu}$. So $\partial_s^k(\iota^* I_{i,j})|_{s=0}$ is not divisible by $t^{\alpha_j - 1}$ for some i, j, k, and we get the assertion. If $n = 2$, we can apply the same argument to $\{\partial_t^{-1} v_1, \ldots, \partial_t^{-1} v_\mu\}$ which is a basis of $\partial_t^{-1}\mathcal{H}''_{f'}$.

It should be noted that MALGRANGE's extension is highly transcendental and it is not easy to calculate it explicitly, because it is obtained as the solution of some connection using the Fourier transformation of the microlocal Gauss-Manin system, cf. [31]. So it is not easy to get a more explicit formula. It is not clear how much (3.10.1) is useful for the calculation of the μ-constant stratum (for example, compare with (3.10.3)).

3.11 Remark. — We can prove the conjecture about the minimal exponent which is proved in [8, (8.12)] in some cases, using the theory of mixed Hodge Module and filtered microlocalization (which does not change φ), as long as the primitive form is associated with a good section in [22]. With the notation and assumption of (2.6), let :

$$\Omega_f := \mathcal{H}''_f / \partial_t^{-1}\mathcal{H}''_f = \mathrm{Gr}^F_{1-n}\mathcal{G}_f\,,$$

with V the filtration induced by V on $\mathcal{G}_f$. Then the minimal exponent has multiplicity one and the corresponding eigenvector by the morphism A_1 generates Ω_f over $\mathcal{O}_{\mathbb{C}^n,0}$, where A_1 is as in (3.6.1). It is enough to show that V is a filtration by $\mathcal{O}_{\mathbb{C}^n,0}$-submodules such that its graded pieces are annihilated by the maximal ideal $\mathfrak{m}_{\mathbb{C}^n,0}$, because we have a surjective morphism of $\mathcal{O}_{\mathbb{C}^n,0}$-modules $\mathcal{O}_{\mathbb{C}^n,0} \to \Omega_f$ by (3.1.2), and the graduation of the induced filtration V on $\mathcal{O}_{\mathbb{C}^n,0}$ is also annihilated by $\mathfrak{m}_{\mathbb{C}^n,0}$. For the proof of the assertion, it is enough to show that $\mathrm{Gr}^F\,\mathrm{Gr}_V^\alpha(\mathcal{D}_{X\times\mathbb{C}}\delta(f-t))$ is annihilated by $\mathfrak{m}_{\mathbb{C}^n,0}$ for $\alpha < 1$, using the commutativity of Gr^F, Gr_V with the direct image of $\mathcal{D}$-Modules by a compactification of f in [3] (cf. [34, 3.3.17], [35, 2.14]) together with the filtered microlocalization which does not change Gr_V^α for $\alpha < 1$. Then the assertion follows from [34, 3.2.6], because $\mathrm{supp}\,\mathrm{Gr}_V^\alpha(\mathcal{D}_{X\times\mathbb{C}}\delta(f-t)) \subset \{0\}$ for $\alpha < 1$.

3.12 Remark. — The period mappings used in this paper and [8] are quite different from that in [19] which is defined on the complement of the discriminant and depends on the choice of primitive form in general. This was first introduced by BRIESKORN in the case of rational double singularity, and was extended to simple elliptic singularity by LOOIJENGA, K. SAITO. The formalism of K. SAITO [19] works well in these two cases where the non-negativity of the degree of the $\mathbb{C}^*$-action on the base space was essentially used. For the moment, it is not clear whether it gives good information about the complement of the discriminant in the general case. The next test is the fourteen exceptional singularities (e.g. $x^7 + y^3 + z^2$) where we cannot compactify the fibers simultaneously in a natural way (using a weighted projective space) and the Milnor fibration is not defined algebraically, i.e., a Milnor fiber of the universal family is not topologically equivalent to the affine hypersurface containing it, because there is a cycle vanishing toward the divisor at infinity (this can be seen in the one-parameter family $\{x^7 + y^3 + z^2 + sx^5y = 1\}$).

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