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THE PROBLEM OF $L^p$-SIMPLE SPECTRUM
FOR ERGODIC GROUP AUTOMORPHISMS

BY

A. IWANIK (*)

1. Introduction

Let $T$ be an invertible measure preserving transformation of a probability space $(X, B, m)$. The associated unitary operator $U_T f(x) = f(Tx)$ acts on $L^2(m)$. The same formula defines an invertible isometry

$$U_T : L^p(m) \longrightarrow L^p(m)$$

for any $1 \leq p \leq \infty$. A function $f \in L^p(m)$ is said to be $L^p$-cyclic if the linear span of the functions $U^n_T f$ $(n \in \mathbb{Z})$ is dense in $L^p(m)$. If there exits an $L^2$-cyclic function then $T$ is said to have simple spectrum. Analogously, we say that $T$ has $L^p$-simple spectrum if there exists an $L^p$-cyclic vector for $U_T$ in $L^p(m)$.

J.-P. THOUVENOT raised the question whether the Bernoulli automorphism has $L^1$-simple spectrum. Without solving the problem we present some related results. We shall show that, like for $p = 2$, the ergodic group automorphisms have no $L^p$-cyclic vectors for $p > 1$ (Theorem 1). Next we prove that there does exist a cyclic vector for a certain norm weaker than the $L^1$-norm (Theorem 2).

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2. $L^p(G)$ is not finitely generated

Thoughout the paper we consider an ergodic continuous group automorphism $T$ of a compact metric abelian group $G$ endowed with its probability Haar measure $dx$. Let $\widehat{G}$ be the dual group. The dual automorphism $\widehat{T}$ is defined by the formula

$$(\widehat{T}\gamma)(x) = \gamma(Tx), \quad (\gamma \in \widehat{G}).$$

By the ergodicity assumption each $\widehat{T}$-orbit

$$O(\gamma) = \{\widehat{T}^n\gamma : n \in \mathbb{Z}\}, \quad (\gamma \in \widehat{G} \setminus \{1\}),$$

is infinite.

It is known that each $O(\gamma)$ is a Sidon set in $\widehat{G}$, hence a $\Lambda(p)$-set for any $1 \leq p < \infty$ (see [K], Lemma 3 and [L-R]). Consequently, the set

$$E = O(\gamma_1) \cup \cdots \cup O(\gamma_k),$$

where $\gamma_1, \ldots, \gamma_k \in \widehat{G}$, is a $\Lambda(p)$-set so, for any $2 \leq q < \infty$, there exists a constant $C_q$ such that

$$\|g\|_q \leq C_q \|g\|_2$$

whenever $g \in L^q(G)$ with $\text{supp} \hat{g} \subset E$.

Now let $1 < p \leq 2$ and $q \geq 2$ with $p^{-1} + q^{-1} = 1$. We define

$$L^q_E(G) = \{g \in L^q(G) : \text{supp} \hat{g} \subset E\}.$$

If $f \in L^p(G)$ and $g \in L^2_E(G)$ then by Parseval’s identity and Hölder inequality we get

$$\left| \sum_{\gamma \in E} \hat{f}(\gamma) \hat{g}(\gamma) \right| \leq C_q \|f\|_p \|\hat{g}\|_2.$$

It follows that $\|\hat{f}\|_p E_2 \leq C_q \|f\|_p < \infty$. Consequently, if $P_E f$ denotes the function determined by the formula

$$(P_E f)(\gamma) = \begin{cases} \hat{f}(\gamma) & \gamma \in E, \\ 0 & \text{otherwise}, \end{cases}$$

then $P_E$ becomes a continuous projection from $L^p(G)$ onto $L^2_E(G)$. Clearly, $P_E$ is well defined on $L^p(G)$ for any $p > 1$. 

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Apart from $U_T$ we shall consider the operator $\hat{U}_T$ acting on $c_0(\hat{G})$ by

$$\hat{U}_T\xi(\gamma) = \xi(\hat{T}^{-1}\gamma).$$

By a direct computation we have $(U_Tf)^* = \hat{U}_T\hat{f}$ for any $f \in L^1(G)$. Since $E$ is $T$-invariant, we obtain

$$U_TP_E f = P_E U_T f, \quad (f \in L^p(G)).$$

In other words, the following diagram commutes

\[
\begin{array}{ccc}
L^p(G) & \xrightarrow{U_T} & L^p(G) \\
| P_E \downarrow & & \downarrow P_E \\
L^2_E(G) & \xrightarrow{U_T} & L^2_E(G)
\end{array}
\]

**Theorem 1.** Let $p > 1$ and $f_1, \ldots, f_r$ be any finite collection in $L^p(G)$. Then the linear span of the functions $U^n_T f_j$ ($n \in \mathbb{Z}$, $j = 1, \ldots, r$) is not dense in $L^p(G)$.

**Proof.** Fix any $k > r$ and let $E$ be the union of $k$ disjoint orbits,

$$E = O(\gamma_1) \cup \cdots \cup O(\gamma_k), \quad (\gamma_1, \ldots, \gamma_k \in \hat{G} \setminus \{1\}).$$

The unitary operator $U_T$ restricted to $L^2_{O(\gamma_j)}(G)$ has simple Lebesgue spectrum since $U_T\gamma = \hat{T}\gamma$. Consequently,

$$U_T|L^2_E(G)$$

has Lebesgue spectrum of multiplicity $k$ so the invariant subspace generated by the $r < k$ vectors $P_E f_1, \ldots, P_E f_r$ is not dense in $L^2_E(G)$. By looking at the diagram we infer that the functions $U^n_T f_j$, ($n \in \mathbb{Z}$, $j = 1, \ldots, r$) cannot be linearly dense in $L^p(G)$.
3. Cyclic function for a weaker norm

For the rest of this paper we consider the spectral norm
\[ \|f\|_F = \|\hat{f}\|_\infty \]
on $L^1(G)$. The convergence in $\| \cdot \|_F$ is simply the uniform convergence of Fourier coefficients, and clearly $\|f\|_F \leq \|f\|_1$ for any $f \in L^1(G)$. Evidently, $U_T$ is a $\| \cdot \|_F$ isometry.

Our aim is to prove the existence of a $\| \cdot \|_F$-cyclic function for $U_T$ acting on $L^1(G)$.

First we shall identify $\hat{G} \setminus \{1\}$ with the product space $\mathbb{N} \times \mathbb{Z}$ where $(i,j)$ represents the character $\hat{T}^{-j} \gamma_i$ for a fixed cross section $\gamma_1, \gamma_2, \ldots$ of the infinite $\hat{T}$-orbits in $\hat{G}$. Now $\hat{U}_T$ restricted to $c_0(\mathbb{N} \times \mathbb{Z})$ becomes the translation operator $S$ on $c_0(\mathbb{N} \times \mathbb{Z})$,
\[ (S\xi)(i,j) = \xi(i,j+1). \]
We shall often write $\xi(j) = \xi(i,j)$.

**Lemma.** — A vector $\xi \in c_0(\mathbb{N} \times \mathbb{Z})$ is $c_0$-cyclic with respect to $S$ iff for every $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$
\[ \sum \mu_i * \xi_i = 0 \implies \mu = 0. \]

**Proof.** — First note that $\xi$ is cyclic iff the operator
\[ K : \ell^1(\mathbb{Z}) \rightarrow c_0(\mathbb{N} \times \mathbb{Z}) \]
defined by $(K\lambda)(i,j) = (\lambda * \xi_i)(j)$ has a dense range. Equivalently, $\xi$ is cyclic iff the adjoint operator
\[ K^* : \ell^1(\mathbb{N} \times \mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z}) \]
is one-to-one. But for any $\lambda \in \ell^1(\mathbb{Z})$ and $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ we have
\[ \langle K\lambda, \mu \rangle = \sum_{i,j} (\lambda * \xi_i)(j) \mu(i,j) \]
\[ = \sum_i \sum_j \lambda(n) \xi_i(j-n) \mu_i(j) \]
\[ = \sum_n \sum_i \lambda(n) (\tilde{\xi}_i * \mu_i)(n) \]
\[ = \langle \lambda, \sum_i \tilde{\xi}_i * \mu_i \rangle, \]
where $\tilde{\xi}_i(j) = \xi_i(-j)$. This means
\[ K^* \mu = \sum_i \tilde{\xi}_i * \mu_i. \]
Since $\xi$ is cyclic iff $\tilde{\xi}$ is cyclic, we obtain the desired condition.
Corollary. — If \( f \in L^1(G) \) has absolutely convergent Fourier series then \( f \) is not \( L^1 \)-cyclic for \( U_T \).

Proof. — Suppose to the contrary that \( \hat{f} \in \ell^1(\hat{G}) \) and \( f \) is \( L^1 \)-cyclic. Then \( \hat{f} \) is \( c_0(\hat{G}) \)-cyclic for \( \hat{U}_T \). By identifying \( \hat{G} \setminus \{1\} \) with \( \mathbb{N} \times \mathbb{Z} \) as above, we would obtain a \( c_0(\mathbb{N} \times \mathbb{Z}) \)-cyclic vector \( \xi = \hat{f}|_{\mathbb{N} \times \mathbb{Z}} \in \ell^1(\mathbb{N} \times \mathbb{Z}) \) for \( S \).

Since clearly \( \xi_i \neq 0 \) for every \( i \in \mathbb{N} \), we can define a nonzero vector \( \mu \) in \( \ell^1(\mathbb{N} \times \mathbb{Z}) \) by letting \( \mu_1 = \xi_2, \mu_2 = -\xi_1 \) and \( \mu_i = 0 \) for \( i \geq 2 \). Now

\[
\sum \xi_i \mu_i = 0
\]

which contradicts the Lemma.

We prove now the existence of a \( \| \cdot \|_F \)-cyclic function.

Theorem 2. — There exists \( f \in L^2(G) \) such that the linear span of the functions \( U^n f \ (n \in \mathbb{Z}) \) is dense in \( \| \cdot \|_F \).

Proof. — Since \( U_T 1 = 1 \) and

\[
\frac{1}{n} (f + U_T f + \cdots + U_T^{n-1} f) \rightarrow \int f(x) \, dx
\]

in \( L^1(G) \), it suffices to find a \( \| \cdot \|_F \)-cyclic vector for the subspace

\[
\{ f \in L^1(G) : \int f(x) \, dx = 0 \}.
\]

Equivalently, we shall find a \( c_0(\mathbb{N} \times \mathbb{Z}) \)-cyclic vector \( \xi \in \ell^2(\mathbb{N} \times \mathbb{Z}) \) for \( S \).

Let \( Q_1, Q_2, \ldots \) be disjoint countable dense subsets of the unit interval \((0,1)\). For each \( Q_n \) pick an atomic probability measure \( \nu_n \) whose set of atoms coincides with \( Q_n \). Now fix a convergent series \( \sum a_n < \infty \), with \( a_n > 0 \), and define

\[
g_n(t) = a_n \nu_n([0,t])
\]

for \( 0 \leq t < 1 \). The functions \( g_n \) are right continuous and the set of discontinuity points of \( g_n \) coincides with \( Q_n \).

Moreover, the functions

\[
h_n(e^{2\pi it}) = g_n(t), \quad (0 \leq t < 1),
\]

satisfy the conditions

\[
\sum \| h_n \|_2 \leq \sum \| h_n \|_\infty = \sum a_n < \infty.
\]
(We can identify [0,1) with T and $g_n$ with $h_n$.)

Now we let $\xi_n = \hat{h}_n$, where the Fourier transform is taken in the sense of the T-Z duality. We shall show that $\xi$ is $c_0(\mathbb{N} \times \mathbb{Z})$-cyclic. By the Lemma it suffices to prove that any $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ which satisfies

$$\sum \mu_n * \xi_n = 0$$

must in fact vanish. Let $u_n \in C(T)$ be such that $\hat{u}_n = \mu_n$. Then

$$(h_n u_n) = \hat{h}_n * \hat{u}_n = \xi_n * \mu_n.$$

The condition $\sum \xi_n * \mu_n = 0$ now implies

$$\sum h_n u_n = 0 \quad a.e.,$$

where the series converges in $L^2(T)$. Since $|h_n| \leq a_n$ and $|u_n| \leq ||\mu||$, the series converges uniformly. By the right continuity of the $g_n$ the sum $\sum h_n u_n$ is also right continuous. This implies

$$\sum h_n(x) u_n(x) = 0$$

everywhere. To end the proof we show that the latter condition forces

$$u_1 = u_2 = \cdots = 0,$$

whence $\mu = 0$. To see this suppose, to the contrary, that e.g. $u_1 \neq 0$. Then there exists an arc $J \subset T$ with

$$|u_1(x)| \geq \varepsilon > 0$$

for $x \in J$. We have

$$h_1 = - \sum_{n \geq 2} \frac{u_n}{u_1} h_n$$
on $J$. The latter series is uniformly convergent en $J$, so its sum is continuous at each continuity point of all the $h_n$'s, $n \geq 2$, in particular on $Q_1 \cap J$. On the other hand, each of these points is an atom of $\nu_1$ hence a discontinuity for $h_1$, a contradiction.

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