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THE PROBLEM OF L^p-SIMPLE SPECTRUM FOR ERGODIC GROUP AUTOMORPHISMS

BY

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RÉSUMÉ. — Soit T un automorphisme d'un groupe abélien compact métrisable. L'isométrie inversible $U_T f = f \circ T$ n'admet pas de fonction cyclique dans l'espace L^p pour $p > 1$. D'autre part, il existe une fonction cyclique pour la norme spectrale dans L^1.

ABSTRACT. — Let T be an ergodic automorphism of a compact metric abelian group. Then the invertible isometry operator $U_T f = f \circ T$ admits no L^p-cyclic vector in any L^p space, $p > 1$. There exits a cyclic vector for the spectral norm in L^1.

1. Introduction

Let T be an invertible measure preserving transformation of a probability space (X, B, m). The associated unitary operator $U_T f(x) = f(Tx)$ acts on $L^2(m)$. The same formula defines an invertible isometry

$$U_T : L^p(m) \longrightarrow L^p(m)$$

for any $1 \le p \le \infty$. A function $f \in L^p(m)$ is said to be *L^p-cyclic* if the linear span of the functions $U_T^n f$ $(n \in \mathbb{Z})$ is dense in $L^p(m)$. If there exits an L^2-cyclic function then T is said to have *simple spectrum.* Analogously, we say that T has *L^p-simple spectrum* if there exists an L^p-cyclic vector for U_T in $L^p(m)$.

J.-P. THOUVENOT raised the question whether the Bernoulli automorphism has L^1-simple spectrum. Without solving the problem we present some related results. We shall show that, like for $p = 2$, the ergodic group automorphisms have no L^p-cyclic vectors for $p > 1$ (THEOREM 1). Next we prove that there does exist a cyclic vector for a certain norm weaker than the L^1-norm (THEOREM 2).

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2. $L^p(G)$ is not finitely generated

Thoughout the paper we consider an ergodic continuous group automorphism T of a compact metric abelian group G endowed with its probability Haar measure dx. Let $\widehat{G}$ be the dual group. The dual automorphism $\widehat{T}$ is defined by the formula

$$(\widehat{T}\gamma)(x) = \gamma(Tx), \qquad (\gamma \in \widehat{G}).$$

By the ergodicity assumption each $\widehat{T}$-orbit

$$O(\gamma) = \{\widehat{T}^n\gamma : n \in \mathbb{Z}\}, \qquad (\gamma \in \widehat{G} \setminus \{1\}),$$

is infinite.

It is known that each $O(\gamma)$ is a Sidon set in $\widehat{G}$, hence a $\Lambda(p)$-set for any $1 \le p < \infty$ (see [K], Lemma 3 and [L-R]). Consequently, the set

$$E = O(\gamma_1) \cup \cdots \cup O(\gamma_k),$$

where $\gamma_1, \ldots, \gamma_k \in \widehat{G}$, is a $\Lambda(p)$-set so, for any $2 \le q < \infty$, there exists a constant C_q such that

$$\|g\|_q \le C_q \|g\|_2$$

whenever $g \in L^q(G)$ with $\operatorname{supp} \hat{g} \subset E$.

Now let $1 < p \le 2$ and $q \ge 2$ with $p^{-1} + q^{-1} = 1$. We define

$$L^q_E(G) = \{g \in L^q(G) : \operatorname{supp} \hat{g} \subset E\}\cdot$$

If $f \in L^p(G)$ and $g \in L^2_E(G)$ then by Parseval's identity and Hölder inequality we get

$$\Big|\sum_{\gamma \in E} \hat{f}(\gamma)\, \hat{g}(\gamma)\Big| \le C_q \|f\|_p \, \|\hat{g}\|_2.$$

It follows that $\|\hat{f}\| \, \|E\|_2 \le C_q \|f\|_p < \infty$. Consequently, if $P_E f$ denotes the function determined by the formula

$$(P_E f)\hat{}\,(\gamma) = \begin{cases} \hat{f}(\gamma) & \gamma \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then P_E becomes a continuous projection from $L^p(G)$ onto $L^2_E(G)$. Clearly, P_E is well defined on $L^p(G)$ for any $p > 1$.

Apart from U_T we shall consider the operator $\widehat{U}_T$ acting on $c_0(\widehat{G})$ by

$$\widehat{U}_T\xi(\gamma) = \xi(\widehat{T}^{-1}\gamma).$$

By a direct computation we have $(U_T f)\check{} = \widehat{U}_T\hat{f}$ for any $f \in L^1(G)$. Since E is T-invariant, we obtain

$$U_T P_E f = P_E U_T f, \qquad (f \in L^p(G)).$$

In other words, the following diagram commutes

$$\begin{array}{ccc} L^p(G) & \xrightarrow{\ U_T\ } & L^p(G) \\ {\scriptstyle P_E}\big\downarrow & & \big\downarrow{\scriptstyle P_E} \\ L^2_E(G) & \xrightarrow{\ U_T\ } & L^2_E(G) \end{array}$$

THEOREM 1. — *Let $p > 1$ and $f_1, \dots, f_r$ be any finite collection in $L^p(G)$. Then the linear span of the functions $U_T^n f_j$ $(n \in \mathbb{Z}, j = 1, \dots, r)$ is not dense in $L^p(G)$.*

Proof. — Fix any $k > r$ and let E be the union of k disjoint orbits,

$$E = O(\gamma_1) \cup \cdots \cup O(\gamma_k), \qquad (\gamma_1, \dots, \gamma_k \in \widehat{G} \setminus \{1\}).$$

The unitary operator U_T restricted to $L^2_{O(\gamma_j)}(G)$ has simple Lebesgue spectrum since $U_T\gamma = \widehat{T}\gamma$. Consequently,

$$U_T \,|\, L^2_E(G)$$

has Lebesgue spectrum of multiplicity k so the invariant subspace generated by the $r < k$ vectors $P_E f_1, \dots, P_E f_r$ is not dense in $L^2_E(G)$. By looking at the diagram we infer that the functions $U_T^n f_j$, $(n \in \mathbb{Z}, j = 1, \dots, r)$ cannot be linearly dense in $L^p(G)$.

3. Cyclic function for a weaker norm

For the rest of this paper we consider the spectral norm

$$\|f\|_F = \|\hat{f}\|_\infty$$

on $L^1(G)$. The convergence in $\|\cdot\|_F$ is simply the uniform convergence of Fourier coefficients, and clearly $\|f\|_F \le \|f\|_1$ for any $f \in L^1(G)$. Evidently, U_T is a $\|\cdot\|_F$ isometry.

Our aim is to prove the existence of a $\|\cdot\|_F$-cyclic function for U_T acting on $L^1(G)$.

First we shall identify $\widehat{G} \setminus \{1\}$ with the product space $\mathbb{N} \times \mathbb{Z}$ where (i,j) represents the character $\widehat{T}^{-j}\gamma_i$ for a fixed cross section $\gamma_1, \gamma_2, \ldots$ of the infinite $\widehat{T}$-orbits in $\widehat{G}$. Now $\widehat{U}_T$ restricted to $c_0(\mathbb{N} \times \mathbb{Z})$ becomes the translation operator S on $c_0(\mathbb{N} \times \mathbb{Z})$,

$$(S\xi)(i,j) = \xi(i,j+1).$$

We shall often write $\xi_i(j) = \xi(i,j)$.

LEMMA. — *A vector $\xi \in c_0(\mathbb{N} \times \mathbb{Z})$ is c_0-cyclic with respect to S iff for every $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$*

$$\sum \mu_i * \xi_i = 0 \implies \mu = 0.$$

Proof. — First note that ξ is cyclic iff the operator

$$K : \ell^1(\mathbb{Z}) \longrightarrow c_0(\mathbb{N} \times \mathbb{Z})$$

defined by $(K\lambda)(i,j) = (\lambda * \xi_i)(j)$ has a dense range. Equivalently, ξ is cyclic iff the adjoint operator

$$K^* : \ell^1(\mathbb{N} \times \mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z})$$

is one-to-one. But for any $\lambda \in \ell^1(\mathbb{Z})$ and $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ we have

$$\begin{aligned}
\langle K\lambda, \mu\rangle &= \sum_{i,j} (\lambda * \xi_i)(j)\, \mu(i,j) \\
&= \sum_{i,j}\sum_{n} \lambda(n)\, \xi_i(j-n)\, \mu_i(j) \\
&= \sum_{n}\sum_{i} \lambda(n)\, (\tilde{\xi}_i * \mu_i)(n) \\
&= \Big\langle \lambda, \sum_i \tilde{\xi}_i * \mu_i \Big\rangle,
\end{aligned}$$

where $\tilde{\xi}_i(j) = \xi_i(-j)$. This means

$$K^*\mu = \sum_i \tilde{\xi}_i * \mu_i .$$

Since ξ is cyclic iff $\tilde{\xi}$ is cyclic, we obtain the desired condition.

COROLLARY. — *If $f \in L^1(G)$ has absolutely convergent Fourier series then f is not L^1-cyclic for U_T.*

Proof. — Suppose to the contrary that $\hat{f} \in \ell^1(\widehat{G})$ and f is L^1-cyclic. Then $\hat{f}$ is $c_0(\hat{G})$-cyclic for $\widehat{U}_T$. By identifying $\widehat{G}\setminus\{1\}$ with $\mathbb{N}\times\mathbb{Z}$ as above, we would obtain a $c_0(\mathbb{N}\times\mathbb{Z})$-cyclic vector $\xi = \hat{f}_{|\mathbb{N}\times\mathbb{Z}} \in \ell^1(\mathbb{N}\times\mathbb{Z})$ for S. Since clearly $\xi_i \neq 0$ for every $i \in \mathbb{N}$, we can define a nonzero vector μ in $\ell^1(\mathbb{N}\times\mathbb{Z})$ by letting $\mu_1 = \xi_2$, $\mu_2 = -\xi_1$ and $\mu_i = 0$ for $i \geq 2$. Now

$$\sum \xi_i * \mu_i = 0$$

which contradicts the Lemma.

We prove now the existence of a $\|\cdot\|_F$-cyclic function.

THEOREM 2. — *There exists $f \in L^2(G)$ such that the linear span of the functions $U_T^n f$ $(n \in \mathbb{Z})$ is dense in $\|\cdot\|_F$.*

Proof. — Since $U_T 1 = 1$ and

$$\frac{1}{n}\left(f + U_T f + \cdots + U_T^{n-1} f\right) \longrightarrow \int f(x)\,\mathrm{d}x$$

in $L^1(G)$, it suffices to find a $\|\cdot\|_F$-cyclic vector for the subspace

$$\Big\{f \in L^1(G) : \int f(x)\,\mathrm{d}x = 0\Big\}\cdot$$

Equivalently, we shall find a $c_0(\mathbb{N}\times\mathbb{Z})$-cyclic vector $\xi \in \ell^2(\mathbb{N}\times\mathbb{Z})$ for S.

Let $Q_1, Q_2, \ldots$ be disjoint countable dense subsets of the unit interval $(0,1)$. For each Q_n pick an atomic probability measure ν_n whose set of atoms coincides with Q_n. Now fix a convergent series $\sum a_n < \infty$, with $a_n > 0$, and define

$$g_n(t) = a_n \nu_n\big([0,t]\big)$$

for $0 \leq t < 1$. The functions g_n are right continuous and the set of discontinuity points of g_n coincides with Q_n.

Moreover, the functions

$$h_n\big(\mathrm{e}^{2\pi i t}\big) = g_n(t), \qquad (0 \leq t < 1),$$

satisfy the conditions

$$\sum \|h_n\|_2 \leq \sum \|h_n\|_\infty = \sum a_n < \infty.$$

(We can identify $[0,1)$ with $\mathbb{T}$ and g_n with h_n.)

Now we let $\xi_n = \hat{h}_n$, where the Fourier transform is taken in the sense of the $\mathbb{T}$-$\mathbb{Z}$ duality. We shall show that ξ is $c_0(\mathbb{N}\times\mathbb{Z})$-cyclic. By the LEMMA it suffices to prove that any $\mu \in \ell^1(\mathbb{N}\times\mathbb{Z})$ which satisfies

$$\sum \mu_n * \xi_n = 0$$

must in fact vanish. Let $u_n \in C(\mathbb{T})$ be such that $\hat{u}_n = \mu_n$. Then

$$(h_n u_n)\hat{} = \hat{h}_n * \hat{u}_n = \xi_n * \mu_n.$$

The condition $\sum \xi_n * \mu_n = 0$ now implies

$$\sum h_n u_n = 0 \qquad \text{a.e.},$$

where the series converges in $L^2(\mathbb{T})$. Since $|h_n| \le a_n$ and $|u_n| \le \|\mu\|$, the series converges uniformly. By the right continuity of the g_n the sum $\sum h_n u_n$ is also right continuous. This implies

$$\sum h_n(x)\, u_n(x) = 0$$

everywhere. To end the proof we show that the latter condition forces

$$u_1 = u_2 = \cdots = 0,$$

whence $\mu = 0$. To see this suppose, to the contrary, that *e.g.* $u_1 \neq 0$. Then there exists an arc $J \subset \mathbb{T}$ with

$$\big|u_1(x)\big| \ge \varepsilon > 0$$

for $x \in J$. We have

$$h_1 = -\sum_{n\ge 2} \frac{u_n}{u_1} h_n$$

on J. The latter series is uniformly convergent en J, so its sum is continuous at each continuity point of all the h_n's, $n \ge 2$, in particular on $Q_1 \cap J$. On the other hand, each of these points is an atom of ν_1 hence a discountinuity for h_1, a contradiction.

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