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THE PROBLEM OF L^p -SIMPLE SPECTRUM FOR ERGODIC GROUP AUTOMORPHISMS

BY

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RÉSUMÉ. — Soit T un automorphisme d'un groupe abélien compact métrisable. L'isométrie inversible $U_T f = f \circ T$ n'admet pas de fonction cyclique dans l'espace L^p pour $p > 1$. D'autre part, il existe une fonction cyclique pour la norme spectrale dans L^1 .

ABSTRACT. — Let T be an ergodic automorphism of a compact metric abelian group. Then the invertible isometry operator $U_T f = f \circ T$ admits no L^p -cyclic vector in any L^p space, $p > 1$. There exists a cyclic vector for the spectral norm in L^1 .

1. Introduction

Let T be an invertible measure preserving transformation of a probability space (X, B, m) . The associated unitary operator $U_T f(x) = f(Tx)$ acts on $L^2(m)$. The same formula defines an invertible isometry

$$U_T : L^p(m) \longrightarrow L^p(m)$$

for any $1 \leq p \leq \infty$. A function $f \in L^p(m)$ is said to be L^p -cyclic if the linear span of the functions $U_T^n f$ ($n \in \mathbb{Z}$) is dense in $L^p(m)$. If there exists an L^2 -cyclic function then T is said to have *simple spectrum*. Analogously, we say that T has L^p -*simple spectrum* if there exists an L^p -cyclic vector for U_T in $L^p(m)$.

J.-P. THOUVENOT raised the question whether the Bernoulli automorphism has L^1 -simple spectrum. Without solving the problem we present some related results. We shall show that, like for $p = 2$, the ergodic group automorphisms have no L^p -cyclic vectors for $p > 1$ (THEOREM 1). Next we prove that there does exist a cyclic vector for a certain norm weaker than the L^1 -norm (THEOREM 2).

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2. $L^p(G)$ is not finitely generated

Throughout the paper we consider an ergodic continuous group automorphism T of a compact metric abelian group G endowed with its probability Haar measure dx . Let \widehat{G} be the dual group. The dual automorphism \widehat{T} is defined by the formula

$$(\widehat{T}\gamma)(x) = \gamma(Tx), \quad (\gamma \in \widehat{G}).$$

By the ergodicity assumption each \widehat{T} -orbit

$$O(\gamma) = \{\widehat{T}^n\gamma : n \in \mathbb{Z}\}, \quad (\gamma \in \widehat{G} \setminus \{1\}),$$

is infinite.

It is known that each $O(\gamma)$ is a Sidon set in \widehat{G} , hence a $\Lambda(p)$ -set for any $1 \leq p < \infty$ (see [K], Lemma 3 and [L-R]). Consequently, the set

$$E = O(\gamma_1) \cup \dots \cup O(\gamma_k),$$

where $\gamma_1, \dots, \gamma_k \in \widehat{G}$, is a $\Lambda(p)$ -set so, for any $2 \leq q < \infty$, there exists a constant C_q such that

$$\|g\|_q \leq C_q \|g\|_2$$

whenever $g \in L^q(G)$ with $\text{supp } \widehat{g} \subset E$.

Now let $1 < p \leq 2$ and $q \geq 2$ with $p^{-1} + q^{-1} = 1$. We define

$$L_E^q(G) = \{g \in L^q(G) : \text{supp } \widehat{g} \subset E\}.$$

If $f \in L^p(G)$ and $g \in L_E^2(G)$ then by Parseval's identity and Hölder inequality we get

$$\left| \sum_{\gamma \in E} \widehat{f}(\gamma) \widehat{g}(\gamma) \right| \leq C_q \|f\|_p \|\widehat{g}\|_2.$$

It follows that $\|\widehat{f}\| \|E\|_2 \leq C_q \|f\|_p < \infty$. Consequently, if $P_E f$ denotes the function determined by the formula

$$(P_E f)^\wedge(\gamma) = \begin{cases} \widehat{f}(\gamma) & \gamma \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then P_E becomes a continuous projection from $L^p(G)$ onto $L_E^2(G)$. Clearly, P_E is well defined on $L^p(G)$ for any $p > 1$.

Apart from U_T we shall consider the operator \widehat{U}_T acting on $c_0(\widehat{G})$ by

$$\widehat{U}_T \xi(\gamma) = \xi(\widehat{T}^{-1}\gamma).$$

By a direct computation we have $(U_T f)^\vee = \widehat{U}_T \hat{f}$ for any $f \in L^1(G)$. Since E is T -invariant, we obtain

$$U_T P_E f = P_E U_T f, \quad (f \in L^p(G)).$$

In other words, the following diagram commutes

$$\begin{array}{ccc} L^p(G) & \xrightarrow{U_T} & L^p(G) \\ P_E \downarrow & & \downarrow P_E \\ L^2_E(G) & \xrightarrow{U_T} & L^2_E(G) \end{array}$$

THEOREM 1. — *Let $p > 1$ and f_1, \dots, f_r be any finite collection in $L^p(G)$. Then the linear span of the functions $U_T^n f_j$ ($n \in \mathbb{Z}, j = 1, \dots, r$) is not dense in $L^p(G)$.*

Proof. — Fix any $k > r$ and let E be the union of k disjoint orbits,

$$E = O(\gamma_1) \cup \dots \cup O(\gamma_k), \quad (\gamma_1, \dots, \gamma_k \in \widehat{G} \setminus \{1\}).$$

The unitary operator U_T restricted to $L^2_{O(\gamma_j)}(G)$ has simple Lebesgue spectrum since $U_T \gamma = \widehat{T} \gamma$. Consequently,

$$U_T|_{L^2_E(G)}$$

has Lebesgue spectrum of multiplicity k so the invariant subspace generated by the $r < k$ vectors $P_E f_1, \dots, P_E f_r$ is not dense in $L^2_E(G)$. By looking at the diagram we infer that the functions $U_T^n f_j$, ($n \in \mathbb{Z}, j = 1, \dots, r$) cannot be linearly dense in $L^p(G)$.

3. Cyclic function for a weaker norm

For the rest of this paper we consider the spectral norm

$$\|f\|_F = \|\hat{f}\|_\infty$$

on $L^1(G)$. The convergence in $\|\cdot\|_F$ is simply the uniform convergence of Fourier coefficients, and clearly $\|f\|_F \leq \|f\|_1$ for any $f \in L^1(G)$. Evidently, U_T is a $\|\cdot\|_F$ isometry.

Our aim is to prove the existence of a $\|\cdot\|_F$ -cyclic function for U_T acting on $L^1(G)$.

First we shall identify $\widehat{G} \setminus \{1\}$ with the product space $\mathbb{N} \times \mathbb{Z}$ where (i, j) represents the character $\widehat{T}^{-j} \gamma_i$ for a fixed cross section $\gamma_1, \gamma_2, \dots$ of the infinite \widehat{T} -orbits in \widehat{G} . Now \widehat{U}_T restricted to $c_0(\mathbb{N} \times \mathbb{Z})$ becomes the translation operator S on $c_0(\mathbb{N} \times \mathbb{Z})$,

$$(S\xi)(i, j) = \xi(i, j + 1).$$

We shall often write $\xi_i(j) = \xi(i, j)$.

LEMMA. — *A vector $\xi \in c_0(\mathbb{N} \times \mathbb{Z})$ is c_0 -cyclic with respect to S iff for every $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$*

$$\sum \mu_i * \xi_i = 0 \implies \mu = 0.$$

Proof. — First note that ξ is cyclic iff the operator

$$K : \ell^1(\mathbb{Z}) \longrightarrow c_0(\mathbb{N} \times \mathbb{Z})$$

defined by $(K\lambda)(i, j) = (\lambda * \xi_i)(j)$ has a dense range. Equivalently, ξ is cyclic iff the adjoint operator

$$K^* : \ell^1(\mathbb{N} \times \mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z})$$

is one-to-one. But for any $\lambda \in \ell^1(\mathbb{Z})$ and $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ we have

$$\begin{aligned} \langle K\lambda, \mu \rangle &= \sum_{i,j} (\lambda * \xi_i)(j) \mu(i, j) \\ &= \sum_{i,j} \sum_n \lambda(n) \xi_i(j - n) \mu_i(j) \\ &= \sum_n \sum_i \lambda(n) (\tilde{\xi}_i * \mu_i)(n) \\ &= \left\langle \lambda, \sum_i \tilde{\xi}_i * \mu_i \right\rangle, \end{aligned}$$

where $\tilde{\xi}_i(j) = \xi_i(-j)$. This means

$$K^* \mu = \sum_i \tilde{\xi}_i * \mu_i.$$

Since ξ is cyclic iff $\tilde{\xi}$ is cyclic, we obtain the desired condition.

COROLLARY. — If $f \in L^1(G)$ has absolutely convergent Fourier series then f is not L^1 -cyclic for U_T .

Proof. — Suppose to the contrary that $\hat{f} \in \ell^1(\hat{G})$ and f is L^1 -cyclic. Then \hat{f} is $c_0(\hat{G})$ -cyclic for \hat{U}_T . By identifying $\hat{G} \setminus \{1\}$ with $\mathbb{N} \times \mathbb{Z}$ as above, we would obtain a $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic vector $\xi = \hat{f}|_{\mathbb{N} \times \mathbb{Z}} \in \ell^1(\mathbb{N} \times \mathbb{Z})$ for S . Since clearly $\xi_i \neq 0$ for every $i \in \mathbb{N}$, we can define a nonzero vector μ in $\ell^1(\mathbb{N} \times \mathbb{Z})$ by letting $\mu_1 = \xi_2, \mu_2 = -\xi_1$ and $\mu_i = 0$ for $i \geq 2$. Now

$$\sum \xi_i * \mu_i = 0$$

which contradicts the Lemma.

We prove now the existence of a $\|\cdot\|_F$ -cyclic function.

THEOREM 2. — There exists $f \in L^2(G)$ such that the linear span of the functions $U_T^n f$ ($n \in \mathbb{Z}$) is dense in $\|\cdot\|_F$.

Proof. — Since $U_T 1 = 1$ and

$$\frac{1}{n}(f + U_T f + \dots + U_T^{n-1} f) \longrightarrow \int f(x) dx$$

in $L^1(G)$, it suffices to find a $\|\cdot\|_F$ -cyclic vector for the subspace

$$\{f \in L^1(G) : \int f(x) dx = 0\}.$$

Equivalently, we shall find a $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic vector $\xi \in \ell^2(\mathbb{N} \times \mathbb{Z})$ for S .

Let Q_1, Q_2, \dots be disjoint countable dense subsets of the unit interval $(0, 1)$. For each Q_n pick an atomic probability measure ν_n whose set of atoms coincides with Q_n . Now fix a convergent series $\sum a_n < \infty$, with $a_n > 0$, and define

$$g_n(t) = a_n \nu_n([0, t])$$

for $0 \leq t < 1$. The functions g_n are right continuous and the set of discontinuity points of g_n coincides with Q_n .

Moreover, the functions

$$h_n(e^{2\pi it}) = g_n(t), \quad (0 \leq t < 1),$$

satisfy the conditions

$$\sum \|h_n\|_2 \leq \sum \|h_n\|_\infty = \sum a_n < \infty.$$

(We can identify $[0, 1)$ with \mathbb{T} and g_n with h_n .)

Now we let $\xi_n = \hat{h}_n$, where the Fourier transform is taken in the sense of the \mathbb{T} - \mathbb{Z} duality. We shall show that ξ is $c_0(\mathbb{N} \times \mathbb{Z})$ -cyclic. By the LEMMA it suffices to prove that any $\mu \in \ell^1(\mathbb{N} \times \mathbb{Z})$ which satisfies

$$\sum \mu_n * \xi_n = 0$$

must in fact vanish. Let $u_n \in C(\mathbb{T})$ be such that $\hat{u}_n = \mu_n$. Then

$$(h_n u_n)^\wedge = \hat{h}_n * \hat{u}_n = \xi_n * \mu_n.$$

The condition $\sum \xi_n * \mu_n = 0$ now implies

$$\sum h_n u_n = 0 \quad \text{a.e.,}$$

where the series converges in $L^2(\mathbb{T})$. Since $|h_n| \leq a_n$ and $|u_n| \leq \|\mu\|$, the series converges uniformly. By the right continuity of the g_n the sum $\sum h_n u_n$ is also right continuous. This implies

$$\sum h_n(x) u_n(x) = 0$$

everywhere. To end the proof we show that the latter condition forces

$$u_1 = u_2 = \dots = 0,$$

whence $\mu = 0$. To see this suppose, to the contrary, that *e.g.* $u_1 \neq 0$. Then there exists an arc $J \subset \mathbb{T}$ with

$$|u_1(x)| \geq \varepsilon > 0$$

for $x \in J$. We have

$$h_1 = - \sum_{n \geq 2} \frac{u_n}{u_1} h_n$$

on J . The latter series is uniformly convergent on J , so its sum is continuous at each continuity point of all the h_n 's, $n \geq 2$, in particular on $Q_1 \cap J$. On the other hand, each of these points is an atom of ν_1 hence a discontinuity for h_1 , a contradiction.

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