On the cohomology of the classifying space of the gauge group over some 4-complexes


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ON THE COHOMOLOGY OF THE CLASSIFYING
SPACE OF THE GAUGE GROUP
OVER SOME 4-COMPLEXES

BY

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1. Introduction

We consider pairs \((X, [X])\), where \(X\) is a space having the homotopy type of a bouquet of a finite number of 2-spheres with one 4-cell attached, and \([X]\) is a generator of \(H^4(X; \mathbb{Z}) \approx \mathbb{Z}\). For example, it is well known (see for instance [MH]) that any oriented closed simply-connected 4-manifold \(X\), with fundamental class \([X]\), is of this type. The algebraic invariants of the pair \((X, [X])\) are \((L, \varphi)\), where \(L = H_2(X; \mathbb{Z})\) is a free \(\mathbb{Z}\)-module of finite rank, and \(\varphi \in \text{BS}(L^*)\) is the symmetric bilinear form on \(L^* = H^2(X; \mathbb{Z})\) given by the cup product and evaluation on \([X]\). We call \(\varphi\) the "intersection form" of \(X\), even though \(X\) in general cannot be realized as a manifold.

Consider a principal SU(2)-bundle \(P \to X\), with second Chern number \(k\). Let \(G_k(X)\) be the gauge group of \(P\), that is the group of automorphisms of the bundle inducing the identity on \(X\). It is well known [D2]}

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that the classifying space \(BG_k(X)\) has the (weak) homotopy type of the function space \(C(X,BS^3)_k\) of continuous maps \(f : X \to BS^3 = \text{BSU}(2)\) of degree \(k\), i.e. such that \(\langle f^*(c_2), [X] \rangle = k\). We are interested in the cohomology of this space.

As in [D2], consider the linear map

\[
\mu : H_i(X; \mathbb{Z}) \to H^{4-i}(C(X,BS^3)_k; \mathbb{Z})
\]

defined by the slant product \(\mu(\alpha) = ev^*(c_2)/\alpha\), where :

\[
ev : X \times C(X,BS^3)_k \to BS^3
\]

is the evaluation map. As observed by DONALDSON, the map \(\mu\) generates all of the \textit{rational} cohomology of \(C(X,BS^3)_k\). More precisely, the rational cohomology of \(C(X,BS^3)_k\) is isomorphic to the polynomial algebra

\[
\mathbb{Q}[\mu([\text{base point}]), \mu(\alpha_1), \ldots, \mu(\alpha_s)],
\]

where \(\alpha_1, \ldots, \alpha_s\) is a basis of \(L\).

To analyze the situation, and study integral cohomology, we can proceed as follows. There is a natural isomorphism \(BS(L^*) \approx \pi_3(M(L,2))\), where \(M(L,2)\) denotes the 2-dimensional Moore space over \(L\). Viewing \(\varphi\) as an element of \(\pi_3(M(L,2))\) via this isomorphism, we can replace \(X\), up to (oriented) homotopy, by the cofibre of \(\varphi : X \sim M(L,2) \cup \varphi D^4\). This induces a fibration :

\[
(1) \quad \Omega^4 \hat{B} \to C(X,BS^3)_k \xrightarrow{r} C(M(L,2),BS^3).
\]

Here \(r\) denotes restriction of maps, \(\hat{B}\) is the 4-connective covering of \(BS^3\), and \(\Omega\) is the loop space functor.

Set \(A(L) = H^*(C(M(L,2),BS^3); \mathbb{Z})\). This algebra is a covariant functor of \(L\), and was determined in [M1].

**Theorem 1.1.**

\[
A(L) = \bigoplus_{i \geq 0} A_i(L)
\]

\[
= \mathbb{Z}[p][\{\mu_i(\alpha) \mid i \geq 0, \alpha \in L\}]/I.
\]
Here $p$ has degree 4, $\mu_i(\alpha)$ has degree $2i$, and the ideal $I$ is given by the following relations:

(i) $\mu_0(\alpha) = 1$;
(ii) $\mu_n(\alpha + \alpha') = \sum_{i+j=n} \mu_i(\alpha)\mu_j(\alpha')$;
(iii) $\mu_i(\alpha)\mu_j(\alpha) = \sum_k \binom{i + j - k - 1}{i - k} \mu_{i+j-2k}(\alpha)p^k$.

Moreover, we have $\mu ([\text{base point}]) = r^*(p)$, and $\mu(\alpha) = r^*(\mu_1(\alpha))$, $\alpha \in L = H_2(X; \mathbb{Z})$. Consider then Serre's spectral sequence of fibration (1):

$$E_2^{**} = A(L) \otimes H^*(\Omega^4 \tilde{B}; \mathbb{Z}) \Longrightarrow H^*(C(X, BS^3)_k; \mathbb{Z}).$$

Note that the $E_2$-terms is independent of $\varphi$ and $k$. Moreover, $A(L)$ has no torsion, whereas $\tilde{H}^*(\Omega^4 \tilde{B}; \mathbb{Z})$ is torsion since $\pi_i(\Omega^4 \tilde{B}) = \pi_{i+3}(S^3)$ is finite for $i \geq 1$. Thus the restriction map $r$ induces an inclusion

$$r^*: A(L) \hookrightarrow H^*(C(X, BS^3)_k; \mathbb{Z})$$

whose cokernel is torsion. From now on, we will identify $A(L)$ with its image under $r^*$.

Here is a brief outline of this paper.

In paragraph 2, we define and study some "natural" cohomology classes on the space $C(X, BS^3)_k$. In particular, the intersection form $\varphi$ defines an integral class $Q$ of degree 4, and as a corollary we show that the class $(kp+n\Omega)p^{n-1} \in H^{4n}(C(X, BS^3)_k; \mathbb{Z})$ is divisible by $2n+1$. This also shows that in general $A(L)$ is not a direct summand in the integral cohomology of the space $C(X, BS^3)_k$.

In paragraph 3, we use some results on Dyer-Lashof-operations to describe explicitly the homology of $\Omega^4 \tilde{B}$, the fiber of fibration (1).

Paragraph 4 is devoted to studying a certain map $j : \Omega^4 \tilde{B} \to BO$ in homology, which will be used later. We also describe the mod 2 cohomology algebra of $\Omega^4 \tilde{B}$ as a quotient of $H^*(BO; \mathbb{F}_2)$.

In paragraph 5, we put together the results of the previous sections to obtain some divisibility properties in the cohomology of $C(X, BS^3)_k$ that depend heavily on the second Chern number $k$. For example, in
PROPOSITION 5.4 we show that in the integral cohomology of the space $C(S^4, BS^3)_k$, for any odd prime $\ell$, the element $p^{(\ell-1)/2}$ is divisible by $\ell$ if and only if $k \neq 0 (\ell)$. The results of this section allow to distinguish some of the topological group extensions:

$$1 \to G \approx C_\bullet(X, S^3) \to G_k(X) \to S^3 \to 1,$$

where $G_\bullet$ is the subgroup of gauge transformations that act as the identity on one fibre (see Remark 5.6).

In paragraph 6, we study integral cohomology modulo torsion in the special case $X = S^4$, $k = 1$. The main result of this section is stated in Proposition 6.1, where we completely determine the subring of $H^*(C(S^4, BS^3)_1; \mathbb{Z})$/torsion generated by $p$ and the natural classes of paragraph 2. It is possible that this subring is actually equal to $H^*(C(S^4, BS^3)_1; \mathbb{Z})$/torsion. We show this to be the case at least in low degrees, and after inverting 2 (see Corollary 6.3).

Finally, the main result of paragraph 7 is Theorem 7.1 where we show that in the case of base-point-preserving maps, the analogue of fibration (1) is a product when localised at a prime $\geq 5$. This gives an upper bound on divisibility of classes of the form $\mu(\alpha)^n$ (see Corollary 7.2).

REMARK. — Gauge Theory has been used by Donaldson to prove striking results on smooth 4-manifolds (see [D1] for an overview). These results are obtained by studying moduli spaces of anti-self-dual connections, using non-linear analysis and algebraic geometry. The definition of Donaldson’s “polynomial invariants” [D3] makes use, at least formally, of the cohomology of the moduli space of all (irreducible) connections on a SU(2)-bundle over a compact smooth 4-manifold $X$. This space has the (weak) homotopy type of the classifying space of the group $G_k(X)$, the quotient of the gauge group $G_k(X)$ of the bundle by its center $\{\pm 1\}$ (cf. [D2]). Hence this space is at odd primes the same as the space $BG_k(X) \approx C(X, BS^3)_k$ studied in this paper. This relationship originally motivated our interest in divisibility properties in the cohomology ring of $BG_k(X)$.

2. Natural cohomology classes on $C(X, BS^3)_k$

Suppose we can associate to each $(X, [X])$ a cohomology class $\omega(X)$ on $C(X, BS^3)_k$ such that for any degree one map $f : X \to X'$ (i.e. such that $f_*[X] = [X']$) we have $F^*(\omega(X)) = \omega(X')$, where

$$F : C(X', BS^3)_k \to C(X, BS^3)_k$$
is composition with \( f \). Then we will call \( \omega(X) \) a natural cohomology class. For example, \( p = \mu \) ([base point]) is natural. The intersection form \( \varphi \) of \( X \) defines another natural class \( \Omega \) as follows.

Recall that the universal quadratic module \( \Gamma_2(L) \) is defined as \( F/R \), where \( F \) is the free \( \mathbb{Z} \)-module generated by \( L \), and \( R \) is the smallest submodule such that the map \( \gamma_2 : L \to \Gamma_2(L) \) defined in the obvious way satisfies:

1) \( \gamma_2(n\alpha) = n^2\gamma_2(\alpha) \) for \( n \in \mathbb{Z} \);
2) the map \( (\alpha, \beta) \mapsto \gamma_2(\alpha + \beta) - \gamma_2(\alpha) - \gamma_2(\beta) \) is bilinear.

There is a well known natural isomorphism \( \Gamma_2(L) \approx BS(L^*) \), given by sending \( \gamma_2(\alpha) \) to the bilinear form \( (\ell_1, \ell_2) \mapsto \ell_1(\alpha)\ell_2(\alpha) \). Next observe that \( \Gamma_2(L) \) is also the degree 4 part of the classical divided power algebra \( \Gamma(L) = \bigoplus_{i \geq 0} \Gamma_i(L) = \mathbb{Z}[\gamma_i(\alpha) \mid i \geq 0, \alpha \in L]/J \),

where \( \gamma_i(\alpha) \) has degree \( 2i \), and the ideal \( J \) is given by relations (i), (ii) and (iii) of Theorem 1.1 with \( \mu_i \) replaced by \( \gamma_i \), and \( p = 0 \). (Note that (iii) becomes simply \( \gamma_i(\alpha)\gamma_j(\alpha) = \binom{i+j}{i}\gamma_{i+j}(\alpha) \).) The correspondence \( \mu_n(\alpha) \mapsto \gamma_n(\alpha) \) defines a ring homomorphism \( A(L) \to \Gamma(L) \), whose kernel is the ideal generated by \( p \) (cf. [M1]). Moreover, the exact sequence

\[
0 \to \mathbb{Z} \cdot p \to A_2(L) \to \Gamma_2(L) \to 0
\]
is canonically split, upon lifting \( \gamma_2(\alpha) \) to \( \mu_2(\alpha) \). Here is then the promised definition: the class \( \Omega \in A_2(L) \subset H^4(C(X, BS^3)_k; \mathbb{Z}) \) is the canonical lift of the intersection form \( \varphi \in BS(L^*) \), where the latter group is identified with \( \Gamma_2(L) \) as explained above.

Here is the main result of this section:

**Theorem 2.1**

(i) There are natural classes \( \tilde{p}_n(X) \in H^{4n}(C(X, BS^3)_k; \mathbb{Z}[\frac{1}{2}]) \), verifying:

\[
2(2n + 1)s_n(\tilde{p}_1(X), \tilde{p}_2(X), \ldots) = (-1)^{n+1}(kp + n\Omega)p^{n-1}.
\]

(ii) If the intersection form of \( X \) is even, there are natural classes \( \tilde{w}_i(X) \in H^i(C(X, BS^3)_k; \mathbb{F}_2) \), verifying:

\[
s_n(\tilde{w}_1(X), \tilde{w}_2(X), \ldots)^4 = (k\tilde{p} + n\tilde{\Omega})p^{n-1}.
\]

Moreover in this case the \( \tilde{p}_n(X) \) are integral classes, and they verify the relations given above in integral cohomology modulo an element of order 2.

Here \( s_n \) is the \( n \)-th Newton polynomial, and "\( \equiv \)" means reduction mod 2.

Before defining these classes and proving their properties, let us point out the following corollary:

**Corollary**

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COROLLARY 2.2. — The class \((kp + n\Omega)p^{n-1} \in H^{4n}(C(X, BS^3)_k; \mathbb{Z})\) is divisible by \(2n + 1\).

Note that if \(\varphi \in BS(L^*)\) is indivisible (e.g. if \(\varphi\) is non-degenerate), and if \((k, n) = 1\), then \((kp + n\Omega)p^{n-1}\) is indivisible in \(A_{2n}(L)\). (Indeed, it is obvious from the definition of the class \(\Omega\) that \(kp + n\Omega\) is indivisible in \(A_2(L)\) if \((k, n) = 1\). Moreover, it is not hard to see that \(A(L)\) is isomorphic as a \(\mathbb{Z}[p]\)-module (but not as a ring, cf. [M1]), to \(\mathbb{Z}[p] \otimes \Gamma(L)\). Hence multiplication by \(p\) preserves indivisible elements, and the statement follows.)

Thus the corollary implies that the subalgebra

\[ A(L) \subset H^*(C(X, BS^3)_k; \mathbb{Z}) \]

is not a direct summand in this case.

REMARK 2.3. — Note that \(H^*(C(X; BS^3)_k; \mathbb{Z})/\text{torsion} \) injects into \(A(L) \otimes \mathbb{Q}\). A calculation shows that modulo torsion, we have:

\[
1 - \tilde{p}_1 + \tilde{p}_2 - \cdots = (1 + p)^{-k/2} \exp \left[ \left( k - \frac{\Omega}{2p} \right) \left( 1 - \frac{\arctan \sqrt{p}}{\sqrt{p}} \right) \right]
\]

\[
= 1 - \frac{1}{6}(kp + \Omega) + \frac{1}{360} \left( 18k + 5k^2 \right)p^2 + \left( 10k + 36 \right)p\Omega + 5\Omega^2 + \cdots
\]

To define the classes appearing in THEOREM 2.1, we need the following lemma, whose proof is left to the reader.

LEMMA 2.4. — The homology Chern character of \(X\) is injective. Moreover, for all \(X\) of the considered type, we have

\[
\text{ch}_*(K_0(X)) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \approx H_*(X; \mathbb{Z}\left[\frac{1}{2}\right]) \subset H_*(X; \mathbb{Q}),
\]

and if \(X\) has even intersection form, then

\[
\text{ch}_*(K_0(X)) = H_*(X; \mathbb{Z}) \subset H_*(X; \mathbb{Q}).
\]

We introduce the following notation. Let :

\[
[X]_K = (\text{ch}_*)^{-1}[X] \in K_0(X; \mathbb{Z}\left[\frac{1}{2}\right]).
\]

Define \(\eta_X \in \tilde{K}^0(X \times C(X, BS^3)_k; \mathbb{Z}\left[\frac{1}{2}\right])\) by the evaluation map

\[
X \times C(X, BS^3)_k \to BS^3 = BSU(2) \to BSU
\]
and put \( \xi_X = \eta_X/ [X]_K \in K^0(\mathcal{C}(X, BS^3)_k; \mathbb{Z}[\frac{1}{2}]) \). We now define:

\[
\tilde{p}_n(X) = (-1)^n c_{2n}(\xi_X) \in H^{4n}(\mathcal{C}(X, BS^3)_k; \mathbb{Z}[\frac{1}{2}]).
\]

Note that, by Lemma 2.4, we have \([X]_K \in K_0(X) \subset K_0(X; \mathbb{Z}[\frac{1}{2}])\) if \(X\) has even intersection form. Hence \(\xi_X \in K^0(\mathcal{C}(X, BS^3)_k)\) in this case, and \(\tilde{p}_n(X) \in H^{4n}(\mathcal{C}(X, BS^3)_k; \mathbb{Z})\). Moreover, we can then define \(\tilde{w}_i(X) = w_i(\xi_X) \in H^4(\mathcal{C}(X, BS^3)_k; \mathbb{F}_2)\).

It is not hard to see that \(\xi_X\) qualifies as natural in our sense, hence the classes \(\tilde{p}_n(X)\) and \(\tilde{w}_i(X)\) are natural. Moreover, after inverting 2, a space \(X\) which is the cofiber of \(\varphi \in \pi_3(M(L, 2))\) has the same homotopy type as a space \(X'\) which is the cofiber of \(4\varphi\) because there is an obvious degree one map \(X \to X'\) induced by multiplication by 2 on \(L\). Hence, to prove Theorem 2.1 we may suppose that \(X\) has even intersection form.

Consider \(S_g = M(\mathbb{Z}^{2g}, 2) \cup_{\varphi_g} D^4\), where \(\varphi_g = \sum [e_i, e'_i]\), the standard basis of \(\mathbb{Z}^{2g}\) being \((e_1, e'_1, \ldots, e_g, e'_g)\). (Here, \([\alpha, \beta] = \gamma_2(\alpha + \beta) - \gamma_2(\alpha) - \gamma_2(\beta)\) is the Whitehead product.) Note that \(S_g\) has the homotopy type of a connected sum of \(g\) copies of \(S^2 \times S^2\). If \(X\) has even intersection form \(\varphi\), then we can write \(\varphi = \sum [\alpha_i, \alpha'_i]\) where \(\alpha_i, \alpha'_i \in L\). Clearly the map \(f : \mathbb{Z}^{2g} \to L\), defined by \(f(e_i) = \alpha_i, f(e'_i) = \alpha'_i\), extends to a degree one map \(f : S_g \to X\). Since the classes \(p, \Omega, \tilde{p}_n, \tilde{w}_i\) are all natural, this shows that it suffices to prove Theorem 2.1 in the case \(X = S_g\).

From now on, we consider \(X = S_g\). The idea of proof is as follows. The stabilisation map \(j : S^3 = SU(2) \to SU\) induces a commutative diagram:

\[
\begin{array}{ccc}
S_g \times \mathcal{C}(S_g, BS^3)_k & \xrightarrow{\eta} & BS^3 \\
\downarrow 1 \times j & & \downarrow j \\
S_g \times \mathcal{C}(S_g, BSU)_k & \xrightarrow{\tilde{\eta}} & BSU.
\end{array}
\]

Here \(\eta\) and \(\tilde{\eta}\) are the evaluation maps. Let \(c_n \in H^{2n}(BSU; \mathbb{Z})\) be the \(n\)-th Chern class. For \(n \geq 3\) we have \(j^*(c_n) = 0\), hence \((1 \times j)^*(\eta(\tilde{\eta})) = 0\). Writing this equation explicitely will prove the theorem.

In order to calculate the total Chern class of \(\tilde{\eta}\), we will first decompose the space \(\mathcal{C}(S_g, BSU)_k\) as a product. Let \(\mathcal{C}_\bullet(S_g, BSU)_k\) be the subspace formed by the base-point preserving maps. The restriction map

\[
r : \mathcal{C}_\bullet(S_g, BSU)_0 \to \mathcal{C}_\bullet(M(\mathbb{Z}^{2g}, 2), BSU)
\]

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admits a canonical section $s$ defined as follows: thinking of $M(\mathbb{Z}^2,2)$ as a bouquet of $2g$ copies of the 2-sphere, we have:

$$C_\bullet(M(\mathbb{Z}^2,2), \text{BSU}) = (\Omega^2 \text{BSU})^{2g}.$$ 

Let $\varepsilon_i, \varepsilon'_i : S^2 \to M(\mathbb{Z}^2,2) \to S_g$ correspond to $e_i, e'_i \in \mathbb{Z}^2$, and define retractions $r_i, r'_i : S_g \to S^2 \times S^2 \to S^2$ by first contracting to the base point those parts of the 2-skeleton corresponding to an index different from $i$, identifying the result in a standard way with $S^2 \times S^2$, and then projecting onto one of the two factors. Then the section $s$ is defined by the formula

$$s(f_1, f'_1, \ldots, f_g, f'_g)(x) = f_1(r_1(x)) \cdot f'_1(r'_1(x)) \cdots f_g(r_g(x)) \cdot f'_g(r'_g(x)).$$

(Here we use the multiplication on BSU induced by Whitney sum of bundles.) Next, define a map $\tilde{Q} : C(S_g, \text{BSU})_k \to C(S_g, \text{BSU})_k$ by the formula

$$\tilde{Q}(f) = (s(r(f(pt)^{-1} \cdot f)))^{-1} f(pt)^{-1} \cdot f.$$ 

We may suppose that the multiplication on BSU has a strict identity. Then the restriction of $\tilde{Q}(f)$ to $M(\mathbb{Z}^2,2)$ is the trivial map, hence $\tilde{Q}$ factors in the obvious way over a map $Q : C(S_g, \text{BSU})_k \to \Omega^2 \text{BSU}$. Moreover, the following is a homotopy equivalence:

$$C(S_g, \text{BSU})_k \xrightarrow{\cong} \text{BSU} \times C_\bullet(M(\mathbb{Z}^2,2), \text{BSU}) \times \Omega^2 \text{BSU}$$

$$f \mapsto (f(pt), r(f(pt)^{-1} \cdot f), Q(f)).$$

Let $F : S^2 \times \text{BU} \to \text{BSU}, \tilde{F} : S^4 \times \text{BU} \times k \to \text{BSU}$ be adjoint to the Bott equivalences $\text{BU} \approx \Omega^2 \text{BSU}, \text{BU} \times k \approx \Omega^2 \text{BSU}$. Using the inverse of the above homotopy equivalence, the evaluation map $\tilde{\eta}$ becomes:

$$S_g \times \text{BSU} \times (\text{BU})^{2g} \times \text{BU} \times k \approx S_g \times C(S_g, \text{BSU})_k \longrightarrow \text{BSU}$$

$$(x, z, (y_1, y'_1, \ldots, y_g, y'_g), y) \mapsto z \cdot F(r_1(x), y_1) \cdot F(r'_1(x), y'_1) \cdots$$

$$\cdots F(r_g(x), y_g) \cdot F(r'_g(x), y'_g) \cdot \tilde{F}([x], y).$$

(Here, $[x]$ means the image of $x \in S_g$ in $S_g/M(L,2) \approx S^4$.) Let $c$ be the total Chern class. A standard calculation using the splitting principle shows:

$$F^*(c) = 1 + \sigma_2 \otimes A, \quad \text{where } A = \sum_{n \geq 1} (-1)^{n+1} s_n(c_1, c_2, \ldots);$$

$$\tilde{F}^*(c) = 1 + \sigma_4 \otimes B, \quad \text{where } B = k + \sum_{n \geq 1} (-1)^{n+1} (n+1)s_n(c_1, c_2, \ldots).$$
(Here, \( \sigma \) is the standard generator of \( H^i(S^1; \mathbb{Z}) \).) Let \( (a_1, a_1', \ldots, a_g, a_g') \) be the basis of \( H^2(S_g; \mathbb{Z}) = (\mathbb{Z}^{2g})^* \) dual to \( (e_1, e_1', \ldots, e_g, e_g') \), and let \( \sigma = [S_g]^* \in H^4(S_g; \mathbb{Z}) \) be the standard generator of \( H^4(S_g; \mathbb{Z}) \). Since our multiplication on BSU is induced by Whitney sum of bundles, the total Chern class of \( \eta \) is given by:

\[
c(\eta) = (1 \otimes c)(1 + a_1 \otimes A_1) (1 + a_1' \otimes A_1') \cdots (1 + a_g \otimes A_g) (1 + a_g' \otimes A_g')(1 + \sigma \otimes B)
\]

\[
= 1 \otimes c + \sum a_i \otimes cA_i + \sum a_i' \otimes cA_i' + \sigma \otimes (B + \sum A_iA_i').
\]

(Here, the classes \( c, A_i, A_i' \) and \( B \in H^* (\mathcal{C}(S_g, BSU)_k; \mathbb{Z}) \) are meant to correspond in the obvious way to the different components of \( \mathcal{C}(S_g, BSU)_k \approx BSU \times (BU)^{2g} \times BU \times k \). We also used \( a_ia_j' = \delta_{ij}\sigma \) and \( a_ia_j = 0 = a_i'a_j \).)

Now consider diagram (2). Clearly the total Chern class of \( \eta \) is of the form:

\[
c(\eta) = 1 \otimes (1 + p) + \sum a_i \otimes b_i + \sum a_i' \otimes b_i' + \sigma \otimes k.
\]

Since \( H^*(S_g; \mathbb{Z}) \) has no torsion, we deduce:

\[
j^*(c) = 1 + p, \quad j^*(cA_i) = b_i, \quad j^*(cA_i') = b_i', \quad j^*(c(B + \sum A_iA_i')) = k.
\]

Multiplying by \( \sum_{n \geq 0} (-p)^n = 1/(1 + p) \), we deduce \( j^*(A_i) = b_i/(1 + p), \quad j^*(A_i') = b_i'/(1 + p) \). Hence

\[
j^*(B) = \frac{k}{1 + p} - \frac{\Omega}{(1 + p)^2} = k + \sum_{n \geq 1} (-1)^n(kp + n\Omega)p^{n-1},
\]

where we used \( \sum b_ib_i' = \Omega \). Thus, the following lemma immediately implies Theorem 2.1.

**Lemma 2.5.** — We have:

(i) \( 2j^*(B) = 2\left(k - 2 \sum_{n \geq 1} (2n + 1)s_n(\bar{p}_1, \bar{p}_2, \ldots) \right) \in H^* \left( \mathcal{C}(S_g, BS^3)_k; \mathbb{Z} \right) \);

(ii) \( j^*(B) = \sum_{n \geq 1} s_n(\tilde{w}_1(X), \tilde{w}_2(X), \ldots)^4 \in H^* \left( \mathcal{C}(S_g, BS^3)_k; \mathbb{F}_2 \right) \).

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Proof. — The main point here is that $\eta/[S_g] \in \tilde{K}^0(C(S_g,BSU)_k)$ is represented by the map $Q : C(S_g,BSU)_k \to \Omega^1_k BSU \approx BU$. This can be seen as follows. Put $\pi_i = \varepsilon_i \circ r_i$, $\pi'_i = \varepsilon'_i \circ r'_i$, and let $\pi : S_g \to S_g$ be the constant map to the base point. Then for $f \in C(S_g,BSU)_k$, $\tilde{Q}(f)$ can be written:

\[
\left((f \circ \pi)^{-1} \cdot (f \circ \pi_1) \cdot (f \circ \pi)^{-1} \cdot (f \circ \pi'_1) \cdots \right) \left((f \circ \pi)^{-1} \cdot (f \circ \pi_g) \cdot (f \circ \pi)^{-1} \cdot (f \circ \pi'_g) \right)^{-1} \cdot (f \circ \pi)^{-1} \cdot f.
\]

Define $\Phi : S_g \times C(S_g,BSU)_k \to BSU$ by the formula

\[
\Phi(x,f) = \tilde{Q}(f)(x) = \tilde{\eta}(x,\tilde{Q}(f)).
\]

Since $\tilde{\eta}(x,f \circ \pi_i) = f(\pi_i(x)) = \tilde{\eta}(\pi_i(x),f)$, we see that in $K$-theory we can write:

\[
\Phi = (q \times 1)(\tilde{\eta}) \in \tilde{K}^0(S_g \times C(S_g,BSU)_k),
\]

where $q = K^0(S_g) \to K^0(S_g)$ is given by $q = 1 - \sum \pi_i - \sum \pi_i^* + (2g-1)\pi^*$. Clearly, $q$ is a projector onto $\tilde{K}^0(S^4) \subset K^0(S_g)$. Applying the Chern character, it is not hard to see that $q$ corresponds to $[S_g]_K = ch^{-1}_*([S_g])$ under the canonical isomorphism:

\[
\text{Hom}(K^0(S_g),\tilde{K}^0(S^4)) \approx \text{Hom}(K^0(S_g),\mathbb{Z}) \approx K_0(S_g).
\]

It follows

\[
\Phi = \theta \otimes (\eta/[S_g]_K),
\]

where $\theta \in \tilde{K}^0(S^4) \subset K^0(S_g)$ denotes the canonical generator. Since $\Phi$ is essentially the adjoint of $Q$, this shows $Q = \eta/[S_g]_K$ as required.

Thus, we have from the very definition of $B$:

\[
B = k + \sum_{n \geq 1} (-1)^{n+1}(n+1)s_n(c_1(\eta/[S_g]_K),c_2(\eta/[S_g]_K),\ldots).
\]

Since $\xi_{S_g} = \eta_{S_g}/[S_g]_K = j^*(\eta/[S_g]_K) \in K^0(C(X,BS^3)_k)$, it follows:

\[
j^*(B) = k + \sum_{n \geq 1} (-1)^{n+1}(n+1)s_n(c_1(\xi_{S_g}),c_2(\xi_{S_g}),\ldots).
\]

Now recall that we have defined $\tilde{p}_n = (-1)^nc_{2n}(\xi_{S_g})$, $\tilde{w}_i = w_i(\xi_{S_g})$. Of course, the reason for this definition is that $\xi_{S_g}$ is in the image of
the complexification $KO^0 \to K^0$, since the stabilisation map $S^3 \to SU$ factors over $Sp$. Thus, it follows from the well known description of the complexification map $BO \to BU$ in integral cohomology that the odd Chern classes of $\xi_{S_g}$ are torsion of order 2. This implies:

$$s_{2n}(c_1(\xi_{S_g}), c_2(\xi_{S_g}), \ldots) = 2s_n(\tilde{p}_1, \tilde{p}_2, \ldots) + \text{an element of order 2},$$

whence part (i) of the lemma. Part (ii) is proved similarly.

This completes the proof of Theorem 2.1.

Remark 2.6. — Let $M_g$ a closed orientable (real) surface of genus $g$. Note that $M_g$ has the homotopy type of a bouquet of circles with one 2-cell attached. The analogy of this with the homotopy type of $S_g$ may be used to apply the above method to study the cohomology algebra of $C(M_g, BS^3) \approx BG(M_g)$, the classifying space of the gauge group of a (necessarily trivial) $SU(2)$-bundle over $M_g$. This generalizes [M1]. Here we only state the result; details may be found in [M2].

Let $\alpha_1, \ldots, \alpha_g, \alpha'_1, \ldots, \alpha'_g$ be a symplectic basis of $H_1(M_g; \mathbb{Z})$. Define

$$p = \mu(\text{base point}), \quad \beta_i = \mu(\alpha_i), \quad \beta'_i = \mu(\alpha'_i), \quad t = \mu([M_g]),$$

where $\mu : H_1(M_g; \mathbb{Z}) \to H^{4-i}(C(M_g, BS^3); \mathbb{Z})$ is defined as in paragraph 1. Set $\Phi = \sum \beta_i \beta'_i \in H^6(C(M_g, BS^3); \mathbb{Z})$. Let $\eta \in K^0(M_g \times C(M_g, BS^3))$ correspond to the evaluation map, set $[M_g]_K = ch^{-1}_*[M_g]$, and define $x_i = c_i(\eta/[M_g]_K) \in H^{2i}(C(M_g, BS^3); \mathbb{Z})$. Note $x_1 = t$. Then

$$H^*(C(M_g, BS^3); \mathbb{Z}) \subset H^*(C_g; \mathbb{Q})$$

$$\approx \mathbb{Q}[p] \otimes \Lambda_{\mathbb{Q}}(\beta_1, \ldots, \beta_g, \beta'_1, \ldots, \beta'_g) \otimes \mathbb{Q}[t]$$

is the subalgebra generated $p, \beta_1, \ldots, \beta_g, \beta'_1, \ldots, \beta'_g$, and the $x_i$. (This fact was already shown in [AB].) Calculating as in [M1], we find:

$$\sum_{n=0}^{\infty} x_n = \exp \left[ \left( t - \frac{\Phi}{2p} \right) \frac{\arctan \sqrt{p}}{\sqrt{p}} + \frac{\Phi}{2p(1 + p)} \right].$$

(This power series can be written $\exp(tf(p) + \Phi f'(p))$, where $f(p) = \arctan(\sqrt{p})/(\sqrt{p})$.)

Here is a description of this algebra analogous to Theorem 1.1. As an algebra over $\mathbb{Z}[p] \otimes \Lambda_{\mathbb{Z}}(\beta_1, \ldots, \beta'_g)$ (which is the cohomology algebra corresponding to the 1-skeleton of $M_g$), $H^*(C(M_g, BS^3); \mathbb{Z})$ is isomorphic
to the algebra generated by the \( x_i \), divided by an ideal of relations of the form:

\[
x_i x_j = \sum_{k, \ell = 0}^{\infty} A_{ijkl} x_{i+j-2k-3\ell} p_k \Phi^\ell.
\]

(Note that \( \Phi^\ell \) is divisible by \( \ell! \) in \( \Lambda_\mathbb{Z}(\beta_1, \ldots, \beta_g'). \)) Here is a formula for the numbers \( A_{ijkl} \):

\[
A_{ijkl} = \sum_{s=0}^{k} (-1)^s \left( \begin{array}{c} i + j - k - s - 3\ell - 1 \\ k - s \end{array} \right) \times \\
\sum_{-s \leq h \leq s \mod 2} \left( \begin{array}{c} i + j - 2k - 2\ell \\ i - k - \ell + h \end{array} \right) \left( \begin{array}{c} \ell + \frac{1}{2}(s - h) - 1 \\ \frac{1}{2}(s - h) \end{array} \right) \left( \begin{array}{c} \ell + \frac{1}{2}(s + h) - 1 \\ \frac{1}{2}(s + h) \end{array} \right).
\]

Note that, as they must, the numbers \( A_{ijk0} \) coincide with the \( A_{ijk} \) given in Theorem 1.1. It also follows from this description that \( x_1^n \in H^{2n}(C(M_g, BS^3); \mathbb{Z}) \) is divisible precisely by the power of 2 contained in \( n! \). This generalizes Corollary 1 of [M1].

### 3. The classifying space of the based gauge group on \( S^4 \)

The subgroup of the gauge group formed by those gauge transformations whose restriction to the fiber over the base point is the identity, is called the \textit{based} gauge group, and denoted by \( G_b(X) \). It is well known that for any \( S^3 \)-bundle, it is isomorphic to the group \( C_\ast(X, S^3) \) of base-point preserving maps \( X \to S^3 \). Hence the classifying space of the based gauge group on \( S^4 \) has the homotopy type of the space \( \Omega^4 \hat{B} \), the fiber of fibration (1).

The space \( \Omega^4 \hat{B} \) is the zero component of \( \Omega^4 BS^3 \approx \Omega^3 S^3 \approx \Omega^3 \Sigma^3 S^0 \), and it is well known how to describe the homology of the latter in terms of Dyer-Lashof-operations acting on \([1] \in H_0(\Omega^3 S^3)\) (see for example [CLM]). However, since we are ultimately interested in cohomology, it is more convenient to restrict attention to the zero component. We proceed as follows. From the definition of \( \hat{B} \), we deduce a fibration:

\[
S^1 \approx K(\mathbb{Z}, 1) \to \Omega^2 \hat{B} \to \Omega^2 BS^3 \approx \Omega S^3.
\]

An easy calculation with the Serre spectral sequence then shows:

\[
H_\ast(\Omega^2 \hat{B}; F_\ell) \approx P(z_{2\ell}) \otimes E(\beta z_{2\ell}).
\]
(Here, $\ell$ is a prime, $P$ means polynomial algebra, $E$ means exterior algebra, $z_n$ is an element of degree $n$, and $\beta$ is the Bockstein operator in (mod $\ell$) homology.) Proceeding as in [CLM, p. 229], we see that $H_*(\Omega^4 \hat{B}; F_\ell)$ is the free graded commutative algebra on generators obtained by certain Dyer-Lashof-operations acting on an element $y_{2\ell-2} \in H_*(\Omega^4 \hat{B}; F_\ell)$ obtained from $z_2$ by transgression. (Note however that if $\ell = 2$, $y_2$ is well defined only modulo $(\beta y_2)^2$.) Here is the result:

**Proposition 3.1.**

a) $H^B(\Omega^4 \hat{B}; F_2) \approx P[(Q_1)^i \beta y_2, (Q_1)^j (Q_2)^j y_2; i, j \geq 0]$;

b) for $\ell \geq 3$, $H_*(\Omega^4 \hat{B}; F_\ell)$ is the free graded commutative algebra on generators $\beta^e (Q_{\ell-1})^j \beta^e (Q_{2(\ell-1)})^i y_{2\ell-2}$, where $i, j \geq 0$, $e, \bar{e} \in \{0, 1\}$, $\varepsilon \leq j$ and $(j \geq 1 \Rightarrow \bar{e} = 1)$.

(See [CLM, p. 7] for a definition of the operations $Q_n$. Compare also [Mi].)

Note that $|(Q_1)^i \beta y_2| = 2^{i+1} - 1$, $|(Q_1)^i (Q_2)^j y_2| = 2^{i+j+2} - 2^i - 1$, and that $|\beta^e (Q_{\ell-1})^j \beta^e (Q_{2(\ell-1)})^i y_{2\ell-2}| = 2\ell^j (\ell^{i+1} - 1) - e - \bar{e}$.

For $\ell \geq 3$, $y_{2\ell-2}$ is clearly primitive, hence it follows from the Cartan formula that $H_*(\Omega^4 \hat{B}; F_\ell)$ is primitively generated. This implies that the mod $\ell$ cohomology algebra $H^*(\Omega^4 \hat{B}; F_\ell)$ is simply a tensor product of an exterior algebra (on odd-dimensional generators) with a divided power algebra (on even-dimensional generators), the generators being the duals of the homology generators given above. The analogous statement is not true for mod 2 cohomology. In the next section, we will obtain a presentation of $H^*(\Omega^4 \hat{B}; F_2)$.

The relations between Dyer-Lashof-operations and the higher Bockstein operators can also be found in [CLM]. This allows to determine the additive structure of $H_*(\Omega^4 \hat{B}; Z)$ as follows. Set $n(i, j; \ell) = 2\ell^j (\ell^i - 1)$ and $\varphi_n(t) = (1 + t^{n-1})/(1 - t^n)$.

**Proposition 3.2.** — For any prime $\ell$, the Poincaré series of

$$E^r H_*(\Omega^4 \hat{B}; F_\ell), \quad r \geq 2,$$

is given by

$$f_r(t) = \prod_{i \geq 1} \varphi_{n(i, r-1; \ell)}(t).$$

We leave it to the reader to write down $f_1(t)$, i.e. the Poincaré series of $E^1 H_*(\Omega^4 \hat{B}; F_\ell) = H_*(\Omega^4 \hat{B}; F_\ell)$, using Proposition 3.1.
Now recall that $H^B_\ell(\Omega^4 \tilde{B}; \mathbb{Z}) = \bigoplus_{r \geq 1} H^B_\ell(\Omega^4 \tilde{B}; \mathbb{Z}/\ell^r)^{a_{nr}}$, since the space $\Omega^4 \tilde{B}$ is rationally contractible. Moreover, if we write

$$H_n(\Omega^4 \tilde{B}; \mathbb{Z}_\ell) \approx \bigoplus_{r \geq 1} (\mathbb{Z}/\ell^r)^{a_{nr}},$$

then the $a_{nr}$ are given by

$$\sum_{n \geq 1} a_{nr} t^n = \frac{f_r(t) - f_{r+1}(t)}{1+t}.$$

This determines the additive structure of $H_*(\Omega^4 \tilde{B}; \mathbb{Z})$. For later use, we record the following

**Corollary 3.3.** — Let $\ell = 2m + 1$ be an odd prime, and set $N(\ell) = \ell^2 - \frac{1}{2}(\ell + 3)$ if $\ell \geq 5$, and $N(3) = 536$. Suppose $1 \leq n < N(\ell)$. If $n \equiv 0 (m)$, then $H^{4n}(\Omega^4 \tilde{B}; \mathbb{Z}_\ell)$ has exponent $\ell^{1+\nu_\ell(n/m)}$. If $n \not\equiv 0 (m)$, then $H^{4n}(\Omega^4 \tilde{B}; \mathbb{Z}_\ell) = 0$.

Here $\nu_\ell : \mathbb{Q}^* \to \mathbb{Z}$ is $\ell$-adic valuation.

**Sketch of proof.** — The Bockstein spectral sequence of $H_*(\Omega^4 \tilde{B}; F_\ell)$ has a direct summand of the form $P(y_{2\ell-2}) \otimes E(\beta y_{2\ell-2})$, with $\beta_{r+1} y_{2\ell-2} = y_{2\ell-2} - \beta y_{2\ell-2}$. The $\mathbb{Z}_\ell$-cohomology corresponding to this direct summand verifies the statement of the corollary for all $n$. Moreover, it turns out that for $n < N(\ell)$, the exponent of $H^{4n}(\Omega^4 \tilde{B}; \mathbb{Z}_\ell)$ stems from this direct summand. Details are left to the reader.

### 4. The map $j : \Omega^4 \tilde{B} \to BO$

The stabilisation map $S^3 = SU(2) \to SU$ factors over the inclusion $Sp \subset SU$. Thus, the induced map $\mathcal{C}(X, BS^3) \to \mathcal{C}(X, BSU)$ factors over $\mathcal{C}(X, BSp)$. Restricting to base-point preserving maps, and using real Bott periodicity, we have a map $\Omega^4 BS^3 \to \Omega^4 BSp \approx BO \times \mathbb{Z}$. In this section, let us denote by $j : \Omega^4 \tilde{B} \to BO$ the map obtained by restricting to the zero degree component. Clearly, this is a morphism of 4-fold loop spaces.

**Proposition 4.1.** — $j_* : H_*(\Omega^4 \tilde{B}; \mathbb{F}_2) \to H_*(BO; \mathbb{F}_2)$ is injective.

**Proof.** — Recall $H_* (BO; \mathbb{F}_2) = P(a_1, a_2, \ldots)$, where $|a_i| = i$. Since the inclusion $S^3 \to Sp$ is 6-connected, $j_*$ is an isomorphism in degrees $\leq 2$. Replacing, if necessary, $y_2$ by $y_2 + (\beta y_2)^2$, it follows $j_*(y_2) = a_2$, $j_*(\beta y_2) = a_1$. From [K], Theorem 36, we know that in $H_*(BO; \mathbb{F}_2)$, we have $Q_n(a_k) =$
\((n+k-1)a_{n+2k}\) modulo decomposable elements. Since \(j_*\) commutes with \(Q_1\) and \(Q_2\), it follows that \(j_*\) sends the generators of \(H_*(\Omega^4 \tilde{B}; \mathbb{F}_2)\) given in Proposition 3.1 to indecomposable elements. This implies the proposition.

**Corollary 4.2.** — \(H^*(\Omega^4 \tilde{B}; \mathbb{F}_2) \approx H^*(BO; \mathbb{F}_2)/(\ker j^*)\).

The Hopf algebra structure of \(H^*(BO; \mathbb{F}_2)\) is given by

\[
\Delta a_n = \sum a_i \otimes a_{n-i}.
\]

Since \(j_*\) is injective, it follows \(\Delta y_2 = 1 \otimes y_2 + \beta y_2 \otimes \beta y_2 + y_2 \otimes 1\). This and the Cartan formula for Dyer-Lashof-operations completely determine the Hopf algebra structure of \(H_*(\Omega^4 \tilde{B}; \mathbb{F}_2)\). Note that generators of the form \((Q_1)^i \beta y_2\) are primitive, whereas those of the form \((Q_1)^i (Q_2)^j y_2\) are not.

For \(n\) a positive integer, let \(\varepsilon_0(n)\) be the number of zeros of \(n\) when written in binary form. Note that \(H_*(\Omega^4 \tilde{B}; \mathbb{F}_2)\) has a generator precisely in those degrees \(n\) such that \(\varepsilon_0(n) \leq 1\). Recall that \(H^*(BO; \mathbb{F}_2)\) is a polynomial algebra on the Stiefel-Whitney-classes \(w_i\). The following proposition will be proved in the appendix:

**Proposition 4.3**

(i) For each \(n\) such that \(\varepsilon_0(n) \geq 2\), the ideal \(\ker(j^*) \subset H^*(BO; \mathbb{F}_2)\) contains an element \(r_n\) of degree \(n\), such that if \(n = 2^m \sqrt{m}\) where \(m\) is odd, then \(r_n\) is indecomposable if \(\varepsilon_0(m) \geq 2\), and \(r_n\) is the square (the fourth power) of an indecomposable element if \(\varepsilon_0(m) = 1\) (\(\varepsilon_0(m) = 0\)).

(ii) The ideal \(\ker(j^*) \subset H^*(BO; \mathbb{F}_2)\) is freely generated by any system of elements \(r_n\) verifying the indecomposability properties of part (i).

Note that the proposition implies \(w_n^4 \in \ker(j^*)\) for all \(n\). Here are generators for \(\ker(j^*)\) in degrees \(\leq 16\): \(w_1^4, w_2^4, s_9, s_8^2, w_3^4, w_4^4\). \((s_n\) means the \(n\)-th Newton polynomial of the \(w_i\).) In the appendix, we will give an algorithm to construct generators \(r_n\) in terms of Stiefel-Whitney-classes.

We now study the map \(j\) at an odd prime \(\ell\). Recall that \(H_*(\Omega^4 \tilde{B}; \mathbb{F}_\ell)\) is the free graded commutative algebra on certain elements of the form \(\beta^{i}(Q_{\ell-1})^{i} \beta^{i}(Q_{2(\ell-1)})^{i} y_{2\ell-2}\).

**Proposition 4.4.** — The kernel of \(j_* : H_*(\Omega^4 \tilde{B}; \mathbb{F}_\ell) \to H_*(BO; \mathbb{F}_\ell)\) is the ideal generated by those of the above elements whose degree is not divisible by \(4\).

Note that these are precisely the generators not of the form \((Q_{2(\ell-1)})^{i} y_{2\ell-2}, i \geq 0\).
Proof. — Clearly these elements are in the kernel of $j_*$, since $H_n(BO; F_\ell)$ is zero unless $n$ is divisible by 4. To complete the proof, it suffices to show that the subalgebra of $H_*(\Omega^4\tilde{B}; F_\ell)$ generated by the classes $(Q_{2(\ell - 1)})^i y_{2\ell - 2}$ injects into $H_*(BO; F_\ell)$. To see this, we proceed as follows. Write

\[ H_*(\Omega^3S^3; F_\ell) = H_*(\Omega^4\tilde{B}; F_\ell) \otimes F_\ell[Z], \]
\[ H_*(BO \times \mathbb{Z}; F_\ell) = H_*(BO; F_\ell) \otimes F_\ell[Z]. \]

From [CLM] we know that in $H_*(\Omega^3S^3; F_\ell)$, one has $Q_1(1 \otimes [1]) \neq 0$. Hence $y_{2\ell - 2}$ may be chosen such that $Q_1(1 \otimes [1]) = y_{2\ell - 2} \otimes [\ell]$. From [K], Theorem 33, we know that in $H_*(BO \times \mathbb{Z}; F_\ell)$, we have $Q_1(1 \otimes [1]) = p_{(\ell - 1)/2} \otimes [\ell]$. Here $p_n \in H_{4n}(BO; F_\ell)$ is the dual of $\tilde{p}_n$, the mod $\ell$ reduction of the $n$-th Pontryagin class. (The dual is taken with respect to the obvious basis of $H^{4n}(BO; F_\ell)$ given by monomials in the $\tilde{p}_j$, $j \leq n$.) Since the map $\Omega^3S^3 \to BO \times \mathbb{Z}$ is a morphism of 3-fold loop spaces, and respects components, it follows $j_*(y_{2\ell - 2}) = p_{(\ell - 1)/2}$. From [K], Theorem 25, it follows:

\[ j_*((Q_{2(\ell - 1)})^i y_{2\ell - 2}) = (Q_{2(\ell - 1)})^i p_{(\ell - 1)/2} = \pm p_{(\ell + i + 1)(\ell - 1)/2}. \]

It is well known that $p_n$ is, up to scalar multiples, the unique primitive element in $H_{4n}(BO; F_\ell)$. (Recall that $H_*(BO; F_\ell) \approx \mathbb{P}(a_n$; $n \geq 1)$, with $|a_n| = 4n$, and $\Delta a_n = \sum a_i \otimes a_{n-i}$. From the Newton formula, we see that $p_{(\ell - 1)/2}$ is indecomposable, since $\frac{1}{2}(\ell^i - 1)$ is not divisible by $\ell$. Thus, $\text{Im}(j_*)$ is freely generated by $\{p_n \mid n = \frac{1}{2}(\ell^i - 1), i \geq 1\}$. This implies the proposition.

5. Divisibility properties depending on $k$

In this section, we study the fibration (1) in cohomology. First, we study the situation at the prime 2.

Proposition 5.1. — If $X$ has even intersection form, then the mod 2 cohomology spectral sequence of fibration (1) degenerates at the $E_2$-level

Proof. — It suffices to prove this in the case $X = S_g$, since there is a degree one map $S_g \to X$ (cf. the proof of Theorem 2.1). The stabilisation map $S^3 \to Sp$ induces a morphism of fibrations $C(S_g, BS^3)_k \to C(S_g, BSp)_k$ whose restriction to the fiber is the map $j : \Omega^4\tilde{B} \to BO$ studied in paragraph 4. Proceeding as in the proof of Theorem 2.1, we can decompose $C(S_g, BSp)_k$ as a product $BSp \times (\Omega Sp)^{2g} \times BO$. Hence the spectral sequence of this fibration degenerates at the $E_2$-level. Since $j^* : H^*(BO; F_2) \to H^*(\Omega^4\tilde{B}; F_2)$ is surjective by Proposition 4.1, the result follows.
COROLLARY 5.2. — If $X$ has even intersection form, then
\[ H^*(\mathcal{C}(X,BS^3)_k;\mathbb{F}_2) \]
is an extension of the algebra $H^*(BO;\mathbb{F}_2)/\ker(j^*)$ determined in Proposition 4.3 by $A(L) \otimes \mathbb{F}_2$.

Note that $\tilde{w}_1^4 = k\bar{p} + \bar{p}$ by Theorem 2.1, hence the above extension of algebras is non-trivial if $k$ is odd.

COROLLARY 5.3. — If $X$ has even intersection form, then
\[ H^*(\mathcal{C}(X,BS^3)_k;\mathbb{Z}(2)) \cong A(L) \otimes \mathbb{Z}(2) \oplus \text{torsion}. \]

Here, $\mathbb{Z}(2)$ is $\mathbb{Z}$ localized at 2. Note that this is not true at odd primes, cf. Corollary 2.2.

Now let $\ell$ be an odd prime. Consider first the case $X = S^4$.

PROPOSITION 5.4. — In $H^*(\mathcal{C}(S^4,BS^3)_k;\mathbb{Z})$, the element $p^{(\ell-1)/2}$ is divisible by $\ell$ if and only if $k \neq 0 (\ell)$.

Proof. — To simplify notation, set $C_k = \mathcal{C}(S^4,BS^3)_k$ and $m = \frac{1}{2}(\ell - 1)$.

From Proposition 3.1, it follows $H^i(\Omega^4\hat{B};\mathbb{F}_\ell) = 0$ for $1 \leq i \leq 4m - 2$, $H^{4m-1}(\Omega^4\hat{B};\mathbb{F}_\ell) \approx \mathbb{F}_\ell$, $H^{4m}(\Omega^4\hat{B};\mathbb{F}_\ell) \approx \mathbb{F}_\ell$. Moreover, the latter is generated by $i^*(\tilde{p}_m)$ where $i : \Omega^4\hat{B} \to C_k$ is the inclusion of the fiber. This follows from Proposition 4.4 since $i^*(\tilde{p}_m) = j^*(p_m)$ where $j : \Omega^4\hat{B} \to BO$ is the map studied in paragraph 4. Also, in the mod $\ell$ cohomology spectral sequence of the fibration $\Omega^4\hat{B} \to C_k \to BS^3$, the first non-trivial differential is:
\[ d_{4m} : H^{4m-1}(\Omega^4\hat{B};\mathbb{F}_\ell) \to H^{4m}(BS^3;\mathbb{F}_\ell). \]

Clearly, $p^m$ is divisible by $\ell$ if and only if $d_{4m} \neq 0$.

If $k \neq 0 (\ell)$, then it follows immediately from Corollary 2.2 that $p^m$ is divisible by $\ell$. Now suppose $k = \ell k'$. Consider:
\[ z = 4(-1)^{m+1}s_m(\tilde{p}_1,\tilde{p}_2,\ldots) - 2k'p^m \in H^{4m}(C_k;\mathbb{Z}). \]

Since $i^*(z) = \pm 4s_m(i^*(\tilde{p}_1),i^*(\tilde{p}_2),\ldots) = \pm 4mi^*(\tilde{p}_m)$, we have $\bar{z} \neq 0 \in H^{4m}(C_k;\mathbb{F}_\ell)$. On the other hand, Theorem 2.1 implies $\ell z = 0$. It follows that $\bar{z}$ is in the image of the mod $\ell$ cohomology Bockstein operator. In particular, we have $H^{4m-1}(C_k;\mathbb{F}_\ell) \neq 0$. This implies $d_{4m} = 0$ in the spectral sequence, hence $p^m$ is not divisible by $\ell$.

This completes the proof of Proposition 5.4.

For general $X$, we have:
PROPOSITION 5.5. — Suppose \( C(X, BS^3)_k \) and \( C(X, BS^3)_{k'} \) have isomorphic cohomology algebras. Then for each prime \( \ell \geq 5 \), one has:

\[
k \equiv 0 \ (\ell) \iff k' \equiv 0 \ (\ell).
\]

Moreover, if the intersection form of \( X \) is even, or divisible by 3, then this is also true for \( \ell = 2 \), or \( \ell = 3 \), respectively.

As an example where the last condition is satisfied, one may take \( X = S^4 \).

Proof. — Let \( \varphi \in BS(L^*) \) be the intersection form of \( X \), and set \( C_k = C(X, BS^3)_k \). As before, set \( m = \frac{1}{2}(\ell - 1) \). We distinguish three cases.

Case 1 : \( \ell = 2 \). Then \( \varphi \) is even by hypothesis, hence by THEOREM 2.1, we have:

\[
\bar{w}_1^8 = s_2(\bar{w}_1, \bar{w}_2)^4 = k\bar{p}^2.
\]

PROPOSITION 5.1 implies \( \bar{p}^2 \neq 0 \). Hence we have \( \bar{w}_1^8 = 0 \) if and only if \( k \equiv 0 \ (2) \). Since \( \bar{w}_1 \) generates \( H^1(C_k; F_2) \approx F_2 \), the result follows.

Case 2 : \( \ell \) an odd prime, and \( \varphi \equiv 0 \ (\ell) \). Then \( \bar{\omega} \), the mod \( \ell \) reduction of \( \omega \), is zero. In this case, we proceed as in the case \( X = S^4 \) to see that \( p^m \) is divisible by \( \ell \) if and only if \( k \neq 0 \ (\ell) \). Actually the proof shows that \( H^{4m-1}(C_k; F_\ell) = 0 \) if and only if \( k \neq 0 \ (\ell) \). The result follows.

Case 3 : \( \ell \) a prime \( \geq 5 \), and \( \varphi \neq 0 \ (\ell) \). We consider again the fibration \( \Omega^4\tilde{B} \to \tilde{C}_k \to \tilde{C}(M(L,2), BS^3) \). In this proof, all cohomology classes will be reduced modulo \( \ell \). But here we will distinguish between \( \bar{\omega}, \bar{\Omega} \) as cohomology classes on \( \tilde{C}(M(L,2), BS^3) \), and their images \( r^* (\bar{\omega}), r^* (\bar{\Omega}) \) on \( \tilde{C}_k \). THEOREM 2.1 implies \( r^* ((k\bar{p} + m\bar{\Omega})\bar{p}^{m-1}) = 0 \). Since \( \varphi \neq 0 \ (\ell) \), it follows easily from the description of \( A(L) \) that for all \( k \), the element \( (k\bar{p} + m\bar{\Omega})\bar{p}^{m-1} \) is non-zero (compare the reasoning following COROLLARY 2.2). Arguing as in the proof of PROPOSITION 5.4, we see from the spectral sequence that in degree \( 4m \), \( \ker(r^*) = \text{Im}(d_{4m}) \) is one-dimensional. Hence \( \ker(r^*) \) is generated by \( (k\bar{p} + m\bar{\Omega})\bar{p}^{m-1} \).

Now let \( \alpha^* : H^*(\tilde{C}_k; F_\ell) \approx H^*(\tilde{C}_k; F_\ell) \) be a (graded) algebra isomorphism. Affect all objects concerning \( \tilde{C}_k \) with \( \alpha \). Since \( \ell \geq 5 \), \( r^* \) and \( r^* \) are isomorphisms in degree 4. Hence, there are elements \( \bar{q}, \bar{\Lambda} \in A(L) \) of degree 4 such that:

\[
\alpha^*(r^*(\bar{p})) = r^*(\bar{q}), \quad \alpha^*(r^*(\bar{\Omega})) = r^*(\bar{\Lambda}).
\]

Again, THEOREM 2.1 implies \( r^*((k'\bar{p} + m\bar{\Omega})\bar{p}^{m-1}) = 0 \). Applying \( \alpha^* \), it follows \( (k'\bar{q} + m\bar{\Lambda})\bar{q}^{m-1} \in \ker r^* \). Since \( \ker r^* \) is one-dimensional, there is \( \lambda \neq 0 \) such that \( (k'\bar{q} + m\bar{\Lambda})\bar{q}^{m-1} = \lambda (k\bar{p} + m\bar{\Omega})\bar{p}^{m-1} \). It then follows easily from the description of \( A(L) \) that \( k \equiv 0 \ (\ell) \) if and only if \( k' \equiv 0 \ (\ell) \).

This completes the proof.
Remark 5.6. — Proposition 5.5 was motivated by the following amusing application. Consider the family of topological group extensions

\[ 1 \to G_\bullet \approx C_\bullet(X, S^3) \to G_k(X) \to S^3 \to 1 \]

depending on the second Chern number \( k \). Here the map \( G_k(X) \to S^3 \) is given by restriction to the fiber over the base point. One wants to conjecture that these extensions are distinguished by \( k \). Since \( BG_k(X) \approx C(X, BS^3)_k \), Proposition 5.5 gives a partial answer. In the literature, there seems to be only the following invariant: if \( X \) has even intersection form, then the central element \(-1 \in G_k(X)\) is homotopic to 1 if and only if \( k \) is even [FU].

6. The classifying space \( BG_1(S^4) \)

We now consider the special case

\[ X = S^4, \quad k = 1. \]

Set \( C_1 = C(S^4, BS^3)_1 \approx BG_1(S^4) \). Theorem 2.1 implies that

\[ H^*(C_1; \mathbb{Z})/\text{torsion} \subset H^*(C_1; \mathbb{Q}) = \mathbb{Q}[p] \]

contains classes \( \tilde{p}_i \) such that:

\[ 1 + \tilde{p}_1 + \tilde{p}_2 + \cdots = \exp \left( \sum_{i=1}^{\infty} \frac{p^i}{2i(2i+1)} \right) = 1 + \frac{1}{6} p + \frac{23}{360} p^2 + \frac{1493}{45360} p^3 + \cdots \]

We introduce the following notation. If \( \ell \) is a prime, set \( m = \frac{1}{2}(\ell - 1) \) if \( \ell \) is odd, and \( m = 1 \) if \( \ell = 2 \). For \( n \in \mathbb{N} \), set \( \mu_\ell(n) = \nu_\ell([\ell/m] \cdot n !) \), where \( \nu_\ell : \mathbb{Q}^* \to \mathbb{Z} \) is \( \ell \)-adic valuation, and \([x]\) means the greatest integer \( \leq x \). The main result of this section is:

Proposition 6.1. — The subring of \( H^*(C_1; \mathbb{Z})/\text{torsion} \) generated by \( p \) and the \( \tilde{p}_i \) is generated in degree \( 4n \) by \( p^n/\alpha_n \), where

\[ \alpha_n = \prod_\ell \ell^{\mu_\ell(n)}. \]

Before giving the proof, we point out that it is tempting to conjecture that \( H^*(C_1; \mathbb{Z})/\text{torsion} \) is actually equal to this subring. Here is a proof for this conjecture in low degrees, and after inverting 2. From fibration (1), we have an exact sequence:

\[ 0 \to \mathbb{Z} \cdot p^n \to H^{4n}(C_1; \mathbb{Z}) \to Q^{4n} \to 0, \]
where $Q^{4n}$ is torsion. Moreover, it follows easily from the spectral sequence that the exponent of $Q^{4n}$ is less or equal than the product of the exponents of $H^i(\Omega^4 B; \mathbb{Z})$ for $1 \leq i \leq n$. Now let $\ell$ be an odd prime. An easy calculation using COROLLARY 3.3 shows that for $n < N(\ell)$, the exponent of the $\ell$-primary part of $Q^{4n}$ is less or equal than $\ell \mu_\ell(n)$. (Recall $N(3) = 536$, and $N(\ell) = \ell^2 - \frac{1}{2}(\ell + 3)$ if $\ell \geq 5$.) On the other hand, PROPOSITION 6.1 implies that $p^n$ is divisible by $\ell \mu_\ell(n)$ in $H^{4n}(C_1; \mathbb{Z})$/torsion. Putting things together, one easily deduces the following corollary.

**Corollary 6.2.** — Let $\ell$ be an odd prime, and $n < N(\ell)$. Then $p^n \in H^{4n}(C_1; \mathbb{Z})$ is divisible by $\ell \mu_\ell(n)$, and $H^{4n}(C_1; \mathbb{Z}[\ell])$/torsion is generated by $p^n/\ell \mu_\ell(n)$.

Note that the smallest $N(\ell)$ is $N(5) = 21$. Since by PROPOSITION 5.1 $p^n \in H^{4n}(C_1; \mathbb{Z})$ is not divisible by 2, it follows:

**Corollary 6.3.** — For $n < 21$, $p^n \in H^{4n}(C_1; \mathbb{Z})$ is divisible precisely by $\prod_{\ell \geq 3} \ell \mu_\ell(n)$. Moreover, in degrees less than $4 \times 21 = 84$, $H^*(C_1; \mathbb{Z}[\frac{1}{2}])$/torsion coincides with the subring generated by $p$ and the $p_i$.

We now prove PROPOSITION 6.1. Write $\tilde{p}_n = b_np^n \in H^{4n}(C_1; \mathbb{Q})$. We leave it to the reader to deduce PROPOSITION 6.1 from the following lemma, using the easily verified inequality $\mu_\ell(n_1) + \mu_\ell(n_2) \leq \mu_\ell(n_1 + n_2)$.

**Lemma 6.4.** — For $n \geq 1$, one has $\nu_\ell(b_n) \geq -\mu_\ell(n)$. Moreover, equality holds if $n \equiv 0 \pmod{m}$.

To prove **Lemma 6.4**, recall that by definition:

$$\exp\left(\sum_{i=1}^{\infty} \frac{p^i}{2i(2i + 1)}\right) = \sum_{n=0}^{\infty} b_np^n.$$  

Differentiating this expression, we obtain:

$$b_{n+1} = \frac{1}{2(n+1)} \sum_{i=0}^{\infty} \frac{b_i}{2n - 2i + 3}.$$  

Using the well known fact $\nu_\ell(x + y) = \min(\nu_\ell(x), \nu_\ell(y))$ whenever $\nu_\ell(x) \neq \nu_\ell(y)$, it is not hard to deduce $\nu_3(b_n) = -\nu_2((2n)!)$ and $\nu_3(b_n) = -\nu_2((3n)!)$ by induction on $n$. This proves **Lemma 6.4** for $\ell \in \{2, 3\}$.

In the general case, we proceed as follows. We have the following expression:

$$b_n = \sum_{k} \sum_{n_1 + 2n_2 + \cdots + kn_k = n} \frac{1}{\prod_{i=1}^{k} n_i!(2i(2i + 1))^{n_i}}.$$
Let $E_n$ denote the set of sequences $(n_1, n_2, \ldots)$ such that $n_1+2n_2+\cdots \leq n$. Define $f : E_n \to \mathbb{Z}$ by the formula:

$$f(n_1, n_2, \ldots) = \nu_{\ell} \left( \prod_i n_i !(2i (2i + 1))^{n_i} \right).$$

Note that $E_n$ contains the sequence $(n_1^{(0)}, n_2^{(0)}, \ldots)$ defined by $n_i^{(0)} = [n/m]$, $n_i^{(0)} = 0$ for $i \neq m$. Moreover,

$$f(n_1^{(0)}, n_2^{(0)}, \ldots) = \nu_{\ell} ([n/m]!) + [n/m] = \mu_{\ell}(n).$$

Clearly, it follows from the expression for the $b_n$ given above that the following LEMMA 6.5 implies LEMMA 6.4.

**LEMMA 6.5.** — For all sequences $(n_1, n_2, \ldots) \in E_n$, one has the inequality $f(n_1, n_2, \ldots) \leq \mu_{\ell}(n)$. Moreover, if $n \equiv 0 \pmod{m}$, then equality holds if and only if:

$$(n_1, n_2, \ldots) = (n_1^{(0)}, n_2^{(0)}, \ldots).$$

We now prove LEMMA 6.5. Consider $(n_1, n_2, \ldots) \in E_n$. Set $h_i = [in_i/m]$.

**SUBLEMMA 1.** — If $n_i \geq 0$, then $\nu_{\ell}(n_i !) < h_i$ unless $h_i = 0$.

Indeed, $\nu_{\ell}(n_i !) \leq \frac{n_i - 1}{2m} < \frac{n_i}{2m} \leq \frac{mh_i + m - 1}{2m} < \frac{h_i + 1}{2} \leq h_i$.

**SUBLEMMA 2.** — If $n_i > 0$ and $i > m$, then $n_i \nu_{\ell}(2i(2i + 1)) < h_i$.

Since $h_i = [in_i/m] \geq n_i > 0$, this is obvious unless $i \equiv 0 \pmod{\ell}$ or $2i + 1 \equiv 0 \pmod{\ell}$. First, suppose $i \equiv 0 \pmod{\ell}$. Then we have

$$n_i \nu_{\ell}(2i(2i + 1)) = n_i \nu_{\ell}(i) \leq n_i \log_{\ell}(i) \leq \frac{mh_i + m - 1}{i} \log_{\ell}(i),$$

hence it suffices to show $((mh + m - 1)/i) \log_{\ell}(i) < h_i$, which is equivalent to

$$(*) \quad i^{mh_i + m - 1} < \ell^{th_i}.$$  

We will show this inequality by induction on $h_i$, keeping $i$ fixed. Observe that we may suppose $h_i \geq 2$. Indeed, since $i \geq \ell$, we have

$$1 \leq n_i \leq \frac{mh_i + m - 1}{i} \leq \frac{mh_i + m - 1}{\ell},$$
which is impossible if \( h_i \leq 1 \).

Letting \( h_i = 2 \) in (*), we obtain :

\[
\ell^{m-1} < \ell^2i.
\]

Observe that once we know (**), it follows \( i^m \leq \ell^{(3m-1)/2} < \ell^i \), which implies the induction. Thus, it only remains to show (**), which is equivalent to :

\[
i < \ell^{2i/(3m-1)} = \ell^{4i/(3\ell-5)}.
\]

Now this is obvious if \( i = \ell \), moreover, differentiating with respect to \( i \) yields :

\[
1 < \frac{4}{3\ell - 5} \log(\ell) \ell^{4i/(3\ell-5)} = \frac{4\ell}{3\ell - 5} \log(\ell) \ell^{(4i-3\ell+5)/(3\ell-5)}
\]

which is true for \( i \geq \ell \). This implies (**), hence SUBLEMMA 2 in the case \( i \equiv 0 \) (\( \ell \)).

The case \( 2i + 1 \equiv 0 \) (\( \ell \)) is similar and left to the reader.

**Sublemma 3.** \( \sum_{i \geq 1} h_i \leq [n/m] \), \( \sum_{i \geq m} n_i \leq [n/m] \).

This is obvious since \( n = \sum_{i \geq 1} in_i \).

Applying these sublemmas, we have :

\[
f(n_1, n_2, \ldots) = \sum_{i \geq 1} (\nu_\ell(n_1!) + n_i \nu_\ell(2i(2i + 1))) \leq \sum_{i \geq m} \nu_\ell(n_i!) + \sum_i h_i
\]

\[
\leq \nu_\ell([n/m]!) + [n/m] = \mu_\ell(n) = f(n_1(0), n_2(0), \ldots).
\]

This implies the first part of LEMMA 6.5. Now suppose we have equality here. Then it follows from sublemmas 1 and 2 that \( n_i = 0 \) for all \( i > m \), and \( h_i = 0 \) for all \( i < m \). But this implies :

\[
f(n_1, n_2, \ldots) = n_m + \nu_\ell(n_m!),
\]

hence \( n_m = [n/m] \). If \( n \equiv 0 \) (\( m \)), then this is impossible unless :

\[
(n_1, n_2, \ldots) = (n_1(0), n_2(0), \ldots).
\]

This completes the proof of LEMMA 6.5.
7. The classifying space of the based gauge group

For \( \varphi \in \pi_2(L) = \pi_3(M(L,2)) \), define \( F_\varphi : C_\bullet(M(L,2), BS^3) \to \Omega^3 BS^3 \) by \( F_\varphi(f) = f \circ \varphi \). Clearly, the map:

\[
F : \Gamma_2(L) \to [C_\bullet(M(L,2), BS^3), \Omega^3 BS^3], \quad \varphi \mapsto F_\varphi
\]

is a homomorphism of abelian groups. (Here, the notation \([A,B]\) means based homotopy classes of based maps \( A \to B \).) The main result of this section is the following theorem.

**Theorem 7.1.** — \( \ker F = 12 \Gamma_2(L) \).

We apply this as follows. It is not hard to see that, up to homotopy, \( C_\bullet(X, BS^3) \) is the total space of the fibration induced by \( F_\varphi \) from the path fibration over \( \Omega^3 BS^3 \). Thus Theorem 7.1 implies that for any prime \( \ell \geq 5 \), we have an \( \ell \)-equivalence:

\[
C_\bullet(X, BS^3) \sim_{(\ell)} C_\bullet(M(L,2), BS^3) \times \Omega^4 BS^3.
\]

Moreover, this is still true for \( \ell = 3 \), or \( \ell = 2 \), if we suppose \( \varphi \equiv 0 \pmod{3} \), or \( \varphi \equiv 0 \pmod{4} \), respectively. On the other hand, if \( \varphi \equiv 0 \pmod{3} \), then \( C_\bullet(X, BS^3(3)) \) is not a product, as follows from Theorem 2.1. Similarly, if \( \varphi \equiv 0 \pmod{4} \), then \( C_\bullet(X, BS^3(2)) \) is not a product (see also Remark 7.8).

Since \( H * (C_\bullet(M(L,2), BS^3); Z) \) is the divided power algebra \( \Gamma(L) \), we deduce:

**Corollary 7.2.** — Let \( \alpha \in L \) be indivisible. If

\[
\mu(\alpha)^n \in H^{2n}(C(X, BS^3)_k; \mathbb{Z}[\frac{1}{2}])
\]

is divisible by \( N \), then \( N \) divides \( n! \). Moreover, if \( \varphi \equiv 0 \pmod{3} \), then this is true with coefficients in \( \mathbb{Z}[\frac{1}{2}] \).

Note that if \( \varphi \) is even (as a bilinear form), then Corollary 5.3 together with Corollary 1 of [Ml] imply that \( \mu(\alpha)^n \in H^{2n}(C(X, BS^3)_k; \mathbb{Z}(2)) \) is divisible exactly by \( n! \).

We now prove Theorem 7.1. We start with two lemmas whose proof is left to the reader.

**Lemma 7.3.** — The suspension \( \Sigma C_\bullet(M(L,2), BS^3) \) has the homotopy type of a bouquet of spheres.

**Lemma 7.4.** — There is a natural filtration (induced by a Postnikov decomposition of \( BS^3 \)):

\[
[C_\bullet(M(L,2), BS^3), \Omega^3 BS^3] = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots,
\]
where $\mathcal{F}_{n-1}/\mathcal{F}_n \cong \Gamma_n(L) \otimes \pi_{2n+2}(S^3)$.

Using this filtration, the map $F$ defines natural linear maps:

$$
\theta_1 : \Gamma_2(L) \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \cong \Gamma_1(L) \otimes \pi_4(S^3) = L \otimes \mathbb{Z}/2,
$$

$$
\theta_2 : \ker(\theta_1) \rightarrow \mathcal{F}_1/\mathcal{F}_2 \cong \Gamma_2(L) \otimes \pi_6(S^3),
$$

where $\theta_i(\varphi) = F\varphi \mod \mathcal{F}_i$. It is not hard to see that $\theta_1$ corresponds to the suspension $\Gamma_2(L) = \pi_3(M(L,2)) \xrightarrow{\Sigma} \pi_4(\Sigma M(L,2)) = L \otimes \mathbb{Z}/2$. Alternatively, $\theta_1$ is given by the formula $\theta_1(\gamma_2(x)) = \bar{x}$, $(x \in L)$. Thus, $\ker(\theta_1)$ consists exactly of the even forms.

The following two lemmas will imply Theorem 7.1.

**Lemma 7.5.** Let $w = [i_1, i_2] \in \pi_3(S^2 \vee S^2) = \Gamma_2(\mathbb{Z} \oplus \mathbb{Z})$ be the Whitehead product of the obvious inclusions $i_1, i_2$. If we localize at a prime $\ell \geq 5$, then $Fw$ becomes null-homotopic.

**Lemma 7.6.** Let $h \in \pi_3(S^2) = \Gamma_2(\mathbb{Z})$ be a generator. Then $\theta_2(2h)$ is the double of a generator of $\Gamma_2(\mathbb{Z}) \otimes \pi_6(S^3) \cong \mathbb{Z}/12$.

Granting these lemmas, here is a proof of the theorem.

First, we show $12\Gamma_2(L) \subset \ker F$. Suppose $\varphi \in 12\Gamma_2(L)$. It follows from Lemma 7.3 that the abelian group $[\mathcal{C}_*(M(L,2), BS^3), \Omega^3 BS^3]$ is (non-naturally) isomorphic to $\prod_{j \in J} \pi_{n_j}(S^3)$ for some integers $n_j$. It clearly suffices to show that the image of $F\varphi$ in each of the $\pi_{n_j}(S^3)$ is zero. Now it is well known [S] that the $\ell$-primary part of $\pi_i(S^3)$ ($i \geq 4$) has exponent $\ell$, for $\ell$ an odd prime, and exponent 4, for $\ell = 2$. Thus, the 2- and 3-primary parts of $F\varphi$ are zero. To study the $\ell$-primary part for $\ell \geq 5$, we may as well localise at $\ell$. The image of $w = [i_1, i_2]$ in $\pi_3(S^2)$ under the obvious sum map $S^2 \vee S^2 \to S^2$ is $2h$, where $h$ is a generator of $\pi_3(S^2)$. Thus, Lemma 7.5 implies that $F_{2h}$ is null-homotopic (after localization at $\ell$). But $F_{2h}$ is homotopic to $2 \circ F_h$, where $2$ means the self-map of $\Omega^3 BS^3_{(2)}$ induced by multiplication by 2 on $S^3$. Since this map is a homotopy equivalence, it follows that $F_h$ is null-homotopic. By naturality, this implies that (the $\ell$-primary part of) $F\varphi$ is null-homotopic for any $\varphi \in \Gamma_2(L)$. This shows $12\Gamma_2(L) \subset \ker F$.

Next, we show $\ker F \subset 12\Gamma_2(L)$. By naturality, Lemma 7.6 implies that there is a generator $\epsilon \in \pi_6(S^3)$ such that $\theta_2(2\varphi) = 2\varphi \otimes \epsilon \in \Gamma_2(L) \otimes \pi_6(S^3)$ for any $\varphi \in \Gamma_2(L)$. Now, suppose we have $\varphi \in \ker F$. Then $2\varphi \in \ker F$, whence $2\varphi \otimes \epsilon = \theta_2(2\varphi) = 0$. Thus $\varphi$ must be divisible by 6. In particular, we have $\varphi = 2\varphi'$, thus we can repeat the argument to find $\varphi \otimes \epsilon = \theta_2(\varphi) = 0$. Thus $\varphi$ must be divisible by 12.
It remains to prove Lemmas 7.5 and 7.6.

*Proof of Lemma 7.5.* — Set $G = S^3_{(t)}$. We must show that

$$F_w : C_*(S^2 \vee S^2, BG) = \Omega^2 BG \times \Omega^2 BG \to \Omega^3 BG$$

is null-homotopic.

Recall that the join $X*Y$ of two spaces $X$, $Y$ is defined as the quotient of the product $X \times I \times Y$ by the identifications $(x,0,y) = (x',0,y)$, $(x,1,y) = (x,1,y')$. Think of $S^3$ as $S^1 \ast S^1$. Think of $S^2$ as $S^1 \wedge S^1$. For $t \in I = [0,1]$, let $[t]$ be its image in $S^1 = I/(0 = 1)$. Then the map $w : S^3 = S^1 \ast S^1 \to S^2 \vee S^2$, defined by:

$$w(x,t,y) = \begin{cases} i_1([2t] \wedge x) & \text{if } t \leq \frac{1}{2}, \\ i_2([2 - 2t] \wedge y) & \text{if } t \geq \frac{1}{2}, \end{cases}$$

represents the Whitehead product $[i_1,i_2]$.

Similarly, define $\bar{w} : G \ast G \to \Sigma G = S^1 \wedge G$ by the formula:

$$\bar{w}(a,t,b) = \begin{cases} [2t] \wedge a & \text{if } t \leq \frac{1}{2}, \\ [2 - 2t] \wedge b & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Let $s : G \to \Omega \Sigma G$ be the canonical map, sending $x \in G$ to the loop $t \mapsto t \wedge x$. Let $i : \Sigma G \to BG$ be the map classifying the principal $G$-bundle whose clutching function is the identity $G \to G$. Then it is well known that the composition

$$G \xrightarrow{s} \Omega \Sigma G \xrightarrow{\Omega i} \Omega BG$$

is a homotopy equivalence.

The key observation is that the following diagram is homotopy commutative:

$$\begin{array}{ccc}
\Omega G \times \Omega G & \xrightarrow{\Omega s \times \Omega s} & \Omega^2 \Sigma G \times \Omega^2 \Sigma G \\
\downarrow* & & \downarrow\Omega^2 i \times \Omega^2 i \\
\Omega^3(G \ast G) & \xrightarrow{\Omega^3 \bar{w}} & \Omega^3 \Sigma G \\
\downarrow F_w & & \downarrow \Omega^3 i \\
\Omega^3 BG & & \Omega^3 BG
\end{array}$$
Here, the map \( \ast \) is the join, that is \( \ast(f, g) = f \ast g \), where \( f \ast g(x, t, y) = (f(x), t, g(y)) \). Now \( i \circ \tilde{w} \in [G \ast G, BG] \approx \pi_7(BS^3) \). But this group is zero, since \( \pi_7(BS^3) = \pi_6(S^3) = \mathbb{Z}/12 \), and \( \ell \geq 5 \). This implies that \( F_{\tilde{w}} \) is null-homotopic, since \( (\Omega) \circ s \) is a homotopy equivalence. This proves Lemma 7.5.

Proof of Lemma 7.6: recall that the image of \( w = [i_1, i_2] \) in \( \pi_3(S^2) \) under the obvious sum map \( S^2 \vee S^2 \to S^2 \) is \( 2h \). Thus, diagram (3) gives a homotopy commutative diagram:

\[
\begin{array}{ccc}
\Omega S^3 & \xrightarrow{\Delta} & \Omega S^3 \times \Omega S^3 \\
\downarrow \sim & & \downarrow \Omega^3(i \circ \tilde{w}) \\
\Omega^2 BS^3 & \xrightarrow{F_{2h}} & \Omega^3 BS^3.
\end{array}
\]

Here, \( \Delta \) is the diagonal map, and \( \ast \) is the join. As in Lemma 7.4, we have a filtration:

\[
[\Omega S^3, \Omega^3 S^7] = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots,
\]

where \( \mathcal{F}_n / \mathcal{F}_n^- \approx \Gamma_n(\mathbb{Z}) \otimes \pi_{2n+3}(S^7) \). Since \( \mathcal{F}_0 / \mathcal{F}_1 = 0 \), the map \( \ast \circ \Delta \) defines an element

\[
\eta \in \mathcal{F}_1 / \mathcal{F}_2 \approx \pi_7(S^7).
\]

Moreover, identifying \( \pi_6(S^3) = \pi_7(BS^3) \), we have by naturality:

\[
\theta_2(2h) = (i \circ \tilde{w})_\ast(\eta).
\]

As is well known \([T]\), \( i \circ \tilde{w} \) is a generator of \( \pi_7(BS^3) \). Thus, identifying \( \pi_7(S^7) = \mathbb{Z} \), we are reduced to prove the following:

Claim: \( \eta = \pm 2 \).

To prove the claim, let \( A : \Sigma^3 \Omega S^3 \to S^7 \) be the the map adjoint to \( \ast \circ \Delta \). Note that the induced map \( H_7(A; \mathbb{Z}) \) is of the form \( \mathbb{Z} \to \mathbb{Z} \), and it is not hard to see that this is actually multiplication by \( \eta \).

I owe P. Vogel the following argument. Represent a generator of \( H_4(\Omega S^3; \mathbb{Z}) \approx \mathbb{Z} \) by a map \( g : M^4 \to \Omega S^3 \), where \( M^4 \) is a closed oriented 4-manifold. Call \( F \) the composition

\[
F : S^3 \times M \to \Sigma^3 M \xrightarrow{\Sigma^3 g} \Sigma^3 \Omega S^3 \xrightarrow{A} S^7.
\]
Clearly \( \eta \) is equal to \( d^0(F) \), where \( d^0(F) \) means the degree of \( F \) as a map between smooth compact oriented manifolds. Now let \( f \) be the map \( S^1 \times M \rightarrow \Sigma M \rightarrow S^3 \) adjoint to \( g \). Identifying \( S^3 = S^1 * S^1 \), \( S^7 = S^3 * S^3 \), we see that \( F \) is given by the formula

\[
F((a, t, b), x) = (f(a, x), t, f(b, x)).
\]

We may suppose \( f \) is smooth. Then \( F \) is also smooth, and has a regular value of the form \((z, t_0, z') \in S^3 * S^3\), where \( 0 < t_0 < 1 \). Thus

\[
d^0(F) = \#\left\{((a, t_0, b), x) \mid f(a, x) = z, \ f(b, x) = z'\right\}
= \pm d^0(\tilde{F}),
\]

where \( \tilde{F} : S^1 \times S^1 \times M \rightarrow S^3 \times S^3 \) is given by \( \tilde{F}(a, b, x) = (f(a, x), f(b, x)) \).

Finally, we can calculate \( d^0(\tilde{F}) \) as follows. Let \( \sigma \in H^3(S^3; \mathbb{Z}) \) and \( \theta \in H^1(S^1; \mathbb{Z}) \) be the standard generators. Then \( f^*(\sigma) = \theta \otimes g^*(\alpha) \), with \( \alpha \) a generator of \( H^2(\Omega S^3; \mathbb{Z}) \). Hence \( \tilde{F}^*(\sigma \otimes \sigma) = \pm \theta \otimes \theta \otimes g^*(\alpha^2) \), and since \( \frac{1}{2} \alpha^2 \) generates \( H^4(\Omega S^3; \mathbb{Z}) \), we see \( d^0(\tilde{F}) = \pm 2 \).

This proves Lemma 7.6, and completes the proof of Theorem 7.1.

**Corollary 7.7.** — Let \( \varphi \in \Gamma_2(L) \). Then \( F_{\varphi} : \mathcal{C}_*(M(L, 2), BS^3) \rightarrow \Omega^3 BS^3 \) is homotopy linear if and only if \( F_{\varphi} \) is null-homotopic.

**Proof.** — Let \( i_1, i_2 : M(L, 2) \rightarrow M(L \oplus L, 2) \) be induced by the obvious inclusions \( L \rightarrow L \oplus L \). For \( \varphi \in \Gamma_2(L) \), define

\[
d(\varphi) = (i_1 + i_2) \circ \varphi - i_1 \circ \varphi - i_2 \circ \varphi \in \pi_3\left(M(L \oplus L, 2)\right) = \Gamma_2(L \oplus L).
\]

Then \( F_{\varphi} \) is homotopy linear if and only if

\[
F_{d(\varphi)} \in [\mathcal{C}_*(M(L \oplus L, 2), BS^3), \Omega^3 BS^3]
\]

is zero. But the linear map \( \Gamma_2(L) \rightarrow \Gamma_2(L \oplus L), \ \varphi \mapsto d(\varphi) \) is injective. This implies the corollary.

**Remark 7.8.** — Consider the fibration

\[
\Omega^4 \tilde{B} \rightarrow \mathcal{C}_*(X, BS^3)_k \rightarrow \mathcal{C}_*(M(L, 2), BS^3)
\]

obtained from fibration (1) by restricting to base point preserving maps. It is not hard to see that in the homology spectral sequence, the differential:

\[
d_{2,0}^2 : E_{2,0}^2 \approx H_2(\mathcal{C}_*(M(L, 2), BS^3); \mathbb{Z}) \approx L^*
\]

\[
\longrightarrow E_{0,1}^2 \approx H_1(\Omega_k^4 BS^3; \mathbb{Z}) \approx \pi_4(S^3) \approx \mathbb{Z}/2
\]

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corresponds to $\theta_1(\varphi)$ via the natural isomorphism

$$\text{Hom}(L^*, \mathbb{Z}/2) \cong L \otimes \mathbb{Z}/2.$$  

Recall that $H_*(C_*(M(L, 2), BS^3); \mathbb{Z})$ is a polynomial algebra on 2-dimensional generators. Thus, if the homology spectral sequence were multiplicative, then the condition $\theta_1(\varphi) = 0$ would imply that the whole spectral sequence degenerates at the $E^2$-level. However, the only geometric condition to ensure multiplicativity of the spectral sequence we can think of is that $F_\varphi$ be homotopy linear. Curiously enough, if $\theta_1(\varphi) = 0$, then the mod 2 spectral sequence does degenerate by Proposition 5.1, although $F_\varphi$, even localised at 2, need not be homotopy linear as follows from Corollary 7.7.

**Appendix : proof of Proposition 4.3.** — Write:

$$A_* = H_*(BO; F_2) = P(a_i; i \geq 1), \quad B_* = \text{Im}(j_*) = P(b_n; \varepsilon_0(n) \leq 1).$$

Here $b_n$ is the image of the generator of degree $n$ appearing in Proposition 3.1. Recall that the $b_n$ are indecomposable, and their expression in terms of the $a_i$ can be found in [K]. We will use the following notation. When $I = (i_1, i_2, \ldots, i_s)$ is a partition of $n$, then $a(I) = a_{i_1}a_{i_2}\cdots a_{i_s}$, $b(I) = b_{i_1}b_{i_2}\cdots b_{i_s}$, and $a(I)^*$ is the dual of $a(I)$ with respect to the basis of $A_*$ given by the monomials in the $a_i$. We need the following lemma:

**Lemma.** — Let $I = (i_1, i_2, \ldots, i_s)$ be a partition of $2^\lambda m$, where $\lambda \geq 1$ and $m$ is an odd integer. Suppose all $i_y = 0(m)$. Then $a(I)^*$ is indecomposable if and only if $I = (m, m, \ldots, m)$.

**Proof.** — Recall $H^*(BO; F_2) = P(w_i; i \geq 1)$, where $w_i$ is the mod 2 reduction of the $i^{th}$ symmetric polynomial $\sigma_i$ in formal indeterminates $t_1, t_2, \ldots$. In terms of symmetric polynomials, the element $a(I)^*$ can be written

$$a(I)^* = s_{i_1, \ldots, i_s} = \sum t_{i_1}^{i_{i_1}} \cdots t_{i_s}^{i_{i_s}}$$

(cf. [MS] for this notation). We will also use the notation

$$s_m^{(i)} = s_{m, \ldots, m} = (a_m^i)^*.$$

Observe that $s_m^{(i)} = \sigma_i(t_1^m, t_2^m, \ldots)$. Finally, recall the Newton formula:

$$s_n - \sigma_1 s_{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} s_1 + (-1)^n n \sigma_n = 0.$$
Consider a partition of $2^\lambda m$ of the form $I = (j_1m, \ldots, j_r m)$. First, suppose $r < 2^\lambda$. Define an algebra homomorphism $\Phi : H^*(BO) \to \mathbb{F}_2$ by setting $\Phi(t_\nu) = 1$ for $1 \leq \nu \leq 2^\lambda$, $\Phi(t_\nu) = 0$ for $\nu > 2^\lambda$. Then $\Phi(w_j) = \binom{2^\lambda}{r}$, hence $\Phi(w_j) = 0$ for $j < 2^\lambda$, and $\Phi(w_{2^\lambda}) = 1$. Similarly, $\Phi(s_{j_1, \ldots, j_r}) = \binom{2^\lambda}{r} = 0$, since $r < 2^\lambda$. Hence $s_{j_1, \ldots, j_r}$ is a polynomial in $w_1, \ldots, w_{2^\lambda - 1}$. This implies that $s_{j_1 m, \ldots, j_r m}$ is a polynomial in $s_m, s_m(2), \ldots, s_m(2^\lambda - 1)$. Thus $a(I)^* = s_{j_1 m, \ldots, j_r m}$ is decomposable.

Now suppose $r = 2^\lambda$, that is $a(I)^* = (a_m(2^\lambda))^* = s_m(2^\lambda)$. We must show that this is indecomposable. To see this, we work in the ring of symmetric polynomials with integral coefficients. By the Newton formula, $s_{2^\lambda m} + 2^\lambda s_m$ is decomposable. Applying the Newton formula with the formal variables $t_i$ replaced by $t_i^m$ shows that $s_{2^\lambda m} + 2^\lambda s_m$ is also decomposable. Hence $s_m(2^\lambda) \equiv m \sigma_{2^\lambda m}$ modulo decomposable elements. Since $m$ is odd, the result follows.

This completes the proof of our lemma.

We now prove Proposition 4.3. For each $n$ such that $\varepsilon_0(n) \geq 2$, we define $r_n \in \ker(j^*)$ as follows. Write $n = 2^\ell m$ where $m$ is odd. Also, write $n = 2^\lambda \mu$ where $\mu = 4m$ if $\varepsilon_0(m) = 0$, $\mu = 2m$ if $\varepsilon_0(m) = 1$, and $\mu = m$ if $\varepsilon_0(m) \geq 2$. Set $r_n^{(0)} = (a_m(2^\lambda))^*$. Define inductively

$$r_n^{(i)} = r_n^{(i-1)} + \sum \langle r_n^{(i-1)}, b(I) \rangle a(I)^*$$

where the sum is over all partitions $I = (i_1, i_2, \ldots, i_s)$ of $n$ such that $s \geq 2^\lambda - i$ and all $i_\nu \equiv 0 (\mu)$. Then set $r_n = r_n^{(2^\lambda)}$.

We now show $r_n \in \ker(j^*)$. It suffices to show that $\langle r_n, b(I) \rangle = 0$ for all possible monomials $b(I)$ of degree $n$. By the very definition of $r_n$, it is clear that we only have to consider those monomials $b(I)$ where the partition $I = (i_1, i_2, \ldots, i_s)$ is such that all $i_\nu \equiv 0 (\mu)$. Call these partitions admissible, and call $s$ the length of such a partition. Observe that since $\varepsilon_0(\mu) \geq 2$, there is no generator $b_\mu$. Hence there is no admissible partition of length $2^\lambda$. It then follows from the definition of $r_n^{(1)}$ that $\langle r_n^{(1)}, b(I) \rangle = 0$ for all admissible partitions of length $\geq 2^\lambda - 1$. Similarly, since $\langle a(I)^*, b(I') \rangle = 0$ whenever the length of $I'$ is greater than the length of $I$, we see by induction on $i$ that $\langle r_n^{(i)}, b(I) \rangle = 0$ for all admissible partitions of length $\geq 2^\lambda - i$. This shows $r_n \in \ker(j^*)$.

Next we show that the $r_n$ verify the indecomposability properties claimed in Proposition 4.3. First suppose $\varepsilon_0(m) \geq 2$. Then $\mu = m$ is
odd, and the above lemma implies that \( r_n \) is indecomposable. Second, suppose \( \varepsilon_0(m) = 1 \). Then \( \mu = 2m \), hence \( r_n \) admits a unique square root \( x_{n/2} \). (Indeed, \( r_n \) is a sum of terms of the form \( s_{j_1, \ldots, j_r} \), and we have \( s_{j_1, \ldots, j_r} = (s_{j_1, \ldots, j_r})^2 \). Moreover, the lemma implies that \( x_{n/2} \) is indecomposable. Similarly, if \( \varepsilon_0(m) = 0 \), then \( \mu = 4m \), and \( r_n \) is the fourth power of an indecomposable element \( x_{n/4} \).

This completes the proof of part (i) of Proposition 4.3. For part (ii), suppose given a system of elements \( r_n \in \ker(j^*) \) with the above indecomposability properties. For \( n = 2^\ell m \) where \( m \) is odd, define \( x_n = r_n \) if \( \varepsilon_0(m) \geq 2 \), \( x_n = (r_{2n})^{1/2} \) if \( \varepsilon_0(m) = 1 \), and \( x_n = (r_{4n})^{1/4} \) if \( \varepsilon_0(m) = 0 \). Since all \( x_n \) are indecomposable, we have \( H^*(BO; F_2) = P(x_n; n \geq 1) \). This shows that no \( r_n \) is in the ideal generated by the \( r_i \) with \( i < n \). Using this, an easy calculation shows that that the Poincaré series of \( \ker(j^*) \) coincides with the Poincaré series of the ideal freely generated by the \( r_n \). This proves part (ii) of Proposition 4.3.

Remarks:

1) If \( \varepsilon_0(m) = 0 \), then \( r_n \) is a fourth power, and we may replace \( r_n \) by \( w_n^{4/4} \).

2) If \( \varepsilon_0(m) \leq 1 \), then \( r_n = r_n^{(0)} = (a_\mu^{2^\lambda})^* \). This is obvious if \( \varepsilon_0(m) = 0 \), since in this case \( \mu \equiv 0 \pmod{4} \), and there are no generators \( b_n \) in degrees divisible by 4. If \( \varepsilon_0(m) = 1 \), the argument is as follows. We must show \( \langle (a_\mu^{2^\lambda})^*, b(I) \rangle = 0 \) for all possible monomials \( b(I) \) of degree \( n = 2^{\ell} m = 2^{\ell - 1} \mu \). Suppose \( I = (j_1, \ldots, j_r) \) is a partition of \( n \) such that there is a monomial \( b(I) \). Then \( \sum j_\nu = n \), and all \( \varepsilon_0(j_\nu) \leq 1 \). If \( \langle (a_\mu^{2^\lambda})^*, b(I) \rangle = 1 \), then all \( j_\nu \) must be divisible by \( \mu = 2m \). But we will show that this is impossible. Indeed, suppose that all \( j_\nu \) are divisible by \( \mu = 2m \). Set \( k_\nu = \frac{1}{2} j_\nu \). Then \( \sum k_\nu = \frac{1}{2} n = 2^\lambda m \). Moreover, we have \( \varepsilon_0(k_\nu) = 0 \), hence we can write \( k_\nu = 2^{\ell_0} - 1 \). Let \( \ell_0 \) denote the order of \( 2 \) in \((\mathbb{Z}/m)^*\). Since each \( k_\nu \) is divisible by \( m \), each \( \ell_\nu \) is divisible by \( \ell_0 \). This implies that each \( k_\nu \) is divisible by \( 2^{\ell_0 - 1} \), hence so is \( \sum k_\nu = \frac{1}{2} n = 2^\lambda m \). It follows that \( m \) is divisible by \( 2^{\ell_0 - 1} \). On the other hand, \( m \) divides \( 2^{\ell_0 - 1} \) by definition. Thus \( m = 2^{\ell_0 - 1} \). But this implies \( \varepsilon_0(m) = 0 \), thus contradicting our hypothesis.

3) It turns out that the smallest \( n \) such that \( r_n \neq (a_\mu^{2^\lambda})^* \), is \( n = 144 \). In this case, the algorithm yields \( r_{144} = (a_0^{16})^* + (a_2^{27}a_{63})^* \).
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BIBLIOGRAPHY


