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**A DIOPHANTINE PROBLEM ON ALGEBRAIC
CURVES OVER FUNCTION FIELDS OF
POSITIVE CHARACTERISTIC**

BY

J.F. VOLOCH (*)

RÉSUMÉ. — Soit K un corps de fonctions d'une variable sur un corps fini de caractéristique p . On détermine les courbes algébriques sur K ayant une fonction K -rationnelle dont leurs valeurs dans une infinité de points K -rationnels sont des puissances p -èmes. On en déduit la finitude de l'ensemble des points rationnels des courbes sur K qui changent de genre sous une extension de corps de base.

ABSTRACT. — Let K be a function field in one variable over a finite field of characteristic p . We determine the algebraic curves over K having a K -rational function on it whose value at infinitely many K -rational points is a p -th power. From this we deduce the finiteness of the set of K -rational points of curves over K that change genus under ground-field extension.

1. Introduction

Let K be a function field in one variable over a finite field of characteristic p . The purpose of this paper is to characterize the algebraic curves X/K and the rational functions $f \in K(X)$ such that $f(P) \in K^p$ for infinitely many rational points $P \in X(K)$. This problem ties up with a question left open by SAMUEL [2] in his extension to positive characteristic of GRAUERT's proof of MORDELL's conjecture for function fields of characteristic zero. The question occurs when the relative genus of X/K is different from the absolute genus of X in the sense of [2] (or equivalently when $K(X)$ is a non-conservative function field in the sense of [1]).

The genus of a curve X defined over K , relative to K , can be defined as follows. It is the integer g for which $\ell(D) = \deg D + 1 - g$, for divisors D , defined over K , with degree $\deg D$ sufficiently large, where $\ell(D)$ is the dimension, as a K -vector space, of the space of rational functions on X ,

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defined over K , with polar divisor bounded by D . The above definition of the genus depends on K . The genus of X , relative to K , does not change under separable extensions of K but may decrease under inseparable extensions. The absolute genus of X is thus defined as the genus of X relative to the algebraic closure of K . A standard example, for $p \geq 3$, is the curve $y^2 = x^p - a$. If $a \in K \setminus K^p$, then its genus, relative to K , is $\frac{1}{2}(p-1)$ and its absolute genus is 0.

SAMUEL showed that, with notation as above, $X(K)$ is finite if the absolute genus of X is at least two [2, Chapitre III, Theorem 1 and app. 2] and therefore the problem above is trivial for those curves. The question left open by SAMUEL [2, page 3] is whether curves with relative genus at least two and absolute genus 0 or 1 have finitely many rational points and we solve this question in the affirmative. Note that we have shown previously [4] that curves with relative genus 1 and absolute genus 0 have finitely many rational points (this will also follow from THEOREM 1 below). Hence all curves that admit genus change have finitely many rational points.

The paper is organized as follows. In sections 2 and 3 we solve our basic problem for rational curves and elliptic curves, respectively, and in section 4 we use these results to show that curves that admit genus change have finitely many rational points. Finally, we obtain the general solution to our problem.

2. Rational curves

Recall that K is a function field in one variable over a finite field of characteristic p . Let $t \in K \setminus K^p$ and $\delta = d/dt$, a derivation of K . If x is a variable over K , we extend δ to $K(x)$ by $\delta(x) = 0$. We shall also use the notation $r^\delta(x)$ for the action of δ on $r(x) \in K(x)$.

THEOREM 1. — *Let $r(x) \in K(x)$ be a rational function such that the set $\{a \in K \mid r(a) \in K^p\}$ is infinite. Then, there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(K)$ such that $r((\alpha x + \beta)/(\gamma x + \delta)) \in K^p(x)$.*

Proof. — Multiplying, if necessary, $r(x)$ by the p -th power of its denominator, we can assume that $r(x)$ is a polynomial. Let n be the degree of $r(x)$ and assume first that $p \nmid n$.

Let $r(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$. By changing, if necessary, the variable x to $a_0^m x$, where $mn + 1 \equiv 0 \pmod{p}$, we can assume that $a_0 \in K^p$. Further, dividing $r(x)$ by a_0 , we can also assume that $a_0 = 1$. Finally, changing x to $x - a_1/n$, we can assume that $a_1 = 0$.

If $a \in K$ is such that $r(a) \in K^p$, then

$$(*) \quad 0 = \delta(r(a)) = r'(a)\delta a + r^\delta(a).$$

Note that $r^\delta(a) = \delta a_2 x^{n-2} + \dots + \delta a_n$ is of degree at most $(n-2)$. If $r^\delta(x)$ is identically zero, then $r(x) \in K^p[x]$, as desired. Assume then that $r^\delta(x) \neq 0$.

Let v be a place of K with $v(a_i) \geq 0$, $i = 0, \dots, n$ and $v(dt) = 0$. If $a \in K$ is such that $v(a) < 0$ then, clearly, $v(r^\delta(a)) \geq (n-2)v(a)$ and $v(r'(a)) = (n-1)v(a)$, whence $v(\delta a) \geq 0$, from (*). If $v(a) \geq 0$ then, obviously, $v(\delta a) \geq 0$, as well. Thus $v(\delta a) \geq 0$ for all but finitely many places of K .

Further, the rational function $-r^\delta(x)/r'(x)$ has a zero at infinity. Thus, for any place v of K , if a has a sufficiently large pole at 0 then $\delta a = -r^\delta(a)/r'(a)$ satisfies $v(\delta a) \geq 0$, say. On the other hand, if $v(a)$ is bounded below, then $v(\delta a)$ is also bounded below. The conclusion of the above discussion is that there exists a divisor D of K such that $\delta a \in L(D)$ for any $a \in K$ with $r(a) \in K^p$. Hence, δa can assume finitely many values b_1, \dots, b_N for those a . The polynomial equations $r'(x)b_i + r^\delta(x) = 0$, $i = 1, \dots, N$, have finitely many solutions unless one of them is identically zero. In the latter case, looking at the coefficient in x^{n-1} , it follows that $b_i = 0$ (recall that $p \nmid n$) and so $r^\delta(x) = 0$, contrary to the hypothesis. This proves the result when $p \nmid n$.

Let now $r(x)$ be a polynomial of degree $n \equiv 0 \pmod{p}$ satisfying the hypothesis of the theorem. Let $a \in K$ be such that $r(a) \in K^p$. To prove the theorem for $r(x)$ it suffices to prove the theorem for the polynomial $x^n(r(1/x+a) - r(a))$, which has degree strictly less than n . The theorem now follows by induction on n .

REMARK 1. — Let $r(x) \in K(x)$ be such that there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(K)$ with $r((\alpha x + \beta)/(\gamma x + \delta)) \in K^p(x)$. Then $r(a) \in K^p$ for infinitely many $a \in K$. Indeed $r((\alpha x^p + \beta)/(\gamma x^p + \delta)) = (s(x))^p$ for some $s(x) \in K(x)$. This also shows that the curve $y^p = r(x)$ is parametrizable over K , that is, has relative genus zero over K .

REMARK 2. — THEOREM 1 contains, as special cases, the results of [4]. The proof of THEOREM 1 is an extension of the techniques of [4].

3. Elliptic curves

We keep the notation of section 2. In particular, recall the derivation δ of K . If E/K is an elliptic curve given by the Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, let $E^{(p)}/K$ be the elliptic curve with Weierstrass equation $y^2 + a_1^p xy + a_3^p y = x^3 + a_2^p x^2 + a_4^p x + a_6^p$ and $F : E \rightarrow E^{(p)}$ be the Frobenius map defined by $F(x, y) = (x^p, y^p)$. Let also $V : E^{(p)} \rightarrow E$ be the isogeny dual to F . We extend δ to a derivation on $K(E^{(p)}) = K(x, y)$ by $\delta(x) = \delta(y) = 0$. As in section 2 we also denote by r^δ the action of δ on $r \in K(E^{(p)})$.

THEOREM 2. — *Notation as above. If $r \in K(E^{(p)})$ is such that the set $\{P \in E^{(p)}(K) \mid r(P) \in K^p\}$ is infinite, then there exists $P_0 \in E^{(p)}(K)$ such that the function $P \mapsto r(P + P_0)$ belongs to $K^p(E^{(p)})$. If $r \in K(E)$ is such that the set $\{P \in E(K) \mid r(P) \in K^p\}$ is infinite, then there exists $P_0 \in E(K)$ such that the function $P \mapsto r(V(P) + P_0)$ belongs to $K^p(E^{(p)})$.*

Proof. — Let $r \in K(E^{(p)})$ satisfy the hypothesis of the theorem. As $E^{(p)}(K)/F(E(K))$ is finite (by the Mordell-Weil theorem) it follows that there exists $P_0 \in E^{(p)}(K)$ such that, for infinitely many $P \in F(E(K))$, $r(P + P_0) \in K^p$. Let $s \in K(E^{(p)})$ be defined by $s(P) = r(P + P_0)$. If $P \in F(E(K))$, its x, y coordinates are p -th powers, hence $\delta(s(P)) = s^\delta(P)$. If, furthermore, $s(P) \in K^p$ then $s^\delta(P) = 0$. But s^δ has finitely many zeros unless it is identically zero. We therefore conclude that $s^\delta = 0$, that is, $s \in K^p(E^{(p)})$, as desired.

Let $r \in K(E)$ satisfy the hypothesis of the theorem. Again by Mordell-Weil, $E(K)/V(E^{(p)}(K))$ is finite : there exists $P_1 \in E(K)$ such that there exists infinitely many $P \in V(E^{(p)}(K))$ with $r(P + P_1) \in K^p$. Thus, the function $P \mapsto r(V(P) + P_1)$ on $E^{(p)}$, satisfies the hypothesis of the theorem and, by what was proved above, there exists P_2 such that the function $P \mapsto r(V(P + P_2) + P_1)$ belongs to $K^p(E^{(p)})$ and the theorem follows with $P_0 = P_1 + V(P_2)$.

REMARK 3. — If $r \in K(E^{(p)})$ is such that $P \mapsto r(P + P_0)$ belongs to $K^p(E^{(p)})$ for some $P_0 \in E^{(p)}(K)$ then $r(P) \in K^p$ for all $P \in E^{(p)}(K)$, $P - P_0 \in F(E(K))$. Indeed $r(F(P) + P_0) = (s(P))^p$ for some $s \in K(E)$. Thus the cover of $E^{(p)}$ defined by the equation $z^p = r$ has genus 1 over K , since it is covered by E by the map $P \mapsto (F(P) + P_0, s(P))$. A similar phenomenon occurs for $r \in K(E)$ such that $P \mapsto r(V(P) + P_0)$ belongs to $K^p(E^{(p)})$. Indeed, $r(pP + P_0) = s(P)^p$ for some $s \in K(E)$, since $V \circ F$ is multiplication by p on E .

4. Mordell's conjecture for non-conservative curves

An algebraic curve X/K is said to be conservative if its genus does not change under base-field extension from K to its algebraic closure \bar{K} (see [2, page 3] or [1], [3]). Otherwise X is called non-conservative. The genus of X over K is called the relative genus of X and the genus of X over \bar{K} , the absolute genus of X .

We retain the notation as above, in particular K is a function field in one variable over a finite field.

THEOREM 3. — *A non-conservative algebraic curve X/K has finitely many K -rational points.*

Proof. — Let X_n/K , $n = 0, 1, \dots$ be the algebraic curve whose function field is $K \cdot (K(X))^{p^n}$ and denote by g_n the genus of X_n .

The sequence g_n is non-increasing and the constant value of g_n for all n sufficiently large is the absolute genus \bar{g} of X (see *e.g.* [3]). If $\bar{g} \geq 2$, then the theorem was already proved by SAMUEL. Assume that $\bar{g} = 0$ or 1 and let n be such that $g_{n-1} > g_n = \bar{g}$. Let $z \in K(X_{n-1}) \setminus K(X_n)$, then $r = z^p \in K(X_n)$ and $K(X_{n-1}) = K(X_n)(z)$. This means that X_{n-1} is the cover of X_n given by $z^p = r$ and it is easy to see that all but finitely many rational points of X_{n-1} correspond to rational points $P \in X_n$ for which $r(P) \in K^p$. From THEOREM 1 (and REMARK 1) and THEOREM 2 (and REMARK 3) it follows that there exists only finitely many such points and thus $X_{n-1}(K)$ is finite. It then follows by a similar and easier argument that $X_{n-2}(K), \dots, X_0(K) = X(K)$ are all finite, proving the theorem.

REMARK 4. — Returning to our basic problem, staded in the introduction, if X/K is an algebraic curve and $f \in K(X)$ is such that $\{P \in K(K) \mid f(P) \in K^p\}$ is infinite, then by THEOREM 3 and the Grauert-Samuel Theorem (*i.e.* the Mordell conjecture in characteristic p) it follows that X is rational or elliptic, and in these cases the problem is solved by THEOREMS 1 and 2.

BIBLIOGRAPHY

- [1] ARTIN (E.). — *Algebraic Numbers and Algebraic Functions*. — Gordon and Bleach, New York-London-Paris, 1967.

- [2] SAMUEL (P.). — *Lectures on old and new results on Algebraic Curves.*
— Tata Institute of Fundamental Research, Bombay, 1966.
- [3] STICHTENOTH (H.). — Zur Konservativität algebraischer Funktio-
nenkörper, *Crelle*, t. **301**, 1978, p. 30–45.
- [4] VOLOCH (J.F.). — Mordell's equation in characteristic three, *Bull.*
Austral. Math. Soc., t. **41**, 1990, p. 149–150.