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ON THE SUPERADDITIVITY OF SECANT DEFECTS

BY

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RÉSUMÉ. — Dans son article [Z2], M. ZAK a énoncé un théorème de super-additivité pour les défauts sécants de variétés projectives lisses ; ensuite, M. ÅDLANDSVIK a donné un contre-example. Dans cet article nous prouvons un théorème semblable à celui de ZAK, mais avec des hypothèses plus fortes ; nous remarquons que tous les corollaires énoncés par ZAK restent vrais, et que le contre-exemple de ÅDLANDSVIK ne satisfait pas les hypothèses supplémentaires.

ABSTRACT. — In his paper [Z2], ZAK stated a theorem of superadditivity for secant defects of smooth projective varieties ; subsequently, ÅDLANDSVIK gave a counterexample. In this paper we state and prove a theorem similar to ZAK's but with stronger hypotheses, we show that these do not hold for Ådlandsvik's counterexample, and we point out that all of ZAK's corollaries are still implied by our version of the theorem.

0. Introduction

The extrinsic properties of an embedded projective variety X, especially concerning linear projections, are related with the properties of its secant varieties. We recall that the k-th (or the $(k-1)$-th, depending on the notation) secant variety of a given projective variety X in $\mathbf{P}^N$ is the closure in $\mathbf{P}^N$ of the union of the $(k-1)$-dimensional linear subspaces generated by k points of X.

The interest for the properties of secant varieties arose first at the beginning of the century, e.g. we can mention TERRACINI's paper [T]. In recent years, some of these properties have been used in the proof of various results in projective algebraic geometry ; the most striking example is ZAK's classification of Severi varieties.

Another field of application is a new proof of linear normality for smooth low-codimension projective varieties, (i.e. such that the dimension is bigger or equal to twice the codimension) ; this result is important as a first step in the direction of Hartshorne's conjecture (see [H]) that low

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codimension varieties should be complete intersections.

In his paper [Z2, p.168] ZAK stated a theorem of superadditivity for secant defects of projective varieties. The “expected” dimension of the k-th secant variety is equal to 1 plus the dimension of X plus the dimension of the $(k-1)$-th secant variety, as long as the latter is not all of $\mathbf{P}^N$. The difference beween the actual dimension and the expected dimension for the k-th secant variety is called the k-th secant defect, and denoted by δ_k for short. (For more precise definitions, we refer the reader to paragraph 1.) ZAK deduced from his theorem another proof of (corollary, p. 170) linear normality for smooth low-codimension varieties, and more generally he gave estimates for the possible dimension of a nondegenerate embedding for a smooth, projective variety of given dimension and first secant defect.

Subsequently, Ådlandsvik ([Å2], see also remark 3.6) gave a counterexample to ZAK's theorem, thus leaving open the question whether or not his other results were correct.

In this paper we introduce the notion of almost smooth variety; it is a projective variety X such that, for every x in X, its tangent star at x (the union of the limits of secants with endpoints tending to x) is contained in the closure of the union of the secants through x (*cf.* section 2). In general the tangent star contains the tangent cone and is contained in the Zariski tangent space; hence, in particular, every smooth variety is almost smooth.

Following the ideas of ZAK, we prove (for notations see section 1) :

THEOREM 2.5. — *Let X be an irreducible, closed subvariety of $\mathbf{P}^N$ and let k, l, m be positive integers, with $l+m=k$, $k \leq k_0(X)$ and $2m \leq k_0$. Assume $J^m(X)$ is almost smooth. Then $\delta_k \geq \delta_l + \delta_m$.*

Although our statement is a slightly weaker version of ZAK's theorem we remark (3.7) that it is strong enough to ensure the validity of all the main results of ZAK's paper, and in particular the linear normality result, and the estimates of embedding dimensions. We also point out that the extra hypothesis needed to make the proof work fails in fact for Ådlandsvik's counterexample.

This paper goes as follows : in section 1 we recall briefly definitions and properties of secant varieties which will be used in the sequel; we also introduce the necessary notation; in section 2 we state the main theorem and give an outline of the proof and in section 3 we deal with some technical lemmas, we describe Ådlandsvik's counterexample and we recall briefly how ZAK derived the linear normality result and the embedding dimension estimates.

I would like to thank Fabrizio CATANESE, who introduced me to this subject and helped me during my thesis work.

1. Definitions and basic properties of secant varieties

We shall denote by $\mathbf{P}^N$ the N-dimensional projective space over a fixed algebraically closed field k. If $X_1,\ldots,X_n$ are subsets of $\mathbf{P}^N$, we shall denote by $\langle X_1,\ldots,X_n\rangle$ their linear span, i.e. the smallest projective subspace of $\mathbf{P}^N$ containing all the X_j's.

Definition 1.1. — Let $X_1,\ldots,X_n$ be irreducible subvarieties of $\mathbf{P}^N$, and let

$$h = \max\left\{\dim\langle x_1,\ldots,x_n\rangle \mid x_j \in X_j\right\}.$$

We denote by $S(X_1,\ldots,X_n)$ the closure in $X_1\times\cdots\times X_n\times\mathbf{P}^N$ of the set

$$(*)\qquad \Big\{(x_1,\ldots,x_n,z) \mid z\in\langle x_1,\ldots,x_n\rangle,\ \dim\langle x_1,\ldots,x_n\rangle = h\Big\}.$$

The projection of $S(X_1,\ldots,X_n)$ in $\mathbf{P}^N$ will be called the *join* of $X_1,\ldots,X_n$ and will be denoted by $J(X_1,\ldots,X_n)$.

Remark 1.2. — *Let $X_1,\ldots,X_n$ be irreducible subvarieties of $\mathbf{P}^N$. Then the following hold :*

(i) *$S(X_1,\ldots,X_n)$ is irreducible ;*
(ii) *$J(X_1,\ldots,X_n)$ is irreducible ;*
(iii) $\dim S(X_1,\ldots,X_n) = h + \sum\dim X_j$ *;*
(iv) *If $X_1\cup X_2$ contains at least two distinct points*

$$\dim S(X_1,X_2) = \dim X_1 + \dim X_2 + 1.$$

Proof. — Let π denote the natural map from $S(X_1,\ldots,X_n)$ to $X_1\times\cdots\times X_n$.

(i) and (iii) both follow by observing that the open set defined in (*) is a projective bundle of rank h over the set $\{(x_1,\ldots,x_n) \mid \dim\langle x_1,\ldots,x_n\rangle = h\}$, which is open in $X\times\cdots\times X$.

(ii) is a consequence of (i), and (iv) is a consequence of (iii). □

Definition 1.3. — $J(X_1,\ldots,X_n)$ is said to be *nondegenerate* if

$$\dim J(X_1,\ldots,X_n) = \dim S(X_1,\ldots,X_n).$$

LEMMA 1.4 (TERRACINI). — *Let X, Y be irreducible subvarieties of $\mathbf{P}^N$, $x\in X$, $y\in Y$, $z\in\langle x,y\rangle$. Then the following hold :*

(i) $T_{X,x} \subset T_{J(X,Y),z}$; $T_{Y,y} \subset T_{J(X,Y),z}$.

(ii) *If char $k = 0$, there exists U open subset of $J(X,Y)$ such that for all $z \in U$, for all $x \in X$, for all $y \in Y$ such that $z \in \langle x, y\rangle$ we have $\langle T_{X,x}, T_{Y,y}\rangle = T_{J(X,Y),z}$.*

Proof. — We can clearly restrict ourselves to an affine space $\mathbf{A}^N$. We shall use the same symbol to denote both a projective variety and its intersection with $\mathbf{A}^N$. Then we can define a dominant map $\varphi : X \times Y \times \mathbf{A}^1 \to J(X,Y)$ by $\varphi(x,y,\lambda) = \lambda x + (1-\lambda)y$. Now it is enough to show that, for $z \neq x$ and $z \neq y$, the image at (x,y,λ) of the tangent space of $X \times Y \times \mathbf{A}^1$ via $d\varphi$ is $\langle T_{X,x}, T_{Y,y}\rangle$; in fact (i) follows immediately, and (ii) follows by remarking that, in characteristic zero, the differential of a dominant map is surjective. Let $z = \varphi(x,y,\lambda)$, $T_{X,x} = V + x$, $T_{Y,y} = W + y$ with V, W vector subspaces of $\mathbf{A}^N$; then

$$d\varphi(T_{X\times Y\times \mathbf{A}^1,(x,y,\lambda)}) = \Big\{\lambda(v+x) + (1-\lambda)(w+y) + \mu(x-y) \mid \text{such that } v \in V,\ w \in W,\ \mu \in T_{\mathbf{A}^1,\lambda} = \mathbf{A}^1\Big\}.$$

It is easy to verify that this is exactly $\langle T_{X,x}, T_{Y,y}\rangle$. □

TERRACINI's proof can be found in [T]. This lemma can be easily extended to the case of $J(X_1, \dots, X_n)$, for any n.

Definition 1.5. — Let X in $\mathbf{P}^N$ be an irreducible variety. We denote $S(X,\dots,X)$ (k copies) by $S^k(X)$; in the same way we denote $J(X,\dots,X)$ by $J^k(X)$. $J^k(X)$ will be called *k-th secant variety* of X.

Notation 1.6. — We shall denote by $s_k(X)$, or s_k, the dimension of the k-th secant variety; we shall denote by $k_0(X)$, or k_0, the biggest k such that $s_k < N$.

Definition 1.7. — If $a_1, \dots, a_r$ are integers ≥ 1 such that $\sum a_i = k$, clearly

$$J\big(J^{a_1}(X), \dots, J^{a_r}(X)\big) = J^k(X).$$

We define $S^{a_1,\dots,a_r}(X)$ to be $S(J^{a_1}(X), \dots, J^{a_r}(X))$.

We now want to choose in the varieties we just defined some "good" open sets, where we shall be able in the following to construct explicitly some useful maps.

Notation 1.8. — Let $\tilde{S}^k(X)$ be the open set in $S^k(X)$ defined as follows :

$$\tilde{S}^k(X) = \Big\{(x_1, \dots, x_k, u) \text{ such that } x_i \in X,\ u \in \langle x_1,\ \dots, x_k\rangle,\ u \notin J^{k-1}(X)\Big\}$$

and let $\tilde{J}^k(X)$ be its projection in $\mathbf{P}^N$. We remark that, for $k \leq k_0$, $J^k(X) \neq J^{k+1}(X)$, whence $\tilde{S}^{k+1}(X)$ and $\tilde{J}^{k+1}(X)$ are nonempty. From now on, when we talk about secant varieties $S^k(X)$ and $J^k(X)$, we shall always assume, unless explicitly stated, that $k \leq k_0(X) + 1$.

Notation 1.9. — In the same way we define

$$\tilde{S}^{a_1,\dots,a_r}(X) = \Big\{(v_1,\dots,v_r,u) \text{ such that } u \in \langle v_1,\dots,v_r\rangle, \\ v_i \in \tilde{J}^{a_i}(X),\ u \in \tilde{J}^k(X)\Big\}.$$

We remark that $\tilde{S}^k(X)$ is just $\tilde{S}^{\overbrace{1,\dots,1}^{k \text{ times}}}(X)$.

Remark 1.10.

(i) If $(x_1,\dots,x_k,u) \in \tilde{S}^k(X)$, $\dim\langle x_1,\dots,x_k\rangle = k-1$.

(ii) If $(v_1,\dots,v_r) \in \tilde{S}^{a_1,\dots,a_r}(X)$, $\dim\langle v_1,\dots,v_r\rangle = r-1$.

Proof. — We give only the proof of (ii), (i) requiring just a small change in notations. We know that there exist $\lambda_{ij} \in k$, $x_{ij} \in X$ such that $v_i = \sum \lambda_{ij}x_{ij}$ with $x_{ij} \in X$. Assume that the v_i's are linearly dependent, say $\sum \mu_i v_i = 0$. We get $\sum \mu_i\lambda_{ij}x_{ij} = 0$; thus the x_{ij}'s are linearly dipendent and $u \in J^{k-1}(X)$, a contradiction. □

LEMMA 1.11. — *If $\{a_j\}_{j=1,\dots,r}$ are positive integers and if we let*

$$b_j = \begin{cases} a_j & \text{if } 1 \leq j < j_0\,; \\ a_j + a_{j+1} & \text{if } j = j_0\,; \\ a_{j+1} & \text{if } j_0 < j \leq r-1\,; \end{cases}$$

then there exists a natural surjective morphism

$$\psi : \tilde{S}^{a_1,\dots,a_r}(X) \longrightarrow \tilde{S}^{b_1,\dots,b_{r-1}}(X).$$

Proof. — Let $(v_1,\dots,v_r,u)$ be a point of $\tilde{S}^{a_1,\dots,a_r}(X)$, with $u = \sum \lambda_j v_j$. Define

$$w_j = \begin{cases} v_j & \text{if } 1 \leq j < j_0\,; \\ \lambda_j v_j + \lambda_{j+1}v_{j+1}, & \text{if } j = j_0\,; \\ v_{j+1} & \text{if } j_0 < j \leq r-1\,; \end{cases}$$

and set $\psi(v_1,\dots,v_r,u) = (w_1,\dots,w_{r-1},u)$. As $u \notin J^{k-1}(X)$, all of the λ_j's are different from zero (they are also determined as a point of projective $(r-1)$ space). Thus ψ is well defined, and its image lies in the requested variety. Let now $Q = (w_1,\dots,w_{r-1},u)$ be a point in

$\tilde{S}^{b_1,\dots,b_{r-1}}(X)$, and assume that $w_j = \sum_{i=1}^{b_j} \lambda_{ij} x_{ij}$, $x_{ij} \in X$. Then if we define

$$v_j = \begin{cases} w_j & \text{if } 1 \le j < j_0\,; \\ \sum_{i=1}^{a_j} \lambda_{ij} x_{ij} & \text{if } j = j_0\,; \\ \sum_{i=a_{j-1}}^{b_{j-1}} \lambda_{ij} x_{ij} & \text{if } j = j_0 + 1\,; \\ w_{j-1} & \text{if } j_0 < j \le r-1. \end{cases}$$

Clearly $P = (v_1, \dots, v_r, u)$ is a point of $\tilde{S}^{a_1,\dots,a_r}(X)$ such that $\psi(P) = Q$. Thus the map ψ is surjective. □

COROLLARY 1.12.

(i) *If $(a_1, \dots, a_r)$ and $(b_1, \dots, b_s)$ are ordered sets of natural numbers, such that there exist $c_0 = 0 < c_1 < \dots < c_s = r$ with $b_i = \sum_{j=c_{i-1}+1}^{c_i} a_j$, we can define a natural surjective morphism Ψ : $\tilde{S}^{a_1,\dots,a_r}(X) \to \tilde{S}^{b_1,\dots,b_s}(X)$;*

(ii) *In particular, if $\sum_{i=1}^{r} a_i = k$, there is a natural surjective map from $\tilde{S}^k(X)$ to $\tilde{S}^{a_1,\dots,a_r}(X)$.*

(iii) *$\tilde{S}^{a_1,\dots,a_r}(X)$ is irreducible.*

Proof.

(i) follows from lemma 1.11 by an easy inductive argument;

(ii) is just a particular case of (i);

(iii) is an obvious consequence of (ii) and of remark 1.2(i). □

Definition 1.13. — We shall define the *k-th secant defect* of X to be

$$\delta_k(X) = s_k(X) + n + 1 - s_{k+1}(X).$$

Remark 1.14. — $\delta_k = \dim S(J^k(X), X) - \dim J(J^k(X), X)$; thus δ_k "measures" how much the join of X and $J^k(X)$ is degenerate. We now want to identify the δ_k with the dimensions of certain subvarieties of X.

Let u be a point in $J^k(X)$, and let p, φ be the projections of $S^k(X)$ on the first factor X and on $J^k(X)$, respectively. We denote $\varphi^{-1}(u)$ by Z_u, and $p(Z_u)$ by Y_u.

As $\tilde{S}^k(X)$ is open in the irreducible variety $S^k(X)$, if u is generic in $J^k(X)$ (in particular $u \in \tilde{J}^k(X)$) we have that $\tilde{S}^k(X) \cap Z_u$ is a dense open set in Z_u.

LEMMA 1.15. — *If u is generic in $J^k(X)$, $\sum_{i=1}^{r} a_i = k$, $a_1 = 1$, we have $Y_u = p_1(\varphi_a^{-1}(u))$ where p_1, φ_a are the projections of $S^{a_1,\dots,a_r}(X)$ on the first factor X and on $J^k(X)$, respectively.*

Proof. — Let ψ : $\tilde{S}^k(X) \to \tilde{S}^{a_1,\dots,a_r}(X)$ be the surjective morphism of COROLLARY 1.12. If u is generic, then Y_u is the closure in X of

$p(\varphi^{-1}(u)\cap\tilde{S}^k(X))$. In the same way, if u is generic, then $p_1(\varphi_a^{-1}(u)) =$ closure in X of $p_1(\varphi_a^{-1}(u)\cap\tilde{S}^{a_1,\dots,a_r}(X))$. We just have to show that $p(\varphi^{-1}(u)\cap\tilde{S}^k(X)) = p_1(\varphi_a^{-1}(u)\cap\tilde{S}^{a_1,\dots,a_r}(X))$. It is enough to remark that the following diagram is commutative

$$\begin{tikzcd}
& \tilde{S}^k(X) \arrow[dl, "p"'] \arrow[dr, "\varphi"] \arrow[dd, "\psi"] & \\
X & & \tilde{J}^k(X) \\
& \tilde{S}^{a_1,\dots,a_r}(X) \arrow[ul, "p_1"] \arrow[ur, "\varphi_a"'] &
\end{tikzcd}$$

and that ψ is surjective. $\square$

LEMMA 1.16. — *If u is generic in $J^k(X)$, then every component of Y_u has dimension δ_{k-1}.*

Proof. — Let Z'_u be $\varphi_1^{-1}(u)\subset\tilde{S}^{1,k-1}(X)$, where $\varphi_1:\tilde{S}^{1,k-1}(X)\to\tilde{J}^k(X)$ is the natural projection. We know that Y_u is the image of the closure of Z'_u via the projection on the first factor. Computing dimensions, we immediately obtain that, for u generic, Z'_u has pure dimension δ_{k-1}. Thus we are reduced to show that $p: Z'_u\to Y_u$, given by projection on the first factor, is finite.

The inverse image in Z'_u of $x\in p(Z'_u)$ are just the points (x,v,u) with $v\in J^{k-1}(X)$ and $u\in\langle x,v\rangle$. It follows that $v\in J^{k-1}(X)\cap\langle x,u\rangle$; as $u\notin J^{k-1}(X)$, this intersection contains at most a finite number of points. $\square$

THEOREM 1.17. — *The sequence δ_k is nondecreasing.*

Proof. — See [Z2]. $\square$

Remark 1.18. — Our definition of the k-th secant variety coincides with ZAK's definition of $(k-1)$-th secant variety; his definition of secant defect is the same as ours.

2. Statement and proof of the main theorem

Definition 2.1. — Let π and φ be the projections of $S^2(X)$ on $X\times X$ and $J^2(X)$, respectively. The *tangent star* at X in x is defined to be $T'_{X,x} = \varphi(\pi^{-1}(x,x))$. If $Y\subset X$ is a subvariety, we let $T'(X,Y)=\bigcup_{x\in Y}T'_{X,x}$.

Definition 2.2. — X is said to be *almost smooth* if for all $x \in X$ we have $T'_{X,x} \subseteq J(\{x\}, X)$.

Definition 2.3.
(i) $T'_{X,x} \subseteq T_{X,x}$.
(ii) If X is smooth in x, $T'_{X,x} = T_{X,x}$.

The statement can be found also in [Z 1]. The proof is in [J], where the definition of tangent star was given for the first time. □

Remark 2.4. — It is easy to see that the tangent star contains the tangent cone and is contained in the tangent space; further information can be found in [J], and an alternative definition over the field of the complex numbers can be found in [W].

THEOREOM 2.5. — *Let k, l, m be positive integers, with $l + m = k$, $k \leq k_0$ and $2m \leq k_0$. Assume $J^m(X)$ is almost smooth. Then $\delta_k \geq \delta_l + \delta_m$.*

Proof. — Consider the diagram

$$\begin{tikzcd}
 & \tilde{S}^{l,1,m}(X) \arrow[dl, "\lambda"'] \arrow[dr, "\mu"] \arrow[dd, "\varphi"] & \\
\tilde{S}^{l+1,m}(X) \arrow[dr, "\varphi_l"'] & & \tilde{S}^{l,m+1}(X) \arrow[dl, "\varphi_m"] \\
 & \tilde{J}^{k+1}(X) &
\end{tikzcd}$$

where the φ's are the natural projections, and λ and μ are maps of the type described in COROLLARY 1.12. It is easy to see that the generic fibre of μ has pure dimension δ_m, and that the generic fibre of λ has pure dimension δ_l. The proof is divided into two steps.

First step. — Let $P \in \tilde{S}^{l,m+1}(X)$ be any point, $P = (v_l, v_{m+1}, u)$, and let

$$Z = Z(P) = \lambda^{-1}\big(\lambda(\mu^{-1}(P))\big).$$

If we can show that $\lambda : \mu^{-1}(P) \to \lambda(\mu^{-1}(P))$ is finite, then we can deduce that, for a generic P, Z has at least an irreducible component of dimension $\delta_m + \delta_l$.

Second step. — We show that, for P generic, the map from Z to Y_u induced by the projection of $\tilde{S}^{l,1,m}(X)$ on the second factor is generically finite on an irreducible component of dimension $\delta_l + \delta_m$. We already know that, for u generic in $J^{k+1}(X)$, Y_u has pure dimension δ_k, so the theorem follows.

Proof of the first step. — Let $Q \in \lambda(\mu^{-1}(P))$; we want to show that the set $\{R \in \tilde{S}^{l,1,m}(X)$ t.c. $\lambda(R) = Q, \mu(R) = P\}$ is finite. Let $Q = (v_{l+1}, v_m, u)$, the possible R's are 4-tuples (v_l, x, v_m, u) where the only unknown element is x; we must also have $x \in X \cap \langle v_{m+1}, v_m\rangle$ and as $\langle v_{m+1}, v_m\rangle \not\subset X$, we have only a finite number of choices for x.

Proof of the second step. — We must show that the map from Z to Y_u is generically finite on its image. Let $R \in Z$ be the 4-tuple (v_l', y, v_m, u). The result will thus be obtained by showing that the map from Z to Y_u given by $R \to y$ is generically finite. We shall factor this map in the following way :

$$\begin{array}{ccc}
Z & & R \\
\downarrow{\scriptstyle p_1} & & \downarrow \\
\tilde{S}^{1,k}(X) \times \tilde{J}^m(X) & & ((y, v_k, u), v_m) \\
\downarrow{\scriptstyle p_2} & & \downarrow \\
\tilde{S}^{1,k}(X) & & (y, v_k, u) \\
\downarrow{\scriptstyle p_3} & & \downarrow \\
Y_u & & y
\end{array}$$

where the map p_1 is defined by setting $v_k(R) = \langle y, u\rangle \cap \langle v_l', v_m\rangle$ (see again COROLLARY 1.12). We now prove that p_1 and p_3 are finite; LEMMA 3.5 will show that, if P is generic, p_2 is generically finite, at least on an irreducible component of $p_1(Z)$ of dimension $\delta_l + \delta_m$.

The proof of the finiteness of p_1 is elementary. Consider

$$Z' = \{(R, R') \text{ such that } R \in Z,\ \lambda(R) = \lambda(R'),\ \mu(R') = P\}.$$

Let $R' = (v_l, x, v_m, u)$. It is enough to show that $Z' \to \tilde{S}^{1,k}(X) \times \tilde{J}^m(X)$ given by $(R, R') \to p_1(R)$ is finite. We can describe the situation in the figure 1 next page.

We notice that, once y, v_k, u, v_m have been fixed, we have for x only a finite number of possible choices, as we must have $x \in \langle v_m, v_{m+1}\rangle \cap X$,

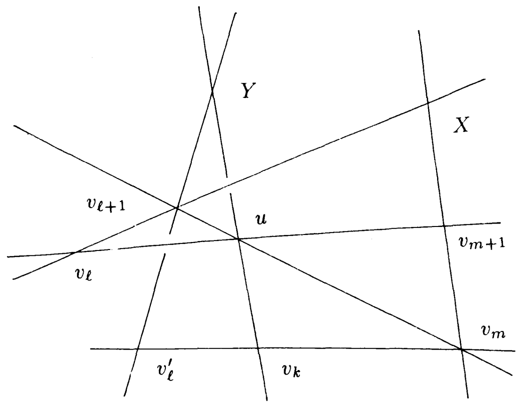

Figure 1

and this set consists of a finite number of points. For a similar reason we have a finite number of choices for v_{l+1}; in fact this point must be in $\langle u, v_m\rangle \cap J^{l+1}(X)$. We finally get $v'_l = \langle v_k, v_m\rangle \cap \langle y, v_{l+1}\rangle$. Thus $Z' \to \tilde{S}^{1,k}(X) \times \tilde{J}^m(X)$ is finite, whence p_1 is. The finiteness of p_3 can be shown in a similar way; in fact, u is fixed, and we must have $v_k \in \langle u, y\rangle \cap J^k(X)$, which is again a finite set. □

3. Some technical lemmas

The basic idea in the proof of LEMMA 3.1 is due to ZAK; we point out the need for the almost smoothness of $J^m(X)$. See also *Remark* 3.6. LEMMAS 3.2 and 3.5 may essentially be found in [Z 2] although in a more implicit form.

LEMMA 3.1. — *Let X be an almost smooth variety in $\mathbf{P}^N$, Y a subvariety. If $J(X,Y)$ is degenerate, $T'(X,Y) = J(X,Y)$.*

Proof. — Let $n = \dim X$, $m = \dim Y$. By hypothesis, $T'(X,Y) \subseteq J(X,Y)$; as $J(X,Y)$ is irreducible, if they do not coincide we must have $\dim T'(X,Y) < \dim J(X,Y)$. Let $t = \dim T'(X,Y)$. If $t = n+m$ the theorem is proved. Thus we can assume $t < n+m$. Let L be a linear subspace of dimension $N-t-1$, not intersecting X nor $T'(X,Y)$; we want to show that L does not meet $J(X,Y)$. It follows that $\dim J(X,Y) = t$ and the theorem is proved.

Let us consider the linear projection with center L. Applying it to both factors we get a regular map $p : Y \times X \to \mathbf{P}^t \times \mathbf{P}^t$. The dimension of the image is $n+m > t$; the theorem of FULTON and HANSEN (see [F-L]) implies that the inverse image of the diagonal $\Delta_{\mathbf{P}}$ is connected.

We now remark that our thesis, $J(X,Y) \cap L \neq \emptyset$, is equivalent to $p^{-1}(\Delta_{\mathbf{P}}) = \Delta_Y$. Arguing by contradiction, assume that W is an irreducible component of $p^{-1}(\Delta_{\mathbf{P}})$ different from Δ_Y; by the connectedness theorem, we can assume that $W \cap \Delta_Y \neq \emptyset$. Let (w,x) be any point in $W \setminus \Delta_Y$; we have $\langle w,x\rangle \cap L \neq \emptyset$. With the same notations as in *Definition* 2.1, this can be expressed as $\varphi(\pi^{-1}(w,x)) \cap L \neq \emptyset$. $W \setminus \Delta_Y$ is an open, dense subset of W; thus for every point Q in W, we have $\varphi(\pi^{-1}(Q)) \cap L \neq \emptyset$. In particular, let (y,y) be a point in $W \cap \Delta_Y$; we have $\varphi(\pi^{-1}(y,y)) \cap L \neq \emptyset$, thus $T'(X,Y) \cap L \neq \emptyset$, contradiction. $\square$

LEMMA 3.2. — *Let l, m, k be positive integers such that $l+m=k$, $k \leq k_0$, $2m \leq k_0$. Assume that $J^m(X)$ is almost smooth and that $v \in J^k(X)$ is generic. We denote by p, φ the projections of $S^{m,l}(X)$ on the first factor $J^m(X)$ and on $J^k(X)$, respectively. We also denote by $Y_v^m = p(\varphi^{-1}(v))$. With these hypothesis, if Y is an irreducible component of Y_v^m, $J(Y,X)$ is nondegenerate.*

Proof. — The proof is divided in two steps.

First step. — It is enough to show that $J(J^m(X),Y)$ is nondegenerate.

Secondstep. — $J(J^m(X),Y)$ is nondegenerate.

Proof of the first step. — As v is generic, we may assume that $\varphi^{-1}(v) \cap \tilde{S}^{m,l}(X)$ is dense in $\varphi^{-1}(v)$. As $2m \leq k_0$, we know that $\tilde{S}^{m,1,m-1}(X)$ is nonempty and dense in $S^{m,1,m-1}(X)$. We denote by Ψ_1 and Ψ_2 the natural maps from $\tilde{S}^{m,1,m-1}(X)$ to $\tilde{S}^{m,m}(X)$ and $\tilde{S}^{m+1,m-1}(X)$, respectively (see COROLLARY 1.12 for the definition). We have a commutative diagram

$$\begin{tikzcd}
 & \tilde{S}^{m,1,m-1}(X) \arrow[dl, "\Psi_1"'] \arrow[dr, "\Psi_2"] & \\
\tilde{S}^{m,m}(X) \arrow[dr, "\Phi_1"'] & & \tilde{S}^{m+1,m-1}(X) \arrow[dl, "\Phi_2"] \\
 & \tilde{J}^{2m}(X) &
\end{tikzcd}$$

where the Φ's are the projections on the last factor. Let $\tilde{S}(Y,X,J^{m-1}(X))$ be the set $\tilde{S}^{m,1,m-1}(X) \cap S(Y,X,J^{m-1}(X))$, and let ψ_1 and ψ_2 be the restrictions of Ψ_1 and Ψ_2 to $\tilde{S}(Y,X,J^{m-1}(X))$.

With similar notations we get the commutative diagram

$$\begin{tikzcd}[column sep=small] & \tilde S(Y,X,J^{m-1}(X)) \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ \tilde S(Y,J^m(X)) \arrow[dr, "\varphi_1"'] & & \tilde S(J(Y,X),J^{m-1}(X)) \arrow[dl, "\varphi_2"] \\ & \tilde J(X,Y) & \end{tikzcd}$$

Now we want to show that ψ_2 is generically finite if φ_1 is. Let Q be a generic point in $\tilde S(J(Y,X),J^{m-1}(X))$. We may assume $\varphi_2(Q)$ generic in $\tilde J(Y,X)$, whence $\varphi_1^{-1}(\varphi_2(Q))$ finite. Thus it is enough to show that if $P \in \tilde S(Y,J^m(X))$ is such that $\varphi_1(P)=\varphi_2(Q)$, then the set $\psi_1^{-1}(P)\cap\psi_2^{-1}(Q)$ is finite. Let $(y,x,u,w)=R\in\tilde S(Y,X,J^{m-1}(X))$ be such that $\psi_1(R)=P$, $\psi_2(R)=Q$, with P, Q fixed. In this case y is determined by P, u by Q and w by both of them. If $Q=(z,u,w)$, we must have $x\in X\cap\langle y,z\rangle$. But by genericity $y\notin X$; this implies that the intersection contains a finite number of points.

Proof of the second step. — Now we must show that $J(J^m(X),Y)$ is nondegenerate. By LEMMA 3.1 it is enough to show that $T'(J^m(X),Y))\neq J(J^m(X),Y)$. By Terracini's lemma 1.4 $T'(J^m(X),Y)\subset T_{J^k(X),v}$. Now v is generic, thus we can assume it is smooth; as $k\le k_0$, $T_{J^k(X),v}$ is a proper linear subspace, while $\langle J(J^m(X),Y)\rangle\supset\langle X\rangle=\mathbf{P}^N$ and therefore $\langle J(J^m(X),Y)\rangle=\mathbf{P}^N$. □

Remark 3.3. — *Let $l<k\le k_0$ be positive integers. The natural map $\tilde S^{l,k-l}(X)\to\tilde J^l(X)$ is dominant.*

Proof. — Assume it is not. This means that if $v_l\in\tilde J^l(X)$ is generic, we have $J(\{v_l\},\tilde J^{k-l}(X))\subseteq J^{k-1}(X)$. Consider $S(J^l(X),J^{k-l}(X))$, and denote by p, φ the projections on $J^l(X)\times J^{k-l}(X)$ and on $J^k(X)$, respectively. Our assumption implies that there is a nonempty open set U in $J^l(X)\times J^{k-l}(X)$ such that $\varphi(p^{-1}(U))\subseteq J^{k-1}(X)$. It follows that $\varphi(S(J^l(X),J^{k-l}(X)))\subseteq J^{k-1}(X)$, thus $J^{k-1}(X)=J^k(X)$; this contradicts the hypothesis $k\le k_0$. □

Remark 3.4. — *Let $\Gamma_1\subset\tilde J^{m+1}(X)\times\tilde J^k(X)\times\tilde J^m(X)$ be defined by*

$$\Gamma_1=\Big\{(v_{m+1},v_k,v_m)\text{ such that }\exists\,x\in X,\ v_l\in\tilde J^l(X),\\ \text{with }(x,v_m,v_{m+1})\in\tilde S^{1,m}(X),\ (v_l,v_m,v_k)\in\tilde S^{l,m}(X)\Big\}.$$

Let Γ *be the image of* Γ_1 *via the natural projection of* $\tilde{J}^{m+1}(X) \times \tilde{J}^k(X) \times \tilde{J}^m(X)$ *on the product of the first two factors. Then* Γ_1 *is irreducible and* $\Gamma_1 \to \Gamma$ *is generically finite.*

Proof.

First step. — Γ_1 is irreducible. Let $\Gamma_2 \subseteq \tilde{S}^{l,m}(X) \times X \times \mathbf{P}^N$ be the closure of the set $\{((v_l, v_m, v_k), x, v_{m+1}),$ such that $v_{m+1} \in \langle v_m, x\rangle\}$. Then Γ_2 is clearly a line bundle over the open set $\{v_m \neq x\}$. Thus Γ_2 is irreducible. Now let π_2 be the map from Γ_2 to $J^{m+1}(X)$ induced by the projection of $\tilde{S}^{l,m}(X) \times X \times \mathbf{P}^N$ on the last factor; its image is dense in $J^{m+1}(X)$ by *Remark* 3.3. Thus there is an open set Γ_2' in Γ_2 such that $\pi_2(\Gamma_2') \subseteq \tilde{J}^{m+1}(X)$. Now the thesis follows, because the natural map $\Gamma_2' \to \Gamma_1$ is surjective by definition.

Second step. — $\Gamma_1 \to \Gamma$ is generically finite. This follows from LEMMA 3.2. In fact, assume that the dimension of the generic fibre is at least one. For v generic in $J^k(X)$, v_{m+1} generic in an irreducible component Y of Y_v^m, we can find a subvariety of $\tilde{J}^m(X)$ (of dimension at least one) such that, for all v_m in the subvariety, the line $\langle v_{m+1}, v_m\rangle$ meets X. It follows that the mapping $S(X,Y) \to J(X,Y)$ is not generically finite; but by the lemma these are two irreducible varieties of the same dimension, a contradiction. $\square$

LEMMA 3.5. — *Let* $P \in \tilde{S}^{l,m+1}(X)$ *be generic. With the notation introduced in the proof of* THEOREM 2.5, *there exists an irreducible component of* $Z(P)$ *of dimension* $\delta_l + \delta_m$ *such that* p_2 *is finite on the image of that component via* p_1.

Proof. — We denote by Λ the subset of $\tilde{S}^{l,m+1}(X) \times \tilde{S}^{l,1,m}(X)$ defined by $\Lambda = \{(P,R)$ such that $\lambda(R) \in \lambda(\mu^{-1}(P))\}$. Let $\pi : \Lambda \to \tilde{S}^{l,m+1}(X)$ be the map induced by the natural projection. We remark that $\pi^{-1}(P) = \{P\} \times Z(P)$. We may associate to R a point $v_k \in \tilde{J}^k(X)$ in the same way we used to define p_1 in the proof of THEOREM 2.5. Thus we define a map $\alpha : \Lambda \to \Gamma$ given by $(P,R) \to (v_{m+1}, v_k)$. Let $\tilde{\Gamma}$ be the open set in Γ over which $\Gamma_1 \to \Gamma$ is finite, and let $U \subseteq \tilde{S}^{l,1,m}(X)$ be an open set such that if an irreducible component of $Z(P)$ meets U its dimension is $\delta_l + \delta_m$.

We define $\Lambda' \subset \Lambda$ by $\Lambda' = \{(P,R)$ such that $R \in U\}$. Our thesis follows by showing that $\pi(\Lambda' \cap \alpha^{-1}(\tilde{\Gamma}))$ is dense in $\tilde{S}^{l,m+1}(X)$.

In fact, if $P \in \tilde{S}^{l,m+1}(X)$ is generic, then $\pi^{-1}(P) \cap \Lambda' \cap \alpha^{-1}(\tilde{\Gamma})$ is nonempty, and thus we can choose $R \in Z(P)$ lying in this intersection. The component of $Z(P)$ containing R has dimension $\delta_l+\delta_m$ by hypothesis; on the other hand, if we let $p_1(R) = ((y, v_k, u), v_m)$, the picture in §2 shows that $(v_{m+1}, v_k, v_m) \in \Gamma_1$. $R \in \alpha^{-1}(\tilde{\Gamma})$ now implies $(v_{m+1}, v_k) \in \tilde{\Gamma}$ so that, for every y, the possible choices for v_m are just a finite number.

Assertion. — There exists an irreducible subvariety Λ_0 *of* Λ *with the following properties :*

(i) $\pi(\Lambda_0) = \tilde{S}^{l,m+1}(X)$;

(ii) $\alpha(\Lambda_0)$ *is dense in* Γ ;

(iii) $\Lambda_0 \cap \Lambda' \neq \emptyset$.

We first show that the lemma follows from the assertion. Let Λ^* be an irreducible component of Λ containing Λ_0. Then $\Lambda^* \cap \alpha^{-1}(\tilde{\Gamma})$ and $\Lambda^* \cap \Lambda'$ are open dense subsets of Λ^*; thus their intersection is dense in Λ^*. It follows that $\pi(\Lambda^* \cap \alpha^{-1}(\tilde{\Gamma}) \cap \Lambda')$ is dense in $\pi(\Lambda^*) = \tilde{S}^{l,m+1}(X)$.

Proof of the assertion. — Let $\Lambda_0 = \{(P, R)$ such that $P = \mu(R)\}$. Λ_0 is clearly irreducible, because it is isomorphic with $\tilde{S}^{l,1,m}(X)$.

(i) follows from the fact that $\mu(\tilde{S}^{l,1,m}(X)) = \tilde{S}^{l,m+1}(X)$.

(ii) Let $\psi : \Gamma_2' \to J^{k+1}(X)$ be defined as

$$\psi((v_l, v_m, v_k), x, v_{m+1}) = \langle v_l, v_{m+1}\rangle \cap \langle v_k, x\rangle.$$

Reasoning as in *Remark* 3.4, we obtain that $\psi(\Gamma_2')$ is dense in $J^{k+1}(X)$. It is easy to see that

$$\alpha(\Lambda_0) \supset \text{ the image in } \Gamma \text{ of } \psi^{-1}(\tilde{J}^{k+1}(X)) \subset \Gamma_2'.$$

(iii) Let $R \in U$; it follows that $(\mu(R), R) \in \Lambda_0 \cap \Lambda'$. □

Remark 3.6. — *The hypothesis X almost smooth in* THEOREM *2.5 is necessary.*

Proof. — We give an explicit example, due to Ådlandsvik (who studied it in [Å1] and pointed it out as a counterexample in [Å2]). It is a smooth variety X such that theorem 2.5 does not hold for X with $k = 4$, $l = m = 2$, and such that $J^2(X)$ is not almost smooth. Let X be the rational normal scroll in $\mathbf{P}^{n+2}$. To construct it explicitly we give a morphism $\varphi : \mathbf{P}^1 \to L$ where L is a line in $\mathbf{P}^{n+2}$ and $\varphi(t_0, t_1) = (t_0, t_1, 0, \ldots, 0)$, and another morphism $\psi : \mathbf{P}^1 \to A$ where A is a rational normal curve of degree n, with $\psi(t_0, t_1) = (0, 0, t_0^n, t_0^{n-1}t_1, \ldots, t_1^n)$. We set

$$X = \bigcup_{t \in \mathbf{P}^1} \langle \varphi(t), \psi(t)\rangle.$$

Clearly $\langle A\rangle$ is a linear subspace of dimension n not meeting L, and X is contained in $J(A, L)$.

It is easy to verify that, for $k \geq 2$, $J^k(X) = J(J^k(A), L)$. Thus $k_0 = [n/2] + 2$, as secant varieties to a curve are nondegenerate (see

[ZAK 2] for a proof). For $2 \le k \le k_0$, we have $s_k = 2k+1$. As $s_1 = 2$, we get $\delta_1 = 0$, $\delta_k = 1$ for $2 \le k \le k_0 - 1$. In particular if $n >> 0$, $\delta_2 = \delta_4 = 1$ contradicting the theorem.

Now we want to show that $J^2(X)$ is not almost smooth. To do this, we shall give explicitly a point $R \in J^2(X)$ and a point $P_1 \in T'_{J^2(X),R}$ not contained in $J(R, J^2(X))$. Let $R \in L$ be any point, $P, Q \in J^2(A)$, $P_1 \in \langle P, Q\rangle$, $P_1 \notin J^2(A)$. Choose a system of coordinates such that $P_1 = \lambda P + Q$. Then for all $(t_0, t_1) \in \mathbf{P}^1 \setminus \{(0,1)\}$ we have

$$\left(t_0 P + t_1 R, -\frac{t_0 Q}{\lambda} + t_1 R, P_1\right) \in S^2(J^2(X)).$$

As $S^2(J^2(X))$ is closed, $(R, R, P_1) \in S^2(J^2(X))$ and thus $P_1 \in T'_{J^2(X),R}$. On the other hand

$$J(R, J^2(X)) = J(R, L, J^2(A)) = J(L, J^2(A)).$$

Now P_1 by hypothesis is not contained in $J^2(A)$; as it lies in $\langle A\rangle$, it cannot be contained in $J(L, J^2(A))$, thus it is not contained in $J(R, J^2(X))$. □

For the reader's convenience, we now briefly recall one of ZAK's estimates and we point out why they all stay valid.

COROLLARY 3.7. — *Let X be an almost smooth projective variety of dimension n in $\mathbf{P}^N$, spanning all of $\mathbf{P}^N$, and let $\delta = \delta_1$. Then the following hold :*

(i) *for $0 \le k \le k_0$, $\delta_k \ge k\delta$;*

(ii) *denoting by square brackets the integer part of a number, we have*

$$N \le f([n/\delta])$$

where

$$f(k) = (k+1)(n+1) - \binom{k+1}{2}\delta - 1.$$

Proof.

(i) This follows from THEOREM 2.5 by induction (apply theorem with $m = 1$); in the smooth case this result is the first corollary, p.170 in [Z 2].

(ii) In the smooth case, this is THEOREM 3 of [Z2]. We recall that our k_0 is the same as the one employed there. Now it is enough to remark that this estimate is in fact deduced by the corollary we just mentioned and not by the superadditivity theorem. This works also for the other estimates (e.g. THEOREM 4 and 4′ of [Z2]). □

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