BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 118, nº 1 (1990), p. 27-54

<http://www.numdam.org/item?id=BSMF_1990__118_1_27_0>

Bull. Soc. math. France,
118, 1990, p. 27–54.

RESONANCE THEORY FOR PERIODIC SCHRÖDINGER OPERATORS

BY

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RÉSUMÉ. — Nous étudions le prolongement analytique de la résolvante $(H-\lambda)^{-1}$ pour un opérateur de Schrödinger périodique H. Nous montrons que $(H-\lambda)^{-1}$ s'étend à travers le spectre de H au complémentaire d'un ensemble discret de points, appelés *singularités de Van Hove* en physique du solide. Les singularités de Van Hove sont les points où la surface de Fermi complexifiée n'est pas lisse et sont en général des points de branchement pour $(H-\lambda)^{-1}$. Nous étudions aussi la relation des singularités de Van Hove avec la structure de bande du spectre, les singularités de la densité d'états et les résonances créées par des impuretés.

ABSTRACT. — We study the problem of analytic extension of the resolvent $(H-\lambda)^{-1}$ for H a periodic Schrödinger operator. We prove that $(H-\lambda)^{-1}$ extends across the spectrum of H to the complementary of a discrete set of points, called *Van Hove singularities* in solid state physics. The Van Hove singularities are roughly the points where the (complex) Fermi surface is not smooth, and are usually branch points of $(H-\lambda)^{-1}$. We study also the relationship of the Van Hove singularities with the band structure of the spectrum, the singularities of the density of states, and the resonances created by impurities.

Introduction

We study in this paper the theory of resonances for Schrödinger operators with periodic potentials. We consider Hamiltonians of the following form :

$$H = -\Delta + V(x) \quad \text{on } \mathbb{R}^n,$$

where V is a real multiplicative potential which is periodic with respect to some lattice T in $\mathbb{R}^n$.

We want to extend the resolvent $(H-\lambda)^{-1}$ from the physical region $\{\lambda \mid \operatorname{Im}\lambda > 0\}$ to the lower half plane across the bands of the spectrum of H, and study the singularities of this extension.

(*) Texte reçu le 9 décembre 1988, révisé le 25 octobre 1989.
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The existence of such an extension is more or less tacitly assumed in solid state physics when one studies the resonances created by localized impurities. (See for example the book of CALLAWAY [C, chapter 5].)

We will consider two kinds of problems :

i) *local extension problem* : given $\lambda_0 \in \sigma(H)$, extend analytically $(H-\lambda)^{-1}$ to a small neighborhood of λ_0, and describe its singularities;

ii) *global extension problem* : given some open set $\mathcal{U}$ extend analytically $(H-\lambda)^{-1}$ to $\mathcal{U}$ and describe its singularities.

For the *local extension problem*, we prove that any $\lambda_0 \in \sigma(H)$, there exist some neighborhood $\mathcal{U}_{\lambda_0}$ of λ_0, and a finite set Σ of points which we call *Landau resonances* such that $(H-\lambda)^{-1}$ extends holomorphically to the universal covering of $\mathcal{U}_{\lambda_0} \setminus \Sigma$. The Landau resonances are usually branch points of $(H-\lambda)^{-1}$ instead of poles. We decided to call the points of Σ *Landau resonances* by analogy with Landau singularities in Feynman integrals. We learned afterwards that these singularities (at least for the density of states) are known in solid state physics as *Van Hove singularities.*

For the *global extension problem*, we have to add to Σ a closed set of measure zero Σ_∞ which corresponds to a complex essential spectrum (see Definition 4.6). Then $(H-\lambda)^{-1}$ extends holomorphically to the universal covering of $\mathcal{U} \setminus \Sigma \cup \Sigma_\infty$.

The Landau resonances can be described geometrically in the following way : in the study of periodic Schrödinger operators one introduces usually the *Fermi surface* S_λ for $\lambda \in \mathbb{R}$: S_λ is the set of Bloch numbers p such that λ is an eigenvalue of the reduced Hamiltonian H_p obtained by the Floquet-Bloch theory.

The *Bloch variety* is then the set $S = \{(p,\lambda) \mid p \in S_\lambda, \lambda \in \mathbb{R}\}$. The Bloch variety has an extension to complex energies and Bloch numbers, and is a complex analytic set. Then roughly Σ is the set of $\lambda \in \mathcal{U}$ such that the (complex) Fermi surface S_λ is not a union of smooth submanifolds. (See Definition 3.2.) So the Landau resonances have a simple geometric interpretation in terms of singularities of complex Fermi surfaces.

Another new feature of the Landau resonances in contrast to the resonances encountered in two-body Hamiltonians is that they are usually branch points of $(H-\lambda)^{-1}$ instead of poles.

Moreover, it can happen that the singular part of $(H-\lambda)^{-1}$ at a Landau resonance is not a finite rank operator. (See THEOREMS 3.5, 3.6.) In simple cases it is however possible to associate resonant eigenfunctions to the leading singularity of $(H-\lambda)^{-1}$ at a Landau resonance. (See COROLLARY 3.7.)

Σ_∞ looks more like essential spectrum in the sense that Σ_∞ acts as a natural boundary for the extension of $(H-\lambda)^{-1}$ between $L^2_a(\mathbb{R}^n)$ and $H^1_{-a}(\mathbb{R}^n)$ for fixed values of a. Σ_∞ comes in part from the fact that we integrate operator-valued functions and that we have to take care of domain considerations. To make this remark more clear, let us compare H with two-body Schrödinger operators with exponentially decreasing potentials.

In the last case, the resolvent can be extended meromorphically to a strip $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > -\alpha\}$ for α depending on the rate of decay of the potential. Σ_∞ plays a role similar to $\operatorname{Im}\lambda = -\alpha$ in this problem.

In the last part of the paper we present some applications.

We study first the relationship of real Landau resonances with the band structure of the spectrum. We recover here some results obtained by Bentosela [B] in his study of time independent impurity scattering. (See Theorem 4.1.)

We study then the analyticity properties of the density of states $d\rho/d\lambda$ and prove that $d\rho/d\lambda$ is analytic outside the real Landau resonances.

Finally, we study the resonances created by a localized impurity. We modelize the impurities by adding to V a potential W which is exponentially decreasing. This is not a severe limitation in view of the phenomenon of dielectric screening. (See [C].)

We prove that the impurities add usual poles on $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$ to the Landau resonances of $(H-\lambda)^{-1}$. As a consequence, we show that the singular continuous spectrum of $H+W$ is empty and that the eigenvalues can accumulate only at the real Landau resonances which play the role of threshold energies.

The plan of the paper is the following :

- In Section I, we recall the Floquet-Bloch reduction which will be used in the next sections.

- In Section II, we prove the meromorphic extension in the energy and Bloch numbers of the reduced resolvent $(H_p-\lambda)^{-1}$ using Fredholm theory.

- In Section III, we prove the main results of this paper using methods from complex analytic geometry.

- In Section IV, we apply these results to the band structure, to the density of states and to resonances created by impurities.

Acknowledgements. — We would like to thank F. Lœser and C. Sabbah for many helpful discussions on complex analytic geometry.

1. The Floquet-Bloch reduction

In this section, we recall the Floquet-Bloch reduction of a periodic operator on $\mathbb{R}^n$ to a family of Schrödinger operators on a n-torus. We will follow the exposition of SKRIGANOV [Sk].

On $\mathbb{R}^n$ we consider the following Hamiltonian :

$$H = -\Delta + V(x),$$

where V is a real multiplicative potential which is T-periodic for some lattice T in $\mathbb{R}^n$, *i.e.* :

$$V(x+\tau) = V(x), \qquad \forall \tau \in T.$$

We will assume that V is $-\Delta$ bounded with relative bound strictly less than 1, so that H is self adjoint with domain $H^2(\mathbb{R}^n)$.

We denote by T^* the dual lattice of T, which is defined as follows : if $(a_1, \dots, a_n)$ is a basis for T, a basis for T^* is given by the $(b_1, \dots, b_n)$ such that $\langle a_i, b_j \rangle = 2\pi\delta_{ij}$, where $\langle\ ,\ \rangle$ is the Euclidean scalar product on $\mathbb{R}^n$.

We denote by F_T a fundamental domain of T, F_{T^*} a fundamental domain of T^* which are chosen to be diffeomorphic to the n-torus $\mathbb{T}^n$. μ_T (resp. μ_{T^*}) will be the Lebesgue measure of F_T (resp. F_{T^*}).

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing C^∞ functions, and for $p \in F_{T^*}$ we set :

$$K_p\varphi(x) = \mu_T^{1/2} \sum_{\tau \in T} \varphi(x+\tau) e^{i\langle p, x+\tau\rangle}. \tag{1.1}$$

The sum in (1.1) is convergent because of the rapid decay of φ and $K_p\varphi$ is T-periodic and satisfies the equations :

$$K_{p+p'}\varphi(x) = e^{i\langle p', x\rangle} K_p\varphi(x) \quad \text{for } p' \in T^*. \tag{1.2}$$

The family of operators K_p gives a unitary operator W_T :

$$\begin{aligned} L^2(\mathbb{R}^n) &\longrightarrow \mathcal{L} = \int_{F_{T^*}}^{\oplus} L^2(F_T)\,dp \\ \varphi &\longmapsto K_p\varphi(x). \end{aligned}$$

Then since V is T periodic, it is well known that we can decompose H as a direct integral of operators :

$$W_T H W_T^{-1} = \int_{F_{T^*}}^{\oplus} H_p\,dp$$

where $H_p = (D_x + p)^2 + V(x)$, with domain $H^2(F_T)$. Here $D_{x_i} = (1/i)(\partial/\partial x_i)$. W_T^{-1} is defined by :

$$W_T^{-1}\varphi_p(x) = (2\pi)^{-n/2}(\mu_{T^*})^{-1/2}\int_{F_{T^*}} e^{-i\langle p,x\rangle}\varphi_p(x)dp. \tag{1.3}$$

With this version of the Floquet-Bloch reduction, the p dependence of the reduced operator is in the operator itself, and not in the domain.

We denote by $H_{0,p} = (D_x + p)^2$ the free reduced operator with domain $H^2(F_T)$. It is an easy exercise to check that V is also $-\Delta$ bounded with relative bound strictly less than 1 if we consider V and $-\Delta$ as operators on $L^2(F_T)$, so that H_p is self-adjoint with domain $H^2(F_T)$.

For $p \in F_T^*$, we will denote by $K_p^{-1} : L^2(F_T) \to L^2_{\mathrm{loc}}(\mathbb{R}^n)$ the operator :

$$u \longmapsto e^{-i\langle p,x\rangle}u(x), \tag{1.4}$$

where $u(x)$ is extended to $\mathbb{R}^n$ by T-periodicity.

2. The resolvent of the reduced operator

In this section we will write down some formulas for the resolvent $(H_p - \lambda)^{-1}$ using Fredholm theory. These formulas have been used for example by Wilcox (see [W]) to construct families of Bloch eigenfunctions. We will assume that p varies in a bounded open set W in $\mathbb{C}^n$, which will be chosen later. We start with an elementary lemma :

LEMMA 2.1. — *For any $\varepsilon > 0$, there exist $C_0 \gg 1$, such that :*

i) $\|-\Delta(H_{0,p} + C_0)^{-1}\| \le 1 + \varepsilon$ *uniformly for* $p \in W$ *;*

ii) $\|(H_{0,p} + C_0)^{-1}\| \le \varepsilon$ *uniformly for* $p \in W$.

Proof. — $H_{0,p}$ is diagonalized on the orthonormal basis $\{\varphi_{\tau^*}\}_{\tau^* \in T^*}$, where $\varphi_{\tau^*}(x) = (\mu_T)^{-1/2}e^{i\langle\tau^*,x\rangle}$, with eigenvalues $(\tau^* + p)^2$. So

$$\|-\Delta(H_{0,p} + C_0)^{-1}\| = \sup_{\tau^* \in T^*} \tau^{*2}\big(|(\tau^* + p)^2 + C_0|\big)^{-1}.$$

Since $\mathrm{Re}((\tau^* + p)^2 + C_0) = (\tau^* + \mathrm{Re}\, p)^2 - (\mathrm{Im}\, p)^2 + C_0$, we can choose C_0 large enough such that :

$$\big|(\tau^* + p)^2 + C_0\big| \ge (1 - \varepsilon)(\tau^*)^2, \quad \text{uniformly for } p \in W$$

which proves i). ii) can be proven similarly. $\square$

PROPOSITION 2.2. — *There exist $C_0 \gg 1$ such that $(H_p + C_0)^{-1}$ exist and is uniformly bounded from $L^2(F_T)$ into $H^2(F_T)$ for $p \in W$.*

Proof. — Since V is $-\Delta$ bounded with relative bound $a < 1$, we get :

$$\big\|V(H_{0,p}+C_0)^{-1}u\big\| \le a\big\|\Delta(H_{0,p}+C_0)^{-1}u\big\| + b\big\|(H_{0,p}+C_0)^{-1}u\big\|$$

for $u \in L^2(F_T)$. Using LEMMA 2.1, we can choose $C_0 \gg 1$ such that $\|V(H_{0,p}+C_0)^{-1}\| \le \tilde{a} < 1$, uniformly for $p \in W$.

Using the second resolvent formula, we get :

$$H_p + C_0 = \big(\mathbf{1} + V(H_{0,p}+C_0)^{-1}\big)(H_{0,p}+C_0).$$

$\mathbf{1} + V(H_{0,p}+C_0)^{-1}$ can be inverted by a Neumann serie, which proves the Proposition.

LEMMA 2.3. — *$R_p = (H_p + C_0)^{-1}$ belongs to the Schatten class $\mathcal{F}_k$ for $k > \frac{1}{2}n$ for $p \in W$.*

Proof. — Since $(H_p + C_0)^{-1}(-\Delta + 1)$ is bounded for $p \in W$, it suffices to show that $(-\Delta+1)^{-1}$ belongs to $\mathcal{F}_k$ for $k > n/2$, which is obvious since the eigenvalues of $(-\Delta+1)^{-1}$ are the $(1+\tau^{*2})^{-1}$ for $\tau^* \in T^*$. □

We can now prove the following theorem :

THEOREM 2.4. — *$(H_p - \lambda)^{-1}$ can be written for $p \in W$, $\lambda \in \mathbb{C}$ as :*

$$(H_p - \lambda)^{-1} = \frac{D(p,\lambda)}{f(p,\lambda)}$$

where $D(p,\lambda)$ (resp. $f(p,\lambda)$) is holomorphic for $p \in W$, $\lambda \in \mathbb{C}$ as a bounded operator from $L^2(F_T)$ in $H^2(F_T)$ (resp. as a function).

Note that $f(p,\lambda)$ is of course not unique, but the only important object is the zero set of f in $W \times \mathbb{C}$, called the complex Bloch variety, which is uniquely determined by the Hamiltonian.

Proof. — From the first resolvent formula, we have :

$$H_p - \lambda = (H_p + C_0)(\mathbf{1} + \mu R_p) \quad \text{for } \mu = -(C_0 + \lambda).$$

So it suffices to get an expression for $(\mathbf{1} + \mu R_p)^{-1}$. We use the theory of regularized determinants. (See [Si].) Let, for $N > \frac{1}{2}n$,

$$\widetilde{R}(p,\mu) = (\mathbf{1} + \mu R_p)\exp\Big(-\sum_{1}^{N-1}(-\mu R_p)^k/k\Big) - \mathbf{1}.$$

To invert $\mathbf{1} + \mu R_p$, it clearly suffices to invert $\mathbf{1} + \widetilde{R}(p,\mu)$. $\widetilde{R}(p,\mu)$ is holomorphic in $(p,\mu) \in W \times \mathbf{C}$ with values in $\mathcal{F}_1$. Then $f(p,\mu) =$

$\sum_{n=0}^{\infty} D_n(p,\mu)$, where $D_n(p,\mu)$ is a polynomial expression of the $\mathrm{Tr}(\widetilde{R}^k(p,\mu))$ for $1 \le k \le n$ satisfying :

$$|D_n(p,\mu)| \le \left(\frac{e}{n}\right)^{n/2} \|\widetilde{R}\|_{\mathrm{Tr}}^n,$$

where $\|\widetilde{R}\|_{\mathrm{Tr}}$ is the trace norm of $\widetilde{R}$ which is bounded by $C(\mu)\|R_p\|_N$, where $\| \ \|_N$ is the norm in $\mathcal{F}_N$. (See for example [S].)

It follows that $f(p,\mu)$ is a holomorphic function of $(p,\mu) \in W \times \mathbb{C}$ as a uniformly convergent series of holomorphic functions.

If $f(p,\mu) \ne 0$, then $\mathbf{1} + \widetilde{R}(p,\mu)$ is invertible and $(\mathbf{1} + \widetilde{R}(p,\mu))^{-1}$ can be written as $D_1(p,\mu)/f(p,\mu)$, where $D_1(p,\mu)$ is of the form :

$$D_1(p,\mu) = \sum_0^{+\infty} D_{1,m}(p,\mu),$$

and $D_{1,m}(p,\mu)$ is a polynomial expression of the $\widetilde{R}^k$, $\mathrm{Tr}(\widetilde{R}^k)$ for $1 \le k \le m$, satisfying :

$$\|D_{1,m}\| \le C^m \frac{1}{m^{m/2}} \|\widetilde{R}\|_{\mathrm{Tr}}^m.$$

As before we see that $D_1(p,\mu)$ is holomorphic for $p \in W$, $\lambda \in \mathbb{C}$ as a bounded operator on $L^2(F_T)$, which proves the theorem. $\square$

PROPOSITION 2.5. — *$f(p,\lambda)$ and $D(p,\lambda)$ satisfy the following properties :*

i) $f(p+\tau^*,\lambda) = f(p,\lambda)$ $\forall \tau^* \in T^*$, $p \in W$, $\lambda \in \mathbb{C}$;

ii) *if u_{τ^*} is the multiplication operator by $e^{i\langle \tau^*, x\rangle}$*, $D(p+\tau^*,\lambda) = u_{\tau^*} D(p,\lambda) u_{\tau^*}^{-1}$, $\forall \tau^* \in T^*$, $p \in W$, $\lambda \in \mathbb{C}$.

Proof.

i) Note first that since T and T^* are dual lattices, $u_{\tau^*}(H^2(F_T)) = H^2(F_T)$. It is then a trivial computation to check that :

$$H_{p+\tau^*} = u_{\tau^*} H_p u_{\tau^*}^{-1}. \tag{2.2}$$

From this, we get $R_{p+\tau^*} = u_{\tau^*} R_p u_{\tau^*}^{-1}$, $\widetilde{R}(p+\tau^*,\mu) = u_{\tau^*}\widetilde{R}(p,\mu)u_{\tau^*}^{-1}$, and finally $f(p+\tau^*,\lambda) = f(p,\lambda)$ since the trace is invariant by unitary conjugation. This proves i).

ii) follows directly from (2.2) and i). $\square$

3. Analytic extension of $(H-\lambda)^{-1}$

In this section, we will prove the existence of an analytic extension for the total resolvent $(H-\lambda)^{-1}$. As explained in the introduction, the singularities of $(H-\lambda)^{-1}$ are different when we consider the *local extension* of $(H-\lambda)^{-1}$ in a small neighborhood of $\lambda_0 \in \sigma(H)$, and when we consider the *global extension* of $(H-\lambda)^{-1}$ to a bounded open set $\mathcal{U}$ in $\mathbb{C}$.

In the local case, $(H-\lambda)^{-1}$ extends holomorphically as a bounded operator between some weighted L^2-spaces to the universal covering of $V \setminus \Sigma$, where V is a neighborhood of λ_0 and Σ is a discrete set of points in $\mathcal{U}$, called *Landau resonances.*

In the global case, in addition to Σ, $(H-\lambda)^{-1}$ can have singularities on a closed set Σ_∞, which corresponds to a kind of complex essential spectrum.

We will prove some properties of $(H-\lambda)^{-1}$ near Σ, like finite determination and moderate growth, which show that $(H-\lambda)^{-1}$ is a (operator-valued) function of the Nilsson class.

Using results of Leray and Pham, we will then study the behavior of $(H-\lambda)^{-1}$ near generic points of Σ and get asymptotic expansions which show that generally a Landau resonance is a branch point rather than a pole.

Let us first fix notations and prove some formulas.

We fix a bounded open set $\mathcal{U}$ in $\mathbb{C}$ intersecting with $\{\operatorname{Im}\lambda > 0\}$ and a bounded open set $\mathcal{W}$ in $\mathbb{C}^n$ such that $F_{T^*} \subset \mathcal{W}$. From THEOREM 2.4 and the results recalled in Section I, we see that, for $\operatorname{Im}\lambda > 0$, $\lambda \in \mathcal{U}$, $(H-\lambda)^{-1}$ can be written as :

$$(3.1) \qquad (H-\lambda)^{-1}u = c(n,T)\int_{F_{T^*}} e^{-i\langle p,x\rangle}\frac{D(p,\lambda)}{f(p,\lambda)}\Big(\sum_{\tau\in T} u(x+\tau)e^{i\langle p,x+\tau\rangle}\Big)\,dp.$$

Here $c(n,T) = (2\pi)^{-n/2}\mu_T^{1/2}\mu_{T^*}^{-1/2}$. We can rewrite (3.1) as :

$$(3.2) \qquad (H-\lambda)^{-1} = \int_{F_{T^*}}^{\oplus} \frac{M(p,\lambda)}{f(p,\lambda)}\,dp.$$

For $a \in \mathbb{R}$, $k \in \mathbb{N}$, we will denote by $L^2_a(\mathbb{R}^n)$ and $H^k_a(\mathbb{R}^n)$ the spaces

$$L^2_a(\mathbb{R}^n) = \big\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \mid e^{a\langle x\rangle}u \in L^2(\mathbb{R}^n)\big\},$$
$$H^k_a(\mathbb{R}^n) = \big\{u \in H^k_{\mathrm{loc}}(\mathbb{R}^n) \mid e^{a\langle x\rangle}u \in H^k(\mathbb{R}^n)\big\}.$$

Here $\langle x\rangle = (1+x^2)^{1/2}$. $L^2_a(\mathbb{R}^n)$ and $H^k_a(\mathbb{R}^n)$ are Hilbert spaces when equipped with their natural norms.

We have the following proposition :

PROPOSITION 3.1. — *If $a > \sup_{p\in W} |\operatorname{Im} p|$, $M(p,\lambda)$ originally defined for $\operatorname{Im}\lambda > 0$, $p \in F_{T^*}$, extends holomorphically to $\mathcal{W}\times\mathcal{U}$ as a bounded operator from $L^2_a(\mathbb{R}^n)$ into $H^2_{-a}(\mathbb{R}^n)$, such that*

$$M(p+\tau^*,\lambda) = M(p,\lambda), \qquad \forall \tau^* \in T^*.$$

Proof. — Using THEOREM 2.4, it is enough to prove that K_p (defined in (1.1)) extends holomorphically to $p\in\mathcal{W}$ as a bounded operator from $L^2_a(\mathbb{R}^n)$ into $L^2(F_T)$ and that K_p^{-1} (defined in 1.4) extends holomorphically to $p\in\mathcal{W}$ as a bounded operator from $H^2(F_T)$ into $H^2_{-a}(\mathbb{R}^n)$.

If $a > \sup_{p\in W}|\operatorname{Im} p|$, we have $|e^{-a\langle x\rangle}e^{i\langle p,x\rangle}| \le e^{-\varepsilon\langle x\rangle}$ for $p\in W$, and it is clear that K_p is holomorphic and bounded from $L^2_a(\mathbb{R}^n)$ into $L^2(F_T)$.

If we consider now the function $e^{-i\langle p,x\rangle}u(x)$ for $u\in H^2(F_T)$, extended by periodicity in x as a function of $H^2_{\mathrm{loc}}(\mathbb{R}^n)$, it is easy to see that $\|e^{-i\langle p,x\rangle}u\|_{H^2_{-a}(\mathbb{R}^n)} \le C\|u\|_{H^2(F_T)}$ and that K_p^{-1} is an holomorphic operator from $H^2(F_T)$ into $H^2_{-a}(\mathbb{R}^n)$. This proves the first part of the Proposition. To prove the last part, we use the fact that $K_{p+\tau^*} = u_{\tau^*}K_p$, $(K_{p+\tau^*})^{-1} = K_p^{-1}u_{\tau^*}^{-1}$. □

We will now use the T^* periodicity of M and f to reduce ourselves to the case when the integration chain in (3.2) is an absolute cycle. By a linear change of coordinates in p, we may assume that $F_{T^*} = [0,2\pi]^n$.

We introduce now the variables $\theta_j = e^{ip_j}$ for $j=1,\dots,n$. Since $M(p,\lambda)$ and $f(p,\lambda)$ are 2π periodic in $p_1,\dots,p_n$, we can write $M(p,\lambda) = \widetilde{M}(\theta,\lambda)$, $f(p,\lambda) = \tilde{f}(\theta,\lambda)$, where $\widetilde{M}$ and $\tilde{f}$ are holomorphic for $(\theta,\lambda)\in\mathcal{W}\times\mathcal{U}$, and $\mathcal{W}$ can be chosen of the form :

$$\mathcal{W} = \Big\{(\theta_1,\dots,\theta_n)\in\mathbb{C}^n\,;\quad a_j \le |\theta_j| \le a_j^{-1},\ a_j<1\Big\}.$$

The integration chain in (3.2) is now the n-torus $\mathbb{T}^n = S^1\times\cdots\times S^1$ in $\mathbb{C}^n$, which has no boundary. By changing the notations, we can rewrite (3.2) as :

$$(3.3) \qquad (H-\lambda)^{-1} = \int_{T^n}\frac{M(\theta,\lambda)}{f(\theta,\lambda)}d\theta_1\wedge\ldots\wedge d\theta_n, \quad \text{for } \operatorname{Im}\lambda > 0.$$

To extend $(H-\lambda)^{-1}$ is a well known problem in complex analysis. It is well known that the integral (3.3) is holomorphic as long as the integration cycle $\mathbb{T}^n$ can be deformed continuously avoiding the singular set $\{f(\theta,\lambda) = 0\}$. The other requirement is that the integration cycle in θ stays in a neighborhood of $\mathbb{T}^n$ where $M(\theta,\lambda)$ is a bounded operator. The singularities of $(H-\lambda)^{-1}$ will come from the following two types of obstructions :

a) *pinching singularities* : the cycle is pinched between some components of the singular set;

b) *pinching singularities at infinity* : the cycle is pinched between a component of the singular set and the boundary of the holomorphy domain of $M(\theta,\lambda)$.

This is the basis of the analysis of Landau singularities in quantum field theory (*see* [F–F–L–P], [B–P], [P]), of ramifications of holomorphic integrals (see [K]) and of functions of the Nilsson class (see [Me]).

We first introduce some notations. We will denote by S the complex Bloch variety $S = \{(\theta,\lambda) \in \mathcal{W} \times \mathcal{U} \mid f(\theta,\lambda) = 0\}$ and for $\lambda \in \mathcal{U}$ by S_λ the complex Fermi surface $S_\lambda = \{\theta \in \mathcal{W} \mid (\theta,\lambda) \in S\}$.

S is a complex analytic set and has a natural stratification with strata consisting of smooth submanifolds. (See [P, Chapter IV].)

Moreover, since the basis $\mathcal{U}$ is one dimensional, there exists a stratification of S satisfying Thom A_π condition, (see [Hil]). In the sequel, we will always consider such a stratification.

We choose a real analytic map $\delta : \overline{\mathcal{W}} \to [0,1]$ such that (the image of) $\mathbb{T}^n$ is given by $\{\theta \mid \delta(\theta) = 0\}$, $\partial\mathcal{W} = \delta^{-1}(1)$, $\mathcal{W} = \delta^{-1}([0,1[)$, and $B_r = \{0 \le \delta(\theta) \le r\}$ is an increasing sequence of neighborhoods of $\mathbb{T}^n$.

If M is a stratum of S in $\mathcal{W}\times\mathcal{U}$, M_λ is a union of smooth submanifolds for $\lambda \in \mathcal{U}$. This is obvious if $d\pi|_{TM} \neq 0$, and follows from Thom A_π condition if $d\pi|_{TM} = 0$. For $\lambda \in \mathcal{U}$, we denote by $D(\lambda)$ the set of $r \in]0,1]$, such that for some stratum M of S, M_λ is tangent to ∂B_r.

Definition 3.2.

i) $\Sigma \subset \mathcal{U}$ is the union of the $\pi(\overline{M})$ for each stratum M of S in $\mathcal{W}\times\mathcal{U}$ such that $d\pi|_{TM} = 0$;

ii) a point $\lambda_0 \in \mathcal{U}$ does not belong to Σ_∞ if there exists a neighborhood V of λ_0 in $\mathcal{U}$, such that for any $r_0 \in]0,1[$, $\exists r > r_0$ with $r \notin D(\lambda)$ $\forall \lambda \in V$.

We can say roughly that $\lambda \in \Sigma$ if the Fermi surface S_λ has a singularity. Indeed if $\lambda \in \mathcal{U}\backslash\Sigma$, S_λ is a union of smooth submanifolds, since all strata of S are transversal to the fibers. We call the points of Σ *Laudau resonances.* Σ corresponds to obstruction a). Σ_∞ corresponds to obstruction b). Arguing as in the proof of Proposition 3.3 below, one can show that

Σ_∞ is included in the set $\{\lambda \in \mathcal{U} \mid 1 \in D(\lambda)\}$.

Using the arguments of [Ge], one can prove that Σ_∞ is included in a *subanalytic set of measure zero.* Of course Σ and Σ_∞ can intersect the upper halfplane $\{\operatorname{Im} \lambda > 0\}$. We have the following Proposition :

PROPOSITION 3.3.

i) Σ *is a finite set of points;*

ii) $\forall \lambda \in \mathcal{U}$, $D(\lambda)$ *is a finite subset of* $]0,1]$;

iii) $K = \cup_{\lambda \in \mathcal{U}} \{\lambda\} \times D(\lambda)$ *is closed in* $\mathcal{U} \times]0,1]$.

Proof. — Let us first prove i). If M is a stratum of S such that $d\pi|_{TM} = 0$, for each $(\theta_0, \lambda_0) \in \overline{M}$ we can find a C^1 path $\gamma : [0, \varepsilon_0[\to \overline{M}$ such that $\gamma(0) = (\theta_0, \lambda_0)$, $\gamma(t) \in M$ $\forall t > 0$. Then $d(\pi \circ \gamma(t)) \equiv 0$, which shows that γ (and $\overline{M}$ par connexity) projects on λ_0. Then i) follows from the fact that the number of strata of S in $\mathcal{W} \times \mathcal{U}$ is finite.

We prove now ii). For $\lambda \in \mathcal{U}$, M a stratum of S, we consider the critical variety $cM_\lambda = \{\theta \in M_\lambda \mid d\delta \mid T_\theta M_\lambda = 0\}$. $c\overline{M}_\lambda$ is a real analytic set. If $\theta_0 \in c\overline{M}_\lambda$, $r_0 = \delta(\theta_0)$, we can apply the curve selection lemma (see [Mi]) to get an analytic path $\gamma : [0, \varepsilon_0[\to c\overline{M}_\lambda$ such that $\gamma(0) = \theta_0$, $\gamma(t) \in cM_\lambda$ $\forall t > 0$. If we consider the path $\delta \circ \gamma(t) in]0,1]$, we have : $d(\delta \circ \gamma(t)) \equiv 0$ since $\gamma(t) \in cM_\lambda$ $\forall t > 0$. So $\delta \circ \gamma(t) = r_0$. Since $c\overline{M}_\lambda$ has a finite number of connected components in $\mathcal{W}$, we get that $D(\lambda)$ is a finite set.

To prove iii), we take $\lambda_0 \in \mathcal{U}$, $r_0 \in]0,1]$ and assume that there exists a sequence $(\lambda_n, r_n) \to (\lambda_0, r_0)$, $\lambda_n \in \mathcal{U}, r_n \in D(\lambda_n)$.

Then we can find a stratum M of S, a sequence $\theta_n \in M_{\lambda_n}$, such that $T_{\theta_n} M_{\lambda_n} \subset T_{\lambda_n} \partial B_{r_n}$. By compactness, we can assume that θ_n tends to some $\theta_0 \in \partial B_{r_0}$, such that $(\theta_0, \lambda_0) \in \overline{M}$. If $(\theta_0, \lambda_0) \in M$, it is clear that $r_0 \in D(\lambda_0)$.

If $(\theta_0, \lambda_0) \in N$, for some stratum N adjacent to M, then using Thom A_π condition, $T_{\theta_0} N_{\lambda_0} \subset T_{\theta_0} \partial B_{R_0}$, and $r_0 \in D(\lambda_0)$, which proves that K is closed. □

We can now state the main result of this Section.

THEOREM 3.4.

i) (local extension problem) : *for any* $\lambda_0 \in \mathcal{U} \cap \mathbb{R}$, *there exists a neighborhood* V *of* λ_0 *in* $\mathcal{U}$, *such that* $(H - \lambda)^{-1}$ *extends holomorphically from* $\{\operatorname{Im} \lambda > 0\} \cap V$ *to the universal covering* $(V \setminus \Sigma)^*$ *of* $V \setminus \Sigma$, *as a bounded operator from* $L^2_a(\mathbb{R}^n)$ *into* $H^2_a(\mathbb{R}^n)$ *for* $a > \operatorname{Sup}_{p \in \mathcal{W}} |\operatorname{Im} p|$;

ii) (global extension problem) : $(H - \lambda)^{-1}$ *extends holomorphically from* $\{\operatorname{Im} \lambda > 0\} \cap \mathcal{U}$ *to the universal covering* $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$ *of* $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)$ *as a bounded operator from* $L^2_a(\mathbb{R}^n)$ *into* $H^2_a(\mathbb{R}^n)$ *for* $a > \operatorname{Sup}_{p \in \mathcal{W}} |\operatorname{Im} p|$.

Proof. — Let us first prove ii). Since $\omega = (M(\theta, \lambda)/f(\theta, \lambda)) d\theta_1 \wedge \ldots \wedge d\theta_n$

is holomorphic of maximal degree outside of S, $\int_\gamma \omega$ depends only on the homology class $[\gamma]$ of γ in $H_n(\mathcal{W})$.

We consider the problem of extending $(H-\lambda)^{-1} = \int_{T^n} \omega$ from $\operatorname{Im}\lambda > 0$ to some point λ_1 along a path $\ell : [0,1] \to \mathcal{U} \setminus \Sigma \cup \Sigma_\infty$, with $\operatorname{Im}\ell(0) > 0$, $\ell(1) = \lambda_1$. Note that for $\theta \in T^n$ and $\operatorname{Im}\lambda > 0$, $f(\theta,\lambda)$ is non zero, since $H_p - \lambda$ is invertible there by the selfadjointness of H_p for $\theta \in \mathbb{T}^n$. $(H-\lambda)^{-1}$ can be extended holomorphically along ℓ if there exists a continuous deformation γ_t of $\mathbb{T}^n$ such that $\gamma_t \subset \mathcal{W}$ and $\gamma_t \cap S_{\ell(t)} = \phi$.

Let $\mathcal{I}$ be the set of $t \in [0,1]$ such that $\mathbb{T}^n$ can be deformed along ℓ between 0 and t satisfying the above conditions. $\mathcal{I}$ is obviously open and since $[0,1]$ is connected it suffices to prove that $\mathcal{I}$ is closed to prove that $\mathcal{I} = [0,1]$.

Let $t_n \in \mathcal{I}$ be a sequence with $t_n \to t_0$ when $n \to +\infty$. Since $\lambda_1 = \ell(t_0) \in \mathcal{U} \setminus \Sigma \cup \Sigma_\infty$, there exists a small neighborhood V of λ_1 such that $V \cap \Sigma = \phi$ and $\forall r_0 \in]0,1[$, $\exists r > r_0$ with $r \notin D(\lambda)$ $\forall \lambda \in V$.

We can take a point $\ell(t_n) \in V$ for n big enough, and a diameter $r_0 \in]0,1[$ such that the cycle γ_{t_n} is included in B_{r_0}.

Then we can apply the local isotopy lemma of [F–K] to $\bar{B}_r \times V$, where V is given as above : $\bar{B}_r \times V \to V$ is a locally trivial fibration with respect to S, which provides the deformation of γ_t along the part of ℓ which stays in V.

Hence $\mathcal{I} = [0,1]$ and $(H-\lambda)^{-1}$ can be extended along any path in $\mathcal{U} \setminus \Sigma \cup \Sigma_\infty$. Then it follows from the monodromy Theorem that $(H-\lambda)^{-1}$ extends as a function on $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$, which proves ii).

Let us now prove i). From PROPOSITION 3.3 i) and ii), we can find some $r_0 > 0$, some neighborhood V of λ_0 in $\mathcal{U}$, such that $r_0 \notin D(\lambda)$ $\forall \lambda \in V$. Then the result follows by applying the arguments above to $\bar{B}_{r_0} \times V$. This proves the Theorem.

Let us remark that one can choose the neighborhood $\mathcal{W}$ of $\mathbb{T}^n$ such that $\Sigma \cap \Sigma_\infty = \phi$. Indeed by PROPOSITION 3.3, one can find some $r_0 > 0$, some neighborhoods V_i of the $\lambda_i \in \Sigma$ such that $r_0 \notin D(\lambda)$ $\forall \lambda \in V_i$. Then we just have to replace $\mathcal{W}$ by B_{r_0}.

It is important to notice that $\Sigma \cup \Sigma_\infty$ is a *maximal set* of obstructions to the analytic extension of $(H-\lambda)^{-1}$. It can happen that some branch of $(H-\lambda)^{-1}$ has a smaller singular set than $\Sigma \cup \Sigma_\infty$.

We will now study some growth and ramification properties of $(H-\lambda)^{-1}$.

- We will say that an operator valued function $M(\lambda)$ is of finite determination near some point λ_0 if there exists a neighborhood V of λ_0 such that the branches of $M(\lambda)$ over any simply connected subset of $V \setminus \{\lambda_0\}$ span a vector space of finite dimension in $\mathcal{L}(L^2_a(\mathbb{R}^n), H^2_{-a}(\mathbb{R}^n))$.

• We will say that $M(\lambda)$ is of *moderate growth* at a point $\lambda_0 \in \Sigma$, if there exists a neighborhood V of λ_0 in $\mathcal{U}$, such that for any simply connected subset $\widetilde{V}$ of $V \setminus \{\lambda_0\}$, for any branch of $M(\lambda)$ on $\widetilde{V}$, denoted by $\widetilde{M}(\lambda)$, there exist C_0, $N_0 > 0$ such that :

$$\|\widetilde{M}(\lambda)\| \leq C_0|\lambda - \lambda_0|^{-N_0} \quad \text{for } \lambda \in \widetilde{V}.$$

Here $\|\widetilde{M}(\lambda)\|$ is the operator norm in $\mathcal{L}(L^2_a(\mathbb{R}^n), H^2_{-a}(\mathbb{R}^n))$.

We first recall a definition. An analytic set S in $\mathcal{W} \times \mathcal{U}$ is called a *divisor* if near any point $(\theta_0, \lambda_0) \in S$, there exist holomorphic coordinates $(z_1, \ldots, z_{n+1})$ such that S is given near (θ_0, λ_0) by the equation $z_1 \cdots z_k = 0$.

To study the growth of $(H-\lambda)^{-1}$ near a point of Σ, we need a geometric hypothesis on S. We first add to S the fiber $\pi^{-1}(\lambda_0)$ for each $\lambda_0 \in \Sigma$, which does not change the set Σ.

Hironaka desingularization theorem says that there exist an analytic space X and a proper morphism $\beta : X \to \mathcal{W} \times \mathcal{U}$ such that :

- $\beta : X \setminus \beta^{-1}(S) \to \mathcal{W} \times \mathcal{U} \setminus S$ is an isomorphism;
- $S' = \beta^{-1}(S)$ is a divisor (see [Hi2]).

We make the same hypothesis than in Mercier [Me] :

(T) *There exists a stratification (M') of the pair (X, S') such that for each stratum M' of (M') there exists a stratum M of $(\mathcal{W} \times \mathcal{U}, S)$ such that $\beta : M' \to M$ is a submersion.*

This hypothesis is made to retain transversality of the strata of S' to $\beta^{-1}(\partial B_{r_0} \times \mathcal{U})$ after the desingularisation process. We have the following result :

Theorem 3.5.

i) $(H - \lambda)^{-1}$ *is of finite determination near any point of* Σ ;

ii) *if condition (T) holds,* $(H - \lambda)^{-1}$ *is of moderate growth near any point of* Σ.

Proof. — Since the properties of finite determination and moderate growth are local, we can consider $(H - \lambda)^{-1}$ near a point $\lambda_1 \in \Sigma$. Let us consider a branch of $(H-\lambda)^{-1}$ near λ_1, obtained by analytic continuation along a path ℓ in $\mathcal{U} \setminus \Sigma \cup \Sigma_\infty$. Since $\lambda_1 \notin \Sigma_\infty$, there exists a neighborhood V of λ_1, and an $r_0' > 0$ such that $\pi : \overline{B}_{r_0'} \times V \setminus \{\lambda_1\} \to V \setminus \{\lambda_1\}$ is a locally trivial fibration and the continuation of $(H - \lambda)^{-1}$ in $(V \setminus \{\lambda_1\})^*$ is obtained by deforming the integration cycle inside $\overline{B}_{r_0'}$.

We claim that there exist some $r_0 > r_0'$ such that $r_0 \notin D(\lambda)$, $\forall \lambda \in V$ and such that the following condition holds :

(T′) *Any stratum of S intersecting $\partial B_{r_0} \times V$ intersect it transversally.*

Indeed arguing as in PROPOSITION 3.3, we see that for any stratum M of S in $\mathcal{W} \times V$ the set of critical values of $\delta \mid M$ is finite.

Let $0 < r_{\max} < 1$ be the maximum of all critical values of $\delta \mid M$ for all strata M of S in $\mathcal{W} \times V$. Since $\lambda_1 \notin \Sigma_\infty$, we can find $r_0 > r_{\max}$, $r_0 > r_0'$ such that $r_0 \notin D(\lambda)$, $\forall \lambda \in V$. Let λ_0 a point in $V \setminus \{\lambda_1\}$, and $r_0 \in]0,1[$ as above.

We introduce the locally finite family of analytic sets given by S_{λ_0} and ∂B_{r_0}. By Lojasiewicz Theorem (see [Me]), we can find a semi-analytic triangulation of $\bar{B}_{r_0}$ which is finite and compatible with this family. This induces a triangulation of $\bar{B}_{r_0} \setminus S_{\lambda_0}$ by K, where K is the simplicial complex made of the simplexes of the previous triangulation which do not intersect S_{λ_0}.

Then $H_{m-1}(\bar{B}_{r_0} \setminus S_{\lambda_0})$ is isomorphic to $H_{m-1}(|K|)$ and we can write $[\gamma_0]$ using the simplexes of the triangulation of

$$|K| : [\gamma_0] = \sum_{j \in J_0} b_j \sigma_j, b_j \in \mathbb{Z},$$

J_0 a finite set.

For λ near λ_0, we can also take $[\gamma_\lambda] = \sum_{j \in J_0} b_j \sigma_j$. Then $(H-\lambda)^{-1} = \sum_{j \in J_0} b_j \int_{\sigma_j} \omega$. Here $\omega = G(\theta, \lambda) d\theta_1 \wedge \ldots \wedge d\theta_n$ is of finite determination on each of the simply connected sets σ_j. So $(H-\lambda)^{-1}$ is a finite sum of the functions $\int_{\sigma_j} \omega$, each of finite determination, which proves i).

We prove now ii) : we consider a point $\lambda_0 \in \Sigma$. Let us denote by $\beta : X \to \mathcal{W} \times \mathcal{U}$ the desingularisation of S in $\mathcal{W} \times \mathcal{U}$. We can write $(H-\lambda)^{-1} = \int_{\gamma_\lambda} \omega = \int_{\beta_* \gamma_\lambda'} \omega = \int_{\gamma_\lambda'} \beta^* \omega$, where $\gamma_\lambda' = (\beta^{-1})_* \gamma_\lambda$. (Here γ_λ' exist because β is an isomorphism outside S.) Using condition (T), we see that $\beta^{-1}(\partial B_{r_0} \times \mathcal{U})$ is transversal to all strata of S'. So denoting again S' by S, we are reduced to the case where $\pi^{-1}(\lambda_0) \subset S$, S is a divisor, and $\partial B_{r_0} \times \mathcal{U}$ is transversal to all strata of S.

We fix a small neighborhood V of λ_0, in which we will estimate the growth of $(H-\lambda)^{-1}$. We can now finish the proof as in [Me].

We will only indicate the principal steps of the proof. Modulo a change of coordinates, we can assume that $\lambda_0 = 0$. The idea of the proof is to lift the radial vector field on $\mathbb{C} : \zeta = -(\lambda \partial/\partial\lambda + \bar{\lambda}\partial/\partial\lambda) = -r\partial/\partial r$ where

$r = |\lambda|$, to a vector field ξ in $\mathcal{W} \times V$ which is tangent to all strata of S and to $\partial B_{r_0} \times V$. This vector field is then used to construct the deformations of γ_λ as λ tends to 0 along an integral curve of ζ, which is of the form : $\alpha_s(\lambda_1) = \lambda_1 e^{-s}$ for $s \in \mathbb{R}^+$.

Step 1 : *Construction of* ξ.

We will construct ξ locally near any point of $B_{r_0} \times V$ and patch together the local vector fields with a partition of unity.

1) Near $(\theta, \lambda) \in \bar{B}_{r_0} \times V \setminus (S \cup \partial B_{r_0} \times V)$, we take :

$$\xi = \Big(\lambda \frac{\partial}{\partial \lambda} + \bar{\lambda} \frac{\partial}{\partial \bar{\lambda}}\Big).$$

2) Near $(\theta, \lambda) \in \bar{B}_{r_0} \times V \cap (S \setminus \partial B_{r_0} \times V)$: since S is a divisor, we can take a coordinate chart near (θ, λ), $(z_1, \ldots, z_{n+1})$ such that S is given near (θ, λ) by the equation $z_1 \cdots z_p = 0$, and $(\theta, \lambda) = (0, \ldots, 0)$. Since $\pi^{-1}(0) \subset S$, by the Nullstellensatz (see [Me, p. 82]) we see that $\pi(z) = r_0(z) z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$, $r_0(0) \neq 0$, $k \leq p$. Changing for example z_1, we can assume that $\pi(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. We can take $\xi = -1/\alpha_1 (z_1 \partial/\partial z_1 + \bar{z}_1 \partial/\partial \bar{z}_1)$, and we have $\pi_* \xi = \zeta$.

3) Near $(\theta, \lambda) \in \bar{B}_{r_0} \times V \cap (S \cap \partial B_{r_0} \times V)$: as before, we can find a coordinate chart such that S has the equation $z_1 \cdots z_p = 0$, and $\pi(z) = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$.

Since ∂B_{r_0} is transversal to each stratum of S, we can extend the set of local coordinates $(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_p, \operatorname{Im} z_p)$ by $u_{p+1}, v_{p+1}, \ldots, u_{n+1}, v_{n+1}$ such that $u_{p+1} = 0$ is a C^∞ equation of ∂B_{r_0} near (θ, λ). We take $\xi = -1/\alpha_1 (z_1 \partial/\partial z_1 + \bar{z}_1 \partial/\partial \bar{z}_1)$. ξ is tangent to S and to ∂B_{r_0}, and $\pi_* \xi = \zeta$.

4) Near $(\theta, \lambda) \in \bar{B}_{r_0} \times V \cap (\partial B_{r_0} \times U \setminus S)$: we can take $\xi = -(\lambda \partial/\partial \lambda) + \bar{\lambda} \partial/\partial \bar{\lambda}$. ξ is tangent to ∂B_{r_0}. We now patch together ξ with a C^∞ partition of unity in $\bar{B}_{r_0+\varepsilon} \times V_\varepsilon$, where V_ε, where V_ε is a small neighborhood of V. We obtain a vector field supported in $\bar{B}_{r_0+\varepsilon} \times V_\varepsilon$.

Step 2 : *Estimates of* $(H - \lambda)^{-1}$.

We want now to control the growth of $(H - \lambda)^{-1} = \int_{\gamma_\lambda} G(\theta, \lambda) d\theta_1 \wedge \ldots \wedge d\theta_n$, when λ tends to 0 along a ray $\alpha_s(\lambda_1) = \lambda_1 e^{-s}$.

Here for $\lambda \in V \setminus \{\lambda_0\}$, γ_λ is a deformation of the cube γ_{λ_1} which stays in the ball $\bar{B}_{r_0}$. The integral curves of the vector field ξ induce a 1-parameter family of diffeomorphisms $j_s = \exp(s\xi)$, from $(\bar{B}_{r_0} \setminus S_{\lambda_1}) \times \{\lambda_1\}$ into $(\bar{B}_{r_0} \setminus S_{\alpha_s(\lambda_1)}) \times \{\alpha_s(\lambda_1)\}$. We use here the fact that ξ is tangent to ∂B_{r_0} and S, and that $\pi_* \xi$ is the radial vector field pointing inwards. Moreover

j_s is a homeomorphism from the pair

$$\bar{B}_{r_0} \setminus S_{\lambda_1} \quad \text{to the pair} \quad \bar{B}_{r_0} \setminus S_{\alpha_s(\lambda_1)}.$$

Since $H_{m-1}(\bar{B}_{r_0} \setminus S_\lambda)$ is a locally constant sheaf over V, (using for example the triangulation in the proof of i)), we see that $[\gamma_{\alpha_s(\lambda_1)}] = (j_s)_*[\gamma_{\lambda_1}]$. So we can write :

$$\left(H - \alpha_s(\lambda_1)\right)^{-1} = \int_{(js)_*\gamma_{\lambda_1}} \omega = \int_{\gamma_{\lambda_1}} j_s^* \omega.$$

To prove ii) it remains to show that $\|j_s^*\omega\| \leq Ce^{cs}$, where $\| \ \|$ is the norm induced by the Riemanian structure on $\mathcal{W}$, uniformly on $\bar{B}_{r_0}$ and when λ_1 is on some arc of circle, $\{\theta_0 \leq \operatorname{Arg}\lambda_1 \leq \theta_1, |\lambda_1| = \varepsilon_0\}$. This can be done as in the proof of [Me, Theorem 3.1], with the modifications of [Me, Theorem 3.2]. $\square$

Behavior of the resolvent near a Landau resonance in generic cases.

We now give asymptotic expansions of $(H-\lambda)^{-1}$ near some points of Σ. We consider the case of Landau resonances $\lambda_0 \in \Sigma$ generated by a pinch of the integration cycle at a point $\theta_0 \in \mathcal{W}$.

We make the following hypotheses :

(3.7) *Near (θ_0, λ_0) S is a union of complex hypersurfaces $S_1, \dots, S_k$, intersecting in general position at (θ_0, λ_0).*

This means that near (θ_0, λ_0) S_i has a irreducible equation $s_i(\theta, \lambda) = 0$, with $s_i(\theta_0, \lambda_0) = 0$, $ds_1, \dots, ds_k$ linearly independent at (θ_0, λ_0).

(3.8) *(θ_0, λ_0) is a non degenerate critical point of π for the stratum $A = \bigcap_{i=1}^k S_i$, and is not a critical point of any other stratum of S.*

We denote by σ the absolute cycle on which we integrate $(M(\theta,\lambda)/f(\theta,\lambda))$ for λ near λ_0, $\lambda \neq \lambda_0$ to obtain one of the branches of $(H-\lambda)^{-1}$ near λ_0. Explicitely we will study one branch of $(H-\lambda)^{-1}$ near λ_0 defined by :

$$(H-\lambda)_\sigma^{-1} = \int_\sigma \frac{M(\theta,\lambda)}{f(\theta,\lambda)} d\theta_1 \wedge \ldots \wedge d\theta_n.$$

We will denote by N the intersection index of σ with the vanishing cell defined by the manifolds $S_1, \dots, S_k$ (See [P, Chapter V]). N is zero if σ is not pinched by the $S_1, \dots, S_k$ when λ tends to λ_0, in which case we expect that this branch of $(H-\lambda)^{-1}$ will have no singularities at λ_0.

Using the hypotheses above, we can write for (θ,λ) near (θ_0,λ_0), $(M(\theta,\lambda)/f(\theta,\lambda))$ as

$$\frac{\widetilde{M}(\theta,\lambda)}{s_1^{\alpha_1}(\theta,\lambda)\dots s_k^{\alpha_k}(\theta,\lambda)}$$

with $\alpha_1,\dots,\alpha_k\in\mathbb{N}$, $\alpha_i\geq 1$, $\alpha=\alpha_1+\cdots+\alpha_k$. We have the following Theorem :

THEOREM 3.6. — *Under the hypotheses* (3.7), (3.8), $(H-\lambda)_\sigma^{-1}$ *can be written for* λ *near* λ_0 *as :*

i) *if* $n+k$ *is even :*

$$(H-\lambda)_\sigma^{-1}=E_0(\lambda)+C_0N(\lambda-\lambda_0)^{((n+k-1)/2)-\alpha}\\ \times\big(M(\theta_0,\lambda_0)+(\lambda-\lambda_0)E_1(\lambda)\big)\ ;$$

ii) *if* $n+k$ *is odd,* $n+k\geq 2\alpha+1$:

$$(H-\lambda)_\sigma^{-1}=E_0(\lambda)+C_0N(\lambda-\lambda_0)^{(n+k-1)/2-\alpha}\\ \times\operatorname{Log}(\lambda-\lambda_0)\big(M(\theta_0,\lambda_0)+(\lambda-\lambda_0)E_1(\lambda)\big)\ ;$$

iii) *if* $n+k$ *is odd,* $n+k<2\alpha+1$:

$$(H-\lambda)_\sigma^{-1}=E_0(\lambda)+C_0N(\lambda-\lambda_0)^{(n+k-1)/2-\alpha}\\ \times\big(M(\theta_0,\lambda_0)+(\lambda-\lambda_0)E_1(\lambda)\big)+N\operatorname{Log}(\lambda-\lambda_0)E_2(\lambda)\ ;$$

iv) *if* $n=k-1$:

$$(H-\lambda)_\sigma^{-1}=E_0(\lambda)+C_0N(\lambda-\lambda_0)^{n-\alpha}\big(M(\theta_0,\lambda_0)+(\lambda-\lambda_0)E_1(\lambda)\big)$$

where : E_0,E_1,E_2 *are holomorphic functions in* $\mathcal{L}(L_a^2(\mathbb{R}^n),H_{-a}^2(\mathbb{R}^n))$*, and* C_0 *is a non-vanishing constant.*

Proof. — We use the results of LERAY [L] and PHAM [P] as stated in the book of Pham (see [P, Chapter VI]).

We first reduce ourselves to a case when $\widetilde{M}(\theta,\lambda)$ is independent of λ. To do this, we use the fact (see [P, Section V.2]) that under the hypotheses (4.7), (4.8), there exist a neighborhood of (θ_0,λ_0) still denoted by $W\times V$ and a holomorphic change of coordinates defined on $W\times V:(\theta,\lambda)\mapsto(\tilde\theta(\theta,\lambda),\lambda)$ such that in the new coordinates $(\tilde\theta,\lambda)$ the functions $s_1,\dots,s_k$ take the simple form :

$$s_1=\lambda-\tilde\theta_1+\cdots+\tilde\theta_{k-1}+\tilde\theta_k^2+\cdots+\tilde\theta_{n-1}^2=\lambda-f(\tilde\theta)\ ;$$
$$s_2=\tilde\theta_1\ ;\ \dots\ ;\ s_k=\tilde\theta_{k-1}.$$

For simplicity, we will still denote by (θ,λ) the new coordinates. We can also assume that the cycle σ is contained in W, by substracting to $(H-\lambda)^{-1}$ some operator holomorphic in λ. This can be done as in the proof of VI.2.1 in [P]. Using Taylor's formula, we can write :

$$
\begin{aligned}
(3.9)\quad & \frac{\widetilde{M}(\theta,\lambda)}{s_1^{\alpha_1}s_2^{\alpha_2}\cdots s_k^{\alpha_k}} \\
&= \sum_{0\le\beta\le\alpha_1-1} \frac{1}{\beta!}\partial_\lambda^\beta \widetilde{M}(\theta,\lambda)|_{\lambda=f(\theta)} s_1^{-\alpha_1+\beta}s_2^{-\alpha_2}\cdots s_k^{-\alpha_k} \\
&\qquad\qquad + \widetilde{R}(\theta,\lambda)s_2^{-\alpha_2}\cdots s_k^{-\alpha_k} \\
&= \sum_{0\le\beta\le\alpha_1-1} \widetilde{M}_\beta(\theta)s_1^{-\alpha_1+\beta}s_2^{-\alpha_2}\cdots s_k^{-\alpha_k} + \widetilde{R}(\theta,\lambda)s_2^{-\alpha_2}\cdots s_k^{-\alpha_k}.
\end{aligned}
$$

From (3.8), it follows that (θ_0,λ_0) is not a critical point of π for the stratum $S_2\cap\cdots\cap S_k$, hence $\int_\sigma \widetilde{R}(\theta,\lambda)s_2^{-\alpha_2}\cdots s_k^{-\alpha_k}d\theta_1\wedge\ldots\wedge d\theta_n$ is holomorphic near λ_0.

So we are reduced to the study of $\int_\sigma \widetilde{M}_\beta(\theta)s_1^{-\alpha_1+\beta}s_2^{-\alpha_2}\cdots s_k^{-\alpha_k}d\theta_1\wedge\ldots\wedge d\theta_n$ for $\beta\le\alpha_1-1$. Then the theorem follows directly by applying VI.2.1 in [P] to each of the terms in (3.9.). In (3.9) only the first term corresponding to $\widetilde{M}(\theta,f(\theta))$ contributes to the leading singularity at $\lambda=\lambda_0$. The only thing that we have to check is that some constant appearing in the formulas VI.2.1 of [P] is non zero. More precisely we can write :

$$d\lambda=\sum_{i=1}^k a_i ds_i \quad \text{at } (\theta_0,\lambda_0).$$

Let us check that $a_i\neq 0$ for $i=1,\ldots,k$. If for example $a_1=0$, then (θ_0,λ_0) would be an element of the critical manifold of the stratum $S_2\cap\ldots\cap S_k$, which is excluded by (3.8). This concludes the proof of the Theorem. □

COROLLARY 3.7. — *Assume that $k=\alpha=1$. Then $M(\theta_0,\lambda_0)$ is a rank one operator π_0, with for $u\in L_a^2(\mathbb{R}^n)$: $\pi_0 u=\langle u,\tilde\varphi_0\rangle\varphi_0$, where if $\theta_0=e^{ip_0}$*

- $\varphi_0\in H^2_{-a}(\mathbb{R}^n)$ *is a p_0-Floquet periodic solution of :*

$$(H-\lambda_0)\varphi_0=0$$

- $\tilde\varphi_0\in H^2_{-a}(\mathbb{R}^n)$ *is a $\bar p_0$-Floquet periodic solution of :*

$$(H-\bar\lambda_0)\tilde\varphi_0=0.$$

Proof. — If $k = \alpha = 1$, λ_0 is a simple eigenvalue of H_{p_0}, so $(H_{p_0} - \lambda)^{-1}$ can be written as : $E_0(\lambda) + (\tilde{\pi}_0)/(\lambda - \lambda_0)$, where $\tilde{\pi}_0$ is a rank one operator with : $\tilde{\pi}_0 u = \langle u, \tilde{\psi}_0 \rangle \psi_0$ for $u \in L^2(F_T)$. It is well known that ψ_0 (resp. $\tilde{\psi}_0$) are solutions of $(H_{p_0} - \lambda_0)\psi_0 = 0$ (resp. $(H_{\bar{p}_0} - \bar{\lambda}_0)\tilde{\psi}_0 = 0$).

From (3.2), it follows that $M(\theta_0, \lambda_0)$ can be written as : $M(\theta_0, \lambda_0)u = C_0\langle u, \tilde{\varphi}_0 \rangle \varphi_0$, where :

$$\varphi_0(x) = e^{-i\langle p_0, x\rangle}\psi_0(x) \qquad \tilde{\varphi}_0(x) = e^{-i\langle \bar{p}_0, x\rangle}\tilde{\psi}_0(x).$$

(Here we extend $\psi_0, \tilde{\psi}_0$ to $\mathbb{R}^n$ by T-periodicity.) This proves the Corollary. □

Remark 3.8. — $M(\theta_0, \lambda_0)$ is always a finite rank operator, but can be equal to zero in some cases. When $M(\theta_0, \lambda_0) \neq 0$, THEOREM 3.6 shows that the leading singularity of $(H - \lambda)^{-1}$ at λ_0 is of finite rank. However $E_1(\lambda)$, $E_2(\lambda)$ are not necessarily of finite rank.

4. Applications

In this section, we present some applications of the results of Section 3.

In the first subsection, we study the relation of real Landau resonances with the band structure of the spectrum of H.

In the second subsection, we prove that the density of states is analytic outside the real Landau resonances.

In the last subsection, we study the resonances created by localized impurities.

a. Relation of Landau resonances with the band structure.

Let us first recall some well known facts about the band structure of $\sigma(H)$, (see [Re–Si]).

If $f(p, \lambda)$ is the determinant defined in THEOREM 2.4, the eigenvalues of H_p for $p \in F_{T^*}$ are the roots $\lambda = E_n(p)$, $n \in \mathbb{N}$ of the equation $f(p, \lambda) = 0$. The n-th band of H is the set $B_n = \bigcup_{p \in F_{T^*}} \{E_n(p)\}$.

In space dimension greater than two, it is well known that the bands can overlap, (see for example [Sk]).

- We will say that a band B_n is simple if $B_n \cap B_m = \phi$, $\forall m \neq n$.
- We will say that two bands B_n and B_m *overlap effectively* if $\exists p \in F_{T^*}$ such that $E_n(p) = E_m(p)$.
- We will say that two bands B_n and B_m *overlap artificially* if $B_n \cap B_m \neq \phi$ but $E_n(p) \neq E_m(p)$ $\forall p \in F_{T^*}$.

If B_n is simple or overlaps only artificially with other bands, it is well known that the functions $E_n(p)$ are holomorphic in a small complex neighborhood of F_{T^*}.

• We will say that $\lambda_0 \in B_n$ is a *critical energy*, if λ_0 is a critical value of $E_n(p)$. λ_0 will be called *non degenerate* if it is associated with non degenerate critical points of $E_n(p)$.

In particular, the extremities of a simple band are critical energies.

We have the following Theorem :

THEOREM 4.1. — *Let B_n be a simple band. Then :*

i) *$\Sigma \cap B_n$ is the set of critical energies in B_n, denoted by Σ_n ;*

ii) *if λ_0 is a non degenerate critical energy in B_n associated with a unique critical point in F_{T^*}, λ_0 is a true singularity of $(H-\lambda)^{-1}$.*

If B_n and B_m overlap artificially, $\Sigma \cap (B_n \cup B_m) = \Sigma_n \cup \Sigma_m$.

Proof. — i) is obvious. To prove ii), we use THEOREM 3.6. It suffices to prove that $(H-\lambda)^{-1}$ is multivalued near λ_0, hence to show that the intersection index N between the cycle F_{T^*} and the vanishing cell of $S_1 = \{(p,\lambda) \mid \lambda = E_n(p)\}$ at (p_0, λ_0) is non zero, if p_0 is the critical point of λ_0. Using Morse lemma, we can find complex coordinates $(y_1, \dots, y_n)$ near p_0 such that $y_i(p_0) = 0$, $i = 1, \dots, n$, and

$$E_n(y) = \lambda_0 + \sum_{i=1}^{n}, y_i^2, \quad \text{for } y \text{ near } 0.$$

The cycle F_{T^*} is transformed in some cycle $\mathbb{R}^k \times i\mathbb{R}^{n-k}$, with k depending on the index of the critical point p_0. In these coordinates, it is easy to check that N is non zero.

The last statement is obvious since the two hypersurfaces $\{\lambda = E_n(p)\}$ and $\{\lambda = E_m(p)\}$ do not intersect for λ near λ_0, p near F_{T^*}, so no additional Landau resonances are created. $\square$

COROLLARY 4.2. — *Let B_n a simple band such that $E_n(p)$ is a simple eigenvalue of H_p for $p \in F_{T^*}$, and let $\lambda_0 \in B_n$ a non degenerate critical energy associated with a unique critical point p_0. Then if n is odd :*

$$(H-\lambda)^{-1} = E_0(\lambda) + C_0(\lambda-\lambda_0)^{n/2-1}\big(K_0 + (\lambda-\lambda_0)E_1(\lambda)\big)$$

for λ near λ_0 ; if n is even :

$$(H-\lambda)^{-1} = E_0(\lambda) + C_0(\lambda-\lambda_0)^{n/2-1}\operatorname{Log}(\lambda-\lambda_0)\big(K_0 + (\lambda-\lambda_0)E_1(\lambda)\big).$$

for λ near λ_0. Here E_0, $E_1(\lambda)$ are holomorphic operators in $\mathcal{L}(L^2_a(\mathbb{R}^n), H^2_{-a}(\mathbb{R}^n))$ and K_0 is a rank one operator having the kernel : $k_0(x,y) = \varphi_0(x)\bar{\varphi}_0(y)$, where φ_0 is a p_0-Floquet periodic solution of $(H-\lambda_0)\varphi_0 = 0$.

Proof. — This follows directly from THEOREM 3.6 and COROLLARY 3.7.

Remark 4.3. — Corollary holds also when λ_0 is associated with several non degenerate critical points, if we add the contributions from the various critical points. It is also interesting to remark that for $n = 2, 3, (H - \lambda)^{-1}$ has exactly the same singularity at λ_0 as a usual two-body Schrödinger operator at $\lambda = 0$.

We have seen in THEOREM 4.1 that artificial overlapping of bands does not create new singularities. The problem of effectively overlapping bands is difficult to treat in general since the geometry of the Bloch variety can be quite complicated. We will just discuss the simplest case of band overlapping making some generic assumptions. We will consider two bands B_n and B_{n+1} which overlap effectively at λ_0. Then there exists $p_0 \in F_{T^*}$ such that $f(p_0, \lambda_0) = (\partial f/\partial\lambda)(p_0, \lambda_0) = 0$.

The simplest case is when $(\partial^2 f/\partial\lambda^2) \neq 0$. Then using Weierstrass preparation Theorem, we can write :

$$f(p, \lambda) = c(p, \lambda)\big((\lambda - a(p)^2 - b(p)\big) \quad \text{for } (p, \lambda) \text{ near } (p_0, \lambda_0),$$

where $c(p_0, \lambda_0) \neq 0$, $a(p_0) = \lambda_0$, $b(p_0) = 0$. Since the roots of $f(p, \lambda) = 0$ are real for $p \in F_{T^*}$, we see that $a(p)$ and $b(p)$ are real for p near p_0, and moreover that $b(p) \geq 0$ for p near p_0. Again the simplest situation is when $b(p_0) = \nabla_p b(p_0) = 0$, and $B = \frac{1}{2}[\partial^2_{p_i p_j} b(p_0)]$ is positive definite.

Then the Bloch variety $S = \{(p, \lambda) \mid (\lambda - a(p))^2 - b(p) = 0\}$ has two strata near (p_0, λ_0) : $M = S \setminus \{(p_0, \lambda_0)\}$ and $N = \{(p_0, \lambda_0)\}$. If $\langle B\nabla_p a(p_0), \nabla_p a(p_0)\rangle \neq 1$, one sees easily using implicit function theorem that $cM = \phi$. On the other hand, $cN = N$ so $\lambda_0 \in \Sigma$. It is quite easy to see (using that the set of selfadjoint matrices with a double eigenvalue has codimension 3) that this hypothesis is sensible only if $n = 3$, *i.e.* the physical case. Then an easy computation using Morse lemma with parameters shows that (under some generic assumptions) $(H - \lambda)^{-1}$ is actually analytic near λ_0.

b. The density of states. — We study now the analyticity properties of the density of states associated with H. The density of states measure ρ is defined as follows (see [Re–Si]) : let Q be a n-cube in $\mathbb{R}^n$ and $\mathbf{1}_Q$ the characteristic function of Q. Then

$$\rho(]-\infty, \lambda]) = \lim_{\mathrm{Vol}(Q)\to+\infty} \frac{2}{\mathrm{Vol}(Q)} \operatorname{Tr}\big(\mathbf{1}_Q E_{]-\infty,\lambda]}\big).$$

Here E_Ω is the spectral projection of H on Ω, for any borelian Ω. Using the translation invariance of H and of the E_Ω, it is very easy to see that :

$$\rho(]-\infty, \lambda]) = \frac{2}{\mu_T} \operatorname{Tr}\big(X_T E_{]-\infty,\lambda]}\big), \tag{4.1}$$

where X_T is the characteristic function of the fundamental domain F_T, and μ_T is the Lebesgue measure of F_T.

Using the fact that the spectrum of H is absolutely continuous, it is easy to show that ρ is absolutely continuous with respect to the Lebesgue measure $d\lambda$.

Then the density of states of H is the Radon-Nikodym derivative of ρ, denotes by $(d\rho/d\lambda)$.

For simplicity, we will assume that the space dimension n is equal to 2 or 3. We have the following Theorem :

THEOREM 4.4. — *Assume in addition to the hypotheses of* Section I *that $\partial_i V$ and $\partial^2_{ij} V$ are bounded from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.* (*See* Appendix *for the precise meaning of this condition*). *Then the density of states $(d\rho/d\lambda)$ is analytic on $\mathbb{R}\setminus\Sigma$.*

Proof. — For $c \in \mathbb{R}$, $c > \inf\sigma(H) + 1$, we consider the following function :

$$f(\lambda) = \operatorname{Tr}\big(X_T(H+c)^{-1}E_{[\lambda_0,\lambda]}\big) = \operatorname{Tr}\big(X_T(H+c)^{-1}E_{[\lambda_0,\lambda]}X_T\big).$$

Here λ_0 and λ belong to the same connected component of $\mathbb{R}\setminus\Sigma$, denoted by I_{λ_0}. Then we have :

$$f(\lambda) = \frac{1}{2}\int_{\lambda_0}^{\lambda}(s+c)^{-1}\frac{d\rho}{ds}\,ds.$$

To prove the theorem, it suffices to prove that $f(\lambda)$ is analytic in I_{λ_0}. If $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$, we have :

$$f(\lambda) = \sum_{i=0}^{\infty}\big((H+c)^{-1}E_{]\lambda_0,\lambda]}X_Te_i, X_Te_i\big) = \sum_{i=0}^{\infty} g_i(\lambda),$$

where the series $\sum_{i=0}^{\infty}|g_i(\lambda)|$ is convergent for each $\lambda\in[\lambda_0,+\infty[$.

Using Stone formula and the absolute continuity of the spectrum of H, we have :

$$g_i(\lambda) = \lim_{\varepsilon\to 0^+}\frac{1}{2i\pi}\int_{\lambda_0}^{\lambda}\big((H+c)^{-1}\big(R(s+i\varepsilon)-R(s-i\varepsilon)\big)X_Te_i, X_Te_i\big)\,ds.$$

From the appendix PROPOSITION A-1, we know that we can choose $c \gg 1$ big enough so that $(H+c)^{-1}$ is bounded from $H^2_{-a}(\mathbb{R}^n)$ into $H^4_{-a}(\mathbb{R}^n)$. So we have :

$$g_i(\lambda) = \frac{1}{2i\pi}\int_{\lambda_0}^{\lambda}\big((H+c)^{-1}\big(R(s+i0)-R(s-i0)\big)X_Te_i, X_Te_i\big)\,ds.$$

We use here the fact that $(H-\lambda)^{-1}$ extends holomorphically as a bounded operator from $L^2_a(\mathbb{R}^n)$ into $H^2_{-a}(\mathbb{R}^n)$, from the upper and lower halfplanes to a small complex neighborhood of $s \in \mathbb{R}$, if $s \notin \Sigma$. We fix a small complex neighborhood $\mathcal{U}_0$ of I_{λ_0}, such that the extensions of $(H-\lambda)^{-1}$ from the upper and lower halfplanes, denoted by $R_\pm(\lambda)$, are holomorphic in $\mathcal{U}_0$.

We consider now the operator :

$$A_\pm(\lambda) = X_T(H+c)^{-1}R_\pm(\lambda)X_T \qquad \text{for } \lambda \in \mathcal{U}_0.$$
$$A_\pm(\lambda) = X_T e^{a\langle x\rangle}(\Delta+i)^{-2}(\Delta+i)^2 e^{-a\langle x\rangle}(H+c)^{-1}R_\pm(\lambda)X_T.$$

From THEOREM 3.4 and PROPOSITION A.1, we get that

$$(\Delta+i)^2 e^{-a\langle x\rangle}(H+c)^{-1}R_+(\lambda)X_T$$

is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Then $X_T e^{a\langle x\rangle}(\Delta+i)^{-2}$ is trace class (see [Re–Si]), so $A_\pm(\lambda)$ is trace class with :

$$(4.2) \qquad \big\|A_\pm(\lambda)\big\|_{\mathrm{Tr}} \le C_1 \quad \text{uniformly for } \lambda \text{ in a compact subset of } \mathcal{U}_0.$$

It follows from (4.2) that the series $\sum_{i=0}^\infty g_i(\lambda)$ is convergent for each λ in a compact subset K of $\mathcal{U}_0$, and that $\sum_{i=0}^\infty |g_i(\lambda)| \le C_1$, uniformly on K. Using for example Lebesgue dominated convergence theorem, we get that the series $\sum_{i=0}^\infty g_i(\lambda)$ converges in $D'(K)$, so that $f(\lambda) = \sum_{i=0}^\infty g_i(\lambda)$ is holomorphic in K. $\square$

c. Perturbation by localized impurities. — We will now study the resonances created by impurities.

We assume that the effect of a localized impurity can be described by an additional real potential decaying exponentially.

The time dependent theory of impurity scattering has been treated by THOMAS (see [T]). An approach closer to ours has been used by BENTOSELA in [B], where a time independent theory of impurity scattering is developped. In particular, BENTOSELA proves that for artificially overlapping bands, the resolvent $(H-\lambda)^{-1}$ extends to the non-physical sheet outside the critical energies of the band, which is contained in our THEOREM 4.1.

We will assume that the effect of localized impurities is described by a real potential W such that :

$$(4.3) \qquad \exists \alpha > 0 \quad \text{such that } e^{\alpha\langle x\rangle}W(\Delta+i)^{-1} \text{ is compact.}$$

We will denote by $\widetilde{H}$ the Hamiltonian $H+W$ with domain $H^2(\mathbb{R}^n)$. Then we have the following Theorem :

THEOREM 4.5.

i) (local extension problem) : *for any* $\lambda_0 \in \mathcal{U} \cap \mathbb{R}$, *there exists a neighborhood* V *of* λ_0 *in* $\mathcal{U}$, *such that* $(\widetilde{H} - \lambda)^{-1}$ *extends meromorphically from* $\{\operatorname{Im}\lambda > 0\} \cap V$ *to the universal covering* $(V \setminus \Sigma)^*$ *of* $V \setminus \Sigma$, *as a bounded operator from* $L^2_a(\mathbb{R}^n)$ *into* $H^2_{-a}(\mathbb{R}^n)$ *for* a *small enough with poles in* $(V \setminus \Sigma)^*$ *having finite rank residues ;*

ii) (global extension problem) : $(\widetilde{H} - \lambda)^{-1}$ *extends meromorphically from* $\{\operatorname{Im}\lambda > 0\} \cap \mathcal{U}$ *to the universal covering* $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$ *of* $\mathcal{U} \setminus \Sigma \cup \Sigma_\infty$ *as a bounded operator from* $L^2_a(\mathbb{R}^n)$ *into* $H^2_{-a}(\mathbb{R}^n)$ *for* a *small enough with poles in* $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$ *having finite rank residues.*

We will call *resonances* of $\widetilde{H}$ the poles of $(\widetilde{H} - \lambda)^{-1}$ in $(V \setminus \Sigma)^*$ or $(\mathcal{U} \setminus \Sigma \cup \Sigma_\infty)^*$, and denote by Γ the set of these resonances.

Proof. — Let us prove i), (ii) can be proved similarly). We use the second resolvent formula. For $\operatorname{Im}\lambda > 0$, we have :

$$(\widetilde{H} - \lambda)^{-1} = (H - \lambda)^{-1}\big(\mathbf{1} + W(H - \lambda)^{-1}\big)^{-1} = (H - \lambda)^{-1}\big(\mathbf{1} + \widetilde{K}(\lambda)\big)^{-1}.$$

We can choose a small enough in THEOREM 3.4 such that $(H - \lambda)^{-1}$ extends analytically to $(V \setminus \Sigma)^*$ as an operator from $L^2_a(\mathbb{R}^n)$ into $H^2_{-a}(\mathbb{R}^n)$ and such that $W(H - \lambda)^{-1}$ is compact on $L^2_a(\mathbb{R}^n)$.

So, $\widetilde{K}(\lambda)$ is holomorphic in $(V \setminus \Sigma)^*$ and compact, and $(\mathbf{1} + \widetilde{K}(\lambda))$ is invertible for $\operatorname{Im}\lambda \gg 1$, using a Neumann serie. The Theorem follows then from the analytic Fredholm theorem. $\square$

COROLLARY 4.6. — *Let us denote by* $\Gamma_{\mathbb{R}} = \Gamma \cap \overline{\{\operatorname{Im}\lambda > 0\}}, \Sigma_{\mathbb{R}} = \Sigma \cap \mathbb{R}$. *Then :*

i) $\sigma_{pp}(\widetilde{H}) \subset \Gamma_{\mathbb{R}} \cup \Sigma_{\mathbb{R}}, \Gamma_{\mathbb{R}} \subset \sigma_{pp}(\widetilde{H})$ *;*

ii) $\sigma_{sc}(\widetilde{H})$ *is empty and the eigenvalues of* $\widetilde{H}$ *can accumulate only at the points of* $\Sigma_{\mathbb{R}}$.

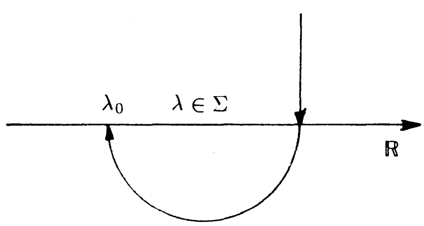

Figure 4.1

Before proving this corollary, let us notice that it may happen that a "real" pole λ_0 of $(\widetilde{H}-\lambda)^{-1}$ is not an eigenvalue of $\widetilde{H}$, if λ_0 is obtained by continuing $(\widetilde{H}-\lambda)^{-1}$ along a path encircling a point of Σ, (see Figure 4.1). This phenomenon is well known in two-body Schrödinger operators.

Proof. — To prove i) we use an idea of BALSLEV–COMBES ([B–C]). We have the following formula, if $d\widetilde{E}_\lambda$ denotes the spectral measure of $\widetilde{H}$:

$$\widetilde{E}_{]-\infty,\lambda]} - \widetilde{E}_{]-\infty,\lambda[} = s \lim_{\substack{z\to\lambda \\ \operatorname{Im} z>0}} (z-\lambda)(\widetilde{H}-\lambda)^{-1}.$$

Let $\lambda_0 \in \sigma_{pp}(\widetilde{H})$. Then $\widetilde{E}_{]-\infty,\lambda_0]} - \widetilde{E}_{]-\infty,\lambda_0[} = \widetilde{E}_{\{\lambda_0\}}$ is non zero. Since $L^2_a(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can find some $\varphi \in L^2_a(\mathbb{R}^n)$, such that

$$\big(\widetilde{E}_{\{\lambda_0\}}\varphi,\varphi\big) = \lim_{\substack{z\to\lambda_0 \\ \operatorname{Im} z>0}} \big((z-\lambda_0)(\widetilde{H}-z)^{-1}\varphi,\varphi\big) \neq 0.$$

Then $(\widetilde{H}-z)^{-1}$ must have a singularity at $z=\lambda_0$, so $\lambda_0 \in \Gamma_{\mathbb{R}} \cup \Sigma_{\mathbb{R}}$.

Suppose now that $\lambda_0 \in \Gamma_{\mathbb{R}}$ and that λ_0 is not an eigenvalue of $\widetilde{H}$. Then for any φ_1, $\varphi_2 \in L^2_a(\mathbb{R}^n)$, we have

$$\lim_{\substack{z\to\lambda_0 \\ \operatorname{Im}_z>0}} ((z-\lambda_0)(\widetilde{H}-z)^{-1}\varphi_1,\varphi_2) = 0,$$

which is impossible if we choose φ_1 and φ_2 correctly with respect to the residues in the Laurent expansion of $(\widetilde{H}-z)^{-1}$ at $z=\lambda_0$. This proves i).

Let us now prove ii). THEOREM 4.6 implies that $\sigma_{sc}(\widetilde{H}) \subset \Sigma_{\mathbb{R}} \cup \Gamma_{\mathbb{R}}$. (See for example [Re–Si].) This set is a set of points having only a locally finite set of accumulation points, so $\Sigma_{\mathbb{R}} \cup \Gamma_{\mathbb{R}}$ cannot support a continuous measure, which proves that $\sigma_{sc}(\widetilde{H}) = \phi$. The properties of eigenvalues of $\widetilde{H}$ follows directly from i). $\square$

Remark 4.7. — Using THEOREM 3.6, one can check that in general the singularity of $(\widetilde{H}-\lambda)^{-1}$ at a Landau resonance $\lambda_0 \in \Sigma$ can be an essential singularity, since the singular part of $\widetilde{K}(\lambda)$ at λ_0 is not always of finite rank. However in the case considered in COROLLARY 4.2, one can prove that the number of poles of $(\widetilde{H}-\lambda)^{-1}$ on each sheet of the (local) Riemann surface of $(\widetilde{H}-\lambda)^{-1}$ near λ_0 is finite in a small neighborhood of λ_0.

Appendix

We prove here the continuity result used in the proof of THEOREM 4.4; let us first make precise the meaning of $\partial_i V$ and $\partial^2_{ij} V$.

We can define $\partial_i V = [\partial_{x_i}, V]$ as a quadratic form with domain $C_0^\infty(\mathbb{R}^n)$, or equivalently as an operator between $C_0^\infty(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$. It is easy to see that $\partial_i V$ is bounded from $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ into $H^{-3}_{\mathrm{loc}}(\mathbb{R}^n)$. We ask that this operator extends continuously from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

Similarly we define $\partial^2_{ij} V = [\partial_{x_i}, \partial_j V]$ as an operator between $C_0^\infty(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$. We ask that $\partial^2_{ij} V$ extends continuously from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

PROPOSITION A.1. — *If $\partial_i V$ and $\partial^2_{ij} V$ are bounded from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, $\exists c \gg 1$ such that $(H+c)^{-1}$ is bounded from $H^2_{-a}(\mathbb{R}^n)$ into $H^4_{-a}(\mathbb{R}^n)$.*

Proof. — We write $e^{-a\langle x\rangle}(H+c)e^{a\langle x\rangle} = H_a + c$ where

$$H_a = \big(D_x + ia\nabla\langle x\rangle\big)^2 + V(x) = -\Delta + V_a,$$

where V_a is a first order differential operator. It is easy to see that V_a is $-\Delta$ bounded with relative bound strictly less than 1, so that $(H_a+c)^{-1}$ exists and is bounded from $L^2(\mathbb{R}^n)$ into $H^2(\mathbb{R}^n)$ for $c \gg 1$ big enough. Hence $(H+c)^{-1}$ is bounded from $L^2_{-a}(\mathbb{R}^n)$ into $H^2_{-a}(\mathbb{R}^n)$.

If $w \in H^2_{-a}(\mathbb{R}^n)$ and $u = (H+c)^{-1}w$ we have :

$$(H+c)\partial_{x_i} u = \partial_{x_i} w - \partial_{x_i} V u$$

in the distribution sense. If $\partial_{x_i} V$ is bounded from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, using the fact that V is a multiplication operator, we see easily that $\partial_{x_i} V$ is bounded from $H^2_{-a}(\mathbb{R}^n)$ into $L^2_{-a}(\mathbb{R}^n)$. This prove that $\partial_{x_i} u \in H^2_{-a}(\mathbb{R}^n)$.

Arguing the same way, we can prove that $\partial_{x_i}\partial_{x_j} u \in H^2_{-a}(\mathbb{R}^n)$, which proves the Proposition.

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