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INDUCED $\mathcal{D}$-MODULES AND DIFFERENTIAL COMPLEXES

BY

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Introduction

Let $f : X \to Y$ be a proper morphism of complex manifolds, or smooth algebraic varieties, and $M^\bullet \in D^b_{\text{coh}}(\mathcal{D}_X)$ a bounded complex of $\mathcal{D}_X$-Modules with coherent cohomologies. Then we have the duality isomorphism (cf. also [B, Be, Sc1-2]) :

$$(0.1) \quad f_* \mathbb{D}M^\bullet \simto \mathbb{D}f_* M^\bullet \quad \text{in} \quad D^b_{\text{coh}}(\mathcal{D}_Y),$$

if $\mathcal{H}^j M^\bullet$ have good filtrations (locally on $Y$). Here $\mathbb{D}$ is the dual functor, and $f_*$ is the direct image of $\mathcal{D}$-Modules. For simplicity, assume $X = \mathbb{P}^n$, $Y = \text{pt}$, and $M^j$ are direct sums of $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(p)$. Then we have the isomorphism (0.1) for each $M^j$ by the Serre duality, but it is not completely trivial that the differentials on the both sides of (0.1) commute with the isomorphism, because we have to prove some relation between the duality isomorphism and the action of differential operators, e. g. the action of the global vector fields on $H^n(\mathbb{P}^n, \omega_{\mathbb{P}^n})$ is zero (this can be easily...
checked, if the distributions are used in the Serre duality). In this note we show that the duality isomorphism (0.1) is naturally defined in general by introducing the notion of induced $\mathcal{D}$-Module and differential complexe.

An induced $\mathcal{D}_X$-Module $M$ is a right $\mathcal{D}_X$-Module isomorphic to $L \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for an $\mathcal{O}_X$-Module $L$. Then we have $f_*M \simeq Rf_*L \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ by definition, where $f_*$ denotes the sheaf theoretic direct image, and $f$ is always assumed to be proper. The dual $\mathcal{D}M$ of $M$ is defined by

$$ (0.2) \quad \mathcal{D}M = R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X[d_X]) \otimes_{\mathcal{O}_X} \mathcal{D}_X. $$

This definition coincides with the usual one,

$$ R\mathcal{H}om_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, \omega_X[d_X] \otimes \mathcal{D}_X), $$

if $L$ is coherent. Then the duality isomorphism (0.1) for such $M$ is defined by

$$ (0.3) \quad f_*\mathcal{D}M = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X[d_X]) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y $$

$$ \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*\omega_X[d_X]) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y $$

$$ \overset{\text{Tr}_f}{\longrightarrow} R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, \omega_Y[d_Y]) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = Df_*M $$

using the analytic, or algebraic, trace morphism $\text{Tr}_f : Rf_*\omega_X[d_X] \to \omega_Y[d_Y]$, where $\omega_X = \Omega^1_X$ and $d_X = \dim X$. But we have to still impose some condition on the trace morphism for the compatibility of the morphism (0.3) with the differential of $M$, if $M$ becomes a complex, and this condition is rather difficult to satisfy in the level of complex, cf. 3.14.

This point can be further simplified by using the associated differential complexes.

For $\mathcal{O}_X$-Modules $L$, $L'$ the differential morphisms of $L$ to $L'$ are the image of the injective morphism defined by $\otimes_{\mathcal{D}_X} \mathcal{O}_X$ :

$$ (0.4) \quad \mathcal{H}om_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, L' \otimes \mathcal{D}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(L, L') $$

where the image is denoted by $\mathcal{H}om_{\text{Diff}}(L, L')$. This injectivity means that the morphisms of the induced Modules are completely recovered by the associated differential morphisms. The direct image of $L$ is defined simply by the sheaf theoretic direct image, i.e. the differential morphisms are stable by the direct image. Then the above duality isomorphism (0.3) corresponds to

$$ (0.5) \quad f_*\mathcal{H}om_{\text{Diff}}^{f}(L, \tilde{K}_X) \longrightarrow \mathcal{H}om_{\text{Diff}}^{f}(Rf_*L, f_*\tilde{K}_X) $$

$$ \overset{\text{Tr}_f}{\longrightarrow} \mathcal{H}om_{\text{Diff}}^{f}(Rf_*L, \tilde{K}_Y) $$

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where \( \tilde{K}_X = DR(K_X) \) with \( \omega_X [d_X] \to K_X \) an injective resolution as \( D_X \)-Modules, and \( \mathcal{H}om^f_{\text{Diff}} \) denotes the subsheaf of \( \mathcal{H}om_{\text{Diff}} \) corresponding to

\[
\bigcup_p \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes F_p D_X) \cong \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes D_X)
\]

\[
= \mathcal{H}om_{D_X}(L \otimes D_X, L' \otimes D_X).
\]

Here the trace morphism \( \text{Tr}_f : f_\bullet \tilde{K}_X \to \tilde{K}_Y \) is defined to be compatible with the filtration \( F \) associated to the de Rham functor \( DR \), and \( \text{Gr}^F_0 \text{Tr}_f \) coincides with the above analytic, or algebraic, trace morphism \( \text{Tr}_f : K_X \to K_Y \). Then the compatibility with the differential of \( L \) is clear, if it is defined in \( \mathcal{H}om^f_{\text{Diff}} \). For the proof of the isomorphism we may assume that \( L \) is a coherent \( \mathcal{O}_X \)-Module. Then (0.5) is quasi-isomorphic to the composition:

\[
(0.6) \quad f_\bullet \mathcal{H}om_{\mathcal{O}_X}(L, K_X) \to \mathcal{H}om_{\mathcal{O}_Y}(Rf_\bullet L, f_\bullet K_X) \to \mathcal{H}om_{\mathcal{O}_Y}(Rf_\bullet L, K_Y)
\]

and the assertion is reduced to the dualities in [H], [RRV]. Here \( Rf_\bullet L \) is defined by choosing some canonical \( f_\bullet \)-acyclic resolution of \( L \).

We also show the compatibility of the dual functor with the de Rham functor \( DRD = DDR \) in the holonomic case using the forgetful functor of the differential complexes, where the proof of the isomorphism is same as in [K1]. Then the compatibility of the duality for proper morphisms (0.1) with the topological (i.e. Verdier) duality by the de Rham functor becomes trivial, because \( \text{Tr}_f : \tilde{K}_X \to \tilde{K}_Y \) represents the topological trace morphism \( \text{Tr}_f : Rf_\bullet C_X[2d_X] \to C_Y[2d_Y] \).

These results were first proved in the filtered case in [S1, paragraph 2], where the arguments are sometimes simplified (e.g. the stability of the coherence by the direct image by proper morphisms) due to the existence of the global filtration. In fact, we have to use the filtrered theory to relate the two trace morphisms \( \text{Tr}_f : \tilde{K}_X \to \tilde{K}_Y \) and \( \text{Tr}_f : K_X \to K_Y \) (see 3.7).

Here it should be noted that

\[
F_p \mathcal{H}om_{\text{Diff}}(L, L') := \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes F_p D_X)
\]

the sheaf of the differential morphisms of order \( \leq p \) coincides with \( \text{Diff}_{\mathcal{O}_X}(L, L') \) the differential operators of order \( \leq p \) in the sense of Grothendieck [BO] (see (1.20.2)).

We also introduce the notion of diagonal pairing to simplify some argument in the proof of the fully faithfulness of the Riemann-Hilbert
In paragraph 1 we define the induced $\mathcal{D}$-Modules and the differential complexes, and prove some equivalence of categories to assure the existence of some resolution. In paragraph 2 we define the dual and prove the compatibility with the de Rham functor in the holonomic case. In paragraph 3 we show the duality for proper morphisms and its compatibility with the topological one. In paragraph 4 we explain about the diagonal pairings.

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1. Induced $\mathcal{D}$-Modules and Differential Complexes

1.0. — In this note $X$ denotes a complex manifold, or a smooth algebraic variety, and $\mathcal{D}_X$ the sheaf of holomorphic, or algebraic, differential operators. In the algebraic case, the $\mathcal{O}_X$- (and $\mathcal{D}_X$-) Modules are assumed quasi-coherent (except in paragraph 4). We identify the left and right $\mathcal{D}_X$-Modules by the functor $M \mapsto \Omega_X^{d_X} \otimes_{\mathcal{O}_X} M$, where $d_X := \dim X$. We use mainly the right $\mathcal{D}$-Modules, because they are more convenient to the definition of dual and the proof of the duality (in fact, the induced $\mathcal{D}$-Modules are naturally defined as right $\mathcal{D}$-Modules).

1.1 Definition. — A $\mathcal{D}_X$-Module $M$ is induced, if it is isomorphic to $L \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for an $\mathcal{O}_X$-Module $L$. $M_i(\mathcal{D}_X)$ denotes the additive category of the induced $\mathcal{D}_X$-Modules, which is a full sub category of the abelian category of $\mathcal{D}_X$-Modules $\mathcal{M}(\mathcal{D}_X)$. Then $C_i^b(\mathcal{D}_X)$, $K_i(\mathcal{D}_X)$, $D_i^b(\mathcal{D}_X)$ and $C^b(\mathcal{D}_X)$, etc. (same for $C_-$, $C_+^-$, etc.) are defined as in [VI], where $D^b(\mathcal{D}_X)$, $\mathcal{C}^b(\mathcal{D}_X)$, etc. are obtained by inverting the quasi-isomorphisms in $K^b(\mathcal{D}_X)$ and $\mathcal{C}^b(\mathcal{D}_X)$, etc.

1.2 Lemma. — For induced $\mathcal{D}_X$-Modules $M, N$, we have $M \otimes_{\mathcal{D}_X} \mathcal{O}_X = M \otimes_{\mathcal{D}_X} \mathcal{O}_X$, i.e. $\text{Tor}_i^{\mathcal{D}_X}(M, \mathcal{O}_X) = 0$ for $i \neq 0$, and the natural morphism

\[
\text{Hom}_{\mathcal{D}_X}(M, N) \longrightarrow \text{Hom}_{\mathcal{O}_X}(M \otimes_{\mathcal{D}_X} \mathcal{O}_X, N \otimes_{\mathcal{D}_X} \mathcal{O}_X)
\]

is injective.

Proof. — The first assertion is clear, and the second is shown in [S1, 2.2.2].

1.3 Definition. — For $\mathcal{O}_X$-Modules $L, L'$, we denote by $\text{Hom}_{\mathcal{D}_X}(L, L')$
and call the \textit{differential morphisms} of \( L \) to \( L' \) the image of the injection (1.2.1):

\[
(1.3.1) \quad \mathcal{H}om_{\mathcal{D}_X}(L \otimes_{\mathcal{O}_X} \mathcal{D}_X, L' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \rightarrow \mathcal{H}om_{\mathcal{D}_X}(L, L')
\]

We denote by \( M(\mathcal{O}_X, \text{Diff}) \) the additive category whose objects are the \( \mathcal{O}_X \)-Modules, and morphisms are the differential morphisms. We have a natural functor

\[
(1.3.2) \quad \overline{\mathcal{D}R}^{-1}: M(\mathcal{O}_X, \text{Diff}) \rightarrow M_i(\mathcal{D}_X)
\]

such that \( \overline{\mathcal{D}R}^{-1}(L) = L \otimes_{\mathcal{O}_X} \mathcal{D}_X \), and it is an equivalence of categories by the injectivity of (1.2.1). We define \( C^b(\mathcal{O}_X, \text{Diff}), K^b(\mathcal{O}_X, \text{Diff}) \) as above, and \( D^b(\mathcal{O}_X, \text{Diff}) \) by inverting the \( D \)-quasi-isomorphisms (same for \( D, D^+, \text{etc.} \)). Here a morphism of \( C^b(\mathcal{O}_X, \text{Diff}) \) is called a \( D \)-\textit{quasi-isomorphism}, if its image by \( \overline{\mathcal{D}R}^{-1} \) is a quasi-isomorphism (similar for \( D \)-acyclic). By definition we have the equivalence of categories

\[
(1.3.3) \quad \overline{\mathcal{D}R}^{-1}: D^b(\mathcal{O}_X, \text{Diff}) \sim D^b_i(\mathcal{D}_X)
\]

(same for \( C^b, K^b, \) and \( D, D^+, \text{etc.} \)).

1.4. \textit{Remark.} — The morphism (1.3.1) is induced by the following commutative diagrams for \( P \in \mathcal{H}om_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, L' \otimes \mathcal{D}_X) = \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes \mathcal{D}_X) \):

\[
\begin{array}{ccc}
L \otimes_{\mathcal{O}_X} \mathcal{D}_X & \xrightarrow{P} & L' \otimes_{\mathcal{O}_X} \mathcal{D}_X \\
\downarrow & & \downarrow \\
L' & \xrightarrow{\mathcal{D}R(P)} & L'
\end{array}
\]

\[
\begin{array}{ccc}
L \otimes_{\mathcal{O}_X} \mathcal{D}_X & \xrightarrow{P} & L' \otimes_{\mathcal{O}_X} \mathcal{D}_X \\
\downarrow & & \downarrow \\
L & \xrightarrow{\mathcal{D}R(P)} & L'
\end{array}
\]

where \( \mathcal{D}R(P) \) is the image of \( P \) by (1.3.1), and the vertical morphisms are defined by the tensor over \( \mathcal{O}_X \) of the natural morphism \( \mathcal{D}_X \rightarrow \mathcal{O}_X \) \((Q \mapsto Q1)\).

1.5. \textit{Lemma.} — For \( M \in M(\mathcal{D}_X) \), put

\[
(1.5.1) \quad \overline{\mathcal{D}R}(M) := [0 \rightarrow M \otimes_{\mathcal{O}_X} \Lambda^{d_1} \mathcal{O}_X \rightarrow \cdots \rightarrow M \otimes_{\mathcal{O}_X} \mathcal{O}_X \rightarrow M \rightarrow 0]
\]
where \( \Theta_X \) is the sheaf of holomorphic or algebraic vector fields, and the degree of \( M \) is zero. Choosing coordinates \((x_1, \ldots, x_{d_X})\) and their dual vector fields \( \partial_1, \ldots, \partial_{d_X} \), (1.5.1) is identified with the Koszul complex \( K(M; \partial_1, \ldots, \partial_{d_X})[d_X] \), (cf. [KK] for the intrinsic definition). Then \( \widehat{\text{DR}}(M) \) is a complex of \( M(\mathcal{O}_X, \text{Diff}) \) by the natural \( \mathcal{O}_X \)-Module structure on \( M \otimes \Lambda^i \Theta_X \), and we have a natural quasi-isomorphism of complexes of \( \mathcal{D}_X \)-Modules

\[
\widehat{\text{DR}}^{-1} \widehat{\text{DR}}(M) \to M,
\]

where \( \widehat{\text{DR}}^{-1} M = M \otimes \mathcal{D}_X \to M \) is induced by \( \mathcal{D}_X \to \mathcal{O}_X \) as in 1.4.

**Proof.** — This is a special case of [S1, 2.1.6, 2.2.8], where \( F \) on \( M \) is trivial.

1.6. Remark. — \( \widehat{\text{DR}}^{-1} M = M \otimes \mathcal{O}_X \mathcal{D}_X \) has two structures of right \( \mathcal{D}_X \)-Modules which are interchangeable by the lemma below. We have a natural isomorphism of \( \widehat{\text{DR}}(\widehat{\text{DR}}^{-1} M) \) with \( \widehat{\text{DR}}^{-1} \widehat{\text{DR}}(M) \), and \( \widehat{\text{DR}}(\widehat{\text{DR}}^{-1} M) \) is just the Koszul complex \( K(M \otimes \mathcal{O}_X \mathcal{D}_X; \partial_1, \ldots, \partial_{d_X})[d_X] \).

1.7. Lemma. — For a right \( \mathcal{D}_X \)-Module \( M \), there is a unique involution of \( M \otimes \mathcal{O}_X \mathcal{D}_X \), which induces the identity on \( M = M \otimes 1 \), and exchanges the two structures of right \( \mathcal{D}_X \)-Module on \( M \otimes \mathcal{O}_X \mathcal{D}_X : \) one is associated with the tensor over \( \mathcal{O}_X \), and the other is the right multiplication of \( \mathcal{D}_X \).

**Proof.** — We denote by \( t(P) \) the action of \( P \in \mathcal{D}_X \) associated with the tensor over \( \mathcal{O}_X \) of right and left \( \mathcal{D}_X \)-Modules, i.e. \((m \otimes n)t(a) = ma \otimes n, (m \otimes n)t(v) = mv \otimes n - m \otimes vn \) for \( a \in \mathcal{O}_X, v \in \Theta_X \). Then the above involution \( \iota \) must satisfy \( \iota(m \otimes P) = (m \otimes 1)t(P) \), and we can easily check \( \iota^2 = \text{id} \) using local coordinates, which completes the proof (cf. [S1, 2.4.2]).

**Remark.** — If we fix the coordinates \((x_1, \ldots, x_{d_X})\), and identify \( \omega_X \otimes \mathcal{O}_X \mathcal{D}_X \) with \( \mathcal{D}_X \) by \( dx \otimes P \leftrightarrow P \), then the involution of \( \omega_X \otimes \mathcal{D}_X \) is identified with the involution \( * \) of \( \mathcal{D}_X \) defined by \((PQ)^* = Q^*P^*, x_i^* = x_i, \partial_i^* = -\partial_i^* \). This involution is essentially used in the proof of \( M^* \cong \text{DD}^{-1} M' \) for \( M^* \in D^h \text{coh}(\mathcal{D}_X) \).

1.8. Proposition. — We have an equivalence of categories

\[
(1.8.1) \quad \widehat{\text{DR}}^{-1} : D^b(\mathcal{O}_X, \text{Diff}) \cong D^b(\mathcal{D}_X) \cong D^b(\mathcal{D}_X)
\]

where a quasi-inverse is given by \( \widehat{\text{DR}} \) (same for \( D, D^+, D^- \)).

**Proof.** — By 1.5 and [VI], \( D^\bullet(\mathcal{D}_X) \to D^b(\mathcal{D}_X) \) is an equivalence of categories, and \( \widehat{\text{DR}}^{-1} \widehat{\text{DR}} \) is isomorphic to id.
1.9. LEMMA. — For $L \in M(O_X, \text{Diff})$, we have a functorial quasi-isomorphism

$$\tilde{DR}(\tilde{DR}^{-1} L) \to L$$

in $C^b(O_X, \text{Diff})$,

where $\tilde{DR}^{-1} L = L \otimes D_X \to L$ is induced by $D_X \to O_X$ as in 1.5.

Proof. — The last morphism $L \otimes D_X \to L$ corresponds to the morphism $L \otimes D_X \otimes O_X \to L \otimes D_X$ defined by $u \otimes P \otimes Q \to u \otimes PQ$, and is defined in $M(O_X, \text{Diff})$. Then the assertion is easily checked using the Koszul complex $K(L \otimes D_X; \partial_1, \ldots, \partial_d)$.

Remark. — $\tilde{DR}^{-1}$ of (1.9.1) is isomorphic to (1.5.2) applied to $M = L \otimes D_X$ using the involution 1.7.

1.10. COROLLARY. — $L^* \in C(O_X, \text{Diff})$ is acyclic, if it is $D$-acyclic, (cf. 1.3), and a morphism of $C(O_X, \text{Diff})$ is an quasi-isomorphism, if it is a $D$-quasi-isomorphism.

1.11 Definition. — We define subsheaves of $\text{Hom}_{\text{Diff}}(L, L') = \text{Hom}_{D_X}(L \otimes D_X, L' \otimes D_X)$ by

$$(1.11.1) \quad F_p \text{Hom}_{\text{Diff}}(L, L') = F_p \text{Hom}_{D_X}(L \otimes D_X, L' \otimes D_X)$$

$$:= \text{Hom}_{r_X}(L, L' \otimes F_p D_X) \hookrightarrow \text{Hom}_{r_X}(L, L' \otimes D_X)$$

$$= \text{Hom}_{D_X}(L \otimes D_X, L' \otimes D_X) = \text{Hom}_{\text{Diff}}(L, L')$$

and put

$$\text{Hom}_{\text{Diff}}^f(L, L') = \bigcup_p F_p \text{Hom}_{\text{Diff}}(L, L').$$

A differential morphism belonging to $\text{Hom}_{\text{Diff}}^f(L, L')$ (resp. to $F_p \text{Hom}_{\text{Diff}}(L, L')$) is called of finite order (resp. of order $\leq p$). Let $M(O_X, \text{Diff})^f$ be the subcategory of $M(O_X, \text{Diff})$ whose morphisms are the differential morphisms of finite orders, and define $C^b(O_X, \text{Diff})^f$, $D^b(O_X, \text{Diff})^f$, etc. as in 1.3.

Remarks.

i) $\text{Hom}_{\text{Diff}}^f(L, L') = \text{Hom}_{\text{Diff}}(L, L')$, if $L$ is coherent.

ii) For a complex of $D_X$-Modules $M^*$, $\tilde{DR}(M^*)$ is a complex of $M(O_X, \text{Diff})^f$, because the differential of $M^*$ is $O_X$-linear, i.e. order 0, and that of $\text{DR}(M^2)$ is order 1.

iii) $\text{Hom}_{\text{Diff}}^f(M, M')$ is not well-defined for induced $D$-Modules $M, M'$ unless the isomorphisms $M = L \otimes D_X$, etc. are chosen.
1.2 Proposition. — We have an equivalence of categories

\[(1.12.1) \quad \overline{DR}^{-1} : D^b(\mathcal{O}_X, \text{Diff}) \overset{f}{\sim} D^b(D_X)\]

with a quasi-inverse \(\overline{DR}\).

Proof. — By 1.5, \(\overline{DR}^{-1} \circ \overline{DR} \simeq \text{id}\), and it is enough to construct a relatively functorial isomorphism \(\overline{DR}(\overline{DR}^{-1}L) \simeq L\) in \(D^b(\mathcal{O}_X, \text{Diff})^f\). With the notations of the two lemmas below, we have \(D\)-quasi-isomorphisms in \(C^b(\mathcal{O}_X, \text{Diff})^f\):

\[(1.12.2) \quad \overline{DR}(\overline{DR}^{-1}L) \leftarrow F_p\overline{DR}(\overline{DR}^{-1}) \rightarrow L\]

for \(p \geq 0\), if we define \(F\) on \(M = L \otimes_{\mathcal{O}_X} D_X\) as in 1.14 below. In fact \(L \otimes D_X \rightarrow L\) in 1.9. has infinite order, because it corresponds to \(\text{id} \in \text{Hom}_{\mathcal{O}_X}(L \otimes D_X, L \otimes D_X)\), but its restriction to \(L \otimes F_p D_X\) has finite order \(p\). Then in the algebraic case, a differential morphism of finite order preserves the filtration \(F\) of \(\overline{DR}(\overline{DR}^{-1}L)\) up to shift globally, and we get the assertion. In the analytic case, for a finite number of \(\mathcal{O}_X\)-Modules and differential morphisms of finite orders between them such that the directions of the morphisms are compatible with some ordering of the \(\mathcal{O}_X\)-Modules, we have a locally finite covering \(\{U_i\}\) of \(X\) such that the morphisms preserves the filtration \(F\) of \(\overline{DR}(\overline{DR}^{-1}L)\) up to shift on \(U_I := \bigcap_{i \in I} U_i\) for any \(I\). Moreover we can choose the shift so that the natural inclusion \((j_I)_! j_I^{-1} \rightarrow (j_{I'})_! j_{I'}^{-1}\) for \(I \supset I'\) preserves the filtration, where \(j_I : U_I \rightarrow X\). Then we replace (1.12.1) by the co-Cech complex using \((j_I)_! j_I^{-1}\), and we get the assertion.

1.13 Lemma. — For a filtered \(D_X\)-Module \((M, F)\), i.e. \((F_p M)_{F_q M} \subset F_{p+q} M, \cup F_p M = M\) and \(F_p M = 0\) for \(p << 0\) locally on \(X\), let \(F, \overline{F}\) be the filtration of \(\overline{DR}(M), \overline{DR}^{-1} \overline{DR}(M) = \overline{DR}(M \otimes D_X)\) (cf. 1.6.) such that \(F_p(M \otimes \Lambda^i \Theta_X) = F_{p-i} (M \otimes \Lambda^i \Theta_X), \overline{F}_p((M \otimes D_X) \otimes_{\mathcal{O}_X} \Lambda^i \Theta_X) = (F_{p-i} M \otimes_{\mathcal{O}_X} D_X) \otimes_{\mathcal{O}_X} \Lambda^i \Theta_X\). Then \(F, \overline{F}\) are filtrations in \(C^b(\mathcal{O}_X, \text{Diff})^f, C^b(D_X)\) such that \(\text{Gr}_{p} F \overline{DR}(M)\) are complexes of \(\mathcal{O}_X\)-Modules and

\[(1.13.1) \quad \overline{F}_p \overline{DR}(M \otimes D_X) = \overline{DR}^{-1} (F_p \overline{DR}(M)), \quad \text{Gr}_{p} \overline{DR}(M \otimes D_X) = (\text{Gr}_{p} \overline{DR}(M)) \otimes_{\mathcal{O}_X} D_X.\]

Proof. — The assertion is clear by definition (cf. 1.6).
1.14. Lemma. — With the notation as above, \( \text{Gr}^p \overline{DR}(M) \) is acyclic for \( p > 0 \) (resp. \( p \gg 0 \) locally on \( X \)), if \((M, F) = L \otimes_{\mathcal{O}_X} \mathcal{D}_X, F)\), i.e. \( F_p M = L \otimes_{\mathcal{O}_X} F_p \mathcal{D}_X \) (resp. \( M \) is coherent and \( F \) is a good filtration, i.e. \( \text{Gr}^F M \) is coherent over \( \text{Gr}^F \mathcal{D}_X \)). If \( \text{Gr}^F \overline{DR}(M) \) are acyclic for \( p > p_0 \), the natural morphisms

\[
(1.14.1) \quad \overline{DR}^{-1} \overline{DR}(M) \leftarrow \overline{DR}^{-1} F_p \overline{DR}(M) \rightarrow M
\]

are quasi-isomorphisms for \( p \geq p_0 \).

Proof. — The first assertion is clear for \( M = L \otimes \mathcal{D}_X \), because \( \text{Gr}^F \overline{DR}(M) \) is the Koszul complex

\[
K(L \otimes_{\mathcal{O}_X} \mathcal{O}_X[\xi_1, \ldots, \xi_{d_X}]; \xi_1, \ldots, \xi_{d_X})[dx]
\]

with \( \xi_i = \text{Gr} \partial_i \).

The coherent case is then reduced to this case using a filtered resolution of \((M, F)\) by filtered free \( \mathcal{D}_X \)-Modules of finite rank, which exists locally (cf. [S1, 2.1.17]). (Here we do not need the finite length of the resolution and Cartan's theorem \( A \) is enough). The last assertion is clear by 1.5 and 1.13.

1.15 Definition. — \( D^{\text{coh}}(\mathcal{D}_X) \) (resp. \( D^{\text{hol}}(\mathcal{D}_X) \)) is the full subcategory of \( D(\mathcal{D}_X) \) defined by the condition : \( \mathcal{H}^i M^\bullet \) are coherent (resp. holonomic) (same for \( D^{\text{coh}}, D^{\text{hol}}_+ \), etc.). We define \( D^{\text{coh}}_{b}(\mathcal{O}_X, \text{Diff}) \), etc. naturally so that

\[
(1.15.1) \quad \overline{DR}^{-1} : D^{\text{coh}}_{b}(\mathcal{O}_X, \text{Diff}) \leftarrow D^{\text{coh}}_{b}(\mathcal{D}_X) \rightarrow D^{\text{coh}}_{b}(\mathcal{D}_X)
\]

(same for hol). A complex in \( D^{\text{coh}}_{b}(\mathcal{O}_X, \text{Diff}) \) (resp. \( D^{\text{hol}}_{b}(\mathcal{O}_X, \text{Diff}) \)) is called \( D \)-coherent (resp. \( D \)-holonomic). We say that \( M \) is a good coherent \( \mathcal{D}_X \)-Module, if for any relatively compact open subset \( U \) of \( X \), there exists a finite filtration \( G \) of \( M|_U \) of \( \mathcal{D}_X \)-Modules such that \( \text{Gr}^G M|_U \) are coherent \( \mathcal{D}_U \)-Modules with good filtrations. In the algebraic case this notion is same as the coherence (cf. [B, Be]). Let \( D^{\text{coh}}_{b}(\mathcal{D}_X) \) be the full subcategory of \( D^{b}(\mathcal{D}_X) \) with good coherent cohomologies, and \( D^{b,\text{coh}}(\mathcal{O}_X, \text{Diff}) \) the full subcategory of \( D^{b}(\mathcal{O}_X, \text{Diff}) \) defined by the condition : \( \overline{DR}^{-1} L^\bullet \in D^{b,\text{coh}}(\mathcal{D}_X) \). Put \( D^{b}_{\text{hol}} = D^{b,\text{coh}} \cap D^{b}_{\text{hol}}. \) Then

\[
(1.15.2) \quad \overline{DR}^{-1} : D^{b,\text{coh}}_{b}(\mathcal{O}_X, \text{Diff}) \leftarrow D^{b,\text{coh}}_{b}(\mathcal{D}_X)
\]

(same for \( D^{b,\text{hol}} \)) (cf. 1.12).
The following lemmas will be used for the definition of dual in paragraph 2.

1.16. LEMMA. — Let $L$, $L'$ and $L''$ be $\mathcal{O}_X$-Modules. Then $u$ in $\text{Hom}^f_{\text{Diff}}(L'', L)$ and $v$ in $\text{Hom}^f_{\text{Diff}}(L', L'')$ induce morphisms of $M(\mathcal{O}_X, \text{Diff})^f$:

$$
\begin{align*}
u^*: \text{Hom}^f_{\text{Diff}}(L, L') &\longrightarrow \text{Hom}^f_{\text{Diff}}(L'', L), \\
v_*: \text{Hom}^f_{\text{Diff}}(L, L') &\longrightarrow \text{Hom}^f_{\text{Diff}}(L, L'')
\end{align*}
$$

by composition, where the $\mathcal{O}_X$-Module structure of $\text{Hom}^f_{\text{Diff}}(L, L')$, etc. is given by the composition of the action of $\mathcal{O}_X$ (viewed as differential morphisms of order 0) on the target of the morphisms $L'$, etc.

Proof. — The assertion for $u^*$ is clear, because it is $\mathcal{O}_X$-linear (i.e. of order 0) by definition of the $\mathcal{O}_X$-structure on $\text{Hom}^f_{\text{Diff}}$. For $v_*$ we use the $\mathcal{O}_X$-Module structure of $L' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ associated to $\otimes_{\mathcal{O}_X}$, and

$$(L' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}_X = (L' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

we define $v': (L' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow (L' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}_X$ such that $DR(v') = v$, where $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is the tensor of left $\mathcal{D}_X$-Modules over $\mathcal{O}_X$ (cf. [S1, 2.4.8-9] for details).

Remark. — The reason why we use the $\mathcal{O}_X$-Module structure on $\text{Hom}^f_{\text{Diff}}$ induced by that on the target of the morphisms is that this structure is related to the definition of dual by the next:

1.17. LEMMA. — With the notation as above, assume $L'$ is a right $\mathcal{D}_X$-Module. Then by the involution 1.7 we have the canonical isomorphism

$$(1.17.1) \quad \text{Hom}^f_{\text{Diff}}(L, L') = \text{Hom}^f_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, L' \otimes \mathcal{D}_X) = \text{Hom}^f_{\mathcal{O}_X}(L, L') \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

as right $\mathcal{D}_X$-Modules, where the $\mathcal{D}_X$-Module structure on the first term is defined by the composition with the action of $\mathcal{D}_X$ on $L$. Moreover we have a natural isomorphism as complexes of $M(\mathcal{O}_X, \text{Diff})^f$:

$$(1.17.2) \quad DR(\text{Hom}^f_{\text{Diff}}(L, L')) = \text{Hom}^f_{\text{Diff}}(L, DR(L'))$$

compatible with the morphism $u^*$, $v_*$ in 1.16, where $v$ is assumed $\mathcal{D}_X$-linear.
Proof. — The first assertion follows from the definition of the right \( D_X \)-Module structures, and the last from 1.6.

1.18. Remark. — For \( u \in F_p \mathcal{H}om^\text{Diff}(L, L') = \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes F_p D_X) \) (cf. 1.11) we define its symbol \( \text{Gr}^F_p u \in \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes \text{Gr}^F_p D_X) \) by its composition with the natural projection. The symbol morphism

\[
(1.18.1) \quad \text{Gr}^F_p : F_p \mathcal{H}om^\text{Diff}(L, L') \rightarrow \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes \text{Gr}^F_p D_X) = \mathcal{H}om_{\mathcal{O}_X}(L, L') \otimes_{\mathcal{O}_X} \text{Gr}^F_p D_X
\]

is not surjective in general, because the filtration \( F \) on \( L' \otimes D_X \) does not splits as right \( \mathcal{O}_X \)-Modules even locally, e.g. \( L' = \mathbb{C}\{x\}/(x^2) \). But if \( L \) or \( L' \) has a connection or a structure of right \( D \)-Module, the symbol morphism (1.18.1) is surjective using the following

1.19 Remark. — For \( u \in F_p \mathcal{H}om^\text{Diff}(L, L') \) and \( \nu = (\nu_1, \ldots, \nu_{d_X}) \in \mathbb{N}^{d_X} \) such that \( |\nu| := \sum \nu_i = p \), define

\[
(1.19.1) \quad u_\nu = (-1)^p \prod_i ((\nu_i!)^{-1}(\text{ad } x_i)^{\nu_i}) u \in \mathcal{H}om_{\mathcal{O}_X}(L, L')
\]

where \((x_1, \ldots, x_{d_X})\) are local coordinates. Here \( (\text{ad } g) u := g \circ u - u \circ g \in F_{p-1} \mathcal{H}om^\text{Diff}(L, L') \) for \( g \in \mathcal{O}_X \), \( u \in F_p \mathcal{H}om^\text{Diff}(L, L') \), because \( \text{Gr}^F_p ((\text{ad } g) u) = 0 \). Then we have

\[
(1.19.2) \quad \text{Gr}^F_p u = \sum u_\nu \otimes \xi^\nu
\]

where \( \xi_i = \text{Gr}^F \partial_i \). In fact we can first reduce to the case \( L = \mathcal{O}_X \) by considering any \( \mathcal{O}_X \)-linear morphisms \( \mathcal{O}_X \rightarrow L \) (locally defined), because (1.19.1) is compatible with the composition with \( \mathcal{O}_X \)-linear morphisms. Then we may assume \( u = u'P \) with \( u' \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, L') \), \( P \in F_p D_X = F_p \mathcal{H}om^\text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \), because \( \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, L') \otimes \text{Gr}^F_p D_X \) is generated by such \( u = u'P \). Therefore the assertion is reduced to the case \( L = L' = \mathcal{O}_X \), and it is clear.

Here note also that \( \text{Gr}^F_{p+q}(vu) = \text{Gr}^F_q v \circ \text{Gr}^F_p u \) for \( u \in F_p \mathcal{H}om^\text{Diff}(L, L') \) and \( v \in F_q \mathcal{H}om^\text{Diff}(L', L'') \), where we use a natural isomorphism

\[
(1.19.3) \quad \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes_{\mathcal{O}_X} \text{Gr}^F_p D_X) \rightarrow \mathcal{H}om_{\text{Gr}^F D_X}(L \otimes_{\mathcal{O}_X} \text{Gr}^F D_X, L' \otimes_{\mathcal{O}_X} \text{Gr}^F[-p] D_X)
\]

with \( F[i] \) the shift of filtration defined by \( F[i] \cdot j = F_{j-i} \).
Remark. — A $\mathcal{C}_X$-linear morphism $u : L \to L'$ of $\mathcal{O}_X$-Modules $L, L'$ is called a differential operator of order $\leq p$ in the sense of Grothendieck (cf. [BO]) and denoted by $u \in \text{Diff}^p_X(L, L')$, iff $u$ is the composition of $L \to P^p_X \otimes_{\mathcal{O}_X} L$ defined by $m \mapsto 1 \otimes m$ and an $\mathcal{O}_X$-linear morphism $\tilde{u} : P^p_X \otimes_{\mathcal{O}_X} L \to L'$, where $P^p_X = \mathcal{O}_X \times X / I^{p+1}$ with $I$ the ideal of the diagonal and $\delta : X \to X \times X$ the diagonal embedding. Here $\tilde{u}$ is uniquely determined by $u$, because the first inclusion is factorized by $L \to \mathcal{O}_X \otimes \mathcal{O}_X = \{ \mathcal{O}_X \otimes \mathcal{O}_X \} \otimes_{\mathcal{O}_X} L \to P^p_X \otimes_{\mathcal{O}_X} L$.

For $u \in \text{Diff}^p_X(L, L')$, we have

$$\prod_{i=0}^{p} (\text{ad} g_i)u = 0$$

for any $g_i \in \mathcal{O}_X$,

because $\tilde{u} : (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \otimes_{\mathcal{O}_X} L \to L'$ annihilates

$$((\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \cap I^{p+1}) \otimes_{\mathcal{O}_X} L.$$

In the algebraic case (1.20.1) is equivalent to $u \in \text{Diff}^p_X(L, L')$, because we can replace $L, L'$, etc. by $\Gamma(U, L)$, etc. with $U$ the affine open subsets of $X$ by the quasi-coherence (but $\delta^* \mathcal{O}_{\times X} \neq \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X$ even in the algebraic case). In the analytic case, $\delta^* I^{p+1} \cap \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X$ is not generated by $\prod (x_1 \otimes 1 - 1 \otimes x_1)^{\nu_i}$ ($|\nu| = p + 1$), where $(x_1, \ldots, x_{d_X})$ are local coordinates, and the Weierstrass preparation theorem does not hold for $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X$.

We have, for $\mathcal{O}_X$-Modules $L, L'$,

$$F^p \text{Hom}_{\text{Diff}}(L, L') = \text{Diff}^p_X(L, L').$$

By the remark below we may assume $L = \mathcal{O}_X$. Then

$$F^p \text{Hom}_{\text{Diff}}(\mathcal{O}_X, L') = \text{Diff}^p_X(\mathcal{O}_X, L')$$

$$= \sum_{|\nu| \leq p} \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L') \otimes \partial^\nu$$

where $\partial^\nu = \prod \partial_i^{\nu_i} \in F^p \mathcal{D}_X = F^p \text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = \text{Diff}^p_X(\mathcal{O}_X, \mathcal{O}_X)$ for $|\nu| \leq p$. In fact we have an exact sequence

$$0 \to \text{Diff}^{p-1}_X(L, L') \to \text{Diff}^p_X(L, L') \to \text{Hom}_{\mathcal{O}_X}(I^p / I^{p+1}, \text{Hom}_{\mathcal{O}_X}(L, L'))$$

where the last morphism is also called the symbol morphism, and corresponds to $u \mapsto u_\nu$ in 1.19 by the isomorphism

$$I^p / I^{p+1} = \bigoplus_{|\nu| = p} \mathcal{O}_X \prod_{i}(x_1 \otimes 1 - 1 \otimes x_i)^{\nu_i}.$$
Then the assertion is clear by definition.

1.21 Remark. — For \( u \in \mathcal{H}om_C(L, L') \) and \( v \in \mathcal{H}om_{\mathcal{O}_X}(L'', L) \) such that \( uv \) belongs to \( F_p \mathcal{H}om_{\text{Diff}}(L'', L') \) (resp. \( \mathcal{D}iff_X(L'', L') \)), the restriction of \( u \) to \( \mathcal{L} := \text{Im} \, v \) belongs to \( F_p \mathcal{H}om_{\text{Diff}}(\mathcal{L}, L') \) (resp. \( \mathcal{D}iff_X(\mathcal{L}, L') \)).

In fact, \( uv \) is uniquely lifted to \( DR^{-1}(uv) \in \mathcal{H}om_{\mathcal{O}_X}(L'', L' \otimes D_p \mathcal{D}X) \) by 1.2, so that \( uv \) is the composition of \( DR^{-1}(uv) \) and the natural projection (cf. 1.4). Let \( w : K \to L'' \) be the kernel of \( v \). Then \( w \) is \( \mathcal{O}_X \)-linear so that

\[
DR^{-1}(uv)w = DR^{-1}(uvw) = 0
\]

by 1.2 and 1.4. Therefore \( DR^{-1}(uv) \) is factorized by the \( \mathcal{O}_X \)-linear morphism \( \mathcal{L} \to L' \otimes F_p \mathcal{D}X \) whose composition with the natural projection is the restriction of \( u \) to \( \mathcal{L} \). This proves the assertion for \( \mathcal{H}om_{\text{Diff}} \). The case of \( \text{Diff}_X \) is similar, because \( P^X \) is flat over \( \mathcal{O}_X \).

---

2. Duality

2.1 Definition. — Let \( \omega_X \) be the analytic, or algebraic, dualizing sheaf \( \Omega_{\mathcal{D}X}^X \), and \( \omega_X[d_X] \rightarrow K^*_X \) a resolution as right \( \mathcal{D}X \)-Modules such that \( K^*_X \) are injective over \( \mathcal{O}_X \) (e.g. take an injective resolution as right \( \mathcal{D}X \)-Modules, because \( \mathcal{D}X \) is flat over \( \mathcal{O}_X \)). For \( L^* \in D^b_{\text{coh}}(\mathcal{O}_X, \text{Diff})^f \), \( M^* \in D^b_{\text{coh}}(\mathcal{D}X) \), we define the dual \( DL^* \), \( DM^* \) by

\[
DL^* = \mathcal{H}om^f_{\text{Diff}}(L^*, \mathcal{D}R(K^*_X)),
\]
\[
DM^* = \mathcal{H}om^f_{\text{Diff}}(\mathcal{D}R(M^*), K^*_X),
\]

(cf. 1.16-17), so that \( \mathcal{D} \mathcal{D}R(M^*) = \mathcal{D}R(DM^*) \). Then \( DL^*, DM^* \) are well-defined in \( D^b(\mathcal{O}_X, \text{Diff})^f \), \( D^b(\mathcal{D}X) \), and \( \mathcal{D} \) is a functor of triangulated categories

\[
\mathcal{D} : D^b_{\text{coh}}(\mathcal{O}_X, \text{Diff})^f \to (D^b_{\text{coh}}(\mathcal{O}_X, \text{Diff})^f)^\circ,
\]
\[
\mathcal{D} : D^b_{\text{coh}}(\mathcal{D}X) \to (D^b_{\text{coh}}(\mathcal{D}X))^\circ
\]

by the following lemmas :

2.2. Lemmas. — \( DL^* \) is \( \mathcal{D} \)-acyclic, if so is \( L^* \), and \( DM^* \) is acyclic, if so is \( M^* \).

Proof. — The second assertion follows from the acyclicity of

\[
\mathcal{H}om^f_{\mathcal{D}X}(M^* \otimes_{\mathcal{O}_X} \mathcal{D}X, K^*_X \otimes_{\mathcal{O}_X} \mathcal{D}X) = \mathcal{H}om_{\mathcal{O}_X}(M^*, K^*_X) \otimes_{\mathcal{O}_X} \mathcal{D}X,
\]

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(cf. 1.17), because \( \overline{DR}^{-1} \overline{DR}(M^*) = \overline{DR}(M^* \otimes_{O_X} D_X) \). Then for the first assertion, it is enough to show the \( D \)-quasi-isomorphisms (1.12.2) in the proof of 1.12 induces \( D \)-quasi-isomorphisms on the dual, but this follows from the next:

2.3 Lemma. — Let \((M, F)\) be a filtered \( D_X \)-Module such that \( \text{Gr}^F_p \overline{DR}(M) \) is acyclic for \( p > p_0 \), (cf. 1.13). Let \( K \) be a right \( D_X \)-Module injective over \( O_X \). Then the natural morphism

\[
\text{Hom}^f_{\text{Diff}}(\overline{DR}(M), K) \longrightarrow \text{Hom}^f_{\text{Diff}}(F_p \overline{DR}(M), K)
\]

is a quasi-isomorphism for \( p \geq p_0 \). If moreover \((M, F) = L \otimes_{O_X} (D_X, F)\) as in 1.14, the morphism induced by \( F_p \overline{DR}(L \otimes_{O_X} D_X) \to L \) in 1.9:

\[
\text{Hom}^f_{\text{Diff}}(L, K) \longrightarrow \text{Hom}^f_{\text{Diff}}(F_p \overline{DR}(L \otimes_{O_X} D_X), K)
\]

is a quasi-isomorphism for \( p \geq 0 \).

Proof. — The second assertion follows from the first, because we have \( F_0 \overline{DR}(M) = L \) by definition. Put \( K^*_p = \text{Hom}^f_{\text{Diff}}(F_p \overline{DR}(M), K) \). Then it is a projective system satisfying the Mittag-Leffler condition [G], and \( K^*_p \to K^*_q \) is a quasi-isomorphism for \( p > q > p_0 \) by assumption, because

\[
K^*_p = \text{Hom}_{O_X} (F_{p-1} M \otimes \Lambda^i \Theta_X, K) \otimes_{O_X} D_X
\]

\[
\text{Ker}(K^*_p \to K^*_{p-1}) = \text{Hom}_{i_X} (\text{Gr}^F_p \overline{DR}(M), K) \otimes_{O_X} D_X.
\]

Then the assertion is clear by [loc. cit.].

2.4. Lemma. — For \( M^* \in D^b_{\text{coh}}(D_X) \), the natural morphism

\[
\text{DM}^* \longrightarrow R \text{Hom}_{D_X} (\overline{DR}^{-1} \overline{DR}(\cdot), \overline{DR}^{-1} K^*_X)
\]

is a quasi-isomorphism. In particular, \( H^j \text{DM}^* \) are coherent (resp. holonomic) \( D_X \)-Modules and \( H^j \text{DM}^* = 0 \) except for \(-d_X \leq j \leq 0 \) (resp. \( j = 0 \)), if \( M \) is a coherent (resp. holonomic) \( D_X \)-Module.

Proof. — The assertion is local, and we have locally a good filtration \( F \) of \( M \) so that \( \text{Gr}^F_p \overline{DR}(M) \) is acyclic for \( p >> 0 \). Then the first assertion follows from 2.3, because we can replace \( M \) by its free resolution, and the second from [K2].

2.5. Remark. — By V.G. Golovin (Soviet Math. Dokl. 16 (1975, p. 854), the injective dimension of \( \omega_X \) over \( O_X \) is \( d_X \).
2.6. Theorem. — Let $X$ be a complex manifold. Let
\[ \text{For} : D^b_{\text{hol}}(\mathcal{O}_X, \text{Diff})^f \to D_c^b(C_X) \]
be the forgetful functor. Put $DR = \text{For} \circ \widetilde{DR} : D^b_{\text{hol}}(\mathcal{D}_X) \to D^b_c(C_X)$. Then for $L^\bullet \in D^b_{\text{hol}}(\mathcal{O}_X, \text{Diff})^f$, the natural morphism
\[ \text{For}(DL^\bullet) = \text{For}(\text{Hom}^f_{\text{Diff}}(L^\bullet, \widetilde{DR}(K^\bullet))) \]
\[ \to \text{Hom}_{C_X}(\text{For}(L^\bullet), \text{For} \circ \widetilde{DR}(K^\bullet)) \]
\[ = \mathbb{R}\text{Hom}_{C_X}(\text{For}(L^\bullet), C_X[2d_X]) = D \text{For}(L^\bullet) \]
is an isomorphism in $D^b_c(C_X)$, i.e. we get a natural functor isomorphism
\[ (2.6.1) \quad \text{For} \circ D = D \circ \text{For} : (D^b_{\text{hol}}(\mathcal{O}_X, \text{Diff}))^f \to (D^b_c(D_X))^\circ. \]

Proof. — Note that the image of the functor For is contained in $D^b_c(C_X)$ the derived category of the bounded complexes with constructible cohomologies by [K1] and 2.4. Because the assertion is local, we can reduce to the case $\mathcal{H}^i\widetilde{DR}^{-1}L^\bullet = 0$ except for one $j$, and $L^j$ are free $\mathcal{O}_X$-Modules of finite type. Then the proof is same as [S1, 2.4.12].

2.7 Corollary. — We have a natural functor isomorphism $DR \circ D \sim D \circ DR$ compatible with (2.6.1) by $DR \circ D \sim D \circ DR$ in 2.1.

2.8. Remark. — The above proof of 2.6. is essentially same as [K1], where the perfectness of the natural pairing
\[ (2.8.1) \quad i^*_x \text{Sol}(M) \times i^!_x DR(M) \to C, \]
i.e. $i^*_x DR(DM) \sim i^*_x DDR(M)$, is proved for a holonomic $D_X$-Module $M$. Here $\text{Sol}(M) := \mathbb{R}\text{Hom}_{\mathcal{D}_X}(M, \omega_X[d_X])$ so that $\text{Sol}(M) = DR(DM)$, and $i_x : \{x\} \to X$. This duality is generalized to $DR(DM) = DDR(M)$ for $M \in D^b_{\text{hol}}(\mathcal{D}_X)$ by [KK], [M1] (cf. [B], [Be] for the algebraic case).

2.9. Lemma. — Let $X$ be a smooth algebraic variety, and $X^{\text{an}}$ the associated complex manifold. Let $\text{An}$ denote the natural functors $D^b(\mathcal{O}_X, \text{Diff})^f \to D^b(\mathcal{O}_X^{\text{an}}, \text{Diff})^f, D^b(\mathcal{D}_X) \to D^b(D_X^{\text{an}})$ induced by $\otimes_{\mathcal{O}_X} \mathcal{O}_X^{\text{an}}$ (where the pull-back by $X^{\text{an}} \to X$ is omitted). Let $K_X^{\text{an}}$ be an injective resolution of $\text{An}(K_X^\bullet)$ as right $D_X^{\text{an}}$-Modules. Then for $L^\bullet \in D_{\text{coh}}^b(\mathcal{O}_X, \text{Diff})^f$ the composition
\[ \text{An}(\text{Hom}^f_{\text{Diff}}(L^\bullet, K_X^\bullet)) \to \text{Hom}^f_{\text{Diff}}(\text{An}(L^\bullet), \text{An}(K_X^\bullet)) \]
\[ \to \text{Hom}^f_{\text{Diff}}(\text{An}(L^\bullet), K_X^{\text{an}}) \]

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is a quasi-isomorphism, and we have the natural functor isomorphisms
\[ \text{AnD} = D\text{An} : D^b_{\text{coh}}(O_X,\text{Diff})^f \rightarrow (D^b_{\text{coh}}(O_{X^{an}},\text{Diff})^f)^\circ. \]
\[ \text{AnD} = D\text{An} : D^b_{\text{coh}}(D_X) \rightarrow (D^b_{\text{coh}}(D_{X^{an}}))^\circ. \]

**Proof.** — By definition of dual, it is enough to show the first assertion. We may assume that \( \mathcal{H}^1 \sim DR^{-1} L^* = 0 \) except for one \( j \), and \( L^j \) are free \( O_X \)-Modules of finite rank. Then the assertion is clear.

**2.10 Theorem.** — Let \( X \) be as above, and \( \text{For} : D^b_{\text{hol}}(O_X,\text{Diff})^f \rightarrow D^b_{\text{hol}}(D_X) \rightarrow D^b_{\text{hol}}(O_{X^{an}}), \text{DR} : D^b_{\text{hol}}(D_X) \rightarrow D^b_{\text{hol}}(D_{X^{an}}) \) the composition of \( \text{An} \) with the functors \( \text{For}, \text{DR} \) in 2.6. Then we have the natural functor isomorphisms \( \text{ForD} = D\text{For} \), \( \text{DRD} = DDR \).

(This is clear by 2.6–7.)

**2.11 Remark.** — For \( L^* \in D^b(O_X,\text{Diff})^f \), the dual \( DL^* \) is represented by the single complex associated with \( \text{Hom}_{O_X}(L^*, K^*_X) \) whose first differential is given by using the identity
\[ \text{Hom}_{O_X}(L^i, K^j_X) = \text{Hom}_{D_X}(L^i \otimes O_X D_X, K^j_X), \]
so that
\[ \text{DR}^{-1} \text{Hom}_{O_X}(L^*, K^*_X) = \text{Hom}_{D_X}(L^* \otimes D_X, K^*_X \otimes D_X). \]

If \( L^i \) are locally free, we can replace \( K^*_X \) by \( \omega_X[d_X] \). Then the differential of the dual is given by the transpose of that of \( L \) (taking local coordinates and local trivializations of \( L^i \)).

**2.12 Remark.** — In [KK], the duality isomorphism \( DRD \sim DDR \) is defined by
\[ \text{Sol}(M) \otimes C DR(M) \rightarrow DR(\omega_X[d_X]) = C_X[2d_X], \quad (\text{cf.} 2.8.) \]
We can check that it coincides with
\[ \text{DRD}(M) = \mathbb{R}\text{Hom}_{D_X}(M, \omega_X[d_X]) \]
\[ \text{DR}(\text{DR}(M), \omega_X[d_X])) = DDR(M) \]
and with that in 2.7. In fact for \( u \in \text{Hom}_{\text{Diff}}(L, K) \), we have a commutative diagram by 1.4:
where the vertical morphisms are as in 1.9.

### 3. Direct Images and Duality

3.1. Let $f : X \to Y$ be a proper morphism of complex manifolds, or smooth algebraic varieties. We denote by $f_*$ and $f^*$ the sheaf theoretic direct image and pull-back. The direct image of $M^* \in \mathcal{D}^b(\mathcal{D}_X)$ is defined by

\[(3.1.1) \quad f_* M^* = Rf_* (M^* \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \in \mathcal{D}^b(\mathcal{D}_X),\]

where $\mathcal{D}_{X \to Y} = \mathcal{O}_X \otimes f_* \mathcal{O}_Y f^* \mathcal{D}_Y$. To define $\otimes_{\mathcal{D}_X}$ we take flat resolution, or factorize $f$ by $X \xrightarrow{i} X \times Y \xrightarrow{p} Y$ and use the relative de Rham $\mathcal{D}R_{X \times Y/Y}$, where $i$ is the embedding by graph and $p$ is the projection. In this note we take the resolution of $M$ by standard induced Modules in (1.5.2). The result is same as the second definition using the canonical factorization. The merit is that we don’t have to work on $X \times Y$, but only on $X, Y$. We also show that for the differential complexes the direct image is defined simply by $f_*$ the sheaf theoretic direct image, i.e. the differential morphisms are stable by the direct image for proper morphisms. This is obtained as a corollary of the above definition of $f_*$ of $\mathcal{D}_X$-Modules. Using this we can prove rather easily the duality for proper morphisms.

3.2. Lemma. For an induced $\mathcal{D}_X$-Module $M = L \otimes_{\mathcal{O}_X} \mathcal{O}_X$, we have $\text{Tor}^i_{\mathcal{D}_X} (M, \mathcal{D}_{X \to Y}) = 0$ for $i \neq 0$ so that

\[(3.2.1) \quad M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y} = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y} = L \otimes_{f_* \mathcal{O}_Y} f^* \mathcal{D}_Y.\]

(This est clear.)

3.3. Proposition. For $L^* \in C^b(\mathcal{O}_X, \text{Diff})$ (or $C^b(\mathcal{O}_X, \text{Diff})^1$), put $M^* = \mathcal{D}R^{-1} L^* \in C^b(\mathcal{D}_X)$. Then $f_* L^* \in C^b(\mathcal{O}_Y, \text{Diff})$ (or $C^b(\mathcal{O}_Y, \text{Diff})^1$), i.e. the differential morphisms are stable by $f_*$, and

\[(3.3.1) \quad f_* M^* = \mathcal{D}R^{-1} f_* L^*.\]

if $L^i$ are $f_*$-acyclic.

Proof. We consider a natural morphism $\mathcal{D}_{X \to Y} \to \mathcal{O}_X$ defined by the tensor of $\mathcal{D}_{X \to Y}$ with $f^* \mathcal{D}_Y \to f^* \mathcal{O}_Y$ (cf. 1.4) over $f^* \mathcal{D}_Y$, and take the tensor of $M = L \otimes \mathcal{D}_X$, etc. in 1.4 with the above morphism, which induces another surjection from $L \otimes_{f_* \mathcal{O}_Y} f^* \mathcal{D}_Y = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$, etc.
the first diagram of (1.4.1). Then it is enough to take the direct image of
the new commutative diagram, cf. the remark below.

3.4. Remark. — For the reduction to the $f_\ast$-acyclic case, we use
an affine covering $\{U_i\} \subset X$ and replace $L^i$ by a Čech complex
$\bigoplus_{|I|=k+1} (j_I)_! j_I^! L^i$ in the algebraic case, where $j_I : \bigcap_{i \in I} U_i \to X$ and
$X$ is always assumed separated. In the analytic case, we use a canonical
flabby resolution of Godement (and truncate it), or a Dolbeault resolution,
etc. Here $f$ is proper, and soft or fine resolution is enough. We also
used the commutativity of $f_\ast$ and the inductive limit in the proof of 3.3,
because $\otimes_{\mathcal{O}_Y} D_Y$ is the inductive limit of $\otimes_{\mathcal{O}_X} F_p D_Y$. As a COROLLARY
of 3.3, $f_\ast M^\bullet$ is well-defined in the derived category of quasi-coherent $D_Y$-
Modules in the algebraic case (compare to [B], [Be]).

3.5. Definition. — For $L^\bullet \in D^b(\mathcal{O}_X, \text{Diff})^f$, we define

\begin{equation}
(3.5.1) 
 f_\ast L^\bullet = Rf_\ast L^\bullet \in D^b(\mathcal{O}_X, \text{Diff})^f
\end{equation}

so that $f_\ast \widetilde{D}R^{-1} = \widetilde{D}R^{-1} f_\ast$, where $Rf_\ast$ is constructed as above.

(10,135),(990,185)

3.6. Proposition. — $D^b_{\text{coh}}(\mathcal{O}_X, \text{Diff})^f$, $D^b_{\text{coh}}(\mathcal{D}_X)$ (cf. 1.5) are
stable by the direct images for proper morphisms (same for $D^b_{\text{g,hol}}$).

Proof. — By 1.14, it is enough to show that $H^i \widetilde{D}R^{-1} f_\ast L =
\widetilde{D}R^{-1} H^i Rf_\ast L$ is a coherent $D_Y$-Module with a good filtration for $L$ a
coherent $\mathcal{O}_X$-Module. But this is clear by Grauert's coherence theorem
for $\mathcal{O}_X$-Modules.

3.7 Definition. — For a complex of filtered $D_X$-Modules $(M^\bullet, F)$, we
define the filtration $F$ on $\widetilde{D}R(M^\bullet)$ by $F_p(M^j \otimes \Lambda^i \Theta_X) = F_{p-j} M^j \otimes \Lambda^i \Theta_X$
so that $\widetilde{D}R(M^\bullet, F) \in CF(\mathcal{O}_X, \text{Diff})$ (cf. [S1, 2.2]). We define $F$ on
$\omega_X[dx], K_X^\bullet$ by $Gr^F_p = 0$ for $p \neq 0$ so that $\omega_X[dx] \to K_X^\bullet$ is a
filtered quasi-isomorphism. The filtered trace morphism $\text{Tr}_f$ for a proper
morphism $f : X \to Y$ is a morphism $\text{Tr}_f : Rf_\ast \widetilde{D}R(\omega_X[dx], F) \to
\widetilde{D}R(\omega_Y[dy], F)$ in $DF(\mathcal{O}_Y, \text{Diff})$ such that $\text{Gr}^F_0 \text{Tr}_f$ represents the ana-
lytic, or algebraic, trace morphism $\text{Tr}_f : Rf_\ast \omega_X[dx] \to \omega_Y[dy]$ in $D(\mathcal{O}_X)$
and For(Tr$_f$) the topological one $\text{Tr}_f : Rf_\ast C_X[2dx] \to C_Y[2dy]$, or
$\text{Tr}_f : Rf_\ast C_X^{an}[2dx] \to C_Y^{an}[2dy]$.

3.8. Remark. — If Tr$_f$ is given in the filtered derived category as
above, it can be represented by

\begin{equation}
(3.8.1) \quad f_\ast \widetilde{D}R(K_X^\bullet, F) \leftarrow F_p \widetilde{D}R(\widetilde{D}R^{-1} f_\ast \widetilde{D}R(K_X^\bullet, F))
\end{equation}

$\to \widetilde{D}R(K_Y^\bullet, F) \quad \text{for } p \geq d_X.$
where the first morphism is a filtered quasi-isomorphism as in the proof of 1.14. In fact, $DR^{-1} Tr_f$ is represented by a filtered morphism $DR^{-1} f_0 DR(K^\bullet_X, F) \to (K^\bullet_Y, F)$, if $K^\bullet_Y$ is chosen appropriately (cf. [S1, 2.5.1]), and we apply $DR$ to it, and restrict to $F_p$.

3.9. Proposition. — The filtered trace morphism $Tr_f$ exists canonically so that $Tr_g f = Tr_g \circ Tr_f$ in $DF(O_Z, Diff)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Proof. — In the analytic case, it is represented by the push-down of the distributions (cf. [S1, 2.5.1]). In the algebraic case, we may assume $X, Y$ proper by Nagata-Hironaka, because the independence of the compactification follows from the functoriality for composition. If $f$ is a closed embedding of codimension $d$, we have $f_0 \omega_X = H^d_X(\omega_Y)$ and $f_0 K^\bullet_X \xrightarrow{\sim} \Gamma_X K^\bullet_Y \to K^\bullet_Y$. Then the filtered trace morphism is induced by the identity $f_0(\omega_X, F) \to (f_0 \omega_X, F)$ with $Gr^F_p f_0 \omega_X = 0$ for $p \neq 0$, because $f_0 DR(\omega_X[d_X], F) = DR(f_0(\omega_X[d_X]), F)$ (cf. [S1, 2.3]), and $Gr^F_p Tr_f$ is the composition

$$\omega_X[d_X] = \mathcal{E}xt^d_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)[d_X] \to H^d_X(\omega_Y)[d_X] \to \omega_Y[d_Y].$$

In the case $f$ is the projection $X \times Y \to Y$, we may assume $Y = pt$, because $\omega_{X \times Y} = \omega_X \boxtimes \omega_Y$ and $DR(\omega_{X \times Y}[d_X + d_Y], F) = DR(\omega_X[d_X], F) \boxtimes DR(\omega_Y[d_Y], F)$. By Hodge theory, the filtration $F$ on $R\Gamma(X, DR(\omega_X[d_X]))$ is strict and $H^0(X, DR(\omega_X[d_X])) = Gr^F_0 H^0(X, DR(\omega_X[d_X])) = C$, and we get $Tr_f : f_0 DR(K_X, F) \to (C, F)$. In general we use the canonical factorization $X \to X \times Y \to Y$ of $f$ as in 3.1. The functoriality is clear for the composition of closed embeddings and that of projections, and it is reduced to the commutativity for

$$\begin{array}{ccc}
X \times Y & \xrightarrow{i'} & X \times Z \\
p' \downarrow & & \downarrow p \\
Y & \xrightarrow{i} & Z
\end{array}$$

where $i$ is a closed embedding, $i' = \text{id} \times i$, and $p, p'$ are projections. But it is also clear by definition.

3.10 Remark. — If we use the theory of mixed Hodge Modules in the algebraic case (cf. [S2-3]), the filtered trace morphism is given as follows.
For $f : X \to Y$ a proper morphism of smooth algebraic varieties, we have

\[(3.10.1) \quad f_\#: f_* Q^H_X(dx)[2dx] \to Q^H_Y(dy)[2dy] \]

by the adjoint relation for $f_1, f^!$ [loc. cit.]. Taking the underlying morphism of filtered $\mathcal{D}$-Modules, we get

\[(3.10.2) \quad \text{Tr}_f : \mathcal{R}f_* (\omega_X[dX], F) \to (\omega_Y[dY], F) \quad \text{in} \quad D^b F(D_Y) \]

\[(3.10.3) \quad \text{Tr}_f : \mathcal{R}f_* \mathcal{DR}(\omega_X[dX], F) \to \mathcal{DR}(\omega_Y[dY], F) \quad \text{in} \quad D^b F(O_Y, \text{Diff}). \]

We can check that this trace morphism coincides with the above one in the case of closed embeddings and projections. By this definition, the check of the functoriality is easy.

**3.11 Theorem. —** For $f : X \to Y$ a proper morphisms of complex manifolds, or smooth algebraic varieties, and $L^\bullet \in D^b_{g,\text{coh}}(O_X, \text{Diff})^f$ (cf. 1.15), we have a canonical functor isomorphism

\[(3.11.1) \quad f_* DL^\bullet \sim Df_* L^\bullet \quad \text{in} \quad D^b_{g,\text{coh}}(O_Y, \text{Diff})^f. \]

**Proof. —** By 1.14, we have a $D$-quasi-isomorphism

\[(3.11.2) \quad \text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, f_* \mathcal{DR}(K^\bullet_X)) \leftarrow \text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, f_* \mathcal{DR}(K^\bullet_X))) \]

\[
\begin{align*}
F_p \mathcal{DR}(\mathcal{DR}^{-1}(f_* \mathcal{DR}(K^\bullet_X))) \\
\text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, F_p \mathcal{DR}(\mathcal{DR}^{-1}(f_* \mathcal{DR}(K^\bullet_X))))
\end{align*}
\]

where the filtration $F$ on the differential complex

\[
\text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, f_* \mathcal{DR}(K^\bullet_X))
\]

is given by that of $f_* \mathcal{DR}(K^\bullet_X)$, and the order of the differential morphisms is not counted here. Then the morphism (3.11.1) is given by its composition with

\[
\begin{align*}
f_* DL^\bullet & = f_* \text{Hom}_{\text{Diff}}^f (L^\bullet, \mathcal{DR}(K^\bullet_X)) \to \text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, f_* \mathcal{DR}(K^\bullet_X)) \\
\text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, F_p \mathcal{DR}(\mathcal{DR}^{-1}(f_* \mathcal{DR}(K^\bullet_X)))) & \xrightarrow{\text{Tr}_f} \text{Hom}_{\text{Diff}}^f (\mathcal{R}f_* L^\bullet, \mathcal{DR}(K^\bullet_Y)).
\end{align*}
\]
To show the isomorphism of (3.11.1), we may assume \( L \) is a coherent \( \mathcal{O}_X \)-Module so that \( f_*DL = f_*\mathcal{H}om_{\mathcal{O}_X}(L, K_X^\bullet) \), \( Df_*L = \mathcal{H}om_{\mathcal{O}_X}(Rf_*L, K_Y^\bullet) \) by 1.17 and 1.5. Moreover they are complexes of \( \mathcal{O}_Y \)-Modules and \( DR^{-1} \) is given by \( \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \). We define the filtration \( F \) on the intermediate terms appeared in the construction of the morphism (3.11.1) as in [S1, paragraph 2], where \( F \) on \( L \) is trivial, i.e. \( \text{Gr}_F^p L = 0 \) \( (p \neq 0) \), and the order of the differential morphisms are counted so that \( \text{Gr}_F^p \) are defined in the category of \( \mathcal{O} \)-Modules. Then \( \text{Gr}_p^F \) of \( f_*DL, Df_*L \) are \( D \)-acyclic for \( p \neq 0 \), and \( \text{Gr}_0^F \) of the morphism (3.11.1) is the composition

\[
\begin{align*}
\text{Tr}_f: f_*\mathcal{H}om_{\mathcal{O}_Y}(L, K_X^\bullet) &\rightarrow \mathcal{H}om_{\mathcal{O}_X}(Rf_*L, f_*K_Y^\bullet) \\
\end{align*}
\]

Therefore it is a quasi-isomorphism by [H], [RRV], and a \( D \)-quasi-isomorphism, because it is defined in \( D(\mathcal{O}_Y) \).

3.12 Corollary. — For \( M^\bullet \in D_{g,\text{coh}}(\mathcal{D}_X) \), we have a canonical functor isomorphism

\[
(3.12.1) \quad f_*DM^\bullet \simto Df_*M^\bullet \ \text{in} \ D_{g,\text{coh}}(\mathcal{D}_Y)
\]

compatible with (3.11.1) by \( \widehat{DR} \).

(This is clear.)

3.13. Theorem. — The duality isomorphisms (3.11.1), (3.12.1) are compatible with the topological one [V2] :

\[
(3.13.1) \quad f_*DK \simto Df_*K \ \text{in} \ D^b_c(\mathcal{C}_Y) \ \text{or} \ D^b_c(\mathcal{C}_{Y^{an}})
\]

for \( K \in D^b_c(\mathcal{C}_X) \) or \( D^b_c(\mathcal{C}_{X^{an}}) \) by For, \( DR \) in the holonomic case.

Proof. — This follows from the condition for the filtered trace morphism : For(\text{Tr}_f) represents the topological trace morphism.

3.14. Remark. — If we want to define the duality isomorphism as (0.3) in the introduction, the trace morphism \( \text{Tr}_f : f_*K_X \rightarrow K_Y \) must satisfy some condition for the compatibility with the differential of \( M \).

For example, if the differential \( d : \mathcal{O}_X \rightarrow \mathcal{O}_X \) is given by \( \xi \in f_*\mathcal{O}_X \), put \( \xi_1_{X \rightarrow Y} = \sum g_i \otimes \eta_i \in \mathcal{D}_{X/Y} = \mathcal{O}_X \otimes_{f_*\mathcal{O}_Y} f^*\mathcal{D}_Y \) locally on \( Y \). Then \( \text{Tr}_f \) must satisfy

\[
(3.14.1) \quad \text{Tr}_f(u\xi) = \sum \text{Tr}_f(u g_i)\eta_i \in K_Y^1 \quad \text{for} \quad u \in f_*K_X^1.
\]

In fact the general condition is

\[
(3.14.2) \quad \text{Tr}_f \text{ is extended to a morphism of double complex}
\]

\[
\begin{align*}
\end{align*}
\]

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and it induces the filtered trace morphism taking the filtration associated with $\hat{D}R$. But (3.14.1-2) are rather difficult to realize in the level of complex. In the case $Y = \text{pt}$, (3.14.1) implies $\text{Tr}_f(u\xi) = 0$. But $\xi : K_X^i \to K_X^i$ is surjective, if $K_X^i$ is injective over $D_X$ and $\xi \neq 0$. Therefore we can assume only the injectivity over $O_X$.

3.15 Remark. — The duality $f_*DM \simto Df_*M$ induces the adjoint relation

$$ (3.15.1) \quad \text{Hom}_{D(D_X)}(M, f^!N) \simto \text{Hom}_{D(D_Y)}(f_*M, N) $$

for $M \in D^b_{\text{coh}}(D_X)$, $N \in D^b(D_Y)$ and a proper morphism $f : X \to Y$. In fact it is enough to take $H^0R\Gamma(Y, *)$ of the tensor of $f_*DM \simto Df_*M$ with $(\omega^{-1}) \otimes_{O_Y} N$ over $D_Y$. This relation was used in the proof of the regularity of Hodge Modules (cf. [S1]).

4. Diagonal Pairings and Riemann-Hilbert Correspondence

4.1. — The Riemann-Hilbert correspondence for regular holonomic $D$-Modules was proved by [K3] and [M2] (cf. [B], [Be]) in the algebraic case. In [K3] the inverse functor is constructed using the tempered distribution. In [M2], [B], [Be] the fully faithfulness of the de Rham functor (or solution) was first proved using the commutativity of $\delta^0$ and the de Rham or solution, where $\delta : X \to X \times X$ is the diagonal embedding. But it was not so clear whether the isomorphism constructed between $\text{Hom}_{D(D_X)}(M^\bullet, N^\bullet)$ and $\text{Hom}_{D(D_X)}(DR(M^\bullet), DR(N^\bullet))$ (or $\text{Hom}_{D(D_X)}(\text{Sol}(N^\bullet), \text{Sol}(M^\bullet))$) was really induced by $DR$ (or $\text{Sol}$), because we had to check the commutativity of some diagram which is not quite obvious. In this section we check this point using the diagonal pairings. In the algebraic case we use non quasicoherent $D$-Modules.

4.2. Lemma. — Let $X$ be a complex manifold or a smooth algebraic variety, $\delta : X \to X \times X$ the diagonal embedding, and $\text{pr}_i : X \times X \to X$ the projection ($i = 1, 2$). Then for a right $D_X$-Module $M$, we have a natural isomorphism

$$ (4.2.1) \quad M \otimes_{O_X} D_X = \delta^*\delta_*M $$

as right $D_X$-bi-Modules, where $\text{pr}_i^\bullet$ ($i = 1, 2$) is used to give the first and second structure of right $D_X$-Module on $\delta^*\delta_*M$, and the first (resp. second) structure of $M \otimes_{O_X} D_X$ is the one associated to $\otimes_{O_X}$ (cf. the proof of 1.7) (resp. right multiplication).
Proof. — By definition of direct image for right $\mathcal{D}$-Modules, we have a natural inclusion

\[(4.2.2) \quad \phi : \delta_* M \rightarrow \delta_* M\]

which is $\mathcal{O}_X$-linear for the both structure of $\delta_* M$, and induces a $\mathcal{D}_X$-linear morphism

\[(4.2.3) \quad \psi : \delta_* M \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \delta_* M\]

using the second structure of $\delta_* M$. Then it is bijective by definition, and $\mathcal{D}_X$-linear for the first structure, because $\phi(mv) = \phi(m)(pr_1^* v + pr_2^* v)$ for $m \in M$, $v \in \mathcal{O}_X$ (in fact $v1_{\mathcal{O}_X \times \mathcal{O}_X} = 1_{\mathcal{O}_X \times \mathcal{O}_X} (pr_1^* v + pr_2^* v)$ where $1_{\mathcal{O}_X \times \mathcal{O}_X} = 1 \otimes 1 \in \mathcal{D}_X \otimes_{\mathcal{O}_X \times \mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X \times \mathcal{O}_X} \mathcal{D}_X$).

4.3. Lemma. — For right $\mathcal{D}_X$-Modules $M, N$ and a right $\mathcal{D}_X \times X$-Module $K$ such that $\text{supp} K \subset \text{Im} \delta$, we have a natural isomorphism

\[(4.3.1) \quad \text{Hom}_{\mathcal{D}_X}(M, \text{Hom}_{\mathcal{D}_X}(N, \delta^* K)) = \text{Hom}_{\mathcal{D}_X \times X}(M \boxtimes N, K)\]

where $M \boxtimes N = (pr_1^* M \otimes_{\mathcal{O}_X} pr_2^* N) \otimes_{pr_1^* \mathcal{D}_X \otimes \mathcal{O}_X} pr_2^* \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

Proof. — This is clear by definition.

4.4. Corollary. — If $K$ is an injective right $\mathcal{D}_X \times X$-Module and $\text{supp} K \subset \text{Im} \delta$, then $\delta^* K$ is injective over $\mathcal{D}_X$ for the both structure and $\text{Hom}_{\mathcal{D}_X}(N, \delta^* K)$ is an injective right $\mathcal{D}_X$-Module for any $N$. (Here the injectivity is considered in the not necessarily quasicoherent $\mathcal{D}$-Modules in the algebraic case).

Proof. — The assertion is clear by taking the global sections of (4.3.1), and putting $M = \mathcal{D}_X$ for the injectivity of $\delta^* K$, because $\mathcal{O}_X \otimes_{pr_1^* \mathcal{O}_X} pr_2^* \mathcal{D}_X$ is flat over $\mathcal{O}_X$ by $pr_2^* \mathcal{D}_X$ and $\boxtimes$ is exact for the both factors.

Remark. — The above $K$ may not be quasicoherent if $\dim X \neq 0$, because the action of nonzero vector fields on $K$ is surjective by the injectivity of $K$.

4.5. Corollary. — For $M^\bullet, N^\bullet \in D^b(\mathcal{D}_X)$ and $K^\bullet \in D^b(\mathcal{D}_X \times X)$ such that $\text{supp} H^0 K^\bullet \subset \text{Im} \delta$, we have a natural isomorphism

\[(4.5.1) \quad \text{Hom}_{D(\mathcal{D}_X)}(M^\bullet, R\text{Hom}_{\mathcal{D}_X}(N^\bullet, \delta^* K^\bullet)) = \text{Hom}_{D(\mathcal{D}_X \times X)}(M^\bullet \boxtimes N^\bullet, K^\bullet).\]

Proof. — Replace $K^bull$ by an injective resolution supported in $\text{Im} \delta$ of $\Gamma_{\text{Im} \delta}$ of its injective resolution, and apply $H^0 \Gamma(X, \cdot)$ to (4.3.1).
4.6. PROPOSITION. — For $M^\bullet$, $N^\bullet \in D^b_{\text{hol}}(\mathcal{D}_X)$, we have a commutative diagram

$$
\begin{array}{c}
\text{Hom}_{D(\mathcal{D}_X)}(M^\bullet, DN^\bullet) \\
\downarrow DR \\
\text{Hom}_{D(\mathcal{C}_X)}(DR(M^\bullet), DR(DN^\bullet)) \\
\downarrow \\
\text{Hom}_{D(\mathcal{C}_X)}(DR(M^\bullet), DDR(N^\bullet)) \\
\sim \\
\mathcal{P} \\
\downarrow DR \\
\mathcal{Q} \\
\end{array}
$$

with the abbreviations

$$
\mathcal{P} = \text{Hom}_{D(\mathcal{D}_X \times X)}(M^\bullet \boxtimes N^\bullet, \delta_*\omega_X[d_X]),
$$

$$
\mathcal{Q} = \text{Hom}_{D(\mathcal{C}_X \times X)}(DR(M^\bullet) \boxtimes DR(N^\bullet), \delta_*C_X[2d_X]).
$$

Here $C_X$, $C_{X \times X}$ are replaced by $C_{X, \text{an}}$, $C_{X, \text{an}} \times X, \text{an}$ in the algebraic case.

Proof. — The first horizontal isomorphism follows from 4.5 taking an injective resolution $K^*$ of $\delta_*\omega_X[d_X]$. In the analytic case we apply $DR$ to this isomorphism, and get the second one with the commutative diagram of the assertion by the proof of 4.5 (cf. the correspondence in 4.3), because $\delta^*DR(K^*) = DR(DR(\delta^*K^*))$ is an injective resolution of $C_X[2d_X]$. Here the left vertical morphism is factorized as in the assertion by using the duality isomorphism 2.7 (cf. 2.12):

$$
(4.6.1) \quad DR(\text{Hom}_{D_X}(N^\bullet, \delta^*K^*)) = \text{Hom}_{D_X}(N^\bullet, DR(\delta^*K^*))
$$

$$
\sim \text{Hom}_{C_X}(DR(N^\bullet), DR(DR(\delta^*K^*)).
$$

In the algebraic case we apply the above argument to $\text{An}(M)$, etc. (cf. 2.9), and use 2.9 with the compatibility of 4.5 with $\text{An}$ by taking an injective resolution $K^*_{\text{an}}$ of $\text{An}(K^*)$.

4.7. Remark. — In the case $M^\bullet$, $N^\bullet$ have regular holonomic cohomologies, the right vertical morphism $DR$ is an isomorphism by the commutativity of $\delta^*$, $\delta_*$ with $DR$, and the duality. In fact, we can replace $M^\bullet \boxtimes N^\bullet$ in $\mathcal{P}$ and $DR(M^\bullet) \boxtimes DR(N^\bullet)$ in $\mathcal{Q}$ by $\delta_*\delta^*(M^\bullet \boxtimes N^\bullet)$ and $\delta_*DR(\delta^*(M^\bullet \boxtimes N^\bullet))$. Then the assertion is reduced to

$$
DR : \text{Hom}_{D(\mathcal{D}_X)}(\delta^*(M^\bullet \boxtimes N^\bullet), \omega_X[d_X])
$$

$$
\sim \text{Hom}_{D(\mathcal{C}_X)}(DR(\delta^*(M^\bullet \boxtimes N^\bullet)), C_X[2d_X]).
$$
and follows from $\text{DRD} \xrightarrow{\sim} \text{DDR}$ (cf. 2.7) in the analytic case. In the algebraic case we have to use also the commutativity of $\text{DR}$ (or $\text{An}$) with the direct image onto a point, where the regularity at infinity is essentially used. Here we need also the fully faithfulness of $D^b(\mathcal{D}_X)_{\text{qcoh}} \rightarrow D^b(\mathcal{D}_X)$, where the first (resp. second) is the bounded derived category of quasi-coherent (resp. not necessarily quasicoherent) $\mathcal{D}_X$-Modules, cf. [B]. This property is equivalent to the following (equivalent) left (resp. right) effaceability: for any quasicoherent $\mathcal{D}_X$-Modules $M, N$ and $e \in \mathcal{E}xt^k(M, N)$, there exists a surjection $u : M' \rightarrow M$ (resp. an injection $v : N \rightarrow N'$) such that $u^*e = 0$ in $\mathcal{E}xt^k(M', N)$ (resp. $v_*e = 0$ in $\mathcal{E}xt^k(M, N')$) for $k > 0$.

In the affine case the left effaceability is easily checked using a surjection from a free Module, and the general case is reduced to this case using the right effaceability and an affine covering with the adjunction for $j^*, j_*$, cf. [loc. cit.]. Then by the commutativity of the diagram we get the isomorphism of the left vertical morphism $\text{DR}$ and the fully faithfulness of the de Rham functor. It seems that the check of the commutativity of the above diagram is not so easy if $\text{DR}$ is replaced by $\text{Sol}$ (cf. 2.8), because $\text{Sol}$ is contravariant and the diagonal pairing is not preserved by $\text{Sol}$ as in the proof of 4.6.

4.8. Remark. — We can also define the duality isomorphism using the diagonal pairings. For $M = \mathcal{D}N$, $N \in D^b_{\text{qcoh}}(\mathcal{D}_X)$, we have a canonical element

$$c \in \text{Hom}_{D(\mathcal{D}_X \times X)}(M \boxtimes N, \delta_*\omega_X[d_X])$$

corresponding to $\text{id} \in \text{Hom}_{D(\mathcal{D}_X)}(M, \mathcal{D}N)$. By the direct image we get

$$f_*c \in \text{Hom}_{D(\mathcal{D}_Y \times Y)}(f_*M \boxtimes f_*N, \delta_*f_*\omega_X[d_X])$$

$$\text{Tr}_f \circ f_*c \in \text{Hom}_{D(\mathcal{D}_Y \times Y)}(f_*M \boxtimes f_*N, \delta_*\omega_Y[d_Y]).$$

For the perfectness of $\text{Tr}_f \circ f_*c$, i.e. the isomorphism of the corresponding morphism $f_*\mathcal{D}N \rightarrow f_*\mathcal{D}f_*\mathcal{D}N$, we use the filtered trace morphism $\text{Tr}_f : f_*\omega_X[d_X], F) \rightarrow (\omega_Y[d_Y], F)$ such that $\text{Gr}^0 F \text{Tr}_f$ is the analytic, or algebraic, trace morphism $\text{Tr}_f : \mathcal{R}f_*\omega_X[d_X] \rightarrow \omega_Y[d_Y]$ as in 3.11. Here note that $\text{Gr}^0 F (f_*\omega_X[d_X]) = \mathcal{R}f_*\omega_X[d_X]$ and $\text{Gr}^0 F \omega_Y[d_Y] = \omega_Y[d_Y]$ are invariant by the closed embeddings (in particular by $\delta_*$), because $0 = \min\{p : \text{Gr}^p F \omega_X \neq 0\}$ (same for $Y$).

4.9. Remark. — It would be possible to define the duality isomorphism (0.1) using the trace morphism of bimodules:

$$\text{Tr}_f : \mathcal{R}f_* \left( (\omega_X[d_X] \otimes \mathcal{D}_X) \otimes^{L}_{\mathcal{D}_X} \mathcal{D}_X \otimes_{Y} \mathcal{D}_Y \right) \rightarrow \omega_Y[d_Y] \otimes \mathcal{D}_Y.$$
(cf. [Sc1-2]). But to do so, we have to show a nontrivial isomorphism

\[(4.9.2) \quad (\omega_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y) \otimes_{\mathcal{D}_X} (\mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y) \]

\[\simeq (\omega_X \otimes_{\mathcal{D}_X} (\mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y)) \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y\]

as right $f^* \mathcal{D}_Y$-bi-Modules, and the relation with the trace morphism for $\mathcal{O}$-Modules and relative $\mathcal{D}$-Modules becomes nontrivial. As a consequence, it is not so clear that the duality isomorphism proved in [loc. cit.] is really induced by (4.9.1). Here we have to take the representative of $\mathcal{R}f_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_X \to Y)$, etc. to check the assertion. In fact, $\mathcal{R} \mathcal{H}om_{\mathcal{D}_X}(M, \omega_X[d_X] \otimes \mathcal{D}_X)$ cannot be defined without taking an injective resolution of $\omega_X[d_X] \otimes \mathcal{D}_X$, and $M \otimes_{\mathcal{D}_X} \mathcal{D}_X \to Y$ without a (flat) resolution of $M$ or $\mathcal{D}_X \to Y$, etc. and we have to check some commutativity of diagram in the level of complexes (compare to paragraph 3). In [Sc2], the isomorphism (4.9.2) seems to be reduced to the isomorphism

\[(4.9.3) \quad (\mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y) \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y) \]

\[\simeq (\mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y) \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y\]

as right $f^* \mathcal{D}_Y$-bi- and left $\mathcal{D}_X$-Modules, where the left $\mathcal{D}_X$-Module structure of $\mathcal{D}_X \to Y = \mathcal{O}_X \otimes f^* \mathcal{O}_Y f^* \mathcal{D}_Y$ is used for $\otimes_{\mathcal{O}_X}$ in the left hand side, and its right $f^* \mathcal{D}_Y$-Module structure for $\otimes f^* \mathcal{O}_Y$ in the right. Then we can prove it directly by $f \otimes P \otimes g \otimes Q \mapsto (fg \otimes 1 \otimes Q) t(P)$, where $t$ means the right $f^* \mathcal{D}_Y$-Module structure associated with the tensor of right and left $f^* \mathcal{D}_Y$-Modules over $\mathcal{O}_Y$ (cf. the proof of 1.7). Note that we can also use 4.8 to define (4.9.1), because (4.9.1) should be the direct image by $\delta$ of the trace morphism.

Here it should be also noted that the involution 1.7 must be used in the proof of $\text{id} \to D^2$ in [loc. cit.].

**BIBLIOGRAPHY**


