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SCHWARTZ'S THEOREM ON MEAN PERIODIC VECTOR-VALUED FUNCTIONS

BY

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1. Introduction and preliminaries

The theorem of L. SCHWARTZ on mean periodic functions of one variable states that every closed translation-invariant subspace of the space of continuous complex functions on \( \mathbb{R} \) is spanned by the polynomial-exponential functions it contains [4]. In [2, VII], J.-J. KELLEHER and B.-A. TAYLOR provide a characterization of all closed submodules of \( \mathbb{C}^N \)-valued entire functions of exponential type which have polynomial growth on \( \mathbb{R} \). By duality, their result generalizes Schwartz’s Theorem to \( \mathbb{C}^N \)-valued continuous functions.

Our goal is to provide a simple and a direct proof to this result.

\( C(\mathbb{R}, \mathbb{C}^N) \) denotes the space of continuous \( \mathbb{C}^N \)-valued functions on \( \mathbb{R} \), with the topology of uniform convergence on compact sets. By a vector-valued polynomial exponential in \( C(\mathbb{R}, \mathbb{C}^N) \), we mean a function of the form \( e^{\lambda x}p(x) \), \( x \in \mathbb{R} \), where \( \lambda \in \mathbb{C} \) and \( p \) is a polynomial in \( C(\mathbb{R}, \mathbb{C}^N) \).

THEOREM. — Every translation-invariant closed subspace of \( C(\mathbb{R}, \mathbb{C}^N) \) is spanned by the vector-valued polynomial-exponential functions it contains.

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For the theory of mean-periodic complex functions, we refer the reader to [4], [1], [3]. We need the following notations and results.

Let $M_0(\mathbb{R})$ denote the space of complex Radon measures on $\mathbb{R}$ having compact support. For $\mu \in M_0(\mathbb{R})$, the Laplace transform $\hat{\mu}$ of $\mu$ is the entire function defined by $\hat{\mu}(z) = \int e^{-zx}d\mu(x)$, $z \in \mathbb{C}$.

We remind that $f \in C(\mathbb{R})$ is mean periodic if $\mu * f = 0$ for some $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$. For $f \in C(\mathbb{R})$, $f^-$ is the function defined by $f^-(x) = f(x)$ if $x \leq 0$ and $f^-(x) = 0$ if $x > 0$. If $f$ is mean-periodic, $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$ and $\mu * f = 0$, then the function $\mu * f^-$ has compact support and the meromorphic function

$$F = (\mu * f^-)/\hat{\mu},$$

which does not depend on the choice of $\mu$, is defined to be the Laplace transform of $f$ ([3]).

The heart of our proof is the fact that $F$ is entire only if $f = 0$ (see [3, Theorem X]).

The dual of $C(\mathbb{R}, \mathbb{C}^N)$ is the space $M_0(\mathbb{R}, \mathbb{C}^N)$ of $\mathbb{C}^N$-valued Radon measures on $\mathbb{R}$ having compact supports. One notices that $M_0(\mathbb{R})$ is an integral domain under the convolution product and $M_0(\mathbb{R}, \mathbb{C}^N)$ is a module over $M_0(\mathbb{R})$ with the coordinatewise convolution. We denote the duality by

$$\langle \mu, f \rangle = \sum_{j=1}^{N} (\mu_j * f_j)(0)$$

for $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ and $f = (f_j) \in C(\mathbb{R}, \mathbb{C}^N)$. If $f$ is a vector-valued polynomial-exponential with

$$f_j(x) = \sum_{\ell=0}^{m} \alpha_j^{(\ell)} x^\ell e^{\lambda x} \quad (1 \leq j \leq N),$$

we have

$$\langle \mu, f \rangle = \sum_{j=1}^{N} \sum_{\ell=0}^{m} \alpha_j^{(\ell)} \hat{\mu}_j^{(\ell)}(\lambda).$$

For any subset $A$ of $C(\mathbb{R}, \mathbb{C}^N)$ let

$$A^\perp = \{ \mu \in M_0(\mathbb{R}, \mathbb{C}^N); \langle \mu, f \rangle = 0 \text{ for all } f \in A \}.$$
By duality, $V$ is spanned by $\text{Sp}(V)$ if and only if $\text{Sp}(V) \perp \subset V \perp$. Since $V$ is translation-invariant, $V \perp$ is a submodule of $M_0(\mathbb{R}, \mathbb{C}^N)$ and $\mu = (\mu_j) \in V \perp$ if and only if

$$\sum_{j=1}^{N} \mu_j \ast f_j = 0 \quad \text{for all } f = (f_j) \in V.$$

2. Main result

In this section, $V$ denotes a given translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$. We have to prove $(\mu, f) = 0$ for any $\mu \in \text{Sp}(V) \perp$ and $f \in V$. We need some more notation and three lemmas.

Let $0 \leq r \leq N$ be the rank of $V \perp$ as a module over $M_0(\mathbb{R})$. That means $r$ is the greatest integer for which there exists a system $(\sigma_{\ell,j})_{1 \leq \ell \leq r}$, where $\sigma_{\ell} = (\sigma_{\ell,j})_{1 \leq j \leq N} \in V \perp$ for $1 \leq \ell \leq r$ and with a non-zero determinant of order $r$. We shall suppose given such a system with, say,

$$\rho = \det (\sigma_{\ell,j} ; 1 \leq \ell, j \leq r) \neq 0.$$

One notices that $\rho$ is the non identically zero entire function given by

$$\hat{\rho}(\lambda) = \det (\hat{\rho}_{\ell,j}(\lambda) ; 1 \leq \ell, j \leq r), \quad \lambda \in \mathbb{C}.$$

If $r = 0$, i.e. $V \perp = \{0\}$, we take for $\rho$ the Dirac measure at 0 and $\hat{\rho}(\lambda) = 1$, $\lambda \in \mathbb{C}$.

For $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ let

$$\Delta_j(\mu) = \det \begin{vmatrix} \mu_1 & \cdots & \mu_r & \mu_j \\ \sigma_{1,1} & \cdots & \sigma_{1,r} & \sigma_{1,j} \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{r,1} & \cdots & \sigma_{r,r} & \sigma_{r,j} \end{vmatrix} \quad \text{(for } 1 \leq j \leq N)$$

and

$$\tau_\ell(\mu) = \det \begin{vmatrix} \sigma_{1,1} & \cdots & \sigma_{1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{\ell-1,1} & \cdots & \sigma_{\ell-1,r} \\ \mu_1 & \cdots & \mu_r \\ \sigma_{\ell+1,1} & \cdots & \sigma_{\ell+1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{r,1} & \cdots & \sigma_{r,r} \end{vmatrix} \quad \text{(for } 1 \leq \ell \leq r).$$
From the definition of $r$, for any $\mu \in V^\perp$

(1) \[ \Delta_j(\mu) = 0 \quad (\text{for } 1 \leq j \leq N). \]

By expanding the $\Delta_j(\mu)$ along the last column, (1) is equivalent to

(2) \[ \rho \ast \mu_j = \sum_{\ell=1}^{r} \tau_{\ell}(\mu) \ast \sigma_{\ell,j} \quad (\text{for } 1 \leq j \leq N). \]

**Lemma 1.** — Let $\lambda \in \mathbb{C}$ such that $\hat{\rho}(\lambda) \neq 0$. For $\alpha = (\alpha_j) \in \mathbb{C}^N$, the vector-exponential $e^{\lambda x} \cdot \alpha$ belongs to $V$ if and only if

(3) \[ \sum_{j=1}^{N} \alpha_j \hat{\sigma}_{\ell,j}(\lambda) = 0 \quad 1 \leq \ell \leq r. \]

**Proof.** — Let $\alpha \in \mathbb{C}^N$. We have $e^{\lambda x} \cdot \alpha \in V$ if and only if, for every $\mu = (\mu_j) \in V^\perp$,

(4) \[ \langle \mu, e^{\lambda x} \cdot \alpha \rangle = \sum_{j=1}^{N} \alpha_j \hat{\mu}_j(\lambda) = 0. \]

This proves the "only if" part. Conversely, since $\hat{\rho}(\lambda) \neq 0$, (2) implies that for any $\mu \in V^\perp$ the equation in (4) is a linear combination of the equations (3).

**Lemma 2.** — Let $\mu \in M_0(\mathbb{R}, \mathbb{C}^N)$. If $\langle \mu, e^{\lambda x} \cdot \alpha \rangle = 0$ for all $\lambda \in \mathbb{C}$ such that $\hat{\rho}(\lambda) \neq 0$ and $\alpha \in \mathbb{C}^N$ such that $e^{\lambda x} \cdot \alpha \in V$, then $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.

**Proof.** — Let $\lambda \in \mathbb{C}$ with $\hat{\rho}(\lambda) \neq 0$. If $\mu$ satisfies the hypothesis, the solutions of (3) are solutions of (4), which implies that the determinants $\Delta_j(\mu)(\lambda)$ for $1 \leq j \leq N$ are equal to zero. Then, since $\hat{\rho}$ and the $\Delta_j(\mu)(\lambda)$ are entire functions and $\hat{\rho} \neq 0$, the $\Delta_j(\mu)(\lambda)$ are identically zero. Hence, $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.

**Remark.** — Lemma 2 shows that any $\mu \in \text{Sp}(V)^\perp$ satisfies (1) and (2). If $r = 0$, $\Delta_j(\mu) = \mu_j$ for $1 \leq j \leq N$; hence $\text{Sp}(V)^\perp = \{0\}$ if $V^\perp = \{0\}$.

**Lemma 3.** — Let $\lambda \in \mathbb{C}$, $m \geq 0$ and $\mu \in \text{Sp}(V)^\perp$. There exists $\nu \in V^\perp$ such that

\[ \hat{\nu}_j^{(\ell)}(\lambda) = \hat{\mu}_j^{(\ell)}(\lambda) \quad (\text{for } 1 \leq j \leq N, 0 \leq \ell < m). \]
Proof. — Suppose the element \((\hat{\nu}_j^{(\ell)}(\lambda))_{1 \leq j < N, 0 \leq \ell < m}\) of \(\mathbb{C}^N\) does not belong to the subspace

\[
M(\lambda, m) = \{ (\nu^{(\ell)}(\lambda))_{1 \leq j \leq N, 0 \leq \ell < m} ; \nu \in V_{\perp} \}.
\]

Then there exists \((\alpha_j^{(\ell)})_{1 \leq j \leq N, 0 \leq \ell < m}\) such that

\[
\sum_{j=1}^{N} \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} \hat{\nu}_j^{(\ell)}(\lambda) = 0 \quad \text{for} \ \nu \in V_{\perp}
\]

and

\[
\sum_{j=1}^{N} \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} \hat{\mu}_j^{(\ell)}(\lambda) \neq 0.
\]

Then if

\[
f_j(x) = \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} x^\ell \quad \text{(for} \ 1 \leq j \leq N),
\]

the polynomial-exponential \(f = (f_j)_{1 \leq j \leq N}\) satisfies

\[
\langle \nu, f \rangle = 0 \quad \text{(for} \ \nu \in V_{\perp}),
\]

therefore \(f \in \text{Sp}(V)\), and

\[
\langle \mu, f \rangle \neq 0,
\]

and we have a contradiction, since \(\mu \in \text{Sp}(V)_{\perp}\).

Proof of the Theorem. — Let \(\mu = (\mu_j) \in \text{Sp}(V)_{\perp}, f = (f_j) \in V\) and

\[
g = \sum_{j=1}^{N} \mu_j * f_j.
\]

We have to prove that \(g = 0\). By Lemma 2, \(\Delta_j(\mu) = 0\) for \(1 \leq j \leq N\) and \(\mu\) verifies (2); therefore

\[
\rho * \sum_{j=1}^{N} \mu_j * f_j = \sum_{\ell=1}^{r} (\tau_{\ell}(\mu) * \sum_{j=1}^{N} \sigma_{\ell,j} * f_j).
\]

For \(1 \leq \ell \leq r\), since \(\sigma_{\ell} \in V_{\perp}\), we have \(\sum_{j=1}^{N} \sigma_{\ell,j} * f_j = 0\). So

\[
\rho * g = 0.
\]
Hence $g$ is mean-periodic and the Laplace transform $G$ of $g$ may be defined by

$$G = (\rho * g^-)/\hat{\rho}.$$  

By ([3, Theorem X]) it is enough to prove that $G$ is entire.

If $[a, b]$ is any interval that contains the supports of the $\mu_j$ $(1 \leq j \leq N)$, $\sum \mu_j * f_j^-(x)$ is equal to $g(x)$ for $x < a$ and 0 for $x > b$. Thus the function

$$s = g^- - \sum_{j=1}^{N} \mu_j * f_j^-$$

has compact support. For $1 \leq \ell \leq r$, let

$$h_\ell = \sum_{j=1}^{N} \sigma_{\ell,j} * f_j^-.$$  

By the same argument, the functions $h_\ell$ have compact supports and, by (2),

$$\rho \sum_{j=1}^{N} \mu_j * f_j^- = \sum_{\ell=1}^{r} \tau_\ell(\mu) * h_\ell.$$  

So

$$\rho * g^- = \sum_{\ell=1}^{r} \tau_\ell(\mu) * h_\ell + \rho * s;$$

(5)\[ G = \frac{1}{\hat{\rho}} \sum_{\ell=1}^{r} \tau_\ell(\mu^-) \cdot \hat{h}_\ell + \hat{s}. \]

The functions $\hat{s}$ and $\hat{h}_\ell$ $(1 \leq \ell \leq r)$ are entire, as Laplace transforms of compactly supported functions.

For any $\nu \in V^\perp$, since $\sum \nu_j * f_j = 0$, $\sum \nu_j * f_j^-$ has compact support, and it follows by (2) that the function

(6)\[ \frac{1}{\hat{\rho}} \sum_{\ell=1}^{r} \tau_\ell(\nu^-) \cdot \hat{h}_\ell \quad \text{is entire.} \]

Let $\lambda \in \mathbb{C}$ and let $m$ be the order of $\hat{\rho}$ at $\lambda$. By Lemma 3, we can choose $\nu \in V^\perp$ so that $\hat{\nu}_j^{(k)}(\lambda) = \hat{\mu}_j^{(k)}(\lambda)$ for $1 \leq j \leq N$, $0 \leq k < m$. Then the functions $(\nu_j - \bar{\mu}_j)/\hat{\rho}$ for $1 \leq j \leq N$ and the functions

$$\frac{1}{\hat{\rho}} (\tau_\ell(\nu^-) - \tau_\ell(\mu^-)) \quad \text{(for } 1 \leq \ell \leq r)$$

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are analytic at $\lambda$. It follows from (5) and (6) that $G$ is analytic at $\lambda$.

Since $\lambda$ is arbitrary, $G$ is entire. That completes the proof of the Theorem.

BIBLIOGRAPHY


