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# TWISTED TENSORS AND EULER PRODUCTS

BY

YUVAL Z. FLICKER (*)

RÉSUMÉ. — Soit $L(s, r(\pi), V) = \Pi_v \det[1 - q_v^{-s} r(t(\pi_v))]^{-1}$ $(v \notin V)$ le produit Eulérien attaché à une représentation cuspidale (irréductible, automorphe, unitaire) $\pi$ du groupe adélique $\mathrm{GL}(n, \mathbb{A}_E)$, où $E/F$ est une extension quadratique de corps globaux, et $r$ est la représentation "tenseur tordue" de $\widehat{G} = [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})] \times \mathrm{Gal}(E/F)$ sur $\mathbb{C}^n \otimes \mathbb{C}^n$. La fonction $L(s, r(\pi), V)$ a un prolongement méromorphe au $s$-plan tout entier avec une équation fonctionnelle $s \leftrightarrow 1 - s$; ses singularités sont simples; elle est holomorphe pour chaque $s \neq 0, 1$, et elle a une singularité pour $s = 1$ si et seulement si $\pi$ est distinguée.

ABSTRACT. — Let $L(s, r(\pi), V) = \Pi_v \det[1 - q_v^{-s} r(t(\pi_v))]^{-1}$ $(v \notin V)$ be the Euler product attached to a cuspidal (irreducible automorphic unitary) representation $\pi$ of the adele group $\mathrm{GL}(n, \mathbb{A}_E)$; here $E/F$ is a quadratic extension of global fields, and $r$ is the "twisted tensor" representation of $\widehat{G} = [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})] \times \mathrm{Gal}(E/F)$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. It is shown that $L(s, r(\pi), V)$ has meromorphic continuation to the entire $s$-plane with a functional equation $s \leftrightarrow 1 - s$; its poles are simple; it is holomorphic at any $s \neq 0, 1$; it has a pole at $s = 1$ if and only if $\pi$ is distinguished.

## 0. Introduction

Let $E$ be a cyclic extension of prime degree $e$ of a global field $F$. Denote by $\mathbb{A}, \mathbb{A}_E$ the rings of adeles of $F, E$. Put $G$ for the multiplicative group of a simple algebra of rank $n$, central over $F$ (thus $G$ is an inner form of $\mathrm{GL}(n)$). Fix a cuspidal (irreducible unitary automorphic) representation $\pi$ of the adele group $G(\mathbb{A}_E)$. There is a finite set $V$ of places of $F$, depending on $\pi$, including the places where $G$ or $E/F$ ramify, and the archimedean places, such that : for each place $v'$ of $E$ above a place $v$ outside $V$ the component $\pi_{v'}$ of $\pi$ is unramified. Thus for each such $v'$ there is an unramified character $(a_{ij}) \to \Pi_i \mu_{iv'}(a_{ii})$ $(1 \leq j \leq n)$ of the upper

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triangular subgroup $B(E_{v'})$ of $G(E_{v'})$, and $\pi_{v'}$ is the unique unramified constituent in the composition series of the (unramified) representation $I((\mu_{iv'}))$ unitarily induced from $(\mu_{iv'})$. Let $\boldsymbol{\pi} = \boldsymbol{\pi}_v$ be a uniformizer of $F_v$. Denote by $t_{v'} = t(\pi_{v'})$ the semi-simple conjugacy class in $\mathrm{GL}(n,\mathbb{C})$ with eigenvalues $(\mu_{iv'}(\boldsymbol{\pi}))$. For each $v'$ the map $\pi_{v'} \to t(\pi_{v'})$ is a bijection from the set of equivalence classes of irreducible unramified $G_{v'} = G(E_{v'})$-modules to the set of semi-simple conjugacy classes in $G(\mathbb{C})$. Fix a generator $\sigma$ of the cyclic galois group $\mathrm{Gal}(E/F)$. Put $\widehat{G}$ for the semi-direct product of $G(\mathbb{C}) \times \cdots \times G(\mathbb{C})$ ($e$ copies) with $\mathrm{Gal}(E/F)$, where $\sigma$ acts by $\sigma(x_1, x_2, \ldots, x_e) = (x_e, x_1, x_2, \ldots)$. If $v$ outside $V$ splits into $v', v'', \ldots$ in $E$, the component $\pi_v = \pi_{v'} \times \pi_{v''} \times \cdots$ defines a conjugacy class $t_{v'} \times t_{v''} \times \cdots$ in $G(\mathbb{C}) \times G(\mathbb{C}) \times \cdots$, and a conjugacy class $t_v = t(\pi_v) = (t_{v'} \times t_{v''} \times \cdots) \times 1$ in $\widehat{G}$. If $v$ outside $V$ is inert in $E$, and $v'$ is the place of $E$ above $v$, then we put $\pi_v$ for $\pi_{v'}$. $\pi_v$ defines a conjugacy class $t_{v'}$ in $G(\mathbb{C})$, namely the conjugacy class $t_v = (t_{v'} \times 1 \times \cdots \times 1) \times \sigma$ in $\widehat{G}$.

For any finite $m$-dimensional representation $r$ of $\widehat{G}$, it would be nice to know the existence of an automorphic representation $r(\pi)$ of $\mathrm{GL}(m, \mathbb{A})$ whose component at $v$ outside $V$ is unramified and parametrized by $r(t(\pi_v))$. But this is an aim for the future. Let $q = q_v$ be the cardinality of the residue field $R_v/\boldsymbol{\pi} R_v$ of the ring $R_v$ of integers in $F_v$. In lieu of $r(\pi)$ we associate here to $\pi, V, r$ the function

$$L(s, r(\pi), V) = \prod_v \det\big[1 - q_v^{-s} r(t_v)\big]^{-1} \qquad (v \text{ outside } V)$$

of the complex variable $s$. The existence of $r(\pi)$ would have many consequences, and in particular it would yield much information about the analytic behavior of $L(s, r(\pi), V)$. We shall confine ourselves here to the study of $L(s, r(\pi), V)$ when (1) $G$ is the split group $\mathrm{GL}(n)$, (2) $e = 2$, thus $E$ is quadratic over $F$, and (3) $r$ is the twisted tensor representation of $\widehat{G}$ on $\mathbb{C}^n \otimes \mathbb{C}^n$, which acts by $r((a,b))(x \otimes y) = ax \otimes by$ and $r(\sigma)(x \otimes y) = y \otimes x$. We adopt the convention that if the restriction of the central character $\omega_\pi$ of $\pi$ to the group $\mathbb{A}^\times$ of ideles of $F$ is unramified, then $\omega_\pi$ is trivial on $\mathbb{A}^\times$, since we can replace $\pi$ by its product with an unramified character.

If $G$ is any inner form of $\mathrm{GL}(n, F)$ we can make the following

*Definition.* — $\pi$ is called *distinguished* if its central character is trivial on $\mathbb{A}^\times$ and there is an automorphic form $\phi$ in the space of $\pi$ in $L^2(G(E)\backslash G(\mathbb{A}_E))$ whose integral $\int \phi(g)\,dg$ over the closed subspace $G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ of $G(E)Z(\mathbb{A}_E)\backslash G(\mathbb{A}_E)$ is non-zero.

So when $G = \mathrm{GL}(n)$, $[E : F] = 2$ and $r$ is the twisted tensor, and each archimedean place of $F$ splits in $E$, we prove

THEOREM. — *The product $L(s, r(\pi), V)$ converges absolutely, uniformly in compact subsets, in some right half-plane. It has meromorphic continuation to the entire complex plane, with a functional equation $L(1-s, r(\tilde{\pi}), V) = \epsilon(s)L(s, r(\pi), V)$; $\epsilon(s)$ is a product over $v$ in $V$ of the meromorphic functions $\epsilon(s, \pi_v)$ which are holomorphic on $\mathrm{Re}\, s \geq 1$ and $\mathrm{Re}\, s \leq 0$; $\tilde{\pi}$ is the contragredient of $\pi$. The only possible pole of $L(s, r(\pi), V)$ in $\mathrm{Re}\, s \geq 1$ is simple, located at $s = 1$. $L(s, r(\pi), V)$ has a pole at $s = 1$ precisely when $\pi$ is distinguished. $L(s, r(\pi), V)$ has no zeroes on the edge $\mathrm{Re}\, s = 1$ of the critical strip. If $F$ is a function field then the only possible poles of $L(s, r(\pi), V)$ are simple, located at $s = 1$ and $0$.*

The fact that our $L$-function $L(s, r(\pi), V)$ has meromorphic continuation and functional equation is well-known. Indeed, consider the group $\widehat{H} = \mathrm{GL}(2n, \mathbb{C}) \times \mathrm{Gal}(E/F)$, where $\sigma$ acts on $g$ in $\mathrm{GL}(2n, \mathbb{C})$ by $g \mapsto J^t g^{-1} J^{-1}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $I$ indicates the identity $n \times n$ matrix in $\mathrm{GL}(n, \mathbb{C})$. We may view $\widehat{G} = [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})] \times \mathrm{Gal}(E/F)$ as the diagonal Levi subgroup of type $(n, n)$ in $\widehat{H}$. The adjoint action of $\widehat{H}$ on the unipotent radical of the upper triangular parabolic subgroup with Levi component $\widehat{G}$ is equivalent to our twisted tensor representation $r$, since

$$\begin{pmatrix} x & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} I & m \\ 0 & I \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} I & xmy \\ 0 & I \end{pmatrix},$$

and

$$\sigma \begin{pmatrix} I & m \\ 0 & I \end{pmatrix} \sigma^{-1} = \begin{pmatrix} I & {}^t m \\ 0 & I \end{pmatrix};$$

a basis for the space of $n \times n$ matrices $m$ is given by $v_i \otimes v_j$ $(1 \leq i, j \leq n)$, and with the standard choice of basis the transpose $t$ maps $v_i \otimes v_j$ to $v_j \otimes v_i$. Thus Langlands' technique of applying Eisenstein series to the study of Euler products is applicable in our case. Hence it follows from [Sha, Theorem 4.1], that $L(s, r(\tau), V)$ has meromorphic continuation and functional equation, and from Theorem 5.1 there that $L(s, r(\pi), V)$ *has no zero on the edge* $\mathrm{Re}\, s = 1$ *of the critical strip.* However this technique does not yield the complete information about the location of poles given above.

When $n = 2$ we check in paragraph 4 that a cuspidal representation $\pi$ of $G(\mathbb{A}_E)$ with a trivial central character $\omega$ is distinguished precisely when it is the basechange lift of a cuspidal representation $\pi_0$ of $G(\mathbb{A})$ whose central character $\omega_0$ is the (unique) non-trivial character of $\mathbb{A}^\times/F^\times N\mathbb{A}_E^\times$. It will be interesting to find a similar characterization of distinguished representations for a general $n$.

Let $G'$ be the multiplicative group of a simple algebra of rank $n$, central over $F$, such that $E/F$ splits at each place where $G'$ ramifies. Let $\pi'$ be an irreducible automorphic representation of $G'(\mathbb{A}_E)$ with a trivial central character which corresponds by the Deligne–Kazhdan correspondence [F2] to a cuspidal $G(\mathbb{A}_E)$-module $\pi$ with the following property : there are two places $v'$, $v''$ of $E$ whose restrictions to $F$ are distinct, such that the component $\pi_{v'}$ of $\pi$ is supercuspidal, and $\pi_{v''}$ is square-integrable. Then $L(s, r(\pi'), V)$ is equal, hence has the same analytic properties, to $L(s, r(\pi), V)$, and we have

COROLLARY. — *$L(s, r(\pi'), V)$ has a pole at $s = 1$ if and only if $\pi'$ is distinguished .*

Indeed, the Theorem of [F1] asserts that $\pi'$ is distinguished if and only if so is $\pi$. Note that :

(1) The condition in [F1] that $E/F$ be split at each place where $G'$ ramifies is not hard to remove. We hope to show this in another paper.

(2) The assumption that $\pi$ has a discrete series component at the second place $v''$ of $E$ can be removed on using the methods of [FK1] or [F5]. This is also delayed to another paper.

When $n = 2$, $F = Q$ and $E$ is a totally real quadratic extension with class number one, our Theorem is due to SHIMURA and ASAI [A]. If $n = 2$ our Corollary holds also when the component of $\pi$ at $v'$ is special, not only supercuspidal, by virtue of the Theorem of JACQUET and LAI [JL]. Our proof of the Theorem follows closely the Rankin-Selberg technique of JACQUET-SHALIKA [JS], who established similar properties of the product

$$L(s, \pi \otimes \pi', V) = \prod_v \det\big[1 - q_v^{-s} t(\pi_v) \otimes t(\pi'_v)\big]^{-1}$$

associated with two cusp forms $\pi = \otimes\pi_v$, $\pi' = \otimes\pi'_v$ of $\mathrm{GL}(n, \mathbb{A})$.

Our initial interest in the Theorem was in its possible application to the proof of the Tate conjecture on algebraic cycles, in the case of schemes obtained by restriction of scalars from Drinfeld's moduli schemes of elliptic modules (with arbitrary rank); see [FK2], [FK3]. Here the base field $F$ is a function field, namely a global field of positive characteristic. We do not discuss these applications here, as they require a separate paper. In particular, we do not carry out here a discussion of the Whittaker theory at the archimedean places, although perhaps this would be of some interest. We refer the interested reader to [JS] for a discussion of the archimedean

Whittaker theory at a place of $F$ which splits in $E$, and restrict our attention to global fields $E/F$ such that each archimedean place of $F$ splits in $E$. This is the most interesting case for us.

In computing the Euler factors we use in addition to Shintani's formula [Sh] (see also [CS], and [F3] for a related formula) for the unramified Whittaker function, also standard combinatorial identities (*see* MACDONALD [M]) which are likely to be useful in the study of other Euler products.

Finally note that the case of $[E : F] = 3$ (as well as the case $E = F \oplus F \oplus F$ and $E = E' \oplus F$ with $[E' : F] = 2$) and $n = 2$ was dealt with classically by GARRETT [G], adelically by PIATETSKI-SHAPIRO and RALLIS [PR], using an integral expression on the rank three symplectic group. It will be interesting to establish a higher rank ($n > 2$) analogue.

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## 1. Notations

Identify $\mathrm{GL}(n-1)$ with a subgroup of $G = \mathrm{GL}(n)$ *via* $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let $U$ be the unipotent radical of the upper triangular parabolic subgroup of type $(n-1, 1)$. Put $P = \mathrm{GL}(n-1)U$. Given a local field $F$, let $S(F^n)$ be the space of smooth and rapidly decreasing (if $F$ is archimedean), or locally-constant compactly-supported (otherwise) complex-valued functions on $F$. Denote by $\Phi^0$ the characteristic function of $R^n$ in $F^n$, if $F$ is non-archimedean and $R$ is its ring of integers. For a global field $F$ let $S(\mathbb{A}^n)$ be the linear span of the functions $\Phi = \otimes \Phi_v$, $\Phi_v$ in $S(F_v^n)$ for all $v$, $\Phi_v$ is $\Phi_v^0$ for all but finitely many $v$. Fix a non-trivial additive character $\psi_0 = \otimes \psi_{0v}$ of $\mathbb{A}$ mod $F$. Denote by $x \cdot y$ the scalar product of two $n$-vectors. Fix a product $dy$ of self-dual Haar measures $dy_v$ on $F_v^n$. The Fourier transform of $\Phi = \otimes \Phi_v$ in $S(\mathbb{A}^n)$ is :

$$\hat{\Phi}(x) = \int_{\mathbb{A}^n} \Phi(y)\psi_0(x \cdot y)\, dy = \prod_v \hat{\Phi}_v(x_v), \qquad x = (x_v),$$

where

$$\hat{\Phi}_v(x_v) = \int_{F_v^n} \Phi_v(y_v)\psi_{0v}(x_v \cdot y_v)\, dy_v.$$

Let $N$ be the unipotent radical of the upper triangular subgroup of $G$. Then $\psi_0$ defines the character $\psi_0(n) = \psi_0(\sum_{i=1}^{n-1} n_{ii+1})$ of $N$, locally and globally.

Let $E$ be a quadratic field extension of a local non-archimedean field $F$. Denote by $x \to \bar{x}$ the non-trivial automorphism of $E$ over $F$. Put

$\widetilde{G} = G(E)$. Let $\psi$ be a non-trivial character of $E$ modulo $F$, for example that given by $\psi(x) = \psi_0((x - \bar{x})/(y - \bar{y}))$ for a fixed $y$ in $E - F$. Fix an irreducible algebraic (hence admissible [BZ]) representation $\pi$ of $\widetilde{G}$ on a complex vector space $V$. The triple $(\pi, \widetilde{G}, V)$ is called *generic* if there exists a non-zero linear form $\lambda$ on $V$ with $\lambda(\pi(n)v) = \psi(n)\lambda(v)$ for all $v$ in $V$ and $n$ in $\widetilde{N} = N(E)$. The dimension of the space of such $\lambda$ is bounded by one ([GK]). Let $W(\pi\,;\ \psi)$ be the space of all functions $W$ on $\widetilde{G}$ of the form $W(g) = \lambda(\pi(g)v)$ $(v$ in $V)$. The space $W(\pi\,;\ \psi)$ is invariant under right translations by $\widetilde{G}$. It is equivalent to $(\pi, \widetilde{G}, V)$ as a $\widetilde{G}$-module. For $W$ in $W(\pi\,;\ \psi)$ we have $W(ng) = \psi(n)W(g)$ $(n$ in $\widetilde{N}$, $g$ in $\widetilde{G})$. Let $K(\pi\,;\ \psi)$ be the space of functions $\phi = W|\widetilde{P}$ obtained on restricting to $P$ the $W$ of $W(\pi\,;\ \psi)$. The natural map $W(\pi\,;\ \psi) \to K(\pi\,;\ \psi)$ is a bijection. We may identify $V$ with $K(\pi\,;\ \psi)$; then $(\pi(p)\phi)(p') = \phi(p'p)$ for $p, p'$ in $\widetilde{P}$. Let $\tau_0 = \operatorname{ind}(\psi\,;\ \widetilde{P}, \widetilde{N})$ be the right representation of $\widetilde{G}$ on the space $K_0$ of functions on $\widetilde{P}$ which (1) transform on the left by $\psi$ under $\widetilde{N}$, (2) are compactly supported on $\widetilde{N}\backslash\widetilde{P}$, (3) are right invariant under some open compact subgroup of $\widetilde{P}$. Then $(\tau_0, K_0)$ is independent of $\psi$, and for each generic $\pi$, $(\pi|\widetilde{P}, K(\pi\,;\ \psi))$ contains a copy of $(\tau_0, K_0)$; see [BZ], [GK].

## 2. Eisenstein Series

Let $F$ be a global field. Put $G = G(F)$ and $Z = Z(F)$ for its center, $P = P(F)$, etc. Fix a unitary character $\omega$ of $\mathbb{A}^\times/F^\times$, and $\Phi$ in $S(\mathbb{A}^n)$. For $\epsilon = (0, \ldots, 0, 1)$ in $\mathbb{A}^n$ and $g$ in $G(\mathbb{A})$ the integral in

$$f(g, s) = |g|^s \int_{\mathbb{A}^\times} \Phi(a\epsilon g)|a|^{ns}\omega(a)d^\times a$$

converges absolutely, uniformly in compact subsets of $\operatorname{Re} s \geq 1/n$. We write $|g|$ for $|\det g|$, and the valuation is normalized as usual. It follows from Lemmas (11.5), (11.6) of [GJ] that the Eisenstein series

$$E(g, \Phi, s) = \sum f(\gamma g, s), \qquad (\gamma \text{ in } ZP\backslash G),$$

converges absolutely in $\operatorname{Re} s > 1$.

Let $A$ be the diagonal subgroup of $G$ and $A_\infty$ the subgroup of $a = (a_v)$ in $A(\mathbb{A})$ with $a_v = 1$ if $v$ is finite, and $a_{v'} = a_{v''}$ if $v'$ and $v''$ are any archimedean places of $F$. Let $A_t$ (for $t > 0$) be the group of $a \in A_\infty$ with $\det a = 1$ and $|a_{i,v}/a_{i+1,v}| \geq t$ for each component $a_v = (a_{1v}, \ldots, a_{nv})$ of $a$ at an archimedean place $v$, and $1 \leq i < n$. A continuous function $\Phi(g, s)$

of $g$ in $G\backslash G(\mathbb{A})$ and $s$ in $\operatorname{Re} s > 0$, which is holomorphic in $s$ for each fixed $g$, is called *slowly increasing* if for any compact subsets $C$ in $G(\mathbb{A})$ and $J$ in $\operatorname{Re} s > 0$, and any positive $t$, there exists a positive number $c$ and a positive integer $m$ so that

$$\left|\Phi(ax, s)\right| \leq c \sup_{1\leq i<n} \left|a_{i,v}/a_{i+1,v}\right|^m$$

for all $a$ in $A_t$, archimedean places $v$, $x$ in $C$ and $s$ in $J$.

LEMMA [JS, (4.2), p. 545]. — *The function $E(g, \Phi, s)$ extends to a meromorphic function on* $\operatorname{Re} s > 0$. *If $\omega$ is non-trivial on the group $\mathbb{A}^1$ of ideles $x$ with $|x| = 1$ then $E(g, \Phi, s)$ is holomorphic on* $\operatorname{Re} s > 0$, *and slowly increasing. There is $c \neq 0$ such that if $\omega = \nu^{i\sigma}$, where $\sigma$ is real and $\nu(x) = |x|$, then,*

$$E(g, \Phi, s) = \frac{c\,\hat{\Phi}(0)}{|g|^{i\sigma/n}\left(s - 1 + \dfrac{i\sigma}{n}\right)} + R(g, s),$$

*where $R(g, s)$ is holomorphic in* $\operatorname{Re} s > 0$ *and slowly increasing. Moveover, we have $E(g, \Phi, s) = E({}^t g^{-1}, \hat{\Phi}, 1 - s)$ on $0 < \operatorname{Re} s < 1$, where ${}^t g$ is the transpose of $g$.*

Let $\omega$ be a unitary character of $\mathbb{A}_E^{\times}/E^{\times}$, where $E$ is a quadratic field extension of $F$. Denote by $L_\omega = L_0^2(\omega, G(E)\backslash G(\mathbb{A}_E))$ the space of complex valued functions $\phi$ on $G(E)\backslash G(\mathbb{A}_E)$ which are right $K_E$-finite (where $K_E = \prod K(E_v)$ denotes the standard maximal compact subgroup of $G(\mathbb{A}_E)$), transform under the center $Z(\mathbb{A}_E)$ by $\omega$, and are absolutely square integrable on $Z(\mathbb{A}_E)G(E)\backslash G(\mathbb{A}_E)$, which are cuspidal. Thus for any $x$ in $G(\mathbb{A}_E)$, and any proper parabolic subgroup of $G$ over $E$ whose unipotent radical we denote by $R$, we have $\int \phi(nx)dn = 0$ ($n$ in $R(E)\backslash R(\mathbb{A}_E)$). Let $\pi$ be a cuspidal representation of $G(\mathbb{A}_E)$, namely an irreducible constituent of the representation of $G(\mathbb{A}_E)$ on $L_\omega$ by right translation. It is unitary, since $\omega$ is unitary. Each such $\phi$ is *rapidly decreasing*; namely for any compact set $C$ in $G(\mathbb{A}_E)$, positive $t$ and positive integer $m$, there is a positive constant $c$ such that $|\phi(ax)| \leq c\,(\max_{1\leq i<n} |a_{i,v}/a_{i+1,v}|)^{-m}$ for all $a$ in $A_t$, $x$ in $C$, and archimedean valuation $v$.

For any global field $R$ put $C_R$ for $\mathbb{A}_R^{\times}/R^{\times}$. The restriction to $C_F$ of the character $\omega$ of $C_E$ was used in the definition of the Eisenstein series. If the restriction of $\omega$ to $C_F$ is of the form $\omega(x) = |x|^{i\sigma}$ for a real $\sigma$ ($x$ in $C_F$), then we assume that $\omega$ is 1 on $C_F$ on multiplying $\pi$ by $\nu_E^{-2i\sigma/n}$, to simplify the notations. Here $\nu_E(x) = |x|$ on $C_E$. For $a$ in $Z(\mathbb{A}) = C_F$ we have $E(ag,\Phi,s)=\bar\omega(a)E(g,\Phi,s)$ for large $\operatorname{Re} s$ by definition, and for all $s$ by analytic continuation. Hence we can introduce

$$I(s,\Phi,\phi)=\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} E(g,\Phi,s)\phi(g)\,dg.$$

If $\omega\neq 1$ on $C_F$ the Lemma implies that the integrand is continuous on $[G(F)\backslash G(\mathbb{A})]\times\{s\mid \operatorname{Re} s>0\}$, holomorphic in $s$, and uniformly bounded in compact sets of $\operatorname{Re} s>0$. Since the volume of $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is finite, the integral converges to a holomorphic function of $s$ in $\operatorname{Re} s>0$.

If $\omega=1$ on $C_F$ the Lemma implies that

$$I(s,\Phi,\phi)=\frac{c\,\hat\Phi(0)}{s-1}\int\phi(g)dg+\int R(g,s)\phi(g)dg,$$

the integrals are over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. The second integral is again holomorphic on $\operatorname{Re} s>0$ and we conclude

COROLLARY. — *The only possible pole of $I(s,\Phi,\phi)$ in $\operatorname{Re} s>0$ is simple, and located at $s=1$. $I(s,\Phi,\phi)$ has a pole at $s=1$ if and only if $\omega=1$ on $C_F$, $\hat\Phi(0)\neq 0$ and $\int\phi(g)\,dg\neq 0$.*

Since $\pi$ is irreducible it is a restricted product $\otimes\pi_v$ of local representations $\pi_v$ of $G(E_v)$ $(=G(E_{v'})\times G(E_{v''})$ if $v$ splits into $v'$, $v''$ in $E$, when we put $\pi_v=\pi_{v'}\otimes\pi_{v''})$. They are all generic since $\pi$ is cuspidal, unitary since $\pi$ is unitary, and unramified for almost all $v$. We fix a non-trivial character $\psi=\otimes\psi_v$ of $\mathbb{A}_E$ modulo $E+\mathbb{A}$, for example by setting $\psi(x)=\psi_0((x-\bar x)/(y-\bar y))$ ($x$ in $\mathbb{A}_E$), where $y$ is a fixed element of $E-F$, and $\psi_0\neq 1$ is the character of $\mathbb{A}$ mod $F$ fixed previously. Let $W(\pi\,;\,\psi)$ be the span of the functions $W(g)=\prod W_v(g_v)$, where $W_v$ lies in $W(\pi_v\,;\,\psi_v)$ if $v$ is non-archimedean, in $W_0(\pi_v\,;\,\psi_v)$ (see [JS, § 3] if $v$ is archimedean, and is the unique right $K(E_v)=\mathrm{GL}(n,R_v)$-invariant function $W_v^0$ in $W(\pi_v\,;\,\psi_v)$ with $W_v(e)=1$, for almost all $v$ (where $\pi_v$ is unramified and $\psi_v$ has conductor $R_v$). Note that when $v$ splits then $W_v(g_{v'},g_{v''})=W_{v'}(g_{v'})\,W_{v''}(g_{v''})$, and then $K(E_v)=K(E_{v'})\times K(E_{v''})$, and our requirement is that $W_{v'}(e)=1=W_{v''}(e)$. The function $W_v^0$ is described explicitly below.

For each $W$ in $W(\pi;\psi)$ the function $\phi(g)=\sum W(\gamma g)$ ($\gamma$ in $N(E)\backslash P(E)$) is in the space of $\pi$, and we have

$$W(g)=\int_{N(E)\backslash N(\mathbb{A}_E)}\phi(ng)\bar\psi(n)\,dn.$$

Such $W$ is majorized by a function $\xi$ on $G(\mathbb{A}_E)$ which is left-$N(\mathbb{A}_E)Z(\mathbb{A}_E)$ and right-$K(\mathbb{A}_E)$ invariant, and given on $A(\mathbb{A}_E)$ by

$$\xi(a) = T\left(\frac{a_1}{a_2}, \frac{a_2}{a_3}, \dots, \frac{a_{n-1}}{a_n}\right)\left|\frac{a_1}{a_n}\right|^t, \qquad (a = (a_1, \dots, a_n)),$$

for some $T$ in $S(\mathbb{A}_E^{n-1})$ and $t \leq 0$ [JPS, (2.3.6)]. Consequently for each $\Phi$ in $S(\mathbb{A}^n)$ and $W$ in $W(\pi;\psi)$ the integral

$$\Psi(s, \Phi, W) = \int_{N(\mathbb{A})\backslash G(\mathbb{A})} W(g)\Phi(\epsilon g)|g|^s dg$$

converges for large $\operatorname{Re} s$. Moreover, we have $\int_{N\backslash\mathbb{N}} \phi(ng)\, dn = \sum_{N\backslash P} W(\gamma g)$, where we put $\mathbb{G} = G(\mathbb{A})$, $\mathbb{Z} = Z(\mathbb{A})$, $\mathbb{N} = N(\mathbb{A})$ and $G = G(F)$, $P = P(F)$, $N = N(F)$.

PROPOSITION. — *For $\Phi, W$ and $\phi$ as above we have $I(s,\Phi,\phi) = \Psi(s,\Phi,W)$ for large* $\operatorname{Re} s$.

*Proof.* — We have

$$\begin{aligned} I(s,\Phi,\phi) &= \int_{\mathbb{Z}G\backslash\mathbb{G}} E(g,\Phi,s)\phi(g)\, dg = \int_{\mathbb{Z}P\backslash\mathbb{G}} f(g,s)\phi(g)\, dg \\ &= \int_{P\backslash\mathbb{G}} \Phi(\epsilon g)\phi(g)|g|^s\, dg = \int_{P\mathbb{N}\backslash\mathbb{G}} \Phi(\epsilon g)|g|^s dg \int_{N\backslash\mathbb{N}} \phi(ng)\, dn \\ &= \int_{P\mathbb{N}\backslash\mathbb{G}} |g|^s\Phi(\epsilon g)\Big[\sum_{N\backslash P} W(\gamma g)\Big]\, dg = \int_{\mathbb{N}\backslash\mathbb{G}} |g|^s\Phi(\epsilon g)W(g)\, dg. \end{aligned}$$

COROLLARY. — *$\Psi(s,\Phi,W)$ extends to a meromorphic function of $s$ in* $\operatorname{Re} s > 0$, *which is holomorphic in* $\operatorname{Re} s > 1$, *and its only possible pole is simple, located at $s = 1$. The function $\Psi(s,\Phi,W)$ has a simple pole at $s = 1$ precisely when $\pi$ is distinguished, $W \neq 0$ with $\int \phi\, dg \neq 0$, and* $\hat{\Phi}(0) \neq 0$.

When $W(g)$ is $\prod W_v(g_v)$ and $\Phi(x)$ is $\prod \Phi_v(x_v)$, the expression $\Psi(s,\Phi,W)$ can be written as the product over $v$ of the local integrals $\Psi(s,\Phi_v,W_v)$, which we now describe.

### 3. Euler Factors

To describe the local integrals put $N_v = N(F_v)$ and $G_v = G(F_v)$, and take $\Phi_v$ in $S(F_v^n)$. If $v$ splits into $v'$, $v''$ in $E$, put

$$\Psi(s,\Phi_v,W_v) = \int_{N_v\backslash G_v} W_{v'}(g)W_{v''}(g)\Phi_v(\epsilon g)|g|^s dg$$

for $W_v = W_{v'}W_{v''}$ with $W$ in $W(\pi_u;\psi_u)$, $u = v', v''$. Otherwise, for $W_v$ in $W(\pi_v;\psi_v)$ put

$$\Psi(s,\phi_v,W_v) = \int_{N_v\backslash G_v} W_v(g)\Phi_v(\epsilon g)|g|^s dg.$$

To compute these integrals in the unramified situation let $x_1,\dots,x_n$ be $n$ variables. For the $n$-tuple $\alpha = (\alpha_1,\dots,\alpha_n)$ of integers consider the polynomial $a_\alpha$ obtained by anti-symmetrizing $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$, thus

$$a_\alpha = a_\alpha(x_1,\dots,x_n) = \sum \epsilon(w)w(x^\alpha) \qquad (w \text{ in } S_n),$$

where $\epsilon(w)$ is the sign of the permutation $w$ in the symmetric group $S_n$ on $n$ letters. The permutation $w$ acts on $x^\alpha$ by $w(x^\alpha) = x_{w(1)}^{\alpha_1}\cdots x_{w(n)}^{\alpha_1}$. It is clear that $w(a_\alpha) = \epsilon(w)a_\alpha$ for all $w$ in $S_n$, so that $a_\alpha$ vanishes unless $\alpha_1,\dots,\alpha_n$ are all distinct. Hence we may assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and write $\alpha = \lambda + \delta$ with $\lambda = (\lambda_1,\dots,\lambda_n)$, $\lambda_i \geq \lambda_{i+1}$, and $\delta = (n-1, n-2,\dots,1,0)$. Then $a_\alpha = a_{\lambda+\delta} = \sum_w \epsilon(w)w(x^{\lambda+\delta})$ is equal to the determinant $\det(x_i^{\lambda_j+n-j})$, $1 \leq i,j \leq n$. When $\lambda_n \geq 0$ this determinant is divisible in $\mathbb{Z}[x_1,\dots,x_n]$ by each of the differences $x_i - x_j$ $(1 \leq i < j \leq n)$, hence by their product — the *Vandermonde determinant*

$$a_\delta = \prod_{i<j}(x_i - x_j) = \det\big(x_i^{n-j}\big).$$

The quotient $s_\lambda = a_{\lambda+\delta}/a_\delta$ is called the *Schur function* [M, (3.1), p. 24].

Recall that when $E$ is a non-archimedean local field, the unramified irreducible representation $\pi$ of $G(E)$ is the unique irreducible unramified subquotient of the unitarily induced representation $I((\mu_i))$ from an unramified character $(\mu_i)$ of the Borel subgroup, and it defines a unique conjugacy class $t$ in $G(\mathbb{C})$ with eigenvalues $x_i = \mu_i(\boldsymbol{\pi})$. Assume that $\pi$ is also generic. Then $I((\mu_i))$ is irreducible [Z, Theorem 9.7(b)], hence equal to $\pi$. Denote by $\delta$ the modular function of the upper triangular subgroup [BZ]. Put $K = \mathrm{GL}(n,R)$.

LEMMA (SHINTANI [Sh]; [CS]). — *Up to a scalar there exists a unique right $K$-invariant function $W = W_\mu$ in $W(\pi;\psi)$ given by $W(\boldsymbol{\pi}^\lambda) = 0$ unless $\lambda = (\lambda_1,\dots,\lambda_n)$ in $\mathbb{Z}^n$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where $W(\boldsymbol{\pi}^\lambda) = \delta^{1/2}(\boldsymbol{\pi}^\lambda)s_\lambda(x)$.*

Returning to our global notations we now deal with an irreducible generic unramified representation $\pi_v$ of $G(E_v)$ (it is $\pi_{v'} \otimes \pi_{v''}$ if $v$ splits, as $G(E_v)$ is then $G_{v'} \times G_{v''}$). Let $W_v$ (*resp.* $W_{v'}, W_{v''}$) be the unique element in $W(\pi_v;\psi_v)$ (*resp.* $W(\pi_{v'};\psi_{v'})$, $W(\pi_{v''};\psi_{v''})$) specified by the Lemma. Signify by $\Phi$ the characteristic function of $R_v^n$ in $F_v^n$. Normalizing the measures by $|K_v| = 1$ and $|N_v \cap K_v| = 1$, we prove

PROPOSITION. — *If $\pi_v$ is unitary then $\Psi(s,\Phi,W_v)$ is absolutely convergent in* $\operatorname{Re} s \geq 1$, *uniformly in compact sets, and (2) we have there*

$$\Psi(s,\Phi,W_v) = L(s, r(\pi_v)) = \det\left[1 - q_v^{-s} r(t_v)\right]^{-1}.$$

*Remark.* — A different proof of this Proposition in the case of $v$ which splits in $E/F$ is given in [JS, Proposition 2.3].

*Proof.* — Using the Iwasawa decomposition $G = NAK$, we write the integral $\Psi(s,\Phi,W_v)$ as a sum over $\lambda$ in $\mathbb{Z}^n \simeq A/A \cap K$. If $v$ splits into $v'$, $v''$ we obtain

$$\sum_\lambda W_{v'}(\boldsymbol{\pi}^\lambda) W_{v''}(\boldsymbol{\pi}^\lambda)\delta^{-1}(\boldsymbol{\pi}^\lambda)\Phi(0,\dots,0,\boldsymbol{\pi}^{\lambda_n})\left|\boldsymbol{\pi}^\lambda\right|^s.$$

Since (1) $\Phi(0,\dots,0,\boldsymbol{\pi}^{\lambda_n})$ is 1 if $\lambda_n \geq 0$ and zero otherwise, and (2) the polynomial $s_\lambda$ is homogeneous of degree $\operatorname{tr}\lambda = \lambda_1 + \cdots + \lambda_n$, we put $x = (x_1,\dots,x_n)$ with $x_i = \mu_{iv'}(\boldsymbol{\pi})$ and $y = (y_1,\dots,y_n)$ with $y_i = \mu_{iv''}(\boldsymbol{\pi})$, to obtain

$$\sum_\lambda s_\lambda\left(q_v^{-s/2}x\right) s_\lambda\left(q_v^{-s/2}y\right) ;$$

the sum ranges over the $n$-tuples $\lambda = (\lambda_1,\dots,\lambda_n)$ of non-negative integers with $\lambda_i \geq \lambda_{i+1}$. Identity (4.3) of [M, p. 33], asserts that this sum is equal to

$$\prod_{i,j}\left(1 - q_v^{-s}x_i y_j\right)^{-1} = \det\left[1 - q_v^{-1} t_{v'} \otimes t_{v''}\right]^{-1} = L\left(s, \pi_{v'} \otimes \pi_{v''}\right)$$
$$= \det\left[1 - q_v^{-s} r(t_v)\right]^{-1} = L\left(s, r(\pi_v)\right).$$

It remains to deal with the case of $v$ which is inert and unramified in $E$.

Then the integral $\Psi(s,\Phi,W_v)$ is again a sum over $\lambda$ in $\mathbb{Z}^n \simeq A/A\cap K$, and only $\lambda$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ appear non-trivially in

$$\sum_\lambda W_v(\boldsymbol{\pi}^\lambda)\delta^{-1}(\boldsymbol{\pi}^\lambda)q_v^{-s\cdot tr\,\lambda}.$$

Since $W_v(\boldsymbol{\pi}^\lambda) = \delta_E^{1/2}(\boldsymbol{\pi}^\lambda)s_\lambda(z)$, $\delta_E(\boldsymbol{\pi}^\lambda) = \delta(\boldsymbol{\pi}^\lambda)^2$ and $s_\lambda(qz) = q^{tr\,\lambda}s_\lambda(z)$, if we put $z = (z_1,\ldots,z_n)$ with $z_i = \mu_{iv}(\boldsymbol{\pi})$, then the sum becomes

$$\sum_\lambda s_\lambda\big(q_v^{-s}z\big).$$

But according to (4) of [M, p. 45], this is equal to

$$\prod_i\big(1-q_v^{-s}z_i\big)^{-1}\prod_{j<k}\big(1-q_v^{-2s}z_jz_k\big)^{-1} = L\big(s,r(t_v)\big),$$

and (2) follows.

To prove (1), note that since $\pi_v$ is unitary we have $|\mu_{iv}(\boldsymbol{\pi})| < \sqrt{q_v}$ for all $i$, by [B, Lemma on p. 94]. Hence $|x_iy_j| < q_v$ and $|z_k| < \sqrt{q_{E_v}} = q_v$ for all $i$ , $j$, $k$, $v$, as required.

*Remark.* — A different proof of (1) is qiven in PROPOSITION 4(i) below. It implies in turn that $|\mu_{iv}(\boldsymbol{\pi})| < \sqrt{q_v}$ for all $i$ and unitary unramified generic irreducible $\pi_v$; see the Corollary to PROPOSITION 4(i) below.

## 4. Euler Products

Let $F$ be a local non-archimedean field, $E$ a quadratic extension of $F$, $\psi \neq 1$ a unitary character of $E$ modulo $F$ as in paragraph 1, and $\pi$ a unitary generic irreducible admissible representation of $G(E)$.

LEMMA. — *For any $W$ in $W(\pi;\psi)$, the integral $\int |W(p)|\,d_r(p)$ over $N(F)\backslash P(F)$ is convergent.*

*Proof.* — This is based on the GELFAND–KAZHDAN theory [GK] (see also [BZ], [BZ′] and [B]) of derivatives of representations, and Bernstein's criterion [B, p. 82] of unitarizability of $P$-modules : Let $S(N(E)_\psi\backslash P(E))$ denote the space of locally constant (on the right) complex-valued functions $\phi$ on $P(E)$ with $\phi(np) = \psi(n)\phi(p)$ ($n$ in $N(E)$, $p$ in $P(E)$). The group $P(E)$ acts by right translation. We claim : *if $\tau$ is a submodule of finite length such that the central exponents* (*see* [B, p. 82]) *of all of its derivatives $\tau^{[k]}$* (*see* [B, (7.2), p. 81) *are all strictly positive, then $\int_{N(F)\backslash P(F)} |\phi(p)|\,d_r(p)$ is finite for all $\phi$ in $\tau$.*

This claim implies the lemma on taking $\tau = \pi|P(E)$, and $\phi = W|P(E)$, and using the criterion of [B, p. 82], which asserts that $\pi$ is unitary if and only if all central exponents of all derivatives of $\tau$ are strictly positive. Note that since our $\pi$ is generic the condition (i) of [B, p. 82], always hold with $h = n-1$. Put $G_n$, $P_n$, $N_n$ for $G(E)$, $P(E)$, $N(E)$, and assume by induction that the claim holds for $G_{n-1}$.

Consider the natural projection $N_n\backslash P_n \simeq N_{n-1}\backslash G_{n-1} \to P_{n-1}\backslash G_{n-1}$. Let $V^*$ be the dual of the vector space $V = E^{n-1}$. Then $G_{n-1}$ acts transitively on $V$ by $g : v \mapsto vg$. If $v_0 = (0,\dots,0,1)$ then $P_{n-1}$ is the stabilizer of $v_0$. Hence $P_{n-1}\backslash G_{n-1}$ is isomorphic to $V^* - \{0\}$. Denote by $\Phi$ the functor $\Phi^- : \mathrm{Alg}(P_n) \to \mathrm{Alg}(P_{n-1})$ and by $\Psi$ the functor $\Psi^- : \mathrm{Alg}(P_n) \to \mathrm{Alg}(G_{n-1})$ of [B, (7.2), p. 81]. Then $\Phi$ is simply the natural restriction map $S(N_{n,\psi}\backslash P_n) \to S(N_{n-1,\psi}\backslash P_{n-1})$, and $\Phi\tau$ is a $P_{n-1}$-module of finite length (see [BZ′]) such that all central exponents of its derivatives are strictly positive (by our assumption on $\tau$). Hence our induction assumption implies that $\tilde{\phi}(g) = \int_{N_{n-1}(F)\backslash P_{n-1}(F)} |\phi(pg)|\, d_r(p)$ is finite for every $\phi$ in $\tau$ (and $g$ in $G_{n-1}$, as $G_{n-1}$ acts on $\Phi\tau$).

Write $Z_{n-1}$ for the center of $G_{n-1}$. We have to show the convergence of the integral

$$\int |\phi(p)|\, d_r(p) = \iint \tilde{\phi}(zg)\delta_E(z)\, dg dz,$$

where $p$ in $N_n(F)\backslash P_n(F)$, $z$ in $Z_{n-1}(F)$, $g$ in $Z_{n-1}(F)P_{n-1}(F)\backslash G_{n-1}(F)$. Here $\delta_E(z) = |(Ad(z)|U_n)|$, where $U_n = U_n(E)$ is the unipotent radical of the parabolic subgroup of $G_n$ of type $(n-1,1)$, as in paragraph 1. Note that since $\dim_F U_n = 2\dim_E U_n$, for $z$ in $Z_{n-1}(F)$ we have $\delta_E(z) = \delta_F(z)^2$.

The function $\tilde{\phi}$ on $P_{n-1}\backslash G_{n-1} \simeq V^* - 0$ is compactly supported on $V^*$ (since $W$ is majorized by a function $\xi$ described prior to PROPOSITION 2), and locally constant on $V^* - \{0\}$. Since $Z_{n-1}P_{n-1}\backslash G_{n-1}$ is compact, our integral converges if and only if so does the integral

$$\int \tilde{\phi}(z)\delta_E(z)\, dz = \iint |\phi(zp)|\delta_E(z)\, d_r(p) dz$$

($z$ in $Z_{n-1}(F)$, $p$ in $N_{n-1}(F)\backslash P_{n-1}(F)$). To study the convergence of this integral we need to understand the asymptotic behaviour of the integrand $|\phi(zp)|\delta_E(z)$ near 0 in $V^*$, namely when $z$ is near 0. For this purpose, note that $\Psi$ is the functor of coinvariants, mapping the $P_n$-module $\tau$ to the $G_{n-1}$-module $\tau_U$, where $U = U_n$. Here $\tau_U$ is the quotient of $\tau$ by the linear span of the vectors $\phi - \tau(u)\phi$, $\phi$ in $\tau$ an $u$ in $U$. Note that $\phi(zp) - \phi(zpu) = (1 - \psi_u(z))\phi(zp)$, where $\psi_u$ is a translate of $\psi$ which depends only on $u$, for all $p$ in $P_{n-1}$. For $z$ sufficiently near zero (with respect to $u$), we have $\psi_u(z) = 1$. Hence the function $\tilde{\eta}$ on $P_{n-1}\backslash G_{n-1} \simeq V^* - \{0\}$, where $\eta(g) = \phi(g) - \phi(gu)$ ($g$ in $G_{n-1}$), is zero near zero in $V^*$.

The $G_{n-1}$-module $\Psi\tau$ is admissible of finite length, by [BZ′]. Let $\chi_i$ be the central characters of the finitely many irreducibles in the composition series of $\Psi\tau$. Then, for each vector $\phi$ in $\tau$, there are vectors $\phi_i$ in $\tau$ and a vector $\eta$ in the span of $\{v - \tau(u)v \mid v \in \tau,\ u \in U\}$ with $\phi(z) = \eta(z) + \sum_i \chi_i(z)\phi_i$ for $z$ near zero. Since $\int_{N_n\backslash P_n} |\phi(p)|^2 d_r(p)$ is finite, we have that $\int_{Z_{n-1}} |\phi(z)|^2 \delta_E(z)\, dz$ is finite. Hence for each $i$ we have that $|\chi_i(z)|^2\delta_E(z)$ is less then one for all $z$. Consequently $|\chi_i(z)|\delta_F(z) < 1$ for all $z$ in $Z_{n-1}(F)$, and

$$\iint |\phi(zp)|\delta_F(z)\, dz d_r(p) \le \sum_i \int |\chi_i(z)|\delta_F(z)\, dz \int |\phi_i(p)|d_r(p) + \iint |\eta(zp)|\delta_F(z)\, dz d_r(p)$$

($z$ in $Z_{n-1}(F)$, $|\det z| \le 1$; $p$ in $N_{n-1}(F)\backslash P_{n-1}(F)$) is finite. This implies the convergence of the integral of the claim, and the lemma follows.

We conclude :

PROPOSITION.

(i) *The integral* $\Psi(s, \Phi, W)$ *converges absolutely, uniformly in compact subsets, for* $\operatorname{Re} s \ge 1$ *;*

(ii) *There exists* $W$ *in* $W(\pi;\psi)$ *and* $\Phi$ *in* $S(F^n)$ *with* $\hat{\Phi}(0) \ne 0$*, such that* $\Psi(s, \Phi, W)$ *is identically one.*

*Remark.*

(1) The analogous result in the split non-archimedean case where $E = F \oplus F$ is true and proved in [JS, (1.5)].

(2) If $F = \mathbb{R}$ and $E = \mathbb{R} \oplus \mathbb{R}$ then (i) here is proven in [JS, Prop. 3.17(i)]; in [3.17(ii)] there it is shown that given $s$ with $\operatorname{Re} s \ge 1$ and $W' \ne 0$ there are $W''$ and $\Phi$ such that $\Psi(s, \Phi, W) \ne 0$ for $W = (W', W'')$.

*Proof.* — As usual $N = N(F)$, $P = P(F)$, $K = K(F)$. The integral $\Psi(s, \Phi, W)$ is equal to

$$\int_K dk \int_{N\backslash P} d_r(p)\, |p|^{s-1} W(pk) \int_{F^\times} \Phi(\epsilon ak)|a|^{rs}\omega(a)\, d^\times a.$$

Since $|\Phi(xk)| \le \Phi_0(x)$ for some $\Phi_0$ in $S(F^n)$ (all $x$ in $F^n$, $k$ in $K$), we have

$$\int_{F^\times} |\Phi(\epsilon ak)|\, |a|^{ns} |\omega(a)|\, d^\times a \le \int_{F^\times} \Phi_0(\epsilon a)|a|^{ns} d^\times a$$

for real $s$, and this is uniformly bounded in compact subsets of $s \geq 1/n$. It remains to show that

$$\int_K dk \int_{N\backslash P} |W(pk)|\, |p|^{s-1} d_r(p)$$

converges uniformly in compact subsets of $s \geq 1$. The integral over the subset of $p$ with $|p| \leq 1$ is bounded by $\int_K \int_{N\backslash P} |W(pk)|\, d_r(p) dk$, which is finite by the Lemma. Since $|W(pk)|$ is bounded by a function $\xi$, the integral over the set $|p| \geq 1$ is taken over a compact set, hence it converges. This proves (i).

For (ii) recall that $K(\pi;\psi)$ contains the space $K_0$ of functions $\Phi$ on $P(E)$ which transform on the left by $\psi$ on $N(E)$ (hence trivially on $N$), are right invariant under some open compact subgroup $C$ of $P(E)$, and have compact support modulo $N(E)$. Fix a congruence subgroup $K'$ in $K(E)$. Let $\phi$ be a function in $K_0$ which is supported on $N(E)(K' \cap P(E))$, which is right invariant under $K' \cap P(E) \cap {}^tB(E)$, where ${}^tB(E)$ is the lower triangular subgroup. Fix $W$ in $W(\pi;\psi)$ with $W|P = \phi$. Then

$$\int_{N\backslash P} W(p)\, |p|^{s-1} d_r(p) = \int_{N\backslash P} \phi(p)\, |p|^{s-1} d_r(p)$$

is a non-zero constant. Let $K_m$ denote the group of $k$ in $K'$ with $\epsilon k = (\boldsymbol{\pi}^m x_1, \ldots, \boldsymbol{\pi}^m x_{n-1}, 1 + \boldsymbol{\pi}^m x_n)$, where the $x_i$ are all in the ring $R(E)$ of integers in $E$. Choose $m$ so that $W$ is right invariant under $K_m \cap {}^tU$, and let $\Phi$ be the characteristic function of $K_m$. Then $\Phi$ lies in $S(F^n)$, and $\hat{\Phi}(0) \neq 0$. Moreover

$$\Psi(s, \Phi, W) = \int_K dk \int_{N\backslash P} d_r(p)\, |p|^{s-1} W(pk) \int_{F^\times} \Phi(\epsilon a k)\, |a|^{ns} \omega(a)\, d^\times a$$

is a non-zero constant, as required.

COROLLARY. — *Let $\pi$ be an irreducible unitary generic unramified $G$-module. Then its eigenvalues are bounded by $\sqrt{q}$ in absolute value.*

*Proof.* — This follows at once from Proposition 3(2) and Proposition 4(i), as in [JS, Cor. 2.5].

THEOREM.

(1) *Let $E/F$ be a quadratic extension of global fields. For any cuspidal (irreducible unitary) representation $\pi$ of $G(\mathbb{A}_E)$ the function $L(s, r(\pi), V)$ has a pole at $s = 1$ if and only if $\pi$ is distinguished.*

(2) *If $E/F$ is an extension of function fields then the function $L(s, r(\pi), V)$ has analytic continuation to the entire complex plane as*

*a meromorphic function of $s$ which satistifies the functional equation $L(s, r(\pi), V) = \epsilon(s)L(1-s, r(\tilde{\pi}), V)$. Here $\epsilon(s) = \epsilon(s, \pi_V)$ is the product over $v$ in $V$ of meromorphic functions $\epsilon(s, \pi_v)$ which are holomorphic and non-zero on* $\operatorname{Re} s \geq 1$ *and* $\operatorname{Re} s \leq 0$, *and we have $\epsilon(s, \pi_V)\epsilon(1-s, \tilde{\pi}_V) = 1$. The poles of $L(s, r(\pi), V)$ are at most simple, and may occur only at $s = 0$ and* 1.

*Proof.* — Recall that $V$ is a finite set which contains the archimedean places and those where $E/F$ or $\pi$ ramify. Increasing $V$ we may assume that the conductor of $\psi_v$ is $R_v$ for $v$ outside $V$. Outside $V$ we take $\Phi_v = \Phi_v^0$ and $W_v = W_v^0$. Put $\Phi = \Phi_V \Pi_{v \in V} \Phi_v$ and $W = W_V \Pi_{v \notin V} W_v$. We have

$$\Psi(s, \Phi, W) = A(s, \Phi_V, W_V) L(s, r(\pi), V),$$

for some function $A(s, \Phi_V, W_V)$ given as a finite linear combination of integrals which are convergent for $s$ with $1 \leq \operatorname{Re} s \leq 2$, by (i) of the Proposition. If $\pi$ is distinguished, then for $\phi$ with $\int \phi \, dg \neq 0$ and $\Phi$ with $\hat{\Phi}(0) \neq 0$, the function $\Psi(s, \Phi, W)$ has a pole at $s = 1$. Hence $L(s, r(\pi), V)$ has a pole at $s = 1$, proving one half of (1). To prove the other direction, if $L(s, r(\pi), V)$ has a pole at $s = 1$, the functions $W_V$ and $\Phi_V$ can be chosen (by virtue of (ii) in the Proposition and the subsequent Remark) to have $A(1, \Phi_V, W_V) \neq 1$. Hence $\Psi(s, \Phi, W)$ has a pole at $s = 1$, and $\pi$ is distinguished.

To prove (2) let $F$ be a function field, and put $\tilde{g} = {}^t g^{-1}$, $\tilde{\varphi}(g) = \varphi(\tilde{g})$, $w = ((-1)^i \delta_{i, n+1-i})$ and $\widetilde{W}(g) = W(w\tilde{g})$. At $v$ in $V$ we take $W_v$ and $\Phi_v$ as in (ii) of the Proposition (and the Remark), and put $W = \Pi W_v$ and $\Phi = \Pi \Phi_v$. Since $\pi$ is unitary the product $L(s, r(\pi), V)$ is absolutely convergent in some right half-plane. Due to our choice of functions at $v$ in $V$ and outside $V$ we obtain

$$L(s, r(\pi), V) = \Psi(s, \Phi, W) = I(s, \Phi, \phi) = I(1-s, \hat{\Phi}, \tilde{\phi}) = \Psi(1-s, \hat{\Phi}, \widetilde{W}).$$

The third equality follows from the identity $E(s, \Phi, g) = E(1-s, \hat{\Phi}, \tilde{g})$. Since $\tilde{\phi}$ lies in the space of the contragredient representation $\tilde{\pi}$, which is also cuspidal, we obtain the assertion of (2) concerning the meromorphic continuation and functional equation. Since the only possible poles of $\Psi$ are simple and located at $s = 0$ and 1, the same holds for $L(s, r(\pi), V)$, as required.

COROLLARY. — *Suppose that $\pi$, $\pi'$ are cuspidal representations of $G(\mathbb{A}_E)$ with $L(s, r(\pi_v)) = L(s, r(\pi'_v))$ for almost all $v$. Then $\pi$ is distinguished if and only if $\pi'$ is distinguished.*

## 5. Characterization

Finally recall that a cuspidal representation $\pi = \otimes\pi_v$ of $G(\mathbb{A}_E)$ is called a *base-change lift* if $\pi_v = \pi_{v'} \times \pi_{v''}$ with $\pi_{v'} \simeq \pi_{v''}$ at each place $v$ of $F$ which splits into $v'$, $v''$ in $E$. The theory of base-change (see, *e.g.*, [F4]) asserts that this condition is equivalent to the same condition for almost all such $v$, and if it is held then there exists a cuspidal representation $\pi_0 = \otimes\pi_{0v}$ of $G(\mathbb{A})$ with $\pi_{0v} \simeq \pi_{v'} \simeq \pi_{v''}$ for all split places $v$, and for all $v$ which are unramified in $E/F$ and $\pi_{0v}$, $\pi_v$ are unramified we have $t(\pi_v) = (t(\pi_{0v}), t(\pi_{0v})) \times \sigma$. Moreover, if $\pi$ is the base-change lift of $\pi_0$, then their central characters $\omega$, $\omega_0$ are related by $\omega(x) = \omega_0(x\bar{x})$ ($\bar{x}$ is the conjugate over $E/F$ of $x$ in $\mathbb{A}_E^\times$).

If $\pi$ is distinguished then the linear form $\phi \to \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(g)\,dg$ on the space of $\pi$ is invariant under the action of $G(\mathbb{A})$. In particular if $v$ splits, then $G(F_v)$ embeds diagonally in $G(E_v) = G(E_{v'}) \times G(E_{v''})$, and we obtain by restriction a $G(F_v)$-invariant bilinear form on the space of $\pi_v = \pi_{v'} \times \pi_{v''}$. Hence $\pi_{v''}$ is contragredient to $\pi_{v'}$. We conclude :

PROPOSITION. — *Suppose $n = 2$ and $\pi$ is a cuspidal* $\mathrm{PGL}(2,\mathbb{A}_E)$*-module. Then $\pi$ is distinguished if and only if it is the base-change lift of a cuspidal* $\mathrm{GL}(2,\mathbb{A})$*-module $\pi_0$ whose central character is non-trivial.*

*Proof.* — Since any $\mathrm{PGL}(2,F_v)$-module is self-contragredient, then for a distinguished $\mathrm{PGL}(2,\mathbb{A}_E)$-module, we have $\pi_{v'} \simeq \pi_{v''}$ at each split $v$, and $\pi$ is a base-change lift of some $\pi_0$. The central character $\omega_0$ of $\pi_0$ is either trivial or it is the unique non-trivial character $\chi$ of $\mathbb{A}^\times/F^\times N\mathbb{A}_E^\times$; here $N$ denotes the norm from $E$ to $F$.

Recall, using the notations of paragraph 3, that when $E_v = E_{v'} \oplus E_{v''}$ we have

$$L\big(s, r(\pi_v)\big) = L\big(s, t(\pi_{v'}) \otimes t(\pi_{v''})\big) = \prod_{i,j}\big(1 - q_v^{-s}x_iy_j\big)^{-1}.$$

When $E_v/F_v$ is a quadratic field extension then

$$\begin{aligned} L\big(s, r(\pi_v)\big) &= \prod_i (1 - q_v^{-s}z_i)^{-1} \prod_{j<k}\big(1 - q_v^{-2s}z_jz_k\big)^{-1} \\ &= \prod_{i,j}\big(1 - q_v^{-s}x_ix_j\big)^{-1} \\ &\qquad \times \prod_{j<k}\Big[(1 + q_v^{-s}x_jx_k)/(1 - q^{-s}x_jx_k)\Big]^{-1}, \end{aligned}$$

where $x_i^2 = z_i$. Hence when $n = 2$ we obtain

$$L\big(s, r(\pi), V\big) = L\big(s, \pi_0 \otimes \pi_0, V\big)L(s, \omega_0\chi)/L(s, \omega_0).$$

$L(s,\omega_0)$ and $L(s,\omega_0\chi)$ are defined as a product over $v$ outside $V$ of local factors. Note that the contragredient $\tilde{\pi}_0$ of $\pi_0$ is $\pi_0\omega_0$, hence

$$L\big(s,\pi_0\otimes\pi_0,V\big)=L\big(s,\pi_0\otimes\tilde{\pi}_0\omega_0,V\big),$$

and this has a (necessarily simple) pole at $s=1$ if and only if $\omega_0=1$, by Theorem 4.8 of [JS]. Consequently, if $\omega_0=1$, then $L(s,\pi_0\otimes\pi_0,V)$ and $L(s,\omega_0)$ have simple poles, but $L(s,\omega_0\chi)$ does not have a pole at $s=1$, hence $L(s,r(\pi),V)$ does not have a pole and $\pi$ cannot be distinguished, contrary to our assumption. Hence the central character $\omega_0$ of $\pi_0$ is $\chi\neq 1$. Indeed $L(s,\pi_0\otimes\pi_0,V)$ is regular at $s=1$, $L(s,\chi)$ is regular at $s=1$, and $L(s,1)$ has a simple pole at $s=1$.

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