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HALPHEN'S GAPS FOR SPACE CURVES OF SUBMAXIMUM GENUS

PAR

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RÉSUMÉ. — On détermine les lacunes d'Halphen pour les courbes de $\mathbb{P}^3$ de degré $d > s(s-1)$ et genre $G(d,s) - 1$, $s \geq 4$.

ABSTRACT. — We determine Halphen's gaps for curves of $\mathbb{P}^3$, of degree $d > s(s-1)$, genus $G(d,s) - 1$, $s \geq 4$.

Introduction

For any pair of integers (d, g) $d \geq 3, g \geq 0$, let $s(d,g)$ be the smallest integer n, such that every smooth, connected curve of $\mathbb{P}^3$(1), of degree d, genus g, lies on a surface of degree n. To determine $s(d,g)$ for any (d,g) is an open problem and has deep connections with other questions regarding space curves.

For instance, a smooth, connected curve X of $\mathbb{P}^3$, of degree d, genus g, is said to be *superficially general*, if the least degree of a surface, containing X, is $s(d,g)$.

Given a certain property, we can think, following HARTSHORNE (see [6, p. 21]), that, without evident (numerical) obstruction, this property is verified by the generic superficially general curve. For example :

1. Existence of maximal rank curves. — One can conjecture that sufficient condition so that there exist smooth, connected curves of $\mathbb{P}^3$, of degree d, genus g, of maximal rank is that a convenient numerical condition, depending only on d, g, $s(d,g)$, holds (see [1, Question 2]).

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(1) projective 3–space over an algebraically closed field of characteristic zero

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2. Stability of the normal bundle. — Let d, g be integers, $g \geq 2$, such that the Hilbert scheme $H_{d,g}$ is not empty, $s = s(d, g)$ and let us suppose that $g < d(s-2)+1$ (resp. $\leq$). Then we can conjecture there exists a curve in $H_{d,g}$ with stable normal bundle (*resp.* semistable) (see [6, conj. 4.2]).

In this paper we consider the problem to determine $s(d, g)$, when $g = G(d, s) - 1$ (submaximum genus), where $G(d, s)$ is the maximum genus for smooth, connected curves of $\mathbb{P}^3$, of degree d, genus g, not contained in a surface of degree $s - 1$.

Our point of view (suggested in [1]) is the following (see 1.2 and 1.3) : if X is a curve of degree d, genus g with $G(d, s) \geq g > G(d, s+1)$, then there exists a surface of degree s, containing it. Since X is arbitrary, we have $s(d, g) \leq s$. On the other hand this should be the last condition (indeed $G(d, s)$ is conjectured to be a decreasing function of s). Hence it seems natural to expect $s(d, g) = s$. If the equality does not hold, the triple (d, g, s) is said to be an Halphen's gap.

The aim of this paper consists in proving the following

THEOREM (see 3.3). — *Let d, s be integers, $s \geq 4$, $d > s(s-1)$, and let r be such that $d + r \equiv 0$ (mod s), $1 \leq r \leq s-1$. Then the triple $(d\,;\ G(d, s) - 1\,;\ s)$ is an Halphen's gap except for*

i) $s = 4$;

ii) $s \geq 5$ *and* $2 \leq r \leq 3$ *or* $s - 3 \leq r \leq s - 2$.

The case $r = 0$ is discussed in [1, 3.10].

The methods, we use, are essentially the *liaison* (see [9] in general and [10] for curves in $\mathbb{P}^3$), the numerical character of a curve (see [3]) and the correspondance between curves and rank 2 reflexive sheaves (see [7]).

In paragraph 1 after having defined the numerical character of an integral curve (1.4), we show some results about their genus (1.6, 1.8, 1.9). In particular we give a sufficient condition so that certain curves have the maximal character (see 1.5 iv, 1.10). Furthermore we prove the equality $s(d, g) = s$ in some particular cases, using the properties of the numerical character (see 1.7).

In paragraph 2 we show there are no smooth, connected curves X of degree $ks - r$, genus $G(ks - r, s) - 1$ $(k \geq s \geq 5, 1 \leq r \leq s-1)$, of maximal character, lying on an irreductible surface of degree s, when $r \neq 2, r \neq s-2$ (2.9, 2.4). We first show that $e(X) = k + s - 5$ or $k + s - 6$ (2.2). The first case is solved using reflexive sheaves and T. SAUER's bound (see [11]) of the arithmetic genus of generally local complete intersection, locally Cohen-Macaulay curves (2.3, 2.4). Instead the second case is solved by comparison with the cohomology of curves having maximal character in a natural way : the curves of maximum genus for $(ks - r, s)$ (see 1.1, 2.9).

In paragraph 3 we prove, by liaison, the equality $s(d, g) = s$ in the remaining cases (3.1) and conclude with the THEOREM 3.3. By the way, when $r = 2, 3, s - 3$ or $s - 2$, we give a complete description of the curves of degree d, genus $G(d, s) - 1$, lying on a irreducible surface of degree s. Moreover we determine every $s(d\,;\ G(d, s) - 1)$, when $s = 5$, $d > s(s - 1)$ (3.5).

Finally I wish to thank Philippe ELLIA for the suggestions about the matter of this paper.

1. A few results on the numerical character of a curve

In this paper *curve* means a closed subscheme of $\mathbb{P}^3$, of (pure) dimension 1.

For any integers d, s, $d \geq 3$, $s \geq 2$, $G(d, s)$ is the maximum genus of smooth, connected curves C of degree d, genus g, with $h^0(\mathcal{I}_C(s - 1)) = 0$.

Remark 1.1. (see [2, thm A]). — If $d > s(s - 1)$, then

$$G(d, s) = 1 + \frac{1}{2s}\Big[d(d + s^2 - 4s) - r(s - 1)(s - r)\Big]$$

where $d + r \equiv 0 \pmod{s}$, $0 \leq r \leq s - 1$.

Furthermore the curves of maximum genus for (d, s) (*i.e.* the curves C with $\deg(C) = d$, $g(C) = g$, $h^0(\mathcal{I}_C(s - 1)) = 0$) are linked to a plane curve of degree r, by a complete intersection of two surfaces of degrees s and $(d + r)/s$.

Remark 1.2. — Let X be a smooth connected curve of degree d, genus g, with $G(d, s) \geq g > G(d, s + 1)$ for some s ($G(d, s)$ is a decreasing function of s at least when $d > s(s - 1)$). Then $h^0(\mathcal{I}_X(s)) \neq 0$. From this, if $s(d, g)$ is the minimum integer n, such that any smooth, connected curve of degree d, genus g is contained in a surface of degree n, we get $s(d, g) \leq s$ and we would be induced to expect equality.

Definition 1.3. — If $G(d, s) \geq g > G(d, s + 1)$ and if $s(d, g) < s$, we say that (d, g, s) is an Halphen's gap.

Definition 1.4. — Let X be an integral curve with $\sigma = \sigma(X) := \min\{n \mid h^0(\mathcal{I}_{X \cap H}(n)) \neq 0,\ H \text{ general plane}\}$.

The (connected) numerical character $\chi = \chi(X)$ of X is a sequence of σ integers $(n_0, \ldots, n_{\sigma-1})$ satisfying

i) $n_0 \geq n_1 \geq \cdots \geq n_{\sigma-1} \geq \sigma$,

ii) $n_i \leq n_{i+1} + 1$ (connection);

iii) $\deg(\chi) := \sum_{i=0}^{\sigma-1}(n_i - i) = \deg(X)$,
iv) the function on $\mathbb{Z}$

$$h^1_\chi(t) := \sum_{i=0}^{\sigma-1}\left[(n_i - t - 1)_+ - (i - t - 1)_+\right],$$

where $(x)_+ = \max\{0, x\}$, satisfies

$$h^1_\chi(t) = h^1\big(\mathcal{I}_{X\cap H}(t)\big) \qquad t \geq 1,\ H \text{ general plane.}$$

Remarks 1.5.

i) Any integral curve has a numerical character ([3, 3.2]) and any numerical character is the character of some smooth, connected, projectively normal curve ([3, 2.5]).

ii) Let X be a smooth, connected curve, of character χ and set

$$\Delta_X(t) := h^2\big(\mathcal{I}_X(t-1)\big) - h^2\big(\mathcal{I}_X(t)\big), \qquad t \geq 1.$$

We get : $g(X) = \sum_{r=1}^{e+1}\Delta_X(t)$ ($e = e(X) := \max\{n \mid h^1(\mathcal{O}_X(n)) \neq 0\}$ the index of speciality).

The exact sequence

$$0 \to \mathcal{I}_X(t-1) \to \mathcal{I}_X(t) \to \mathcal{I}_{X\cap H}(t) \to 0$$

yields $h^1_\chi(t) \geq \Delta_X(t)$, hence : $g(\chi) := \sum_{r\geq 1} h^1_\chi(t) \geq g(X)$. Furthermore, X is projectively normal if and only if $g(\chi) = g(X)$ and $s(X) := \min\{n \mid h^0(\mathcal{I}_X(n)) \neq 0\} = \sigma(X)$.

iii) Clearly $s(X) \geq \sigma(X)$ holds. Moreover if X is an integral curve of degree d, with $d > t^2 + 1$, $\sigma(X) \leq t$, then $s(X) = \sigma(X)$ ([2, p. 225]).

iv) If $d > s(s-1)$, the maximal (for the lexicographic order) character of degree d, length s is :

$$\Phi = (k+s-1, \ldots, k+1, k) \qquad \text{if } d = ks\,;$$
$$\Phi = (k+s-2, \ldots, k+s-r-1, k+s-r-1, \ldots, k+1, k) \qquad \text{if } d + r = ks, \text{ with } 1 \leq r \leq s-1.$$

We have : $g(\Phi) = G(d, s) \geq g(\chi)$ for any character χ of degree d, length s ([3, § 2]).

We want a *measure* of the genus of any character $\chi = (\bar{n}_0, \ldots, \bar{n}_{s-1})$ of length s, degree $ks - r$, $k \geq s \geq 4$, $1 \leq r \leq s$.

Let us consider the following characters

$$\begin{aligned}
\Phi\ &= (k+s-2,\dots,k+s-r-1,k+s-r-1,\dots,k+1,k)\,;\\
\Phi_1 &= (k+s-2,\dots,k+s-r-1,k+s-r-2,\\
&\qquad\qquad k+s-r-2,\dots,k+1,k+1) \qquad r\le s-3\,;\\
\Phi_2 &= (k+s-3,k+s-3,\dots,k+s-r-1,\\
&\qquad\qquad k+s-r-1,\dots,k+1,k+1) \qquad r\ge 2\,;\\
\Phi_3 &= (k+s-3,k+s-3,\dots,k+s-r,\\
&\qquad\qquad k+s-r,k+s-r-1,\dots,k+1,k) \qquad r\ge 3.
\end{aligned}$$

LEMMA 1.6. — *Let Φ, Φ_h, $1\le h\le 3$ be as before. Then*

$$\begin{aligned}
g(\Phi_1) &= G(ks-r,s)-(s-r-2),\\
g(\Phi_2) &= G(ks-r,s)-(s-3),\\
g(\Phi_3) &= G(ks-r,s)-(r-2).
\end{aligned}$$

Proof. — Indeed $g(\Phi)=G(ks-r,s)$ (1.5 iv)). We conclude computing $g(\Phi)-g(\Phi_h)$, $1\le h\le 3$, with 1.5 ii).

PROPOSITION 1.7. — *Let d,s be integers, $d>s(s-1)$; with the same notations as in* 1.1, 1.2 *we have :*

i) *If $s=4$ and $d\not\equiv 0 \pmod s$, then $s(d\,;\,G(d,s)-1)=4$.*

ii) *If $s\ge 5$ and $d+3\equiv 0 \pmod s$ or $d+s-3\equiv 0 \pmod s$ then $s(d\,;\,G(d,s)-1)=s$.*

Proof. — In both cases i) and ii) we have $g(\Phi_h)=G(d,s)-1$ for some h (see 1.6). We conclude with 1.5 i), ii).

LEMMA 1.8. — *Let $\Phi=(n_i)$, $\Phi_1=(n_i^{(1)})$, $\chi=(\bar n_i)$ be as before. We have :*

(i) $\bar n_0\le k+s-2$*;*

(ii) *If $\bar n_i=n_i$, $0\le i\le q$, $q\ne r-1$, then $\bar n_{q+1}=n_{q+1}$;*

(iii) *If $\bar n_0=k+s-2$, then $\bar n_i=n_i$ $0\le i\le r-1$. Moreover if $\chi\ne\Phi$, then $r\le s-3$ and $\bar n_r=n_r^{(1)}$.*

(iv) *If $\bar n_0=k+s-2$ and $\chi\ne\Phi$, then $g(\chi)\le g(\Phi_1)$.*

Proof.

(i) If $\bar n_0\ge k+s-1$, from connection we get : $\bar n_i\ge k+s-1-i$, hence $ks-r=\sum_{i=0}^{s-1}(\bar n_i-i)\ge\sum_{i=0}^{s-1}(k+s-1-2i)=ks$, that is absurd.

(ii) Indeed, by maximality of Φ : $\bar{n}_{q+1} \le n_{q+1}$. Since $q+1 \ne r$, $n_{q+1} = n_q - 1$. If $\bar{n}_{q+1} < n_{q+1}$, then $\bar{n}_{q+1} < \bar{n}_q - 1$, which contradicts the connection of χ.

(iii) The first statement follows from (ii), because $\bar{n}_0 = n_0$. If $\chi \ne \Phi$, from (ii) we must have : $\bar{n}_r = n_{r-1} - 1 = n_r^{(1)}$. If $r > s-3$, we get : $\deg(\chi) < \deg(\Phi)$, which is absurd.

(iv) By definition 1.5 ii) : $g(\chi) = \sum_{m \ge 1} h^1_\chi(m)$. So it is enough to show :

$$h^1_\chi(m) \le h^1_{\Phi_1}(m), \qquad m \ge 1.$$

For each character $\psi = (z_0, \dots, z_{s-1})$ let F_ψ be the function defined on $\mathbb{R}^+$ by

$$F_\psi(x) = \begin{cases} [x]+1 & 0 < x < s, \\ \#\{z_i / z_i \ge x\} & x \ge s. \end{cases}$$

We have :

$$\int_0^{+\infty} F_\psi(x)\,dx = \sum_{i=0}^{s-1} (z_i - i) = \deg(\psi)$$

$$h^1_\psi(m) = \int_{m+1}^{+\infty} F_\psi(x)\,dx$$

With these notations it is enough to prove

$$(*) \qquad \int_{m+1}^{+\infty} (F_{\Phi_1}(x) - F_\chi(x))dx \ge 0, \qquad m \ge 1.$$

We have :

$$\int_{m+1}^{+\infty} (F_{\Phi_1} - F_\chi)(x)\,dx = \int_{m+1}^{+\infty} (F_{\Phi_1} - F_\Phi)(x)\,dx + \int_{m+1}^{+\infty} (F_\Phi - F_\chi)(x)\,dx$$
$$= J(m+1) + \int_{m+1}^{+\infty} (F_\Phi - F_\chi)(x)\,dx.$$

One can easily verify : $J(m+1) = -1$ if $k+1 \le m+1 \le k+s-r-2$, $J(m+1) = 0$ otherwise. (Again, we have : $g(\Phi_1) = G(ks-r, s) - (s-r-2)$). Hence it is enough to prove

$$(**) \quad \int_{m+1}^{+\infty} (F_\Phi - F_\chi)(x)\,dx \ge 1, \qquad \text{if } k+1 \le m+1 \le k+s-r-2.$$

It is known ([3, p. 45]) that

$$\int_{m+1}^{+\infty} (F_\Phi - F_\chi)(x)\,dx \ge 0, \qquad m \ge 1.$$

Now if $\int_{m+1}^{+\infty}(F_\Phi - F_\chi)(x)\,dx = 0$ for some m such that $k+1 \le m+1 \le k+s-r-2$, then we get :

$$0 = \int_0^{+\infty}(F_\Phi - F_\chi)(x)\,dx = \int_0^{m+1}(F_\Phi - F_\chi)(x)\,dx$$

(the first equality holds, because $\deg(\Phi) = \deg(\chi)$).

Since $F_\Phi - F_\chi$ is first negative and then positive ([3, p. 45]), one of the following cases holds :

$$(1) \qquad (F_\Phi - F_\chi)(x) = 0, \qquad x \le m+1;$$

$$(2) \qquad (F_\Phi - F_\chi)(x) = 0, \qquad x > m+1.$$

We will show that both cases are impossible.

Case (1) : From $F_\Phi(k) = s$ we have $\bar{n}_i \ge k$, for all i; since $F_\Phi(k+1) = s-1$, $\bar{n}_{s-1} = k$ and $\bar{n}_i \ge k+1$, when $i \ne s-1$, so, by connection, $\bar{n}_{s-2} = k+1$.

From (iii) we get $n_i = \bar{n}_i$ $0 \le i \le r-1$. By connection it must be either $\bar{n}_r = k+s-r-1 = n_r$ or $\bar{n}_r = k+s-r-2 = n_r - 1$.

If $\bar{n}_r = n_r$, from (ii) we have $\chi = \Phi$, which is absurd. If $\bar{n}_r = n_r - 1$, we have :

$$0 = \sum_{i=0}^{s-1}(n_i - \bar{n}_i) = 1 + \sum_{i=r+1}^{s-3}(n_i - \bar{n}_i)$$

(with convention that $\sum_{i=a}^{b} y_i = 0$ if $a > b$).

By connection : $\bar{n}_{s-3-j} \le k+2+j$, $j \ge 0$. But we have : $n_{s-3-j} = k+2+j$, $0 \le j \le s-r-4$. Hence

$$0 = \sum_{i=0}^{s-1}(n_i - \bar{n}_i) \ge 1,$$

which is absurd.

Case (2) : Since $F_\Phi(k+s-r-1) = r+1$, we have :

$$F_\chi(k+s-r-1) = r+1.$$

So $n_i = \bar{n}_i$, $i \le r$ and $\chi = \Phi$ from (ii).

LEMMA 1.9. — *Let $\Phi = (n_i)$, $\Phi_h = (n_i^{(h)})$ $1 \le h \le 3$, $\chi = (\bar{n}_i)$ be as before and let us suppose $\bar{n}_0 \le k+s-3$ (then, in particular : $\chi \ne \Phi$).*

i) *If $r = 1$, then $g(\chi) \le g(\Phi_1)$;*

ii) *If* $r = 2$, *then* $g(\chi) \leq g(\Phi_2)$;
iii) *If* $r \geq 3$, *then* $g(\chi) \leq g(\Phi_3)$.

Proof. — We can repeat the proof of 1.8 (iv), using Φ_1, Φ_2, Φ_3 respectively.

PROPOSITION 1.10. — *Let* X *be a smooth, connected curve of degree* $ks - r$, *genus* $G(ks - r, s) - 1$ *with* $s(X) = s$ (*see* 1.5 ii)). *Assume*
(i) $s \geq 5$ *and* $r = 1$ *or* $4 \leq r \leq s - 4$ *or* $r = s - 1$;
(ii) $s \geq 6$ *and* $r = 2$ *or* $r = s - 2$.
Then the numerical character of X *is* Φ.

Proof. — Under assumptions (i), (ii) we have :

$$G(ks - r, s) - 1 > g(\Phi_h) \qquad 1 \leq h \leq 3. \qquad \text{(see 1.6)}.$$

We conclude with 1.8 and 1.9, remembering 1.5 iii).

2. Curves of maximal character and submaximum genus

In this paper we are interested in smooth, connected space curves, but our results hold more generally for integral curves.

Notations 2.1. — X indicates a smooth, connected curve of $\mathbb{P}^3$, of degree $d = ks - r$, $k \geq s \geq 5$, $1 \leq r \leq s - 1$, genus $g = G(d, s) - 1$, with

$$s(X) := \min\{n \mid h^0(\mathcal{I}_X(n) \neq 0\} = s ;$$

$\mathcal{C}$ indicates a smooth, connected curve of maximum genus for (d, s) (see 1.1).

LEMMA 2.2. — *Let* X *be as in* 2.1, *with* $\chi(X) = \Phi$. *For the index of speciality* $e(X)$ (1.5 ii)) *we have :*
(i) $k + s - 6 \leq e(X) \leq k + s - 5$,
(ii) *If* $e(X) = k+s-6$, *then* $r \geq 2$ *and* $h^1_\Phi(t) = \Delta_X(t)$, $1 \leq t \leq k+s-5$.

Proof.
(i) We have $G(d, s) = \sum_{t\geq 1} h^1_\Phi(t)$, $g(X) = \sum_{t=1}^{e+1} \Delta_X(t)$. Hence

$$(*) \qquad \sum_{t\geq 1} h^1_\Phi(t) - \sum_{t=1}^{e+1} \Delta_X(t) = 1.$$

Since $h^1_\Phi(t) = 0$ for $t \geq k + s - 3$ and $h^1_\Phi(t) \geq \Delta_X(t)$, for $t \geq 1$ we get $e(X) \leq k + s - 5$. If $e \leq k + s - 7$, then

$$\sum_{t=1}^{k+s-6} \big(h^1_\Phi(t) - \Delta_X(t)\big) + h^1_\Phi(k + s - 5) + h^1_\Phi(k + s - 4) - 1$$

is strictly positive, which contradicts $(*)$.

(ii) It follows from $(*)$, because

$$h^1_{\Phi}(k+s-4) = \begin{cases} 1 & r \neq 1, \\ 2 & r = 1. \end{cases}$$

LEMMA 2.3. — *If there exists a smooth, connected curve X of degree $d = ks - r$, $1 \leq r \leq s-1$, $k \geq s \geq 5$, genus $g = G(d,s) - t - 1$ $(t \geq 0)$ and $e(X) = k + s - 5$, then $s - 3 - t \leq r \leq s - 2$.*

Proof. — A non zero element of $H^0(\omega_X(-k-s+5))$ yields an exact sequence :

$$(\dagger) \qquad 0 \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{G} \to \mathcal{I}_X(k+s-1) \to 0$$

where $\mathcal{G}$ is a rank 2 reflexive sheaf with $c_1(\mathcal{G}) = k+s-1$, $c_2(\mathcal{G}) = d$, $c_3(\mathcal{G}) = 2g-2+d(-k-s+5)$ (see [7, thm. 4.1]). Since $h^0(\mathcal{I}_X(s-1)) = 0$ and $h^0(\mathcal{I}_X(s)) \neq 0$, $\mathcal{G}(-k+1)$ has a section vanishing along a localy Cohen-Macaulay, generically local complete intersection curve, Y :

$$(\ddagger) \qquad 0 \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{G}(-k+1) \to \mathcal{I}_Y(-k+s+1) \to 0.$$

Using [7, 4.1,2.2], we find : $\deg(Y) = s - r$ and $p_a(Y) = \frac{1}{2}(s^2 + r^2 - 3s + 3r - 2rs - 2t)$. If $s = r+1$ then $p_a(Y) = -1-t$, which is absurd. If $s - r \geq 2$, from SAUER's bound of the arithmetic genus of locally Cohen-Macaulay, generically local complete intersection curves ([11, 6.2]), we must have :

$$\begin{aligned} &p_a(Y) = \tfrac{1}{2}\big(\deg(Y)-1\big)\big(\deg(Y)-2\big) && \text{or} \\ &p_a(Y) \leq \tfrac{1}{2}\big(\deg(Y)-2\big)\big(\deg(Y)-3\big), && \textit{i.e.} \\ &r^2 + s^2 - 3s + 3r - 2rs - 2t = (s-r-1)(s-r-2) && \text{or} \\ &r^2 + s^2 - 3s + 3r - 2rs - 2t \leq (s-r-2)(s-r-3). \end{aligned}$$

The first condition gives $-2t = 2$, which is impossible. The second one gives the statement of the lemma.

PROPOSITION 2.4. — *Let X be as in 2.1 with $e(X) = k+s-5$. If $\chi(X) = \Phi$, then $r = s-2$ and X is the liaison class (see [10]) of two skew lines.*

Proof. — If we put $t = 0$ in the previous lemma we get $s-3 \leq r \leq s-2$. If $r = s-3$, the curve Y (see proof of 2.3) has degree 3 and $p_a(Y) = 0$. By [5, p. 430] Y is arithmetically Cohen-Macaulay. From the exact sequences $(\dagger)$, $(\ddagger)$ in the proof of 2.3, X should be arithmetically Cohen-Macaulay too. But this is impossible since $g(X) \neq g(\Phi)$ (see 1.5 ii)).

If $r = s - 2$, then $\deg(Y) = 2$, $p_a(Y) = -1$. It is well know that Y is a type $(2,0)$ divisor on a smooth quadric. Since the Rao's modules of Y and X are isomorphic up to twist (exact sequences (†), (‡), by [10, § 2],) we get the lemma.

Remark 2.5. — When $r = s - 2$, we will see (3.1) how to construct curves as in 2.4.

LEMMA 2.6. — *Let X, C be as in 2.1 and let us suppose $e(X) = k+s-6$, $\chi(X) = \Phi$. Then we have :*

(i) $h^1(\mathcal{I}_X(t)) = 0, \qquad t \leq k+s-5;$
(ii) $h^0(\mathcal{I}_X(t)) = h^0(\mathcal{I}_C(t)), \quad t \leq k+s-4;$
(iii) $h^0(\mathcal{O}_X(t)) = h^0(\mathcal{O}_C(t)), \quad t \leq k+s-5.$

Proof. — It is enough to show the results for $t \geq 1$. Since $h^1_\Phi(t) = \Delta_X(t)$ (see 2.2 (ii)) we have the surjections

$$H^1\big(\mathcal{I}_X(t-1)\big) \to H^1\big(\mathcal{I}_X(t)\big) \to 0,$$

so, by induction, $h^1(\mathcal{I}_X(t)) = 0$, $1 \leq t \leq k+s-5$, which proves (i).

From (i) we get the exact sequence

$$0 \to H^0\big(\mathcal{I}_X(t-1)\big) \to H^0\big(\mathcal{I}_X(t)\big) \to H^0\big(\mathcal{I}_{X\cap H}(t)\big) \to 0,$$

$1 \leq t \leq k+s-4$ and H general plane. Comparing it with the same sequence for C, by induction, we prove (ii). Indeed C and X having the same character, $C \cap H$ and $X \cap H$ have the same postulation. The statement (iii) follows from (i) and (ii).

LEMMA 2.7. — *Let Γ be a locally Cohen-Macaulay curve verifying :*

1. $\deg(\Gamma) = r$, $p_a(\Gamma) = p(r) - 1$, *where $p(r)$ is the arithmetic genus of a plane curve of degree r.*
2. $h^0(\mathcal{O}_\Gamma(2)) \leq h^0(\mathcal{O}_Z(2)) + 1$, *where Z is a plane curve of degree r.*

Then Γ is one of the following curves :

(a) $r = 2$, $p_a(\Gamma) = -1$, *Γ is a type $(2,0)$ divisor on a smooth quadric surface,*

(b) $r = 3$, $p_a(\Gamma) = 0$, *Γ is arithmetically Cohen-Macaulay ("twisted cubic").*

Proof. — From 1, $r > 1$ and Γ is not a plane curve. From 1 and 2 we get :

$$h^0\big(\mathcal{I}_\Gamma(2)\big) = 10 - h^0\big(\mathcal{O}_\Gamma(2)\big) + h^1\big(\mathcal{I}_\Gamma(2)\big) \geq 9 - h^0\big(\mathcal{O}_Z(2)\big).$$

If $r = 2$, Γ is the union of two skew lines or a double line (locally Cohen-Macaulay with $p_a(\Gamma) = -1$). It is well known that each such curve is a type $(2,0)$ divisor on a smooth quadric surface. Suppose $r \geq 3$. From $h^0(\mathcal{O}_Z(2)) = 6$, we get : $h^0(\mathcal{I}_\Gamma(3)) \geq 3$. We distinguish two cases :

First case. — There exist two quadrics Q_1, Q_2 containing Γ, without irreducible common components. Let Γ_1 be the residual intersection between Q_1 and Q_2. If Γ_1 is the empty set, then Γ is complete intersection of two quadric surfaces, hence $p_a(\Gamma) = 1$, which is impossible.

If Γ_1 is not empty, it can only be a straight line, hence Γ is arithmetically Cohen-Macaulay of degree 3 and arithmetic genus 0.

Second case. — Any two quadric surfaces, containing Γ, have an irreducible common component (which is necessarily a plane). Let Q_1, Q_2 be two such quadrics ($Q_1 \neq Q_2$). Then $Q_1 = H \cup H_1$, $Q_2 = H \cup H_2$ (H, H_1, H_2 are planes; $H_1 \neq H_2$).

Set $L = H_1 \cap H_2$. If $Q = H \cup \widetilde{H}$, with $\widetilde{H}$ a plane through L, then Q contains Γ. Conversely any quadric, Q, containing Γ, is the union of H and of a plane through L. Indeed, if not, Q has to be $H_1 \cup H_2$ (because it has a common component with Q_1 and with Q_2). Hence Q has no common components with $H \cup H'$, where H' is a plane through L (different from H_1, H_2), which is absurd.

So we have $h^0(\mathcal{I}_\Gamma(2)) = 2$ and this contradicts $h^0(\mathcal{I}_\Gamma(2)) \geq 3$.

LEMMA 2.8. — *Let $X, \mathcal{C}$ be as in* 2.1 *with* $\chi(X) = \Phi$. *If*
(i) $h^0(\mathcal{I}_X(k)) \geq h^0(\mathcal{I}_\mathcal{C}(k))$ *and*
(ii) $h^2(\mathcal{I}_X(k+s-6)) + 1 \geq h^2(\mathcal{I}_\mathcal{C}(k+s-6)) + h^1(\mathcal{I}_X(k+s-6))$, *then* $r = 2$, $e(X) = k+s-6$ *and X is linked to a curve* Γ, *of degree* 2, $p_a(\Gamma) = -1$, *by a complete intersection* (s,k).

Proof. — From 2.1, X lies on an irreducible surface S, of degree s. Because of the degrees, the surfaces, containing X, of degree less or equal to $k-1$, are exactly the *multiples* of S.

Since $h^0(\mathcal{I}_\mathcal{C}(k)) \geq h^0(\mathcal{O}_{\mathbb{P}^3}(k-s)) + 1$ (see 1.1), X lies on an irreducible surface F, of degree k.

The complete intersection $U = F \cap S$ links X to a curve Γ, of degree r, arithmetic genus $p(r) - 1$. Let U' be the complete intersection, linking $\mathcal{C}$ to a plane curve Z, of degree r (see 1.1). We will show

$$(*) \qquad h^0\big(\mathcal{O}_\Gamma(2)\big) \leq h^0\big(\mathcal{O}_Z(2)\big) + 1.$$

From the exact sequence ([9, § 1])

$$0 \to \mathcal{I}_U \to \mathcal{I}_X \to \omega_\Gamma(4-k-s) \to 0$$

we have

$$0 \to H^1\big(\mathcal{I}_X(k+s-6)\big) \to H^1\big(\omega_\Gamma(-2)\big) \to$$
$$\to H^2\big(\mathcal{I}_U(k+s-6)\big) \to H^2\big(\mathcal{I}_X(k+s-6)\big) \to 0.$$

Using Serre duality, we get

$$h^0(\mathcal{O}_\Gamma(2)) = h^2(\mathcal{I}_U(k+s-6)) - h^2(\mathcal{I}_X(k+s-6)) + h^1(\mathcal{I}_X(k+s-6)).$$

In the same way, remembering $h^1(\mathcal{I}_C(t)) = 0$, $t \in \mathbb{Z}$, we have

$$h^0(\mathcal{O}_Z(2)) = h^2(\mathcal{I}_{U'}(k+s-6)) - h^2(\mathcal{I}_C(k+s-6)).$$

Since $h^i(\mathcal{I}_U(t)) = h^i(\mathcal{I}_{U'}(t))$, $t \in \mathbb{Z}$, (*) follows from (ii). Because of (*), we can use the LEMMA 2.7. The case $r = 3$ is impossible, because X is not projectively normal ($g(\chi(X)) \neq g(X)$, see 1.5 ii)). For $r = 2$, see 3.1.

PROPOSITION 2.9. — *Let X be as in 2.1 with $e(X) = k + s - 6$. If $\chi(X) = \Phi$, then $r = 2$ and X is linked to a curve Γ, of degree 2, arithmetic genus -1, by a complete intersection (s, k).*

Proof. — Since $\chi(\mathcal{I}_X(k+s-6)) - \chi(\mathcal{I}_C(k+s-6)) + 1 = 0$ (here χ is the Euler characteristic of a sheaf), then, from 2.6,

$$h^2(\mathcal{I}_X(k+s-6)) + 1 = h^2(\mathcal{I}_C(k+s-6)).$$

Moreover, from 2.6 we have $h^0(\mathcal{I}_X(k)) = h^0(\mathcal{I}_C(k))$, because $k \leq k+s-4$. Hence we can use 2.8.

Remarks 2.10.

(i) When $r = 2$ the existence of X as in 2.9 is proved in 3.1.

(ii) It should be noticed that the arguments, used in the prooves of 2.3, 2.4, do not apply to prove 2.9.

3. The theorem

PROPOSITION 3.1. — *Let d, s be integers, $s \geq 5$, $d > s(s-1)$ and let r be such that $d + r \equiv 0 \pmod s$. Then $s(d\,;\, G(d,s) - 1) = s$, if $r = 2$ or $r = s - 2$.*

Proof.

Case $r = 2$. — Let Y be the union of two skew lines ($\deg(Y) = 2$, $p_a(Y) = -1$). From $h^1(\mathcal{I}_Y(2)) = h^2(\mathcal{I}_Y(1)) = 0$ there exist two smooth surfaces of degrees s and k, linking Y to a smooth curve X, with $\deg(X) = ks - 2$, $p_a(Y) = G(ks-2, s) - 1$ (see [4, III.3]).

X is also connected, because $h^1(\mathcal{I}_X) = h^1(\mathcal{I}_Y(k+s-4)) = 0$. Furthermore $h^0(\mathcal{I}_X(s-1)) = 0$, because of the degree of X.

Case $r = s - 2$. — Let $\overline{Y}$ be a smooth, connected curve of bidegree $(s, s-2)$ on a smooth quadric surface. From the cohomology of a curve on such a surface, it follows

$$h^1(\mathcal{I}_{\overline{Y}}(s-1)) = h^1(\mathcal{O}_{\overline{Y}}(s-2)) = 0.$$

Arguing as in the previous case, by liaison $(s, k+1)$, we can link $\overline{Y}$ to a smooth, connected curve X, with $\deg(X) = ks - (s-2)$, $p_a(X) = G(ks-(s-2), s) - 1$ and $s(X) = s$.

Remarks 3.2.

(i) Using the liaison formulæ we get $e(X) = k + s - 6$ if $r = 2$ (resp. $e(X) = k + s - 5$ if $r = s - 2$) as predicted by 2.9 (resp. 2.6).

(ii) If $r = s - 2$ the curve $\overline{Y}$ of the proof above is linked to the union of two skew lines by a complete intersection $(2, s)$ (see 2.4).

Finally we are able to show the

THEOREM 3.3. — *Let d, s be integers, $s \geq 4$, $d > s(s-1)$, and let r be such that $d + r \equiv 0 \pmod{s}$, $1 \leq r \leq s-1$. Then the triple $(d\,;\ G(d,s)-1\,;\ s)$ is an Halphen's gap (see 1.3) except for*

(i) *$s = 4$;*

(ii) *$s \geq 5$ and $2 \leq r \leq 3$ or $s-3 \leq r \leq s-2$.*

Proof. — From 1.7 and 3.1 it follows that we have no Halphen's gaps in both cases (i) and (ii). Let X be a smooth, connected curve with $\deg(X) = ks - r$, $r \notin \{2, 3, s-3, s-2\}$, $g(X) = G(d,s) - 1$ and $s(X) = s$. From 1.5 iii) we get $\sigma(X) = s$, hence from 1.10, $\chi(X) = \Phi$. Now we conclude with 2.4 and 2.9.

Remarks 3.4.

(i) Actually the proof yields a complete description of the curves of degree d, genus $G(d,s) - 1$, lying on an irreducible surface of degree s. This description can be used to give informations on the Hilbert scheme of these curves.

(ii) If $r = s - 2$ and $k = s$, then any curve, X, of degree d, genus $G(d,s) - 1$, with $s(X) = s$ is of maximal rank, but not projectively normal (see [1, 5.7]).

(iii) It is know that $(d\,;\ G(d,s)-1\,;\ s)$ is an Halphen's gap, when $r = 0$, $s \geq 4$ (see [1, 3.10]). Hence the problem to determine the Halphen's gap of space curves, of degree d, genus $G(d,s) - 1$, is completely solved, when $s \geq 5$, $d > s(s-1)$.

On the other hand it is still an open problem to determine the exact value of $s(d,g)$. At present only the cases $s \leq 5$ are solved.

COROLLARY 3.5. — *If $s \leq 5$, $d > s(s-1)$, $g \geq G(d,s)-1$, then $s(d,g)$ is known.*

Proof. — If $g \geq G(d,5)$, see [1, 3.13]. If $g = G(d,5) - 1$, from 3.3 we get $s(d\,;\ G(d,5)-1) = 5$ if $r = 2$ or 3 and $s(d\,;\ G(d,5)-1) \leq 4$ otherwise. From [8, thm. 1], there exist smooth, connected curves of degree $d > 20$ genus $G(d,5) - 1$, lying on a smooth quartic surface. Such curves do not lie on a cubic surface, because of the condition $d > 20$. Hence we conclude $s(d;\ G(d,5)-1) = 4$, when $0 \le r \le 1$ or $r = 4$.

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