BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 115 (1987), p. 391-395

<http://www.numdam.org/item?id=BSMF_1987__115__391_0>

Bull. Soc. math. France,
115, 1987, p. 391-395.

ODD VALUES OF THE RAMANUJAN τ-FUNCTION

BY

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RÉSUMÉ. — Soit τ la fonction de Ramanujan. Nous prouvons qu'il existe une effectivement calculable constante $c>0$ absolue, tel que si $\tau(n)$ est impaire, alors $|\tau(n)| \geqslant (\log n)^c$. Nous utilisons les résultats sur les formes linéaires des logarithmes.

ABSTRACT. — Let τ denote Ramanujan's function. We prove that there exists an effectively computable absolute constant $c>0$ such that if $\tau(n)$ is odd, then $|\tau(n)| \geqslant (\log n)^c$. We use results on linear forms in logarithms.

Ramanujan's τ-function is defined by the relation

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

It is conjectured by ATKIN and SERRE [6, equation 4.11 k] that for any $\varepsilon > 0$,

$$|\tau(p)| \gg_{\varepsilon} p^{(9/2)-\varepsilon}.$$

In particular this implies that for any a, there are only finitely many primes p such that $\tau(p)=a$. In this note, we study a related, though simpler, question. Our main result is the following.

(*) Texte reçu le 7 avril 1986.

Research of the first two authors was partially supported by NSERC grants U0237 and U0373 respectively.

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BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE — 0037-9484/1987/03 391 05/$2.50

THEOREM. — *There exists an effectively computable absolute constant $c>0$, such that for all positive integers n for which $\tau(n)$ is odd, we have*

$$|\tau(n)| \geqslant (\log n)^c.$$

It follows from the theorem that for an odd integer a, the equation

$$\tau(n)=a \tag{1}$$

has only finitely many solutions.

As $\tau(p)$ is even, all integers satisfying (1) are squarefull (i. e. every prime divisor of n appears to at least the second power). We apply the theory of linear forms in logarithms to obtain lower bounds for $\tau(p^m)$, p a prime and $m \geqslant 2$, which, in particular gives the theorem.

We require several lemmas.

LEMMA 1. — $\tau(p^m)=0$ *if and only if m is odd and* $\tau(p)=0$.

Proof. – Write $\tau(p)=\alpha_p+\bar{\alpha}_p$, $\alpha_p=p^{11/2}e^{i\theta_p}$, $0 \leqslant \theta_p \leqslant \pi$. Set

$$\gamma_m(p)=\begin{cases} 1, & \text{if } m \text{ is even} \\ \tau(p), & \text{if } m \text{ is odd}\end{cases}$$

and $\zeta=\exp(2\pi i/(m+1))$. Then, as in Ramanujan [4],

$$\begin{aligned}\tau(p^m)&=(\alpha_p^{m+1}-\bar{\alpha}_p^{m+1})/(\alpha_p-\bar{\alpha}_p)\\ &=\gamma_m(p)\prod_{r=1}^{[m/2]}(\alpha_p-\zeta^r\bar{\alpha}_p)(\alpha_p-\zeta^{-r}\bar{\alpha}_p)\\ &=\gamma_m(p)\prod_{r=1}^{[m/2]}(\tau(p)^2-4p^{11}\cos^2(\pi r/(m+1))).\end{aligned} \tag{2}$$

If the r-th factor is zero,

$$4\cos^2(\pi r/(m+1))=\zeta^r+\zeta^{-r}+2=\tau(p)^2/p^{11},$$

is both an algebraic integer and a rational number. Thus it is a rational integer and so must be one of 1, 2 or 3. But none of $\tau(p)^2-p^{11}$, $\tau(p)^2-2p^{11}$, $\tau(p)^2-3p^{11}$ can be zero, since $\tau(2)=-24 \neq \pm 2^6$ and $\tau(3)=252 \neq \pm 3^6$. Thus $\tau(p^m)=0$ if and only if $\gamma_m(p)=0$.

The next three lemmas depend on the theory of linear forms in logarithms. They are stronger than needed for the proof of the theorem. They may be of independent interest.

LEMMA 2. — *There is an effectively computable absolute constant $C_1>0$ such that for all $m\geqslant 2$, we have*

$$|\tau(p^m)| \geqslant |\gamma_m(p)|\, p^{(11/2)(m-C_1\log m)}.$$

Proof. — Suppose that m is odd. If $\tau(p)=0$, there is nothing to prove. If $\tau(p)\neq 0$, then we see from (2) that $\tau(p^m)\neq 0$. Then

$$\begin{aligned}|\tau(p^m)/\gamma_m(p)| &= |\alpha_p^{m+1}-\bar{\alpha}_p^{m+1}|\,|\alpha_p^2-\bar{\alpha}_p^2|^{-1}\\ &\geqslant \frac{1}{2}p^{(11/2)(m-1)}|(\alpha_p/\bar{\alpha}_p)^{m+1}-1|\\ &\geqslant p^{(11/2)(m-C_1\log m)},\end{aligned}$$

where in the final step, we used the fact that the height of $\alpha_p/\bar{\alpha}_p$ is bounded by a power of p and estimate of BAKER [1] on linear forms. If m is even, the required estimate follows similarly.

The constant C_1 above is quite large and so the bound is non-trivial only for large m. The next lemma gives a bound which is non-trivial for bounded m.

LEMMA 3. — *Let $m\geqslant 6$. There is an effectively computable number $C_2>0$ depending only on m such that either $\tau(p^m)=0$ or*

$$|\tau(p^m)|>p^{C_2}.$$

Proof. — Let $m\geqslant 6$ and $\tau(p^m)\neq 0$. Observe that $\tau(p^m)/\gamma_m(p)$ is a binary form in $\tau(p^2)$ and p^{11} with at least three distinct linear factors. We apply an estimate of FELDMAN [3] or BAKER [2] on the magnitude of integral solutions of Thue's equation to obtain the assertion of the lemma.

Remark. — In fact, we could have applied a theorem of ROTH [5] on the approximations of algebraic numbers by rationals to obtain the following stronger, but ineffective, version of Lemma 3: for every $\varepsilon>0$, $\tau(p^m)=0$ or

$$|\tau(p^m)| \gg_{\varepsilon,m} p^{(11/2)(m-4)-\varepsilon}.$$

LEMMA 4. — *There is an effectively computable absolute constant $C_3>0$ such that*

$$|\tau(p^m)|\geqslant(\log p)^{C_3},\qquad m=2,\ 4.$$

Further,

$$\min(|\tau(p^3)|, |\tau(p^5)|) \geqslant \frac{1}{2} p^{11/2}$$

whenever $\tau(p) \neq 0$.

Proof. — Observe that

$$\tau(p)^2 = p^{11} + \tau(p^2)$$

and

$$(2\tau(p)^2 - 3p^{11})^2 = 5p^{22} + 4\tau(p^4).$$

Now we apply an estimate of SPRINDZUK [7] on the magnitude of integral solutions of hyperelliptic equations to obtain the first inequality of the lemma. The second inequality follows immediately from the relations.

$$\tau(p^3) = \tau(p)(\tau(p)^2 - 2p^{11})$$

and

$$\tau(p^5) = \tau(p)(\tau(p)^2 - 3p^{11})(\tau(p)^2 - p^{11}).$$

Proof of theorem. — Let n be such that $\tau(n)$ is odd. As remarked in the beginning, we see that n is squarefull. Therefore, if p and m are such that $p^m \| n$, it follows from our lemmas that

$$|\tau(p^m)| \geqslant (\log p^m)^{C_4}$$

where $C_4 > 0$ is an effectively computable absolute constant. Hence,

$$|\tau(n)| = \prod_{p^m \| n} |\tau(p^m)| \geqslant (\log n)^{C_4}$$

which implies the assertion of the theorem.

The same method can be used to study the Fourier coefficients of other modular forms. Indeed, let f be a cusp form of weight $k \geqslant 4$ for $\Gamma_0(N)$ and write

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

for the Fourier expansion at $i\infty$. Suppose that:

(i) *f is a normalized eigenform for all the Hecke operators T_p for $(p, N) = 1$;*

(ii) *f does not have complex multiplication;*

(iii) *the* a_n *are rational integers;*

(iv) $a_2 \neq \pm 2^{k/2}$ *and* $a_3 \neq \pm 3^{k/2}$.

Then, for n squarefull, $a_n = 0$ or

$$|a_n| \geqq (\log n)^D$$

for some effectively computable constant $D > 0$ which depends only on f.

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