BULLETIN DE LA S. M. F.

MICHEL ZINSMEISTER

A distortion theorem for quasiconformal mappings

Bulletin de la S. M. F., tome 114 (1986), p. 123-133
<http://www.numdam.org/item?id=BSMF_1986__114__123_0>

Bull. Soc. math. France,
114, 1986, p. 123-133

A DISTORTION THEOREM FOR QUASICONFORMAL MAPPINGS

BY

MICHEL ZINSMEISTER (*)

RÉSUMÉ. — Un théorème de distorsion pour les applications quasiconformes. Nous généralisons aux applications quasiconformes dans la boule unité de $\mathbb{R}^n$ un théorème de Pommerenke relatif aux transformations conformes du disque unité du plan.

ABSTRACT. — A distortion theorem for quasiconformal mappings. We extend to quasi-conformal mappings in the unit ball of $\mathbb{R}^n$ a theorem of Pommerenke concerning conformal mappings in the unit disk of the plane.

1. Introduction

(*a*) The purpose of this paper is to extend to quasi-conformal mappings the following distortion theorem for conformal mappings in the unit disk B^2 of $\mathbb{R}^2$, a theorem due to Pommerenke. In this statement, $I(z)$ is, for $z \in B^2$, the interval of B^2 centered at $z/|z|$ of length $2\pi(1-|z|)$.

THÉORÈME [9]. — *There is a universal constant $C>0$ such that if $f: B^2 \to \mathbb{R}^2$ is a conformal mapping, then, for every $z \in B^2$, there exists a non euclidean segment γ from z to $I(z)$ such that*

$$\text{Length}(f(\gamma)) \leqq C \,\text{distance}(f(z), \partial f(B^2)).$$

Before giving the precise results, we set some preliminary notations and results.

For $x \in \mathbb{R}^n$, $n \geqq 2$, let $|x|$ be the euclidean norm of x. For $r>0$,

$$B^n(x, r) = \{y \in \mathbb{R}^n;\ |y-x|<r\},$$
$$B^n = B^n(0, 1) \quad \text{and} \quad S^{n-1} = \partial B^n.$$

(*) Texte reçu le 14 mars 1985.

M. ZINSMEISTER, Mathématiques, Université de Rouen, 76130 Mont St-Aignan Cedex.

If $E\subset\mathbb{R}^n$ and $x\in\mathbb{R}^n$, $d(x, E)$ is the distance from x to E. If $E\subset\mathbb{R}^n$ is measurable, we note $m(E)$ its Lebesgue measure; σ stands for Lebesgue measure on S^{n-1} and $\sigma_{n-1}=\sigma(S^{n-1})$. For a real $n\times n$ matrix A, let

$$|A|=\sup_{x\in S^{n-1}}|Ax|.$$

If $\Omega\subset\mathbb{R}^n$ is open and $f:\Omega\to\mathbb{R}^n$ is in the Sobolev space $W^1_{n\,\mathrm{loc}}(\Omega)$, $Df(x)$ will denote the Jacobian matrix of f, defined almost everywhere, and $J(x, f)=\det Df(x)$.

For $K\geqq 1$, a continuous one-to-one mapping $f:\Omega\to\mathbb{R}^n$ is said to be K-*quasiconformal* if $f\in W^1_{n\,\mathrm{loc}}(\Omega)$ and if

$$|Df(x)|^n\leqq KJ(x, f)\quad \text{a. e. on } \Omega.$$

For $x\in S^{n-1}$, we define the cone with vertex x as

$$\Gamma(x)=\{y\in B^n; |y-x|<3(1-|y|)\}.$$

If $F: B^n\to\mathbb{R}^n$ is any function, the non-tangential maximal function of F is defined as

$$\forall x\in S^{n-1},\quad F^*(x)=\sup_{y\in\Gamma(x)}|F(y)|.$$

If $z\in B^n$, we define the « cap » $S(z)$ as

$$S(z)=\{x\in S^{n-1}; z\in\Gamma(x)\}=S^{n-1}\cap B^n(z, 3(1-|z|)).$$

Let M be the group of Möbius self-maps of B^n. If $z\in B^n$, $z\neq 0$, we define

$$T_z(x)=\frac{(1-|z|^2)(x-z)-|x-z|^2 z}{|z|^2\,|x-(z/|z|^2)|^2};$$

then $T_z\in M$ and $T_z(z)=0$.

We will need the following elementary results, the proof of which we omit:

(1) $$S(z)=S^{n-1}\quad \text{if}\quad |z|<1/2.$$

(2) $T_z(S(z))$ always contains an hemisphere,

(3) If $x, y\in S(z)$,

$$(9(1-|z|))^{-1}|x-y|\leqslant|T_z(x)-T_z(y)|\leqslant 2(1-|z|)^{-1}|x-y|,$$

(4) $$\forall z\in B^n,\quad B^n(0, 1/7)\subset T_z\left(B^n\left(z, \frac{1}{4}(1-|z|)\right)\right)\subset B^n(0,1/2).$$

(*b*) The main result will be the following.

THEOREM 1. — *For $K \geqslant 1$ there exists a constant $C(K, n) > 0$ such that if $f\, B^n \to \mathbb{R}^n$ is K-quasiconformal, then, for any $z \in B^n$, there exists a non-euclidean segment γ joining z to $S(z)$ such that*

$$\text{(5)} \qquad \operatorname{Length}(f(\gamma)) \leqslant C(K, n)\, d(f(z), \partial f(B^n)).$$

Adapting an idea of B. DAVIS and J. LEWIS [2], we will show in a moment that Theorem 1 is a corollary of the following, of independent interest:

THEOREM 2. — *For $f : B^n \to \mathbb{R}^n$ k-quasiconformal, define, for $x \in S^{n-1}$, $L_f(x) = \operatorname{Length}[f([0, x])]$ where $[0, x]$ is the radius $\{tx;\ 0 \leqslant t \leqslant 1\}$. Then there exist $C(n, K) > 0$ and $p(n, K) > 0$ such that*

$$\text{(6)} \qquad \left(\int_{S^{n-1}} L_f(x)^p\, d\sigma(x)\right)^{1/p} \leqslant C(K, n)\, d(f(0), \partial f(B^n)).$$

Assuming Theorem 2 is true, let us prove Theorem 1. So let $f : B^n \to \mathbb{R}^n$ be a K-quasiconformal mapping and $z \in B^n$; put $g = f \circ T_z^{-1}$: Applying Theorem 2 to g, we see that for every $M > 0$,

$$\text{(7)} \qquad \sigma(\{x \in S^{n-1};\ \operatorname{Lg}(x) > M\, d(f(z), \partial f(B^n))\}) \leqslant \left(\frac{C(K, n)}{M}\right)^p.$$

Now choose M large enough so that $(C/M)^p \leqslant (\sigma_{n-1})/4$. By (2) there exists then $x \in T^z(S(z))$ such that

$$\operatorname{Lg}(x) \leqslant M\, d(f(z), \partial f(B^n)),$$

and this proves (5) with $\gamma = T_z^{-1}([0, x])$.

The main tools in proving Theorem 2 will be Theorem 3, due to P. Jones, which we discuss in Part 2, and an estimate for the nontangential maximal function f^*, which is proved in part 3.

This paper was written during a stay at the University of Michigan. I would like to thank Professor F. Gehring for his invitation and the constant help he gave me during my stay. I also would like to thank Professor T. Iwaniec for many helpful conversations.

2. Let f be as in Theorem 2. Performing a preliminary translation, we may assume that f does not vanish in B^n and that

$$|f(0)|=d(f(0), \partial f(B^n)).$$

We will say that a function $g: B^n \to \mathbb{R}^n-\{0\}$ satisfies Harmack property if there exists a constant $C(g)>0$, called the Harnack constant of g, such that:

$$(8) \qquad \forall x \in B^n, \quad \forall y, \quad z \in B^n\left(x, \frac{1}{4}(1-|x|)\right), \qquad |g(y)| \leqslant C(g)|g(z)|.$$

LEMMA 1. — *If f: $B^n \mathbb{R}^n-\{0\}$ is K-quasiconformal, then f satisfies Harnack property with a constant depending only on K and n.*

Proof. — By the special distortion theorem for quasiconformal mappings [4], there exists $C(K, n)>0$ such that

$$\forall x \in B^n, \quad \forall y, \quad z \in B^n\left(x, \frac{1}{4}(1-|x|)\right), \qquad \frac{|f(y)-f(z)|}{d(f(z), \partial f(B^n))} \leqslant C(K, n),$$

and Lemma 1 follows, for $d(f(z), \partial f(B^n)) \leqslant |f(z)|$ since f does not vanish in B^n.

Now let f: $B^n \to \mathbb{R}^n-\{0\}$ be K-quasiconformal. For almost every $x \in S^{n-1}$, we may write

$$L_f(x) \leqslant \int_0^1 |Df(tx)| \, dt \leqslant V_f(x)+2^{n-1} f^*(x) H_f(x),$$

where

$$V_f(x)=\int_0^{1/2} |Df(tx)| \, dt$$

and

$$H_f(x)=\int_0^1 \frac{|Df(tx)|}{|f(tx)|} t^{n-1} \, dt.$$

LEMMA 2:

$$V_f \in L^1(S^{n-1}) \qquad \textit{with} \quad \|V_f\|_1 \leqslant C(K, n)|f(0)|.$$

Proof:

$$\| V_f \|_1 = \int_{B^n(0,\,1/2)} \frac{|Df(y)|}{|y|^{n-1}}\, dm(y):$$

By Gehring's inequality [5], there exists $p=p(K, n)>n$ and $C(K, n)>0$ such that

$$\text{(9)}\quad \left(\int_{B^n(0,\,1/2)} |Df(y)|^p\, dm(y)\right)^{1/p} \leqslant C(K, n)\left(\int_{B^n(0,\,1/2)} J(x, f)\, dm(x)\right)^{1/n} = C(K, n)\, m[f(B^n(0, 1/2)]^{1/n} \leqslant C(K, n)\,|f(0)|,$$

the last inequality being a consequence of Lemma 1. We now apply Hölder's inequality to the expression of $\| V_f \|_1$, to obtain

$$\| V_f \|_1 \leqslant \left(\int_{B^n(0,\,1/2)} |Df(y)|^p\, dm(y)\right)^{1/p} \times \left(\int_{B^n(0,\,1/2)} |y|^{-(n-1)p/(p-1)}\, dm(y)\right)^{(p-1)/p} \leqslant C(K, n)\,|f(0)|,$$

by (9) and the fact that $(n-1)p/(p-1)<n$.

An estimate for H_f is given by the following theorem, due to P. Jones:

THEOREM 3 [8]:

$$H_f \in L^1(S^{n-1}) \qquad \textit{with} \quad \| H_f \|_1 \leqslant C(K, n).$$

As P. Jones has observed, Theorem 3 implies that

$$\sup_{T\in M} \| H_{f\circ T} \|_1 \leqslant C(K, n) \quad \Leftrightarrow$$

$$\text{(10)}\qquad \sup_{TCM} \int_{B^n} \frac{|Df(x)|}{|f(x)|}\, |DT(x)|^{n-1}\, dm(x) \leqslant C(K, n),$$

and (10) exactly says that $|Df|/|f|\,dm$ is a Carleson measure in B^n[3]. Since $|\nabla |f|| \leqslant |Df|$, the same is true for $|\nabla u|\,dm$, where $u=\log|f|$. Writing

$$\| u \|_* = \sup_{T\in M} \int_{B^n} |\nabla u(x)|\, |DT(x)|^{n-1}\, dm(x),$$

we can now invoke the following theorem, due to Varopoulos:

THEOREM 4 [10]. — *Let* $u \in W^1_{1\,\mathrm{loc}}(B^n)$ *be a real-valued function having radical limit* $\tilde{u}(x)$ *a. e. on* S^{n-1}. *If* $|\nabla u|\,dm$ *is a Carleson measure in* B^n, *then* $\tilde{u} \in BMO(S^{n-1})$ *with* $\|\tilde{u}\|_{BMO} \leqslant C(n)\|u\|_*$.

(Varopoulos proves Theorem 4 for $n=2$ only, but his argument is easily seen to extend to the general case.)

From Theorems 3 and 4, it follows that if $f: B^n \to \mathbb{R}^n - \{0\}$ is K-quasiconformal, then

$$\|\operatorname{Log}|f(rx)|\|_{BMO(S^{n-1})} \leqslant C(K, n),$$

for $0 \leqslant r \leqslant 1$. If we now apply the John and Nirenverg inequality [7] and Lemma 1, we get

LEMMA 3. — *If* $f: B^n \to \mathbb{R}^n - \{0\}$ *is K-quasiconformal there exist* $C(K, n) > 0$ *and* $p = p(K, n) > 0$ *such that*

$$\sup_{0 < r \leqslant 1} \int_{S^{n-1}} |f(rx)|^p \, d\sigma(x) \leqslant C(K, n)\,|f(0)|^p. \tag{11}$$

In the case $n=2$ and f conformal, (11) implies that f is in the Hardy space $H^p(B^2)$, so that $f^* \in L^p(S^{n-1})$, and Theorem 2 follows in this particular case. In the general case, we need an extra argument, provided by the next section.

3. Non-tangential maximal function

PROPOSITION 1. — *Let* $f: B^n \to \mathbb{R}^n - \{0\} \in W^1_{1\,\mathrm{loc}}(B^n)$, *and* $u = \operatorname{Log}|f|$. *If* $|\nabla u|\,dm$ *is a Carleson measure and if f satisfies Harnack property* (8) *then, for every* $p > 0$,

$$\int_{S^{n-1}} f^*(x)^p \, d\sigma(x) \leqslant C_p \int_{S^{n-1}} |f(x)|^p \, d\sigma(x),$$

where C_p *depends only on n, p,* $\|u\|_*$ *and* $C(f)$.

We first notice that this statement makes sense, since $|f(x)|$ has radial limits a. e. on S^{n-1}, for $|\nabla u|\,dm$ is a Carleson measure.

Before going into the proof of Proposition 1, let us see why it implies Theorem 2. So let $f: B^n \to \mathbb{R}^n - \{0\}$ be K-quasiconformal with

$|f(0)|=\text{dist}\ (f(0),\ f(B^n))$. By the results in Part 2 we can apply Proposition 1 to $f(x)$ and, by Lemma 3, we get

$$\| f^* \|_p \leqslant C(K, n) | f(0) |,$$

for some p depending only on K and n. Recalling now that

$$L_f(x) \leqslant V_f(x) + 2^{n-1} f^*(x) H_f(x),$$

Lemma 2, Theorem 3 and (12) imply that $L_f \in L^{p/p+1}(S^{n-1})$ with

$$\| L_f \|_{p/p+1} \leqslant C(K, n) | f(0) |,$$

and Theorem 2 is proved.

To prove Proposition 1, we need 3 lemmas:

LEMMA 4. — *If f is as in Proposition 1 and $N > C(f)^2$,*

$$\sigma\left(\left\{x \in S^{n-1};\ |f(x)| \leqslant \frac{|f(0)|}{N}\right\}\right) \leqslant \frac{7^n \| u \|_*}{\operatorname{Log} N}.$$

Proof: Put

$$F_N = \left\{x \in S^{n-1}; | f(x) | < \frac{| f(0) |}{N}\right\}$$

and

$$G(x) = \int_0^1 | \nabla u(tx) | t^{n-1}\, dt.$$

If $x \in F_n$,

$$\begin{aligned} G(x) &\geqslant 7^{1-n} \int_{1/7}^1 | \nabla u(tx) |\, dt \\ &\geqslant 7^{1-n} \left| \int_{1/7}^1 | f(tx) |^{-1} \frac{\partial}{\partial t}(| f(tx) |)\, dt \right| \\ &= 7^{1-n} \left| \operatorname{Log} \frac{| f(x) |}{| f(x/7) |} \right|. \end{aligned}$$

By Harnack property, $| f(0) | \leqslant C(f) | f(x/7) |$; so, if $x \in F_N$ and $N > C(f)^2$,

$$G(x) \geqslant 7^{1-n} \left| \operatorname{Log} \frac{N}{C(f)} \right| \geqslant 7^{-n} \operatorname{Log} N,$$

and from this it follows that

$$\sigma(F_N) \leqslant \frac{7^n \| G \|_1}{\operatorname{Log} N} \leqslant \frac{7^n \| u \|_*}{\operatorname{Log} N}.$$

LEMMA 5. — *There is an universal constant* $\alpha > 0$ *such that if* f *is as in Proposition* 1 *and* $z \in B^n$,

$$\sigma\left(\left\{x \in S(z); |f(x)| \leqslant \frac{|f(z)|}{N}\right\}\right) \leqslant \frac{\alpha\, 7^n \| u \|_*}{\operatorname{Log} N} \sigma(S(z)).$$

Proof. — Define $g(x) = f \circ T_z^{-1}(x)$. By (4), if $x \in S^{n-1}$ then $|g(0)| \leqslant C(f) |g(x/7)|$; also, by definition $\| \operatorname{Log} |g| \|_* = \| u \|_*$. By Lemma 4 we then have, for $N \geqslant C(f)^2$,

$$\sigma\left(\left\{x \in S^{n-1}; g(x) \leqslant \frac{|f(z)|}{N}\right\}\right) \leqslant \frac{7^n \| u \|_*}{\operatorname{Log} N},$$

which implies

$$\sigma\left(T_z\left(\left\{y \in S(z); |f(y)| \leqslant \frac{|f(z)|}{N}\right\}\right)\right) \leqslant \frac{7^n \| u \|_*}{\operatorname{Log} N},$$

and the result follows from (3).

LEMMA 6. — *There exist* $0 < C(n) < 1$ *and* $N(n, \| u \|_*, C(f))$ *such that the following inequality holds*:

$$(13) \quad \forall \lambda > 0, \quad \sigma\left(\left\{x \in S^{n-1}; f^*(x) > \lambda, |f(x)| \leqslant \frac{\lambda}{N}\right\}\right) \leqslant C(n)\, \sigma(\{x \in S^{n-1}; f^*(x) \geqslant \lambda\}).$$

Proof. — Let

$$\mathscr{E}(\lambda) = \{z \in B^n, |f(z)| > \lambda\}$$

and

$$\mathscr{U}(\lambda) = \{x \in S^{n-1}; f^*(x) > \lambda\}.$$

Then

$$\mathscr{U}(\lambda) = \cup_{z \in \mathscr{E}(\lambda)} S(z).$$

By Vitali covering lemma, there exists $\alpha(n)\in(0,1)$ and a sequence $\{z_j\}\subset\mathscr{E}(\lambda)$ such that the $S(z_j)$'s are mutually disjoint and

$$\sum_{j\in\mathbb{N}}\sigma(S(z_j))\geqslant\alpha(n)\,\sigma(\mathscr{U}(\lambda)).$$

Now let

$$E_N=\left\{x\in S^{n-1};\ |f(x)|\leqslant\frac{\lambda}{N}\right\}.$$

Then

$$\sigma(E_N\cap\mathscr{U}(\lambda))\leqslant\sum_{j\in\mathbb{N}}\sigma(E_N\cap S(z_j))+(1-\alpha(n))\,\sigma(\mathscr{U}(\lambda)).$$

But

$$E_N\cap S(z_j)\subset\left\{x\in S(z_j);\ |f(x)|\leqslant\frac{|f(z_j)|}{N}\right\}$$

and so

$$\sigma(E_N\cap S(z_j))\leqslant\frac{\alpha\, 7^n\|u\|_*}{\operatorname{Log}N}\,\sigma(S(z_j))\qquad\text{if}\quad N\geqslant C(f)^2,$$

by Lemma 5. So, choosing N so large that

$$\frac{\alpha 7^n\|u\|_*}{\operatorname{Log}N}<\frac{\alpha(n)}{2},$$

we get Lemma 6 with $C(n)=1-\alpha(n)/2$. We can now complete the proof of Proposition 1. For $\lambda>0$, let

$$\chi(\lambda)=\sigma(\{x\in S^{n-1};\ f^*(x)>\lambda\})$$

and

$$\theta(\lambda)=\sigma(\{x\in S^{n-1};\ |f(x)|>\lambda\}).$$

By (13),

$$\chi(\lambda)\leqslant C(n)\,\chi(\lambda)+\theta(\lambda/N)\quad\Leftrightarrow\quad\chi(\lambda)\leqslant\frac{1}{1-C(n)}\,\theta(\lambda/N).$$

Finally,

$$\|f^*\|_p^p=p\int_0^\infty\lambda^{p-1}\chi(\lambda)\,d\lambda\leqslant\frac{p}{1-C(n)}\int_0^\infty\lambda^{p-1}\theta\left(\frac{\lambda}{N}\right)d\lambda=\frac{N^p}{1-C(n)}\|f\|_p^p,$$

which proves Proposition 1.

Remark. — A mapping $f: B^n \to \mathbb{R}^n$ is said to be K-quasiregular if $f \in W^1_{n\,\mathrm{loc}}(B^n)$ and if

$$|Df(x)|^n \leqslant KJ(x, f) \quad \text{a. e. on } B^n.$$

The above methods imply the following:

PROPOSITION 2. — *Let $f: B^n \to \mathbb{R}^n - \{0\}$ be a K-quasiregular mapping and $u = \mathrm{Log}\,|f|$. If $|\nabla u|\,dm$ is a Carleson measure in B^n, there exists an exponent $p = p(n, K, \|u\|_*) > 0$ such that $L_f \in L^p(S^{n-1})$, where L_f has the same meaning as in Theorem* 1.

Sketch of Proof. — Using the same notations as in 2, one may write

$$L_f(x) \leqslant V_f(x) + 2^{n-1} K f^*(x) \tilde{H}_f(x),$$

where

$$\tilde{H}_f(x) = \int_0^1 |\nabla u(tx)|\, t^{n-1}\, dt.$$

From results in [1], Lemma 2 is valid for V_f; also, $\tilde{H}_f \in L^1$ by hypothesis. To prove Proposition 2, it suffices then to show that $f^* \in L^p$ for some $p > 0$. Using Proposition 1 and Theorem 4, this reduces to proving that f satisfies Harnack property. To do this, we use a recent result of IWANIEC and NOLDER [6]: They proved that u is actually in $W^1_{q\,\mathrm{loc}}(B^n)$ for some $q(K, n) > n$ and that

$$\forall x \in B^n, \quad \left(\int_{B^n(x,\, 1/2(1-|x|))} |\nabla u(y)|^q\, dm(y)\right)^{1/q} \leqslant C(K, n)(1-|x|)^{n(1/q-1)} \times \int_{B^n(x,\, 3/4(1-|x|))} |\nabla u(y)|\, dm(y),$$

and the result follows from the Sobolev embedding theorem and the fact that $|\nabla u|\,dm$ is a Carleson measure.

REFERENCES

[1] BOJARSKI (B.) and IWANIEC (T.). – Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$, *Ann. Acad. Sc. Fenn.*8 (1983) p. 257-324.

[2] DAVIS (B.) and LEWIS (J.). – Paths for subharmonic functions, *Proc. London Math. Soc.*, Vol. XLVIII, 1984, p. 401-428.

[3] GARNETT (J.), *Bounded analytic functions*, Academic Press, 1981.

[4] GEHRING (F.), Rings and quasiconformal mappings in space, *Trans. AMS*, Vol. 103, 1962, p. 353-393.

[5] GEHRING (F.). – The L^p integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.*, Vol. 130, 1973, pp. 265-277.

[6] IWANIEC (T.) and NOLDER (C.). – Hardy-Littlewood inequality for quasiregular mappings in certain domains in $\mathbb{R}^n$, *Ann. Acad. Sc. Fenn.* (to appear).

[7] JOHN (F.) and NIRENBERG (L.). – On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, Vol. 14, 1961, pp. 415-426.

[8] JONES (P.). – Extension theorems for BMO, *Indiana J. Math.*, Vol. 29, 1980, pp. 41-66.

[9] POMMERENKE (C.). – *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[10] VAROPOULOS (N.). – *BMO* functions and the $\bar{\partial}$ equation, *Pacific J. Math.*, Vol. 71, 1977, pp. 221-273.