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A distortion theorem for quasiconformal mappings


<http://www.numdam.org/item?id=BSMF_1986__114__123_0>
A DISTORTION THEOREM
FOR QUASICONFORMAL MAPPINGS

BY

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Résumé. — Un théorème de distorsion pour les applications quasiconformes. Nous
généralisons aux applications quasiconformes dans la boule unité de \( \mathbb{R}^n \) un théorème de
Pommerenke relatif aux transformations conformes du disque unité du plan.

Abstract. — A distortion theorem for quasiconformal mappings. We extend to
quasiconformal mappings in the unit ball of \( \mathbb{R}^n \) a theorem of Pommerenke concerning conformal
mappings in the unit disk of the plane.

1. Introduction

(a) The purpose of this paper is to extend to quasi-conformal mappings
the following distortion theorem for conformal mappings in the unit disk
\( B^2 \) of \( \mathbb{R}^2 \), a theorem due to Pommerenke. In this statement, \( I(z) \) is, for
\( z \in B^2 \), the interval of \( B^2 \) centered at \( z/|z| \) of length \( 2\pi (1 - |z|) \).

Théorème [9]. — There is a universal constant \( C > 0 \) such that if
\( f : B^2 \to \mathbb{R}^2 \) is a conformal mapping, then, for every \( z \in B^2 \), there exists a
non euclidean segment \( \gamma \) from \( z \) to \( I(z) \) such that

\[
\text{Length} (f(\gamma)) \leq C \text{ distance} (f(z), \partial f(B^2)).
\]

Before giving the precise results, we set some preliminary notations and
results.

For \( x \in \mathbb{R}^n, n \geq 2 \), let \( |x| \) be the euclidean norm of \( x \). For \( r > 0 \),

\[
B^n(x, r) = \{ y \in \mathbb{R}^n; |y - x| < r \}.
\]

\( B^n = B^n(0, 1) \) and \( S^{n-1} = \partial B^n \).

(*) Texte reçu le 14 mars 1985.

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If $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $d(x, E)$ is the distance from $x$ to $E$. If $E \subset \mathbb{R}^n$ is measurable, we note $m(E)$ its Lebesgue measure; $\sigma$ stands for Lebesgue measure on $S^{n-1}$ and $\sigma_{n-1} = \sigma(S^{n-1})$. For a real $n \times n$ matrix $A$, let

$$|A| = \sup_{x \in S^{n-1}} |Ax|.$$  

If $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^n$ is in the Sobolev space $W^{1}_{n, \text{loc}}(\Omega)$, $Df(x)$ will denote the Jacobian matrix of $f$, defined almost everywhere, and $J(x, f) = \det Df(x)$.

For $K \geq 1$, a continuous one-to-one mapping $f : \Omega \to \mathbb{R}^n$ is said to be $K$-quasiconformal if $f \in W^{1}_{n, \text{loc}}(\Omega)$ and if

$$|Df(x)|^n \leq K J(x, f) \quad \text{a.e. on } \Omega.$$  

For $x \in S^{n-1}$, we define the cone with vertex $x$ as

$$\Gamma(x) = \{ y \in \mathbb{B}^n; |y-x| < 3(1 - |y|) \}.$$  

If $F : \mathbb{B}^n \to \mathbb{R}^n$ is any function, the non-tangential maximal function of $F$ is defined as

$$\forall x \in S^{n-1}, \quad F^*(x) = \sup_{y \in \Gamma(x)} |F(y)|.$$  

If $z \in \mathbb{B}^n$, we define the « cap » $S(z)$ as

$$S(z) = \{ x \in S^{n-1}; \ z \in \Gamma(x) \} = S^{n-1} \cap B^n(z, 3(1 - |z|)).$$  

Let $M$ be the group of Möbius self-maps of $\mathbb{B}^n$. If $z \in \mathbb{B}^n$, $z \neq 0$, we define

$$T_z(x) = \frac{(1 - |z|^2)(x-z) - |x-z|^2 z}{|z|^2 |x - (z/|z|^2)|^2},$$

then $T_z \in M$ and $T_z(z) = 0$.

We will need the following elementary results, the proof of which we omit:

1. $S(z) = S^{n-1}$ if $|z| < 1/2$.
2. $T_z(S(z))$ always contains an hemisphere,
3. If $x, y \in S(z),$

$$|x - y| \leq (9(1 - |z|))^{-1} |T_z(x) - T_z(y)| \leq 2(1 - |z|)^{-1} |x - y|.$$  

4. $\forall z \in \mathbb{B}^n, \quad B^n(0, 1/7) \subset T_z\left( B^n \left( z, \frac{1}{4} (1 - |z|) \right) \right) \subset B^n(0, 1/2).$
(b) The main result will be the following.

**Theorem 1.** — For $K \geq 1$ there exists a constant $C(K, n) > 0$ such that if $f : B^n \to \mathbb{R}^n$ is $K$-quasiconformal, then, for any $z \in B^n$, there exists a non-euclidean segment $\gamma$ joining $z$ to $S(z)$ such that

$$\text{Length}(f(\gamma)) \leq C(K, n) d(f(z), \partial f(B^n)).$$

Adapting an idea of B. Davis and J. Lewis [2], we will show in a moment that Theorem 1 is a corollary of the following, of independent interest:

**Theorem 2.** — For $f : B^n \to \mathbb{R}^n$ $k$-quasiconformal, define, for $x \in S^{n-1}$, $L_f(x) = \text{Length}([f([0, x])])$ where $[0, x]$ is the radius $\{tx; 0 \leq t \leq 1\}$. Then there exist $C(n, K) > 0$ and $p(n, K) > 0$ such that

$$\left(\int_{S^{n-1}} L_f(x)^p \, d\sigma(x)\right)^{1/p} \leq C(K, n) d(f(0), \partial f(B^n)).$$

Assuming Theorem 2 is true, let us prove Theorem 1. So let $f : B^n \to \mathbb{R}^n$ be a $K$-quasiconformal mapping and $z \in B^n$; put $g = f \circ T^{-1}_z$. Applying Theorem 2 to $g$, we see that for every $M > 0$,

$$\sigma(\{x \in S^{n-1}; L_g(x) > M d(f(z), \partial f(B^n))\}) \leq \left(\frac{C(K, n)}{M}\right)^p.$$

Now choose $M$ large enough so that $(C/M)^p \leq (\sigma_{n-1})/4$. By (2) there exists then $x \in T^*(S(z))$ such that

$$L_g(x) \leq M d(f(z), \partial f(B^n)),$$

and this proves (5) with $\gamma = T^{-1}_z([0, x])$.

The main tools in proving Theorem 2 will be Theorem 3, due to P. Jones, which we discuss in Part 2, and an estimate for the nontangential maximal function $f^*$, which is proved in part 3.

This paper was written during a stay at the University of Michigan. I would like to thank Professor F. Gehring for his invitation and the constant help he gave me during my stay. I also would like to thank Professor T. Iwaniec for many helpful conversations.
2. Let $f$ be as in Theorem 2. Performing a preliminary translation, we may assume that $f$ does not vanish in $B^n$ and that

$$|f(0)| = d(f(0), \partial f(B^n)).$$

We will say that a function $g : B^n \to \mathbb{R}^n - \{0\}$ satisfies Harnack property if there exists a constant $C(g) > 0$, called the Harnack constant of $g$, such that:

$$(8) \forall x \in B^n, \forall y, z \in B^n \left( x, \frac{1}{4}(1 - |x|) \right), \quad |g(y)| \leq C(g)|g(z)|.$$  

**Lemma 1.** If $f : B^n \to \mathbb{R}^n - \{0\}$ is $K$-quasiconformal, then $f$ satisfies Harnack property with a constant depending only on $K$ and $n$.

**Proof.** By the special distortion theorem for quasiconformal mappings [4], there exists $C(K, n) > 0$ such that

$$\forall x \in B^n, \forall y, z \in B^n \left( x, \frac{1}{4}(1 - |x|) \right), \quad \frac{|f(y) - f(z)|}{d(f(z), \partial f(B^n))} \leq C(K, n),$$

and Lemma 1 follows, for $d(f(z), \partial f(B^n)) \leq |f(z)|$ since $f$ does not vanish in $B^n$.

Now let $f : B^n \to \mathbb{R}^n - \{0\}$ be $K$-quasiconformal. For almost every $x \in S^{n-1}$, we may write

$$L_f(x) \leq \int_0^1 |Df(tx)|^p dt \leq V_f(x) + 2^{n-1} f^*(x) H_f(x),$$

where

$$V_f(x) = \int_0^{1/2} |Df(tx)| dt$$

and

$$H_f(x) = \int_0^1 \frac{|Df(tx)|}{f(tx)} t^{n-1} dt.$$

**Lemma 2:**

$$V_f \in L^1(S^{n-1}) \quad \text{with} \quad \|V_f\|_1 \leq C(K, n)|f(0)|.$$
Proof:

\[ \|V_f\|_1 = \int_{B^* (0, 1/2)} \frac{|Df (y)|}{|y|^{n-1}} \, dm (y). \]

By Gehring's inequality [5], there exists \( p = p (K, n) > n \) and \( C (K, n) > 0 \) such that

\[ (9) \left( \int_{B^* (0, 1/2)} |Df (y)|^p \, dm (y) \right)^{1/p} \leq C (K, n) \left( \int_{B^* (0, 1/2)} J (x, f) \, dm (x) \right)^{1/n} \]

\[ = C (K, n) m \left[ f (B^* (0, 1/2)) \right]^{1/n} \leq C (K, n) |f (0)|, \]

the last inequality being a consequence of Lemma 1. We now apply Hölder's inequality to the expression of \( \|V_f\|_1 \), to obtain

\[ \|V_f\|_1 \leq \left( \int_{B^* (0, 1/2)} |Df (y)|^p \, dm (y) \right)^{1/p} \]

\[ \times \left( \int_{B^* (0, 1/2)} |y|^{-(n-1)p/(p-1)} \, dm (y) \right)^{(p-1)/p} \]

\[ \leq C (K, n) |f (0)|, \]

by (9) and the fact that \( (n-1)p/(p-1) < n \).

An estimate for \( H_f \) is given by the following theorem, due to P. Jones:

**Theorem 3 [8]:**

\[ H_f \in L^1 (S^{n-1}) \quad \text{with} \quad \|H_f\|_1 \leq C (K, n). \]

As P. Jones has observed, Theorem 3 implies that

\[ \sup_{T \in M} \|H_f \circ T\|_1 \leq C (K, n) \quad \iff \quad (10) \sup_{T \in M} \int_{B^*} \frac{|Df (x)|}{|f (x)|} \, |DT (x)|^{n-1} \, dm (x) \leq C (K, n), \]

and (10) exactly says that \( |Df| \, |f| \, dm \) is a Carleson measure in \( B^* [3] \). Since \( |\nabla f| \leq |Df| \), the same is true for \( |\nabla u| \, dm \), where \( u = \log |f| \). Writing

\[ \|u\|_* = \sup_{T \in M} \int_{B^*} |\nabla u (x)| \, |DT (x)|^{n-1} \, dm (x), \]
we can now invoke the following theorem, due to Varopoulos:

**Theorem 4** [10]. - Let \( u \in W^{1}_{1, \text{loc}}(B^n) \) be a real-valued function having radical limit \( \tilde{u}(x) \) a.e. on \( S^{n-1} \). If \( |\nabla u| \, dm \) is a Carleson measure in \( B^n \), then \( \tilde{u} \in \text{BMO}(S^{n-1}) \) with \( \|\tilde{u}\|_{\text{BMO}} \leq C(n)\|u\|_{*} \).

(Varopoulos proves Theorem 4 for \( n=2 \) only, but his argument is easily seen to extend to the general case.)

From Theorems 3 and 4, it follows that if \( f : B^n \rightarrow \mathbb{R}^n - \{0\} \) is \( \lambda \)-quasiconformal, then

\[
\| \text{Log} |f(x)| \|_{\text{BMO}(S^{n-1})} \leq C(K, n),
\]

for \( 0 \leq r \leq 1 \). If we now apply the John and Nirenberg inequality [7] and Lemma 1, we get

**Lemma 3.** - If \( f : B^n \rightarrow \mathbb{R}^n - \{0\} \) is \( K \)-quasiconformal there exist \( C(K, n) > 0 \) and \( p = p(K, n) > 0 \) such that

\[
\sup_{0 < r \leq 1} \int_{S^{n-1}} |f(rx)|^p \, d\sigma(x) \leq C(K, n) |f(0)|^p.
\]

In the case \( n=2 \) and \( f \) conformal, (11) implies that \( f \) is in the Hardy space \( H^p(B^2) \), so that \( f^* \in L^p(S^{n-1}) \), and Theorem 2 follows in this particular case. In the general case, we need an extra argument, provided by the next section.

3. Non-tangential maximal function

**Proposition 1.** - Let \( f : B^n \rightarrow \mathbb{R}^n - \{0\} \in W^{1}_{1, \text{loc}}(B^n) \), and \( u = \text{Log} |f| \). If \( |\nabla u| \, dm \) is a Carleson measure and if \( f \) satisfies Harnack property (8) then, for every \( p > 0 \),

\[
\int_{S^{n-1}} f^*(x)^p \, d\sigma(x) \leq C_p \int_{S^{n-1}} |f(x)|^p \, d\sigma(x),
\]

where \( C_p \) depends only on \( n, p, \|u\|_{*} \) and \( C(f) \).

We first notice that this statement makes sense, since \( |f(x)| \) has radial limits a.e. on \( S^{n-1} \), for \( |\nabla u| \, dm \) is a Carleson measure.

Before going into the proof of Proposition 1, let us see why it implies Theorem 2. So let \( f : B^n \rightarrow \mathbb{R}^n - \{0\} \) be \( K \)-quasiconformal with
\[ |f(0)| = \text{dist} \left( f(0), f(B^n) \right). \] By the results in Part 2 we can apply Proposition 1 to \( f(x) \) and, by Lemma 3, we get
\[ \|f^*\|_{p} \leq C(K, n) |f(0)|, \]
for some \( p \) depending only on \( K \) and \( n \). Recalling now that
\[ L_f(x) \leq V_f(x) + 2^{n-1} f^*(x) H_f(x), \]
Lemma 2, Theorem 3 and (12) imply that \( L_f \in L^{p+1, p+1}(S^{n-1}) \) with
\[ \|L_f\|_{p+1} \leq C(K, n) |f(0)|, \]
and Theorem 2 is proved.

To prove Proposition 1, we need 3 lemmas:

**Lemma 4.** — If \( f \) is as in Proposition 1 and \( N > C(f)^2 \),
\[ \sigma\left( \left\{ x \in S^{n-1}; |f(x)| \leq \frac{|f(0)|}{N} \right\} \right) \leq \frac{7^n \|u\|_*}{\log N}. \]

**Proof:** Put
\[ F_N = \left\{ x \in S^{n-1}; |f(x)| < \frac{|f(0)|}{N} \right\} \]
and
\[ G(x) = \int_0^1 |\nabla u(tx)| t^{n-1} \, dt. \]
If \( x \in F_N \),
\[ G(x) \geq 7^{1-n} \int_{1/7}^1 |\nabla u(tx)| \, dt \]
\[ \geq 7^{1-n} \left| \int_{1/7}^1 \frac{1}{t} \frac{\partial}{\partial t} \left( |f(tx)| \right) \, dt \right| \]
\[ = 7^{1-n} \left| \log \frac{|f(x)|}{|f(x/7)|} \right|. \]

By Harnack property, \( |f(0)| \leq C(f) |f(x/7)| \); so, if \( x \in F_N \) and \( N > C(f)^2 \),
\[ G(x) \geq 7^{1-n} \left| \log \frac{N}{C(f)} \right| \geq 7^{-n} \log N, \]

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and from this it follows that
\[ \sigma(F_N) \leq \frac{7^n \|G\|}{\log N} \leq \frac{7^n \|u\|_\ast}{\log N}. \]

**Lemma 5.** There is an universal constant \( \alpha > 0 \) such that if \( f \) is as in Proposition 1 and \( z \in B^n \),
\[ \sigma \left( \left\{ x \in S(z); \left| f(x) \right| \leq \frac{\left| f(z) \right|}{N} \right\} \right) \leq \frac{\alpha 7^n \|u\|_\ast}{\log N} \sigma(S(z)). \]

**Proof.** Define \( g(x) = f \circ T_x^{-1}(x) \). By (4), if \( x \in S^{n-1} \) then \( |g(0)| \leq C(f) |g(x/7)| \); also, by definition \( \|\log|g||\|_\ast = \|u\|_\ast \). By Lemma 4 we then have, for \( N \geq C(f)^2 \),
\[ \sigma \left( \left\{ x \in S^{n-1}; g(x) \leq \frac{\left| f(z) \right|}{N} \right\} \right) \leq \frac{7^n \|u\|_\ast}{\log N}, \]
which implies
\[ \sigma \left( T_z \left( \left\{ y \in S(z); \left| f(y) \right| \leq \frac{\left| f(z) \right|}{N} \right\} \right) \right) \leq \frac{7^n \|u\|_\ast}{\log N}, \]
and the result follows from (3).

**Lemma 6.** There exist \( 0 < C(n) < 1 \) and \( N(n, \|u\|_\ast, C(f)) \) such that the following inequality holds:
\[ (13) \quad \forall \lambda > 0, \quad \sigma \left( \left\{ x \in S^{n-1}; f^\ast(x) > \lambda, \left| f(x) \right| \leq \frac{\lambda}{N} \right\} \right) \leq C(n) \sigma \left( \left\{ x \in S^{n-1}; f^\ast(x) \geq \lambda \right\} \right). \]

**Proof.** Let \( \delta(\lambda) = \{ z \in B^n; |f(z)| > \lambda \} \) and \( \Psi(\lambda) = \{ x \in S^{n-1}; f^\ast(x) > \lambda \}. \)

Then
\[ \Psi(\lambda) = \bigcup_{z \in \delta(\lambda)} S(z). \]
By Vitali covering lemma, there exists $\alpha(n) \in (0, 1)$ and a sequence $\{z_j\} \subset \mathcal{S}(\lambda)$ such that the $S(z_j)$'s are mutually disjoint and
\[
\sum_{j \in N} \sigma(S(z_j)) \geq \alpha(n) \sigma(\mathcal{H}(\lambda)).
\]

Now let
\[
E_N = \left\{ x \in S^{n-1}; |f(x)| \leq \frac{\lambda}{N} \right\}.
\]

Then
\[
\sigma(E_N \cap \mathcal{H}(\lambda)) \leq \sum_{j \in N} \sigma(E_N \cap S(z_j)) + (1 - \alpha(n)) \sigma(\mathcal{H}(\lambda)).
\]

But
\[
E_N \cap S(z_j) \subset \left\{ x \in S(z_j); |f(x)| \leq \frac{|f(z_j)|}{N} \right\}
\]
and so
\[
\sigma(E_N \cap S(z_j)) \leq \frac{\alpha 7^n \|u\|_*}{\log N} \sigma(S(z_j)) \quad \text{if } N \geq C(f)^2,
\]
by Lemma 5. So, choosing $N$ so large that
\[
\frac{\alpha 7^n \|u\|_*}{\log N} < \frac{\alpha(n)}{2},
\]
we get Lemma 6 with $C(n) = 1 - \alpha(n)/2$. We can now complete the proof of Proposition 1. For $\lambda > 0$, let
\[
\chi(\lambda) = \sigma(\{x \in S^{n-1}; f^*(x) > \lambda\})
\]
and
\[
\theta(\lambda) = \sigma(\{x \in S^{n-1}; |f(x)| > \lambda\}).
\]

By (13),
\[
\chi(\lambda) \leq C(n) \chi(\lambda) + \theta(\lambda/N) \quad \Rightarrow \quad \chi(\lambda) \leq \frac{1}{1 - C(n)} \theta(\lambda/N).
\]

Finally,
\[
\|f^*\|_p = p \int_0^\infty \lambda^{p-1} \chi(\lambda) d\lambda \leq \frac{p}{1 - C(n)} \int_0^\infty \lambda^{p-1} \theta \left(\frac{\lambda}{N}\right) d\lambda = \frac{N^p}{1 - C(n)} \|f\|_p^p,
\]
which proves Proposition 1.
Remark. — A mapping $f: B^n \to \mathbb{R}^n$ is said to be $K$-quasiregular if $f \in W^{1}_{\text{loc}}(B^n)$ and if

$$|Df(x)|^n \leq KJ(x, f) \quad \text{a.e. on } B^n.$$

The above methods imply the following:

**Proposition 2.** Let $f: B^n \to \mathbb{R}^n - \{0\}$ be a $K$-quasiregular mapping and $u = \log |f|$. If $|\nabla u| \, dm$ is a Carleson measure in $B^n$, there exists an exponent $p = p(n, K, \|u\|_{\infty}) > 0$ such that $L_f \in L^p(S^{n-1})$, where $L_f$ has the same meaning as in Theorem 1.

**Sketch of Proof.** Using the same notations as in 2, one may write

$$L_f(x) \leq V_f(x) + 2^{n-1} K f^*(x) \tilde{H}_f(x),$$

where

$$\tilde{H}_f(x) = \int_0^1 |\nabla u(tx)| t^{n-1} \, dt.$$

From results in [1], Lemma 2 is valid for $V_f$; also, $\tilde{H}_f \in L^1$ by hypothesis. To prove Proposition 2, it suffices then to show that $f^* \in L^p$ for some $p > 0$. Using Proposition 1 and Theorem 4, this reduces to proving that $f$ satisfies Harnack property. To do this, we use a recent result of Iwaniec and Nolder [6]: They proved that $u$ is actually in $W^{1}_{q \overline{\text{loc}}} (B^n)$ for some $q(K, n) > n$ and that

$$\forall x \in B^n, \quad \left( \int_{B^n(x, 1/2 (1 - |x|))} |\nabla u(y)|^q \, dm(y) \right)^{1/q} \leq C(K, n) (1 - |x|)^n (1/2 - 1) \times \int_{B^n(x, 3/4 (1 - |x|))} |\nabla u(y)| \, dm(y),$$

and the result follows from the Sobolev embedding theorem and the fact that $|\nabla u| \, dm$ is a Carleson measure.
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