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A COMPUTATION OF THE EQUIVARIANT INDEX OF THE DIRAC OPERATOR

BY

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TO J. DIXMIER

RÉSUMÉ. — Nous donnons une démonstration des formules de points fixes d'Atiyah-Singer pour l'opérateur de Dirac tordu sur une variété riemanienne M, en utilisant l'équation de la chaleur sur sa variété des repères P. En particulier, le genre $\hat{A}$ fait son entrée en scène comme Jacobien de l'application exponentielle sur la variété P.

ABSTRACT. — We give a proof of the Lefschetz fixed-point formulas of Atiyah-Singer for the twisted Dirac operator on a compact spin manifold M by using the heat equation on the frame bundle P. In particular, the $\hat{A}$ class makes its entrance as the Jacobian of the exponential map on the manifold P.

Introduction

In this article, we give a simple proof of ATIYAH-BOTT-SEGAL-SINGER's ([3], [4], [5]) fixed-point formulae for the twisted Dirac operator $D_{\mathscr{E}}$ on a compact Spin manifold M. Our method relies on the heat equation on the frame bundle P of M.

Recall that the index density $\hat{A}$ was computed independently by GILKEY [14] and PATODI [21] using the heat equation method on M and their familiarity with differential geometry.

Recently, some physicists (ALVAREZ-GAUME [1], FRIEDAN and WINDEY [11], WITTEN (*see* [2])) came up with the idea that the index density $\hat{A}$ and its related mysterious function

$$j^{-1/2}(x)=\left(\frac{\operatorname{sh} x}{x}\right)^{-1/2},$$

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should appear naturally. Indeed, soon after, E. GETZLER ([12], [13]) and J. M. BISMUT [9] gave direct proofs of the formulae of Gilkey-Patodi. In E. Getzler, the class $\hat{A}$ appears due to the approximation of the kernel of $e^{tD^2_{\tilde{}}}$ by the kernel of an harmonic oscillator, while in J. M. Bismut $j^{-1/2}(x)$ is related to the law of random areas in $\mathbb{R}^2$. For us, the class $\hat{A}$ make its entrance as the Jacobian of the exponential map on P. Thus our desire to understand the striking similarity between Kirillov universal formula and index formulae ([7], [8], [22]) is partly fulfilled.

As it should appear clearly to the reader, our proof contains some transfers of the probabilistic approach of J. M. Bismut and uses an idea of E. Getzler. It seems however worthwhile to write our own proof, as our method, bound by our own limitations, requires little. Furthermore, our method applies as well to the case of a group action. It gives the index density on the fixed point submanifold. The only other direct approach to this result is the probabilistic method of J. M. BISMUT [9].

To illustrate the basic simplicity of our method, we now give a quick proof – which should be read before going on with the rest of the paper – of the simplest case, index $D=\int_M \hat{A}$, where D is the Dirac operator on the spin manifold M. [Notions not defined in the introduction may be found in the text of the article.]

Let M be a spin manifold. Let $\Gamma(\mathcal{S})=\Gamma(\mathcal{S}^+)\oplus\Gamma(\mathcal{S}^-)$ be the decomposition of spinor fields in even and odd spinors, D^+ (resp. D^-) the restriction of the Dirac operator D to $\Gamma(\mathcal{S}^+)$ (resp. $\Gamma(\mathcal{S}^-)$). As proposed by ATIYAH-BOTT [3], the use of the fundamental formula of MCKEAN-SINGER [18],

$$\text{index } D=\dim\ker D^+ - \dim\ker D^-$$
$$=\operatorname{tr}(e^{-tD^-D^+})-\operatorname{tr}(e^{-tD^+D^-}),\quad \text{for all } t>0,$$

reduces the calculation of the index of D to the study of the asymptotics of the right hand side. We indicate now how to carry out this calculation by relating D^2 to the scalar Laplacian of the frame bundle of M.

Let M be a spin manifold of dimension $n=2l$, $V=R^n$, $\mathfrak{g}=\mathfrak{so}(n)$, $G=\text{Spin}(n)$ the two-fold cover of $SO(n)$. We also identify $\mathfrak{g}$ with $\Lambda^2 V$. Let $P\xrightarrow{\pi} M$ be the frame bundle of M. Let $\mathcal{A}(P)$ be the algebra of forms on P. Consider the fundamental 1-form $\theta \in \mathscr{A}^1(P) \otimes V$, defined by

$$\theta_u(X) = u^{-1}(\pi_* X) \qquad \text{for} \quad X \in T_u(P).$$

Let $\omega \in \mathscr{A}^1(P) \otimes \mathfrak{g}$ be the Levi-Civita connection. Then $\theta \oplus \omega$ define a trivialization of the tangent bundle to P, $TP = P \times (V \oplus \mathfrak{g})$.

The manifold P has a canonical Riemannian structure by considering $T_u P$ as the direct orthogonal sum of its horizontal tangent space and its vertical tangent space $\mathfrak{g}$.

Let $P' \to P$ be the two-fold cover of P with structure group G. The lifts to P' of the forms θ, ω are still denoted by θ, ω. We have $TP' = P' \times (V \oplus \mathfrak{g})$. For $x \in V \oplus \mathfrak{g}$, we denote by $\tilde{x}$ the vector field on P' such that $(\theta \oplus \omega)(\tilde{x}) = x$. Similarly P' is considered as a Riemannian manifold. The curves $t \to u$ expta for $a \in \mathfrak{g}$, $u \in P'$ are geodesics of P'. Let e_i be the canonical basis of $\mathbb{R}^n$, e_{ij} an orthonormal basis of $\mathfrak{g}$. The Laplacian Δ is given on the space of function on P' by

0.1
$$\Delta = \sum (\tilde{e}_i)^2 + \sum (\tilde{e}_{ij})^2.$$

Let τ be an irreducible representation of the group G in a vector space V_τ, $\mathscr{V}_\tau = P' \times V_\tau / G$ be the associated vector bundle. We identify the space of sections $\Gamma(\mathscr{V}_\tau)$ of $\mathscr{V}_\tau$ to the subspace of $C^\infty(P', V_\tau)$ of V_τ-valued functions on P' satisfying

$$f(ug) = \tau(g)^{-1} f(u), \qquad u \in P', \qquad g \in G.$$

Remark that the restriction of Δ to $\Gamma(\mathscr{V}_\tau)$ differs from the operator $\sum_i (\tilde{e}_i)^2$ by a scalar (image of the Casimir of $\mathfrak{g}$ in the representation τ).

Consider on M, P', G the volume forms dx, du, dg. Let $k(t, u, u')(t > 0)$ be the C^∞-kernel of the operator $e^{t\Delta}$. Let

$$k^\tau(t, u_0, u) = \int_G k(t, u_0, ug^{-1}) \tau(g)\, dg.$$

Then, for $f \in \Gamma(\mathscr{V}_\tau)$, the function $u \to k^\tau(t, u_0, u).f(u)$ is a function on $P'/G = M$ and the kernel of the restriction of $e^{t\Delta}$ to $\Gamma(\mathscr{V}_\tau)$ is given by:

0.2
$$(e^{t\Delta}.f)(u_0) = \int_M (k^\tau(t, u_0, u).f(u))\, dx.$$

In particular, let $S=S^+\oplus S^-$ be the spinor-space (*see* Section 1). We denote by $\rho^\pm$ the representation of G in $S^\pm$. We obtain the spin bundles $\mathscr{S}^+$, $\mathscr{S}^-$. Denote by $\chi^\pm$ the characters of ρ^+, ρ^-.

0.3 Let $A(t, u_0, u)=\int_G k(t, u_0, ug^{-1})(\chi^+-\chi^-)(g)\,dg.$

The function $u\to A(t, u, u)$ is a function on $P'/G=M$, that we denote by $A(t, x)$. Denote by Δ^+, Δ^- the restrictions of Δ to $\Gamma(\mathscr{S}^\pm)$. The operator $\Delta^+\oplus\Delta^-$ coincides with the square of the Dirac operator, except for a 0th-order term [17] which is easily seen to be irrelevant in the calculation of the index density (*see* Section 3). We have then:

0.4 $$\operatorname{tr} e^{t\Delta^+}-\operatorname{tr} e^{t\Delta^-}=\int_M A(t, x)\,dx,$$

and

$$\text{index } D=\int_M \lim_{t\to 0} A(t, x)\,dx.$$

We will indeed see that, for every $x\in M$,

$$I(x)=\lim_{t\to 0} A(t, x)$$

exists. The density $I(x)\,dx$ is called the index density.

Consider $\Omega\in\mathscr{A}^2(P)\otimes\mathfrak{g}$ the Riemannian curvature of the manifold M. Consider the function

$$j_V(a)=\det_V\left(\frac{e^{a/2}-e^{-a/2}}{a}\right) \qquad \text{for} \quad a\in\mathfrak{g}=\mathfrak{so}(V).$$

Let $\hat{A}=j_V^{-1/2}(-(\Omega/2i\pi))$ be the Chern-Weil form on M, associated to the G-invariant analytic function $j_V^{-1/2}$ on $\mathfrak{g}$. Let us prove:

0.5 Let $x\in M$, then $\lim_{t\to 0} A(t, x)\,dx=[\hat{A}_x]^{[\max]}$.

Proof. — Let u above x, then

$$A(t, x)=\int_G k(t, u, ug^{-1})(\chi^+-\chi^-)(g)\,dg.$$

Let $\psi(a)$ be a cut-off function on $\mathfrak{g}$ identically 1 near 0 and of small support. As, for $t \to 0$, $k(t, u, u')$ is very small outside the diagonal, we can compute 0.5 by

0.6 $$\lim_{t \to 0} \int_{\mathfrak{g}} k(t, u, u \exp -a)(\chi^+ - \chi^-)(\exp a)\psi(a) j_{\mathfrak{g}}(a)\, da,$$

where $dg = j_{\mathfrak{g}}(a)\, da$ in exponential coordinates.

Let $u_0 \in P'$, $\exp_{u_0} : T_{u_0}(P') \to P'$ the exponential map. Let $\theta(u_0, u)$ be the tangent map to $\exp_{u_0}$ at the point $\exp_{u_0}^{-1}(u)$. Then $\theta(u_0, u) : T_{u_0}(P') \to T_u(P')$. For any N uniformly on $P' \times P'$,

0.7 $$k(t, u_0, u) = (4\pi t)^{-\dim P'/2} e^{-(1/4t)\, q(u_0, u)} \times \left(\sum_{i=0}^{N} t^i U_i(u_0, u)\right) + O\left(t^{N - \dim P'/2) + 1}\right)$$

with $U_0(u_0, u) = (\det(\theta(u_0, u)))^{-1/2}$. As $T_{u_0}(P')$ and $T_u(P')$ are both identified with $V \oplus \mathfrak{g}$ we can consider $\theta(u_0, u)$ as an endomorphism of $V \oplus \mathfrak{g}$. We denote $J(u, a) = \theta(u, u \exp a)$.

The restriction of Ω_u to the horizontal tangent space at u define an element of $\Lambda^2 V^* \otimes \mathfrak{g}$ (still denoted by Ω_u) *by* $e_i \wedge e_j \to \Omega_u(e_i, e_j)$. If $a = \Sigma a_{ij} e_i \wedge e_j$, we denote by (Ω_u, a) the contraction of Ω_u by a, i. e. $(\Omega_u, a) = \Sigma a_{ij} \Omega_u(e_i, e_j) \in \mathfrak{g}$.

It is easy to prove (*see* 2.20, 2.21) that

0.8 $$J(u, a) \,|\, \mathfrak{g} = \frac{1 - \exp(-\operatorname{ad} a)}{\operatorname{ad} a},$$

0.9 $$J(u, a) \,|\, V = \exp_V(-a) \frac{1 - \exp_V(-\Omega_u, a)/2}{(\Omega_u, a)/2}.$$

Thus we have

0.10 $$\lim_{t \to 0} A(t; x) = \lim_{t \to 0} (4\pi t)^{-l} (4\pi t)^{-\dim \mathfrak{g}/2} \int e^{-\|a\|^2/4t} j_V^{-1/2}(-(\Omega, a)/2) j_{\mathfrak{g}}^{1/2}(a)\, \psi(a)(\chi^+ - \chi^-)(\exp a)\, da.$$

To express the right hand side, we introduce the distribution Super-Dirac on $\mathfrak{g}$

0.11 $$(\delta_{-1}, \varphi) = (\mathrm{Pff}(\partial).\varphi)(0),$$

where Pff is the constant coefficient operator on $\mathfrak{g} = \Lambda^2 V$ corresponding to the polynomial Pfaffian of a, i. e.

$$l!\,\mathrm{Pff}(a)\, e_1 \wedge e_2 \wedge \ldots \wedge e_n = a \wedge \ldots \wedge a.$$

The skew-invariance of Pff under $O(n)$ shows that, for ψ invariant by $O(n)$

0.12 $$\delta_{-1}(\psi\varphi) = \psi(0)\,\delta_{-1}(\varphi).$$

We have

$$(\chi^+ - \chi^-)(\exp a) = (-2i)^l\, \mathrm{Pff}(a) j_V^{1/2}(a).$$

The functions $j_V^{1/2}$, $j_{\mathfrak{g}}^{1/2}$ are $O(n)$-invariant functions on $\mathfrak{g}$.

As $\mathrm{Pff}(a)$ is a harmonic polynomial on $\mathfrak{g}$ homogeneous of degree l, $\mathrm{Pff}(a)\, e^{-\|a\|^2/4}$ is an eigenfunction for the Fourier transform, thus the following formula results:

If φ is a C^∞ function on $\mathfrak{g}$ with compact support:

0.13 $$\lim_{t \to 0} (4\pi t)^{-l} (4\pi t)^{-\dim \mathfrak{g}/2} \times \int e^{-\|a\|^2/4t} (\chi^+ - \chi^-)(\exp a)\, \varphi(a)\, da = (i\pi)^{-l} (\delta_{-1}, \varphi).$$

To prove 0.5, we need only to compute the value of the distribution δ_{-1} on the function $a \to j_V^{-1/2}(-(\Omega, a)/2)$.

More generally, let E be a vector space and P be a polynomial function on E (or a germ at 0 of an analytic function on E). Then, as $\Lambda^{[\mathrm{even}]} V^*$ is a commutative algebra, P extends to a polynomial function:

$$P: \quad \Lambda^{[\mathrm{even}]} V^* \otimes E \to \Lambda^{[\mathrm{even}]} V^*.$$

Let $\Phi \in \Lambda^2 V^* \otimes E$, then $P(\Phi) \in \Lambda^{[\mathrm{even}]} V^*$. For $a \in \mathfrak{g} = \Lambda^2 V$, we denote by $(\Phi, a) \in E$ the contraction of Φ by a. The function $P_\Phi(a) = P((\Phi, a))$ is a function on $\mathfrak{g}$. It is immediate to see that

$$\delta_{-1}(P_\Phi) = [P(\Phi)]^{[\max]}$$

and the formula (0.5) results.

The paper is organized as follows: In Section 1 we specify notations for the Clifford algebra, even and odd spin representations, etc., and we give the technical asymptotic result (Proposition 1.23, generalizing 0.13) which will be essential in our derivation of the index density formula. In Section 2 we make the necessary computations concerning the Riemannian structure on the frame manifold and its two-fold covering P'. In Section 3 we relate the index density of a twisted Dirac operator on M to the corresponding heat kernel on P'. Then, with the help of Morse lemma, the identification of the equivariant index density as a differential form on the fixed point set is reduced to the computation of a Gaussian integral which was done in 1.23. It turns out that all the information needed about the asymptotics of the heat kernel is contained in the graded terms—with respect to the natural filtration on the Clifford algebra, the use of which is a key idea in E. Getzler—and these can be computed explicitly.

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1. Clifford algebra, spinors and the distribution Super Dirac

1.1. Let $V=\mathbb{R}^n$ with its canonical basis $e_1, e_2, \ldots, e_n$, orientation, inner product, etc. In the whole article, we will assume that n is even, $n=2l$.

Let $C(V)$ be the Clifford algebra of V. It is defined as the quotient of the tensor algebra $T(V)$ by the ideal generated by the elements

$$x\otimes y+y\otimes x+2\langle x, y\rangle=0 \qquad \text{for} \quad x, y\in V.$$

The Clifford algebra inherits a natural filtration, where $C^i(V)$ is the subspace of $C(V)$ spanned by products of at most i elements of V. The corresponding graded algebra is isomorphic to the exterior algebra ΛV. We denote by $\mathrm{gr}=\oplus\ \mathrm{gr}^{[i]}$ the canonical map $\oplus_i C^i(V)\to\Lambda V$.

We denote by $\Lambda^+ V$ the (commutative) algebra of even elements in ΛV and by $C^+(V)$ the subalgebra of $C(V)$ generated by products of an even number of elements of V.

1.2. The complexified Clifford algebra $C(V)\otimes_{\mathbb{R}}\mathbb{C}$ has a unique (up to equivalence) irreducible representation in a complex vector space S of dimension 2^l. This representation, denoted by ρ, identifies $C(V)\otimes_{\mathbb{R}}\mathbb{C}$

with End S. The space S is called the space of spinors and ρ the spin representation.

Put $\alpha=e_1 e_2 \ldots e_n \in C(V)$. Then $\alpha^2=(-1)^l$, and $\rho(\alpha)$ decomposes S into the sum $S^+ \oplus S^-$ of the even and odd spinor spaces, where

$$S^+ = \{ v \in S,\ \rho(\alpha)v = i^{-l} v \},$$
$$S^- = \{ v \in S,\ \rho(\alpha)v = -i^{-l} v \}.$$

The subalgebra $C^+(V)$ leaves S^+ and S^- stable. We denote by $\rho^\pm$ the corresponding representations of $C^+(V)$ in $S^\pm$.

Define the supertrace of an element $a \in C(V)$ by

$$\mathrm{st}(a) = i^l \operatorname{tr} \rho(\alpha a).$$

Then, for $a \in C^+(V)$, the supertrace is given by

$$\mathrm{st}(a) = \operatorname{tr} \rho^+(a) - \operatorname{tr} \rho^-(a),$$

and, for $a \in C(V)$, we have

$$\mathrm{gr}^{[n]}(a) = 2^{-l} i^l \, \mathrm{st}(a)\, e_1 \wedge e_2 \wedge \ldots \wedge e_n.$$

1.3. We identify $\Lambda^2 V$ with the subspace $\mathfrak{g} = \oplus_{i<j} \mathbb{R} e_i e_j$ of $C^+(V)$ via the map $e_i \wedge e_j \to e_i e_j$. Then V is invariant under the map $v \to \tau(a)v = av - va$, for $a \in \mathfrak{g}$. The endomorphism $\tau(a)$ is infinitesimally orthogonal and the representation τ of $\mathfrak{g}$ in V defines an isomorphism of $\mathfrak{g}$ onto $\mathfrak{so}(n)$. We note that $\tau(e_i e_j/2) = E_{ij}$ where

$$E_{ij} e_j = -e_i,$$
$$E_{ij} e_i = e_j \qquad (i \neq j),$$
$$E_{ij} e_k = 0 \qquad \text{for} \quad k \neq i, j.$$

We may write $a.v$ for $\tau(a)v$.

1.4. The universal covering group $G = \mathrm{Spin}(n)$ of $SO(n)$ can be realized as a subgroup of the group of invertible elements of $C^+(V)$. The map $\exp : \mathfrak{g} \to G$ coincides with the exponential map in the algebra $C(V)$. We denote by the same letters ρ, $\rho^\pm$ the restriction to G and $\mathfrak{g}$ of the spin representations. Then the spin representation of the Lie algebra $\mathfrak{g}$ is the differential of the spin representation of the group G.

1.5. For $a \in \mathfrak{g}$, we define a linear transformation of $\mathfrak{g}$ by

$$J_{\mathfrak{g}}(a) = \frac{1-\exp(-\operatorname{ad} a)}{\operatorname{ad} a}.$$

Then (as in any Lie algebra) we have for $a, b \in \mathfrak{g}$.

1.6 $$\frac{d}{d\varepsilon} \exp(a+\varepsilon b)\big|_{\varepsilon=0} = (\exp a)\, J_{\mathfrak{g}}(a)\, \mathrm{b}.$$

Define $j_{\mathfrak{g}}(a) = \det J_{\mathfrak{g}}(a)$. Since $\det_{\mathfrak{g}} \exp \operatorname{ad} a/2 = 1$ we have

1.7 $$j_{\mathfrak{g}}(a) = \det_{\mathfrak{g}} \frac{\exp(\operatorname{ad} a/2) - \exp-(\operatorname{ad} a/2)}{\operatorname{ad} a}.$$

Define

$$j_V(a) = \det_V \frac{\exp \tau(a)/2 - \exp - \tau(a)/2}{\tau(a)}.$$

The functions j_V and $j_{\mathfrak{g}}$ have analytic square roots on $\mathfrak{g}$, such that

$$j_V^{1/2}(0) = j_{\mathfrak{g}}^{1/2}(0) = 1.$$

1.8. The Pfaffian $\operatorname{Pff}(a)$ of an element $a \in \Lambda^2 V = \mathfrak{g}$ is defined by

$$\frac{a \wedge a \wedge \ldots \wedge a}{l!} = \operatorname{Pff}(a)\, e_1 \wedge \ldots \wedge e_n.$$

It is easily verified that for $a \in \mathfrak{g}$

$$\operatorname{st}(\exp a) = 2^l i^{-l} \operatorname{Pff}(a) j_V^{1/2}(a).$$

1.9. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the scalar product normalized by $\| e_i \wedge e_j \| = 1$.

Denote by $\operatorname{Pff}(\partial)$ the differential operator with constant coefficients on $\mathfrak{g}$ which corresponds to the polynomial function $a \to \operatorname{Pff}(a)$.

We define the distribution SuperDirac δ_{-1} on $\mathfrak{g}$ by $\delta_{-1}(\varphi) = (\operatorname{Pff}(\partial).\varphi)(0)$. We identify the symmetric algebra $S(\mathfrak{g}) = S(\mathfrak{g}^*)$ with the algebra of polynomial functions on $\mathfrak{g}$.

The injection

$$\mathfrak{g}^* \simeq \Lambda^2 V \to \Lambda^+ V,$$

extends to a canonical algebra homomorphism

$$A:\quad S(\mathfrak{g}^*)\to\Lambda^+ V.$$

Then the distribution Super Dirac, considered as a linear map on $S(\mathfrak{g}^*)$, factors through A.

Precisely, let us define a linear map t_{-1} on ΛV by

1.10 $$w^{[n]}=t_{-1}(w)\,e_1\wedge\ldots\wedge e_n.$$

Then

1.11 $$\delta_{-1}(\varphi)=t_{-1}(A\,\varphi),$$

1.12 Denote by $\hat{S}(\mathfrak{g}^*)$ the algebra of formal power series on $\mathfrak{g}$.

The canonical homomorphism A clearly extends to $\hat{S}(\mathfrak{g}^*)$. We still denote by A the homomorphism

$$C^\infty(\mathfrak{g})\to\Lambda^+ V$$

obtained by composing A with the map

$$C^\infty(\mathfrak{g})\to\hat{S}(\mathfrak{g}^*),$$

given by the Taylor series expansion at the origin. Then 1.11 holds with

$$\varphi\in\hat{S}(\mathfrak{g}^*)\qquad\text{or}\qquad\varphi\in C^\infty(\mathfrak{g}).$$

1.13. The full orthogonal group $O(n)$ acts on V, $\mathfrak{g}$, ΛV, etc. The map A commutes with these actions. Remark that the only $O(n)$ invariant elements in ΛV are the scalars. Thus if φ, $\psi\in C^\infty(\mathfrak{g})$ and φ is $O(n)$-invariant, we have

$$A(\varphi\psi)=\varphi(0)\,A(\psi)$$

and by 1.11.

1.14 $$\delta_{-1}(\varphi\psi)=\varphi(0)\,\delta_{-1}(\psi).$$

1.15. We will compute the distribution δ_{-1} on functions on $\mathfrak{g}$ obtained by the following procedure: let E be a finite dimensional vector space, let $\varphi\in S(E^*)$ be a polynomial function on E. Since the algebra $\Lambda^+ V$ is commutative, φ extends to a polynomial map

$$\varphi:\quad\Lambda^+ V\otimes E\to\Lambda^+ V.$$

In particular, for $\Omega \in \Lambda^2 V \otimes E$, $\varphi(\Omega)$ is an element of $\Lambda^+ V$.

For $a \in \mathfrak{g} = \Lambda^2 V$, we denote by $\langle \Omega, a \rangle \in E$ the contraction defined by the scalar product on $\mathfrak{g}$. We obtain a polynomial function on $\mathfrak{g}$ by

$$\varphi_\Omega(a) = \varphi(\langle \Omega, a \rangle).$$

The following relation is immediate

1.16 $$A(\varphi_\Omega) = \varphi(\Omega),$$

thus

$$\delta_{-1}(\varphi_\Omega) = t_{-1}(\varphi(\Omega)).$$

As in 1.12 all this still holds for φ in $\hat{S}(E^*)$ or in $C^\infty(E)$.

The distribution δ_{-1} will appear in the study of the asymptotics of the heat kernel associated to the Dirac operator on a spin manifold M.

Denote by da the Lebesgue measure on the Euclidean vector space $\mathfrak{g}$. The following proposition is well known:

1.17. PROPOSITION. — (i) *Let φ be a C^∞ function with compact support on the Euclidean space $\mathbb{R}^m$, then*

$$\lim_{t \to 0} (4\pi t)^{-m/2} \int_{\mathbb{R}^m} e^{-\|v\|^2/4t} \varphi(v)\, dv = \varphi(0).$$

(ii) *Let φ be a C^∞ function with compact support on $\mathfrak{g}$, then*

$$\lim_{t \to 0} (4\pi t)^{-l-\dim \mathfrak{g}/2} \int_{\mathfrak{g}} e^{-\|a\|^2/4t}\, \mathrm{Pff}(a)\, \varphi(a)\, da = (2\pi)^{-l} \delta_{-1}(\varphi).$$

Proof. — For the convenience of the reader we give a proof of (ii). Remark that the Pfaffian is a harmonic polynomial homogenous of degree l on $\mathfrak{g}$. Recall the:

LEMMA. — *Let P be a homogeneous harmonic polynomial of degree l on $\mathbb{R}^m$. Then the function $x \to P(x) e^{-\|x\|^2/2}$ is an eigenfunction with eigenvalue i^{-l} for the Fourier transform*

$$(\mathcal{F} f)(x) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle x, y \rangle} f(y)\, dy.$$

Proof. — Consider the harmonic oscillator

$$H=\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}-x_i^2.$$

Then

$$\mathscr{F}=e^{im\pi/4}\,e^{i\pi H/4}.$$

As P is harmonic and homogeneous of degree l, the function $P(x)e^{-\|x\|^2/2}$ is an eigenfunction for H with eigenvalue $-(m+2l)$.

From the lemma we obtain:

$$(4\pi t)^{-l-\dim \mathfrak{g}/2}\int e^{-\|a\|^2/4t}\,\mathrm{Pff}(a)\,\varphi(a)\,da$$

$$=(2\pi)^{-\dim \mathfrak{g}/2}(2i\pi)^{-l}\int e^{-t\|\xi\|^2}\,\mathrm{Pff}(\xi)\,(\mathscr{F}\varphi)(\xi)\,d\xi.$$

This last expression has an asymptotic expansion in powers of t, when $t\to 0$, given by the Taylor expansion of $e^{-t\|\xi\|^2}$. In particular the limit exists and is given by (ii).

1.18 Corollary. — *Let φ be a C^∞ function on $\mathfrak{g}$ with compact support. Then:*

$$\lim_{t\to 0}(4\pi t)^{-l-\dim \mathfrak{g}/2}\int e^{-\|a\|^2/4t}\,\mathrm{st}(e^a)\,\varphi(a)\,da=(i\pi)^{-l}\delta_{-1}(\varphi).$$

Proof. — We have (1.8) $\mathrm{st}(e^a)=(-2i)^l\,\mathrm{Pff}(a)\,j_V^{1/2}(a)$, and $j_V^{1/2}$ is $O(n)$-invariant, with $j_V^{1/2}(0)=1$. The result follows from (ii) and 1.14.

To compute the equivariant index of a twisted Spin complex we will need a generalization of 1.18, Proposition 1.23 below.

1.19. Let E be a finite dimensional complex vector space. We identify $\mathrm{End}(S\otimes E)=\mathrm{End}\,S\otimes \mathrm{End}\,E$ with $C(V)\otimes \mathrm{End}\,E$.

We extend the supertrace to $C(V)\otimes \mathrm{End}\,E$ by setting $\mathrm{st}(a\otimes b)=\mathrm{st}(a)\,\mathrm{tr}_E(b)$ for $a\in C(V)$ and $b\in \mathrm{End}\,E$.

We still denote by tr_E

$$1\otimes \mathrm{tr}_E:\quad \Lambda^+ V\otimes \mathrm{End}\,E\to \Lambda^+ V.$$

We fix a section map $\sigma: \Lambda^+ V\to S(\mathfrak{g})$ of the map A in 1.9. We denote also by σ the map $\sigma\otimes 1$: $\Lambda^+ V\otimes \mathrm{End}\,E\to S(\mathfrak{g})\otimes \mathrm{End}\,E$.

A first generalization of 1.18 is the following.

1.20. PROPOSITION. — *Let* Φ *be a compactly supported* C^∞ *function on* $\mathfrak{g}$ *with values in* $C^{2j}(V) \otimes \operatorname{End} E$.

Then

$$\lim_{t \to 0} (4\pi t)^{-l-\dim \mathfrak{g}/2}\, t^j \times \int e^{-\|a\|^2/4t} \operatorname{st}((e^a \otimes 1)\Phi(a))\, da = (i\pi)^{-l} \delta_{-1}\left(a \to \operatorname{tr}_E \sigma(\operatorname{gr}^{[2j]} \Phi(a))\left(\frac{a}{2}\right)\right).$$

Proof. — Modifying Φ outside a neighborhood of $0 \in \mathfrak{g}$ does not change the asymptotics of the integral. Since the endomorphism $J_\mathfrak{g}(a)$ in 1.6 is invertible for small a, we may suppose that Φ is of the form

$$\Phi(a) = (J_\mathfrak{g}(a) b_1) \ldots (J_\mathfrak{g}(a) b_j) \varphi(a),$$

with $b_1, \ldots, b_j \in \mathfrak{g}$ and $\varphi \in C_c^\infty(\mathfrak{g}, \operatorname{End} E)$. By Campbell-Hausdorff formula, we have for $a, b \in \mathfrak{g}$

$$e^{-a} e^{a+\varepsilon b} = e^{\varepsilon f(\varepsilon, a, b)},$$

where f is a $\mathfrak{g}$-valued function, analytic in a neighborhood of (0, 0, 0), such that

$$f(0, a, b) = J_\mathfrak{g}(a) b.$$

Therefore

$$e^{-a} e^{a+\varepsilon b} = \sum_{j=0}^{\infty} \frac{\varepsilon^i}{j!} \Phi_j(a, b)$$

where

$$\Phi_j(a, b) \in C^{2j}(V) \quad \text{and} \quad \operatorname{gr}^{[2j]} \Phi_j(a, b) = (J_\mathfrak{g}(a) b)^j.$$

Put

$$1.21 \qquad \Psi(a) = \frac{\partial^j}{\partial \varepsilon_1 \ldots \partial \varepsilon_j} e^{-a} e^{a+\varepsilon_1 b_1 + \ldots + \varepsilon_j b_j} \varphi(a) \Big|_{\varepsilon_1 = \ldots = \varepsilon_j = 0}.$$

Then, for Φ as above,

$$\operatorname{gr}^{[2j]} \Phi(a) = \operatorname{gr}^{[2j]} \Psi(a).$$

Using induction on j, we see that if suffices to prove 1.20 for the function Ψ of the form 1.21. For such a Ψ, performing the change of variables $a+\varepsilon_1 b_1+\ldots+\varepsilon_j b_j \to a$, we see easily that the limit of the integral in 1.20 is the same as

$$\lim_{t\to 0}(4\pi t)^{-l-\dim \mathfrak{g}/2}\int_{\mathfrak{g}}\langle b_1, a\rangle\ldots\langle b_j, a\rangle e^{-\|a\|^2/4t}\operatorname{st}(e^a)\operatorname{tr}_E\varphi(a)\,da.$$

For $k\geqslant 1$ we have $\langle(\operatorname{ad} a)^k b_j, a\rangle=0$, thus we have

$$\langle J_{\mathfrak{g}}(a)b, a\rangle=\langle b, a\rangle,$$

and we obtain 1.20 by applying 1.18.

1.22 We will need a further generalization of 1.18 in the following situation:

Let $V=V_0\oplus V_1$ be an orthogonal decomposition of $V=\mathbb{R}^n$, with V_0 identified to $\mathbb{R}^{2l_0}$ and V_1 to $\mathbb{R}^{2l_1}$. For $i=0, 1$ we denote by $\mathfrak{g}_i$, G_i the Lie algebra $\mathfrak{so}(V_i)$ and the corresponding Spin group, considered as subsets of the Clifford algebra $C(V_i)$. We have a canonical identification

$$C(V)=C(V_0)\otimes C(V_1).$$

We denote by $\delta^{\mathfrak{g}}_{-1}$ the Super Dirac distribution on $\mathfrak{g}$ and by $\delta^{\mathfrak{g}_0}_{-1}$ the Super Dirac distribution on $\mathfrak{g}_0$. We denote by st^1 the supertrace on $C(V_1)$.

1.23 Proposition. — *Let γ_1 be an element of $C(V_1)$ and let Φ be a compactly supported C^∞ function on $\mathfrak{g}\times V_1$ with values in $C^{2j}(V)\otimes\operatorname{End} E$.*

Then

$$\lim_{t\to 0}(4\pi t)^{-l-\dim \mathfrak{g}/2}t^j \times\int_{\mathfrak{g}\times V_1}e^{-(\|a\|^2+\|v\|^2)/4t}\operatorname{st}((e^a\gamma_1\otimes 1)\Phi(a, v))\,da\,dv$$

$$=(i\pi)^{-l_0}\operatorname{st}^1(\gamma_1)\,\delta^{\mathfrak{g}_0}_{-1}\left(a\to \operatorname{tr}_E\sigma(\operatorname{gr}^{[2j]}\Phi(a, 0))\left(\frac{a}{2}\right)\right).$$

Proof. — Put $\Psi(a, v)=(\gamma_1 \otimes 1)\Phi(a, v)$. Since $C(V_1)\subset C^{2l_1}(V)$, we have $\psi(a, v)\in C^{2(j+l_1)}(V)$. Applying 1.20 to this function, and 1.17 (i) we obtain

1.24 $$\lim_{t\to 0}(4\pi t)^{-l-\dim \mathfrak{g}/2-l_1}t^{l_1+j} \times \int e^{-(\|a\|^2+\|v\|^2)/4t}\,\mathrm{st}((e^a\gamma_1 \otimes 1)\Phi(a, v))\,da\,dv = (i\pi)^{-l}\delta^{\mathfrak{g}}_{-1}\left(a \to \mathrm{tr}_E\,\sigma(\mathrm{gr}^{[2j+2l_1]}\Psi(a, 0))\left(\frac{a}{2}\right)\right).$$

We have

1.25 $$\mathrm{gr}^{[2j+2l_1]}((\gamma_1 \otimes 1)\Phi(a, 0))=\left(\frac{i}{2\pi}\right)^{l_1}\mathrm{st}^1(\gamma_1)\,\mathrm{gr}^{[2j]}\Phi(a, 0)\wedge\omega_1,$$

where ω_1 is the canonical element of $\Lambda^{2l_1}(V_1)$. Denote by p the projection $\Lambda^+ V \to \Lambda^+ V_0$ and by p_s the projection $S(\mathfrak{g})\to S(\mathfrak{g}_0)$ associated to the decomposition $V=V_0\oplus V_1$. Denote by A_0 the canonical map $S(\mathfrak{g}_0)\to\Lambda^+ V_0$. Then $A_0\circ p_s=p\circ A$ and for $u\in\Lambda^+ V$ we have

$$t^{\mathfrak{g}}_{-1}(u\,\omega_1)=t^{\mathfrak{g}_0}_{-1}(p\,.\,u).$$

Therefore, using 1.11, we deduce 1.23 from 1.24, 1.25 and the obvious relation

$$A\left(a \to \sigma\omega_1\left(\frac{a}{2}\right)\right)=2^{-l_1}\omega_1.$$

2. Riemannian structure on the frame bundle

Let M be a C^∞ manifold. We denote by $\mathcal{A}(M)$ the algebra of exterior differential forms on M, by $\mathcal{A}^+(M)$ the commutative subalgebra of even forms.

2.1. Let M be an oriented Riemannian manifold of dimension $n=2l$. Let P be the principal bundle of oriented orthonormal frames and $\pi : P \to M$ the canonical projection. An element $u\in P$ is an isometry of $V=\mathbb{R}^n$ with $T_x M$, if $x=\pi(u)$. The group $SO(n)$ operates on the right on P by $(u, g)\mapsto ug$. The vertical tangent space to the fibers is then identified with $\mathfrak{g}$.

Consider on M the Levi-Civita connection. Let $\omega \in \mathscr{A}^1(P) \otimes \mathfrak{g}$ be the corresponding 1-form. The horizontal tangent space at $u \in P$ is

$$\{X \in T_u P \text{ such that } \omega_u(X)=0\}.$$

Denote by $\theta \in \mathscr{A}^1(P) \otimes V$ the fundamental 1-form on P defined by $\theta_u(X)=u^{-1}(\pi_* X)$, for $X \in T_u P$. Let $\Omega \in \mathscr{A}^2(P) \otimes \mathfrak{g}$ be the curvature form on M.

Recall the structure equations

2.2 $$d\theta = -\omega \wedge \theta,$$

2.3 $$d\omega = \Omega - \frac{1}{2}[\omega, \omega].$$

2.4. The form $\theta \oplus \omega$ determines a canonical trivialisation of the tangent bundle, $TP = P \times (V \oplus \mathfrak{g})$.

2.5. Suppose that M admits a spin-structure. Thus there exists a principal bundle P' with group $G = \text{Spin}(n)$ such that

$$P = (P' \otimes SO(n))/\text{Spin}(n).$$

For $u \in P'$, $g \in G$ denote by $R(u)g$ or ug the action of G on P'. The lifts to P' of the forms θ, ω, Ω will also be denoted by θ, ω, Ω. The tangent bundle TP' to P' is also trivialized in $TP' = P' \times (V \oplus \mathfrak{g})$, by $\theta \oplus \omega$.

2.6. For $x \in V \oplus \mathfrak{g}$, we denote by $\tilde{x}$ the vector field on P (or P') such that $(\theta \oplus \omega)(\tilde{x}) = x$. It is easy to obtain the following commutation relations:

2.7 $$[\tilde{a}, \tilde{b}] = [a, b]^{\sim} \quad \text{for} \quad a, b \in \mathfrak{g},$$

2.8 $$[\tilde{a}, \tilde{x}] = (ax)^{\sim} \quad \text{for} \quad a \in \mathfrak{g}, \quad x \in V,$$

2.9 $$\left\{ \begin{array}{c} \omega([\tilde{x}, \tilde{y}]) = -\Omega(\tilde{x}, \tilde{y}) \\ \theta([\tilde{x}, \tilde{y}]) = 0 \end{array} \right\} \quad \text{for} \quad x, y \in V.$$

We may also write the formulae 2.7, 2.8 as

2.10 $$[\tilde{a}, \tilde{x}] = (\tau(a)x)^{\sim} \quad \text{for} \quad a \in \mathfrak{g}, \quad x \in V \oplus \mathfrak{g},$$

where τ denotes the canonical representation of $\mathfrak{g}$ in $V \oplus \mathfrak{g}$.

2.11. Consider on $\mathfrak{g}$ the inner product defined by 1.9. As the tangent space $T_u P$ is the direct sum of the horizontal space isomorphic to $T_x M$ and the vertical space isomorphic to $\mathfrak{g}$, it has a canonical inner product. We will then consider P as an oriented Riemannian manifold. The trivialisation $T_u P \simeq V \oplus \mathfrak{g}$ is an isomorphism of Euclidean spaces. Consider the Levi-Civita connection ∇^P (or simply ∇) on the Riemannian manifold P. It is determined by its connection matrix in the trivialisation $TP = P \times (V \oplus \mathfrak{g})$, i. e. by the endomorphism $\Gamma_u(x)$ of $V \oplus \mathfrak{g}$, defined by:

$$(\theta \oplus \omega)_u (\nabla_{\tilde{x}} \tilde{y}) = \Gamma_u(x) . y.$$

We compute now $\Gamma_u(x)$ omitting the subscript u. We need a notation: the restriction of Ω_u to the horizontal space at u defines an element of $\Lambda^2 V^* \otimes \mathfrak{g}$, by $(x, y) \mapsto \Omega_u(\tilde{x}, \tilde{y})$ for $x, y \in V$. We still denote this element by Ω_u.

Let $a \in \mathfrak{g} = \Lambda^2 V$, the contraction (Ω_u, a) of Ω_u by a is then an element of $\mathfrak{g}$. Explicitly, if

$$a = \sum_{1 \leq i \leq j \leq n} a_{ij} e_i \wedge e_j,$$
$$(\Omega_u, a) = \sum a_{ij} \Omega_u(\tilde{e}_i, \tilde{e}_j).$$

LEMMA. — *Let $a, b \in \mathfrak{g}$ and $x, y \in V$. Then*

2.12 $$\Gamma(x) y = -\frac{1}{2} \Omega(\tilde{x}, \tilde{y}),$$

2.13 $$\Gamma(x) a = \frac{1}{4} (\Omega, a) x,$$

2.14 $$\Gamma(a) x = ax + \frac{1}{4} (\Omega, a) x,$$

2.15 $$\Gamma(a) b = \frac{1}{2} [a, b].$$

Proof. — Let x_1, x_2, x_3 be elements of $V \oplus \mathfrak{g}$, then

$$2 \langle \Gamma(x_1) x_2, x_3 \rangle = \langle [\tilde{x}_1, \tilde{x}_2], \tilde{x}_3 \rangle - \langle \tilde{x}_1, [\tilde{x}_2, \tilde{x}_3] \rangle - \langle [\tilde{x}_1, \tilde{x}_3], \tilde{x}_2 \rangle.$$

Thus formulae 2.12, 2.15 are immediate consequences of 2.7, 2.9.

Similarly:

$$2 \langle \Gamma(x) a, y \rangle = - \langle ax, y \rangle - \langle x, ay \rangle + \langle \Omega(\tilde{x}, \tilde{y}), a \rangle.$$

As a is antisymmetric, we have $\langle ax, y\rangle + \langle x, ay\rangle = 0$ and

$$\langle \Omega(\tilde{x}, \tilde{y}), a\rangle = \frac{1}{2}\sum_{1\leq i<j\leq n}\langle \Omega(\tilde{x}, \tilde{y})e_i, e_j\rangle a_{ij},$$

by definition of the scalar product on $\mathfrak{g}$,

$$= \frac{1}{2}\sum_{1\leq i<j\leq n}\langle \Omega(\tilde{e}_i, \tilde{e}_j)x, y\rangle a_{ij},$$

by one of the symmetry property of the curvature tensor. Thus we obtain 2.13.

The relation 2.14 is a consequence of 2.13 and of the no-torsion property of ∇, i.e. $\nabla_{\tilde{a}}\tilde{x} - \nabla_{\tilde{x}}\tilde{a} = [\tilde{a}, \tilde{x}]$.

2.16 Lemma. — *Let $x\in V$ and $a\in\mathfrak{g}$. The integral curves of the vector fields $\tilde{x}$ and $\tilde{a}$ are geodesics of P. In particular, we have, for $u\in P$, $g\in G$*

2.17 $$\exp_u \tilde{a} = u \exp a$$

2.18 $$(\exp_u \tilde{x})g = \exp_{ug}(g^{-1}x)^{\sim}.$$

Proof. — The first assertion follows from the relation $\nabla_{\tilde{x}}\tilde{x} = 0$, $\nabla_{\tilde{a}}\tilde{a} = 0$. The integral curves of $\tilde{a}$ are by definition the trajectoires $s\mapsto u \exp sa$ of the one parameters subgroups of G, hence 2.17. The curve $(\exp_u s\tilde{x})g$ has tangent vector $R(g)_*\tilde{x} = (g^{-1}x)^{\sim}$ hence 2.18.

2.19. Let $u\in P$. The map $\exp_u$ defines a diffeomorphism of a neighborhood of 0 in $T_u P$ on a neighborhood of u in P. Let $X\in T_u P \simeq V\oplus\mathfrak{g}$. The tangent map to $\exp_u$ at the point X is a linear map from $T_u P$ to $T_{\exp_u X} P$. Both spaces being identified with $V\oplus\mathfrak{g}$, we denote by $J(u, X)$ the corresponding automorphism of $V\oplus\mathfrak{g}$. It is possible to compute it explicitly, when $X\in\mathfrak{g}$.

Proposition. — *Let $u\in P$ and $a\in\mathfrak{g}$. Then*

2.20 $$J(u, a)\big|_{\mathfrak{g}} = \frac{1-e^{-\operatorname{ad} a}}{\operatorname{ad} a},$$

2.21 $$J(u, a)\big|_V = \exp(-a) . \frac{1-\exp(-1/2(\Omega_u, a))}{1/2(\Omega_u, a)}.$$

Proof. — Let $x_0\in V\oplus\mathfrak{g}$. Consider the application $\mathbb{R}^2 \to P$ defined by:

$$u(s, t) = \exp_u(s(a+tx_0)^{\sim}).$$

Let

$$Y(s)=\frac{\partial u}{\partial t}(s,\,0) \qquad \text{and} \qquad y(s)=(\theta\oplus\omega)(Y(s)).$$

Then

$$J(u,\,a).x_0=y(s)\big|_{s=1}.$$

Denote by R the curvature of the manifold P. The Jacobi vector field $Y(s)=\partial u/\partial t$ is determined by the differential equation:

$$\frac{\nabla_\partial}{\partial s}\,\frac{\nabla_\partial}{\partial s}\,\frac{\partial u}{\partial t}=R\left(\frac{\partial u}{\partial s},\,\frac{\partial u}{\partial t}\right).\frac{\partial u}{\partial s},$$

and the initial conditions $y(0)=0$, $y'(0)=x_0$ (*see* [6]).

We have, for $t=0$, $u(s,\,0)=u\exp sa$, $\partial u/\partial s=\tilde{a}$. Thus, at the point $u(s)=u\exp sa$, we have:

2.22 $$\nabla_{\tilde{a}}\nabla_{\tilde{a}}Y=R(\tilde{a},\,Y).\tilde{a}.$$

By definition of Γ, if $\Gamma_s(a)=\Gamma_{u(s)}.a$, we have:

2.23 $$(\theta\oplus\omega)(\nabla_{\tilde{a}}\nabla_{\tilde{a}}Y)=\left(\frac{d}{ds}+\Gamma_s(a)\right)\left(\frac{d}{ds}+\Gamma_s(a)\right).y(s)$$

Let us fix s_0, and let $v_0=y(s_0)$. Then:

$$\begin{aligned}R_{u(s_0)}(\tilde{a},\,Y).\tilde{a}&=(\nabla_{\tilde{a}}\nabla_{\tilde{v}_0}-\nabla_{\tilde{v}_0}\nabla_{\tilde{a}}-\nabla_{(\tau(a)v_0)^\sim}).\tilde{a}\\&=\nabla_{\tilde{a}}\nabla_{\tilde{v}_0}.\tilde{a}-\nabla_{(\tau(a)v_0)^\sim}.\tilde{a},\ as\ \nabla_{\tilde{a}}\tilde{a}=0\\&=\nabla_{\tilde{a}}(\nabla_{\tilde{a}}\tilde{v}_0-(\tau(a)v_0)^\sim)-\nabla_{\tilde{a}}(\tau(a).v_0)^\sim+(\tau(a)^2v_0)^\sim,\end{aligned}$$

by no-Torsion property.

Thus:

2.24 $$\begin{aligned}&(\theta\oplus\omega)(R_{u(s_0)}(\tilde{a},\,Y).\tilde{a})\\&\quad=\left(\frac{d}{ds}+\Gamma_s(a)\right)\left(\frac{d}{ds}+\Gamma_s(a)\right)v_0\\&\qquad\qquad-2\,\Gamma_s(a).(\tau(a)v_0+\tau(a)^2v_0)\big|_{s=s_0}.\end{aligned}$$

The equality of 2.23 and 2.24 gives the differential equation:

2.25 $$\frac{d^2 y}{ds^2}+2\Gamma_s(a)\frac{dy}{ds}+2\Gamma_s(a)(\tau(a)y(s))-\tau(a)^2 y(s)=0.$$

Let us also denote by τ the natural representation of G in $V\oplus\mathfrak{g}$.

From the G-invariance of the Riemannian connection of P, it follows that, for $g\in G$, $u\in P$, $x\in V\oplus\mathfrak{g}$

2.26 $$\tau(g)^{-1}\Gamma_u(x)\tau(g)=\Gamma_{ug}(\tau(g)^{-1}x).$$

Let $y(s)=\tau(\exp -sa)x(s)$. Then 2.25 becomes:

2.27 $$\frac{d^2 x}{ds^2}+2(\Gamma_u(a)-\tau(a))\frac{dx}{ds}=0.$$

As $y(0)=0$, $y'(0)=x_0$, the initial conditions are $x(0)=0$, $x'(0)=x_0$. The endomorphism $\Gamma_u(a)-\tau(a)$ preserves the decomposition $V\oplus\mathfrak{g}$, thus $J(u, a)$ also preserves this decomposition. If $x_0\in V$, $x(s)$ for all s and from 2.14

$$\frac{d^2 x}{ds^2}+\frac{1}{2}(\Omega_u, a)\frac{dx}{ds}=0.$$

By integration we obtain 2.21.

If $x_0\in\mathfrak{g}$, then $x(s)\in\mathfrak{g}$ for all s, and we obtain the well known formula 2.20.

We now establish a formula that we will use for the determination of the equivariant index. The reader can come back to read the proof of this proposition when needed. Let $q(u, u')$ be the square of the geodesic distance between two points u, u' of the manifold P.

2.28. PROPOSITION. — *Let $u\in P$, $\gamma\in\operatorname{End} V$ and $a\in\mathfrak{g}$. Define for $v\in V$, v and a sufficiently near 0, $h(v, \gamma, a)=q(\exp_u\tilde{v}, \exp_u(\gamma v)^{\sim}\exp a)$. The map $v\mapsto h(v, \gamma, a)$ admits $0\in V$ as critical point. Its Hessian at 0 is given by the quadratic form:*

$$Q(\gamma, a)(v)=\quad (1-(\exp(\Omega, a)/2)\gamma)v, \frac{(\Omega, a)/2}{1-\exp-(\Omega, a)/2}(1-\gamma)v\quad ,$$

Proof. — Denote $u(t)=\exp_u t\tilde{v}$. Let $X(t)\in V\oplus \mathfrak{g}$ the vector such that

2.29 $$\exp_{u(t)} X(t)^{\sim}=\exp_u (t\gamma v)^{\sim} \exp a.$$

Then $h(tv, \gamma, a)=\|X(t)\|^2$.

We need to compute the development at order two of $\|X(t)\|^2$, at $t=0$. For $t=0$, $X(0)=a$. Let us first calculate $X_1=dX/dt\,|_{t=0}$ by differentiating 2.29. We have:

$$\frac{\partial}{\partial t}\exp_{u(t)} X(t)\,|_{t=0}=\frac{\partial}{\partial t}(\exp_{u(t)}\tilde{a})+\frac{\partial}{\partial t}\exp_u X(t)^{\sim}\,|_{t=0}.$$

As

$$\exp_{u(t)}\tilde{a}=u(t),\qquad \exp a=\exp_u t\tilde{v},$$
$$\exp a=\exp_{u\,\exp a} t(\exp -a).\tilde{v},$$
$$\frac{\partial}{\partial t}\exp_{u(t)}\tilde{a}=((\exp -a).v)^{\sim}.$$

With the notation of 2.19,

$$\frac{\partial}{\partial t}\exp_u X(t)^{\sim}\,|_{t=0}=(J(u, a).X_1)^{\sim}.$$

The right hand side of 2.29 is the integral curve of $((\exp -a)\gamma v)$.

Thus we obtain:

$$(\exp -a).v+J(u, a).X_1=(\exp -a)\gamma v, \quad \text{i.e.}$$

2.30 $$X_1=\frac{1/2(\Omega, a)}{1-\exp -1/2(\Omega, a)}.(\gamma-1)v.$$

In particular $X_1\in V$, thus

$$h(tv, \gamma, a)=\langle a+tX_1, a+tX_1\rangle+o(t)=\langle a, a\rangle+o(t) \quad \text{as } \langle X_1, a\rangle=0.$$

Thus 0 is a critical point of the map $v\mapsto h(v, \gamma, a)$.

$$\text{Put } w(s, t)=\exp_{u(t)}(sX(t)^{\sim}).$$

Then $X(t)^{\sim}=\partial w/\partial s(0, t)$.

$$\frac{d}{dt}\|X(t)\|^2=\frac{d}{dt}\left(\left\langle\frac{\partial w}{\partial s},\frac{\partial w}{\partial s}\right\rangle\Big|_{s=0}\right)$$

$$=2\left\langle\nabla_{\partial/\partial t}\frac{\partial w}{\partial s},\frac{\partial w}{\partial s}\right\rangle\Big|_{s=0}$$

$$=2\left\langle\nabla_{\partial/\partial s}\frac{\partial w}{\partial t},\frac{\partial w}{\partial s}\right\rangle\Big|_{s=0}, \quad \text{by no-torsion,}$$

$$=2\frac{\partial}{\partial s}\left\langle\frac{\partial w}{\partial t},\frac{\partial w}{\partial s}\right\rangle\Big|_{s=0} \quad \text{as } \nabla_{\partial/\partial s}\frac{\partial}{\partial s}=0.$$

The differential equation for the Jacobi vector field $\partial w/\partial t$ implies that

$$\left\langle\frac{\partial w}{\partial t},\frac{\partial w}{\partial s}\right\rangle=A(t)+s\,B(t).$$

Hence

$$\frac{d}{dt}\|X(t)\|^2=2\,B(t).$$

We now compute the linear term in the expansion of $B(t)$ at $t=0$.

For $s=0$, $\partial w/\partial t(0, t)=\tilde{v}$ (2.16).

Thus $A(t)=\langle v, X(t)\rangle=t\langle v, X_1\rangle+o(t)$.

For $s=1$, $A(t)+B(t)=\left\langle\frac{\partial w}{\partial t}(1, t), \frac{\partial w}{\partial s}(1, t)\right\rangle$.

We have already seen, from

$$\exp_{u(t)}X(t)^{\sim}=\exp_u(t\gamma v)^{\sim}\exp a$$

that

$$\frac{\partial w}{\partial t}(1, t)=(\exp(-a)\gamma v)^{\sim}.$$

It remains to calculate $\partial w/\partial s(1, t)$.

We write $\partial w/\partial s(s, t)=(a+t\,Z(s)+o(t))^{\sim}$. The equation $\nabla_{\partial/\partial s}\partial/\partial s=0$ for the geodesic $s\to w(s, t)$ gives us:

$$0\equiv\frac{d}{ds}(a+t\,Z(s))+\Gamma_{w(s,\,t)}(a+t\,Z(s))(a+t\,Z(s))+o(t).$$

As $\Gamma_u(a)a=0$, the linear term in t of this equation, gives the equation:

2.31 $$\frac{d}{ds}Z(s)+\Gamma_{w(s,0)}(a)Z(s)+\Gamma_{w(s,0)}(Z(s))a=0.$$

Let $Z(s)=\exp -saZ_1(s)$. Using 2.26, 2.31 gives us the differential equation:

2.32 $$\frac{dZ_1}{ds}+2\Gamma_u(Z_1(s))a=0,$$

with initial conditions $Z_1(0)=Z(0)=X_1\in V$.

As the application $x\to\Gamma_u(x)a$ preserves the decomposition $V\oplus \mathfrak{g}$, the curve $Z_1(s)$ remain in V, and the integration of 2.32 gives $Z_1(s)=\exp(-s(\Omega_u, a)/2)X_1$.

Finally:

$$A(t)+B(t)=t\left\langle \gamma v, \exp -\frac{1}{2}(\Omega_u, a)X_1\right\rangle +o(t),$$

$$B(t)=t\left\langle \exp\frac{1}{2}(\Omega_u, a)\gamma v, X_1\right\rangle -\langle v, X_1\rangle +o(t).$$

As X_1 is given by 2.30, we obtain our proposition.

3. Supertrace of the heat kernel and the equivariant index of the Dirac operator

We start this section by stating some well known facts on the asymptotics of the Heat kernel of a compact Riemannian manifold. Then we will compare the square of the twisted Dirac operator $D_{\mathscr{E}}$ of a spin-manifold to a horizontal Laplacian $\Delta_{\mathscr{E}}$ on the manifold P'. We will finally deduce the asymptotics of the trace of the kernel of the operator $(\gamma e^{tD^2_{\mathscr{E}}})$, where γ is an isometry of M, from the asymptotics of the kernel of $e^{t\Delta_{\mathscr{E}}}$ and Morse lemma.

3.1. Let Q be a Riemannian manifold, with Levi-Civita connection ∇^Q, volume form dx and Laplacian Δ. Let ξ_i be a local choice of orthonormal vector fields on Q. The Laplacian Δ is the second-order differential operator acting on functions on Q, given locally by the expression

$\Delta=\Sigma_i \xi_i . \xi_i - \nabla^Q_{\xi_i} . \xi_i$. The operator $e^{t\Delta}$ (for $t>0$) is a smoothing operator with a C^∞-kernel $k(t, x_0, x)$, i. e.

$$(e^{t\Delta}.f)(x_0)=\int_Q k(t, x_0, x) f(x)\, dx.$$

Let $\Psi(x, y)$ be a cut-off function on $Q \times Q$ identically 1 in a small neighborhood of the diagonal of Q. Let $q(x, y)$ be the square of the geodesic distance between x, y. Ψ is chosen in order that $q(x, y)$ is well defined on Supp (Ψ). Let $\mathrm{Exp}_x : T_x Q \to Q$ be the exponential map and $\theta(x, y)$: $T_x Q \to T_y Q$ the tangent map to Exp_x at the point $\mathrm{Exp}_x^{-1}(y)$.

Let us fix $x_0 \in Q$ and let

$$j_Q(y)=\det \theta(x_0, y), \qquad r(y)=q(x_0, y)^{1/2}.$$

Let $\mathscr{R}$ be the radial vector field $r\, d/dr$ along the geodesic joining x_0 to y (y varies in a small neighborhood of x_0). Define inductively the functions $y \to U_i(x_0, y)$ by the differential equations:

$$\mathscr{R}.(U_i r^i j_Q(y)^{1/2})=r^i j_Q(y)^{1/2}(\Delta . U_{i-1})$$

and the initial condition $U_0(x_0, x_0)=1$.

We then have

$$U_0(x_0, y)=j_Q(y)^{-1/2}=\det \theta(x_0, y)^{-1/2}.$$

Recall (*see* [6]) the:

3.2. Proposition [19]. — *For every integer N, we have uniformly on $Q \times Q$,*

$$k(t, x_0, y)=(4\pi t)^{-\dim Q/2}\Psi(x_0, y) \times e^{-q(x_0, y)/4t}\left(\sum_{i=0}^N U_i(x_0, y) t^i\right)+O(t^{N-(\dim Q/2)+1}).$$

More generally, let $\mathscr{E} \to Q$ be a complex hermitian vector bundle over Q with a hermitian connection ∇. For ξ a vector field on Q, we denote by $\nabla_\xi : \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ the covariant derivation. The Laplace operator $\Delta_{\mathscr{E}}$ acting on the space of sections $\Gamma(\mathscr{E})$ of the bundle $\mathscr{E}$ is defined locally by the expression:

$$\Delta_{\mathscr{E}}=\Sigma_i \nabla_{\xi_i}\nabla_{\xi_i}-\nabla_{\nabla^Q_{\xi_i} . \xi_i}.$$

Let V be a C^∞ section of the bundle End $(\mathscr{E})$. We suppose that $V(x)=V(x)^*$. Let us consider the operator $\Delta_{\mathscr{E}}-V$ acting on $\Gamma(\mathscr{E})$. The operator $e^{t(\Delta_{\mathscr{E}}-V)}(t>0)$ has a C^∞-kernel $k_{\mathscr{E}}(t, x_0, y)\in \operatorname{Hom}(\mathscr{E}_y, \mathscr{E}_{x_0})$. We fix $x_0\in Q$, and define $E_0=\mathscr{E}_{x_0}$. We identify $\mathscr{E}$ with $\mathscr{E}^*$ using the hermitian structure. We extend the covariant derivative ∇_ξ to $E_0\otimes\Gamma(\mathscr{E}^*)$.

Consider the section $y\mapsto U_i(x_0, y)$ of

$$E_0\otimes\Gamma(\mathscr{E}^*)\ (\text{i. e. } U_i(x_0, y)\in \operatorname{Hom}(\mathscr{E}_y, \mathscr{E}_{x_0})),$$

defined inductively for $i=0, 1, \ldots$, by:

3.3. $\nabla_{\mathscr{R}}(U_i r^i j_Q(y)^{1/2})=r^i j_Q(y)^{1/2}(\Delta_{\mathscr{E}}-V).U_{i-1}$

and the initial condition

$$U_0(x_0, x_0)=\operatorname{Id}\in E_0\otimes E_0^*.$$

We then have:

$$U_0(x, y)=\det\theta(x, y)^{-1/2}\tau(x, y)$$

where

$$\tau(x, y)\in \operatorname{Hom}(\mathscr{E}_y, \mathscr{E}_x)$$

denotes the parallel transport along the geodesic from x to y. The proof of 3.2 ([6], [20]) leads similarly to:

3.4. Proposition. — *For every integer N, we have uniformly on $Q\times Q$:*

$$k_{\mathscr{E}}(t, x_0, y)=(4\pi t)^{-\dim Q/2}\Psi(x_0, y)$$
$$\times e^{-q(x_0, y)/4t}\left(\sum_{i=0}^{N} U_i(x_0, y)t^i\right)+O(t^{N-\dim Q/2+1}).$$

3.5. Consider a spin manifold M and let $P'\to M$ be a principal spin-bundle over M of group $G=\operatorname{Spin}(n)$. As in Section 2, we consider P' itself as a Riemannian manifold.

Let S be the space of the spin representation ρ. Then $P'\times_G S=\mathscr{S}$ is the spin bundle over M. As the decomposition $S=S^+\oplus S^-$ is stable by ρ, the spin bundle $\mathscr{S}$ is the direct sum of the even and odd spin bundles

$$\mathscr{S}^+=P'\times_G S^+,\qquad \mathscr{S}^-=P'\times_G S^-.$$

The group G acts on the space of C^∞ functions on P' by $(g.f)(u)=f(ug)$.

Consider the action of G on the tensor product $S \otimes C^\infty(P')$. The subspace $(S \otimes C^\infty(P'))^G \subset S \otimes C^\infty(P')$ can be identified to the space $\Gamma(\mathscr{S})$ of sections of the bundle $\mathscr{S}$. Explicitly

$\Gamma(\mathscr{S}) = \{ f$, S-valued C^∞ functions on P', such that $f(ug)=\rho(g)^{-1} f(u) \}$.

3.6. More generally, let $\mathscr{E} \to M$ be a complex vector bundle over M of typical fiber $E=\mathbb{C}^q$. Consider the bundle $\pi^*(\mathscr{E})$ over P' and its space of sections $\Gamma(\pi^*\mathscr{E})$. If $x=\pi(u)$, $\pi^*(\mathscr{E})_u=\mathscr{E}_x$, so the group G acts on $\Gamma(\pi^*\mathscr{E})$ by $(g.f)(u)=f(ug)$.

Consider the action of G on the tensor product $S \otimes \Gamma(\pi^*\mathscr{E})$. The subspace $(S \otimes \Gamma(\pi^*\mathscr{E}))^G$ of $S \otimes \Gamma(\pi^*\mathscr{E})$ is then identified with the space of sections of the bundle $\mathscr{S} \otimes \mathscr{E}$ over M.

Explicitly:

3.7 $\Gamma(\mathscr{S} \otimes \mathscr{E}) = \{ f \in S \otimes \Gamma(\pi^*\mathscr{E})$ such that $f(ug)=(\rho(g) \otimes 1)^{-1} f(u)) \}$.

Let ∇ be a hermitian connection on $\mathscr{E}$. Denote its curvature by $\Omega^{\mathscr{E}}$. We will consider $\Omega^{\mathscr{E}}$ as a section of $\Lambda^2 T^* M \otimes \operatorname{End} \mathscr{E}$. We pull back the connection on $\mathscr{E}$ to a connection on $\pi^*\mathscr{E}$. If ξ is a vector field on P' we denote the covariant derivative by $\nabla_\xi : \Gamma(\pi^*\mathscr{E}) \to \Gamma(\pi^*\mathscr{E})$. We also denote by ∇_ξ the operator $1 \otimes \nabla_\xi$ on the space $S \otimes \Gamma(\pi^*\mathscr{E})$.

Let us note that, for $a \in \mathfrak{g}$, $\varphi \in \Gamma(\pi^*\mathscr{E})$.

3.8 $$(\nabla_{\tilde{a}}\varphi)(u) = \frac{d}{d\varepsilon}\varphi(u \exp \varepsilon a)\big|_{\varepsilon=0},$$

3.9. Let $D : \Gamma(\mathscr{S} \otimes \mathscr{E}) \to \Gamma(\mathscr{S} \otimes \mathscr{E})$ be the Dirac operator. In the identification 3.7, D is given by the formula:

$$D = \sum_{1 \leq i \leq n} (c(e_i) \otimes \nabla_{\tilde{e}_i}).$$

The operator D maps $\Gamma(\mathscr{S}^+ \otimes \mathscr{E})$ into $\Gamma(\mathscr{S}^- \otimes \mathscr{E})$ and vice versa. We denote by

$$D^+ : \quad \Gamma(\mathscr{S}^+ \otimes \mathscr{E}) \to \Gamma(\mathscr{S}^- \otimes \mathscr{E})$$

and

$$D^- : \quad \Gamma(\mathscr{S}^- \otimes \mathscr{E}) \to \Gamma(\mathscr{S}^+ \otimes \mathscr{E}),$$

the restrictions of D.

The operator D^2 leaves stable the decomposition

$$\Gamma(\mathcal{S}\otimes\mathcal{E})=\Gamma(\mathcal{S}^{+}\otimes\mathcal{E})\oplus\Gamma(\mathcal{S}^{-}\otimes\mathcal{E}).$$

We denote by D^2_+ (resp. D^2_-) the restrictions of D^2 to $\Gamma(\mathcal{S}^{+}\otimes\mathcal{E})$ (resp. $\Gamma(\mathcal{S}^{-}\otimes\mathcal{E})$).

3.10. Denote by $\Delta_{\mathcal{E}}$ the Laplace operator of the bundle $\pi^*\mathcal{E}$. Let E_{ij}, $1\leqslant i<j\leqslant n$, be the basis of $\mathfrak{g}$ introduced in 1.3.

Then:

$$\{\tilde{e}_i,\ 1\leqslant i\leqslant n,\ 2\tilde{E}_{ij},\ 1\leqslant i<j\leqslant n\},$$

is an orthogonal frame of the Riemannian manifold P′. From 2.12 and 2.15, $\Delta_{\mathcal{E}}$ is given by:

$$\Delta_{\mathcal{E}}=\sum_{1\leqslant i\leqslant n}\nabla_{\tilde{e}_i}\nabla_{\tilde{e}_i}+\sum_{1\leqslant i<j\leqslant n}4\,\nabla_{\tilde{E}_{ij}}\nabla_{\tilde{E}_{ij}}.$$

The operator $\Delta_{\mathcal{E}}$ commutes with the action of G on $\Gamma(\pi^*\mathcal{E})$.

We also denote by $\Delta_{\mathcal{E}}$ the operator $1\otimes\Delta_{\mathcal{E}}$ on $S\otimes\Gamma(\pi^*\mathcal{E})$.

3.11 LEMMA. — *The operator* $\sum_{1\leqslant i<j\leqslant n}4\,\nabla_{\tilde{E}_{ij}}\nabla_{\tilde{E}_{ij}}$ *operates on* $\Gamma(\mathcal{S}\otimes\mathcal{E})\subset S\otimes\Gamma(\pi^*\mathcal{E})$ *by the scalar*

$$\lambda=-(\dim\mathfrak{g})=-\frac{n(n-1)}{2}.$$

Proof. — For $a\in\mathfrak{g}$ and $f\in S\otimes\Gamma(\pi^*\mathcal{E})$, we have

$$(\nabla_{\tilde{a}}f)(u)=\frac{d}{d\varepsilon}f(u\exp\varepsilon a)\big|_{\varepsilon=0}.$$

Thus, if f verifies 3.7, $\nabla_{\tilde{a}}f=-(\rho(a)\otimes 1)f$. The element e_ie_j of the Clifford algebra is such that $(e_ie_j)^2=-1$. As $E_{ij}=\tau(1/2\,e_ie_j)$, we obtain our lemma.

3.12. Let $k(u)=-\sum_{1\leqslant i<j\leqslant n}<\Omega_u(\tilde{e}_i,\tilde{e}_j)e_i,e_j>$ be the scalar curvature of the manifold M ($k(u)$ depends only of $x=\pi(u)$). Define a section of Ends $(S)\otimes$ End $(\pi^*\mathcal{E})$ by

3.13 $$V_{\mathcal{E}}(u)=\left(\lambda+\frac{1}{4}k(u)\right)\mathrm{Id}+\sum_{1\leqslant i<j\leqslant n}c(e_i)c(e_j)\otimes\Omega^{\mathcal{E}}_x(ue_i,ue_j).$$

Remark that:

3.14 $$V_{\mathscr{E}}(ug)=(\rho(g)^{-1}\otimes 1)\,V_{\mathscr{E}}(u)(\rho(g)\otimes 1).$$

Thus the multiplication of $V_{\mathscr{E}}$ commutes with the action of G on $S\otimes\Gamma(\pi^*\mathscr{E})$. The formula of Lichnerowicz for D^2 is given by the following:

3.15. PROPOSITION [17] :

$$D^2=(-\Delta_{\mathscr{E}}+V_{\mathscr{E}})\,|\,\Gamma(\mathscr{S}\otimes\mathscr{E}).$$

Proof:

$$D^2=\sum_{i,\,j}c(e_i)c(e_j)\otimes\nabla_{\tilde e_i}\nabla_{\tilde e_j}$$

$$=-1\otimes\Sigma_i\nabla_{\tilde e_i}\nabla_{\tilde e_i}+\frac{1}{2}\Sigma\,c(e_i)c(e_j)\otimes[\nabla_{\tilde e_i},\nabla_{\tilde e_j}],$$

using the commutation relations of the Clifford algebra.

Now

$$[\nabla_{\tilde e_i},\nabla_{\tilde e_j}]=\nabla_{[\tilde e_i,\,\tilde e_j]}+\Omega_x^{\mathscr{E}}(ue_i,\,ue_j).$$

From 2.9 $[\tilde e_i,\tilde e_j]=-(\Omega(\tilde e_i,\tilde e_j))^{\sim}$ is a vertical vector field on P' thus, if $f\in\Gamma(\mathscr{S}\otimes\mathscr{E})$

$$\nabla_{[\tilde e_i,\,\tilde e_j]}f(u)=(\rho(\Omega(\tilde e_i,\tilde e_j)\otimes 1)\,f(u).$$

and

$$D^2 f(u)=-\Sigma_i\nabla_{\tilde e_i}\nabla_{\tilde e_i}f(u)$$

$$+\frac{1}{2}(\sum_{i,\,j}c(e_i)c(e_j)\,\rho(\Omega(\tilde e_i,\tilde e_j))\otimes 1)\,f(u)$$

$$+(\sum_{i<j}c(e_i)c(e_j)\otimes\Omega_x^{\mathscr{E}}(ue_i,\,ue_j))\,f(u).$$

A classical formula, using the symmetries of the Riemanian curvature Ω, shows that

$$\sum_{i,\,j}c(e_i)c(e_j)\,\rho(\Omega_u(\tilde e_i,\tilde e_j))=\frac{1}{2}k(u)\quad\text{Id}.$$

From 3.11, we obtain 3.15.

3.16. Consider the Riemannian volume forms dx, du, dg, on the oriented Riemannian manifolds M, P', G. For a continuous function φ on P', we have:

$$\int_P \varphi(u)\,du = \int_M \left(\int_G \varphi(ug)\,dg\right) dx.$$

3.17. The operator $e^{t(\Delta_{\mathscr{E}} - V_{\mathscr{E}})}(t>0)$ on $S \otimes \Gamma(\pi^* \mathscr{E})$ is given by a C^∞ kernel with respect to the measure du on P'. We denote this kernel by

$$k_{\mathscr{E}}(t, u_0, u) \in \operatorname{End} S \otimes \operatorname{Hom}(\mathscr{E}_x, \mathscr{E}_{x_0}),$$

for

$$u_0,\ u \in P', \qquad x_0 = \pi(u_0), \qquad x = \pi(u).$$

Since $\Delta_{\mathscr{E}}$ and $V_{\mathscr{E}}$ commute with the action of G on $S \otimes \Gamma(\pi^* \mathscr{E})$ we have

3.18 $$k_{\mathscr{E}}(t, u_0 g, ug) = (\rho(g)^{-1} \otimes 1)\, k_{\mathscr{E}}(t, u_0, u)\, (\rho(g) \otimes 1)$$

for $g \in G$.

Define, for u_0, $u \in P'$, $\pi(u) = x$, $\pi(u_0) = x_0$.

3.19 $$\bar{k}_{\mathscr{E}}(t, u_0, u) = \int_G k_{\mathscr{E}}(t, u_0, ug^{-1})(\rho(g) \otimes 1)\, dg.$$

Then, for $f \in \Gamma(\mathscr{S} \otimes \mathscr{E})$, $\bar{k}_{\mathscr{E}}(t, u_0, u).f(u) \in S \otimes \mathscr{E}_{x_0}$ and depends only of $x = \pi(u)$.

3.20. Proposition. — *For $f \in \Gamma(\mathscr{S} \otimes \mathscr{E})$, we have:*

$$(e^{tD^2}.f)(u_0) = \int_M \bar{k}_{\mathscr{E}}(t, u_0, u).f(u))\, dx.$$

Proof. — This follows immediately from the formula 3.16 and the property 3.7.

3.21. Let H be a group of orientation preserving isometries of M. If $\gamma \in H$, the natural lift of γ to P is an isometry of the Riemannian manifold P commuting with the action of G. We will assume that the action of H on P lifts to an action on P' which will be denoted by $\gamma.u$ for $\gamma \in H$ and $u \in P'$.

Let us assume that the action of H on M lifts to an action on the vector bundle $\mathscr{E}$ preserving the connection.

For $\gamma \in H$, we denote by $l(\gamma)$ the action of γ on $\mathcal{E}$. The restriction $l(\gamma)_x$ of $l(\gamma)$ to $\mathcal{E}_x$ is an isomorphism of the Hermitian vector spaces $\mathcal{E}_x \to \mathcal{E}_{\gamma x}$. The group H acts on $\Gamma(\pi^* \mathcal{E})$, thus on $S \otimes \Gamma(\pi^* \mathcal{E})$ by

$$(\gamma . f)(u) = (1 \otimes l(\gamma)) f(\gamma^{-1} u).$$

As γ is an isometry of P' and its action on $\mathcal{E}$ preserves the connection, the action of H on $S \otimes \Gamma(\pi^* \mathcal{E})$, commutes with $\Delta_{\mathcal{E}}$, $V_{\mathcal{E}}$. Thus the kernel $k_{\mathcal{E}}(t, u_0, u)$ satisfies the relation:

3.22 $$(1 \otimes l(\gamma)_{\gamma^{-1} x_0}) k_V(t, \gamma^{-1} u_0, u) = k_V(t, u_0, \gamma u)(1 \otimes l(\gamma)_x).$$

As the action of H on P' commutes with G, H preserves $\Gamma(\mathcal{S} \otimes \mathcal{E})$, and for $f \in \Gamma(\mathcal{S} \otimes \mathcal{E})$.

3.23 $$\begin{aligned}(\gamma e^{-tD^2} . f)(u_0) &= \int_M (\bar{k}_{\mathcal{E}}(t, u_0, \gamma u)(1 \otimes l(\gamma)_x) f(u)) \, dx \\ &= \int_M \left(\int_G k_{\mathcal{E}}(t, u_0, \gamma u g^{-1})(\rho(g) \otimes l(\gamma)_x) . f(u) \, dg \right) dx.\end{aligned}$$

For $g \in G$ and $u_0 = u \in P'_x$, the map

$$k_{\mathcal{E}}(t, u, \gamma u g^{-1})(\rho(g) \otimes l(\gamma)_x),$$

is an endomorphism of $S \otimes \mathcal{E}_x$ which preserves $S^+ \otimes \mathcal{E}_x$ and $S^- \otimes \mathcal{E}_x$. We define its supertrace as in 1.19.

Let

3.24 $$A(t, \gamma, x) = \int_G \operatorname{st}(k_{\mathcal{E}}(t, u, \gamma u g^{-1}) \rho(g) \otimes l(\gamma)_x) \, dg$$

(the right hand side depends only on $x = \pi(u)$).

Recall that we can compute the equivariant index $\operatorname{tr}_{\ker D^+}(\gamma)$-$\operatorname{tr}_{\ker D^-}(\gamma)$ of the Dirac operator by the fundamental formula of McKean-Singer [18]:

3.25. For every $t > 0$,

$$\operatorname{tr}_{\ker D^+}(\gamma) - \operatorname{tr}_{\ker D^-}(\gamma) = \operatorname{tr}(\gamma e^{-tD_+^2}) - \operatorname{tr}(\gamma e^{-tD_-^2}) = \int_M A(t, \gamma, x) \, dx.$$

Let M_γ be the manifold of fixed points by the action of γ on M. If $x \notin M_\gamma$, the elements u, $\gamma u g^{-1}$ are distinct for all g in G. Let $\Psi(x)$ be a cut-off function equal to 1 in a small neighborhood of M_γ. As, for $t \to 0$ $k_g(t, u_0, u)$ is small at all orders outside the diagonal of $P' \times P'$ a partitition of unity gives:

3.26 $$\lim_{t \to 0} \int_M A(t, \gamma, x)\, dx = \lim_{t \to 0} \int_M A(t, \gamma, x) \Psi(x)\, dx.$$

Assume that M_γ is oriented. Let $\mathcal{N} M_\gamma$ be the normal bundle to M_γ in M. Let U be a tubular neighborhood of M_γ. By the exponential map, U is isomorphic to a neighborhood of the zero section of $\mathcal{N} M_\gamma$. We denote by $p : U \to M_\gamma$ the corresponding fibration, and by

$$p_* : \quad \mathcal{A}^{[\max]}_{\mathrm{cpt}}(U) \to \mathcal{A}^{[\max]}(M_\gamma),$$

the integration along the fibers of compactly supported forms.

We then have

$$\int_M A(t, \gamma, x) \Psi(x)\, dx = \int_{M_\gamma} p_*(\Psi(x) A(t, \gamma, x)\, dx).$$

We will see that $\lim_{t \to 0} p_*(\Psi(x) A(t, \gamma, x)\, dx)$ exists and we will identify the limit $I(\gamma)$ as an element of $\mathcal{A}^{[\max]}(M_\gamma)$ (Theorem 3.34). It is clear, from 3.25, that we have the:

3.27. THEOREM:

$$\operatorname{tr}_{\ker D^+}(\gamma) - \operatorname{tr}_{\ker D^-}(\gamma) = \int_{M_\gamma} I(\gamma).$$

The density $I(\gamma)$ on M is called the equivariant index density.

We now proceed towards its calculation in terms of characteristic forms.

3.28. Let M_0 be a connected component of M_γ. We fix $x_0 \in M_0$. The tangent space $T_{x_0}(M)$ is the direct sum of the tangent space $T_{x_0}(M_0)$ to M_0 and of the normal space N_{x_0}. The space $T_{x_0}(M_0)$ is the eigenspace of eigenvalue 1 for the action of γ on $T_{x_0}(M)$. As $\det_{T_{x_0}(M)} \gamma = 1$, the dimension of N_{x_0} is even. We denote by $n_0 = 2 l_0$ the dimension of M_0, $n_1 = 2 l_1$ the dimension of $\mathcal{N}_{x_0}$.

Let us decompose as in 1.22 $V=\mathbb{R}^n$ in

$$V=V_0\oplus V_1 \qquad \text{with} \quad V_0=\mathbb{R}^{n_0}, \qquad V_1=\mathbb{R}^{n_1}.$$

We consider as in 1.22, $C(V_0)$, $C(V_1)$, $\mathfrak{g}_0$, $\mathfrak{g}_1$, G_0, G_1. The groups G_i are included in $C^+(V_i)$.

Let P_0 be the subbundle of $P\,|\,M_0$ consisting of frames $u:\mathbb{R}^n\to T_{x_0}M$ which are compatible with the orthogonal decompositions

$$V=V_0\oplus V_1, \qquad T_{x_0}(M)=T_{x_0}(M_0)\oplus N_{x_0}$$

and the given orientations. We write $u=(u_0, u_1)$. Let $P'_0\subset P'\,|\,M_0$ be the pull back of P_0. Let $u\in P'_0$. There exists an element $\gamma_1=\gamma_1(\mathrm{u})\in \mathrm{G}_1$ such that $\gamma u=u\gamma_1$. Remark that γ_1 depends only on the image of u in P_0.

The invariance of the Riemannian connection of M by the isometry γ implies that for $v\in V_0$ and $u\in P_0$ the element $\omega_u(\tilde{v})$ commutes with γ_1, thus decomposes

$$\omega_u(\tilde{v})=\omega_u^0(\tilde{v})+\omega_u^1(\tilde{v}),$$

with $\omega_u^i(\tilde{v})\in\mathfrak{g}_i$ for $i=(0,1)$. The forms ω^0, ω^1 define canonical connections on the tangent bundle TM_0 and the normal bundle $\mathcal{N}M_0$. (The connection on TM_0 coincides of course with the Riemannian connection of the manifold M_0.)

Similarly, for v, $v'\in V_0$, the curvature $\Omega_u(\tilde{v}, \tilde{v}')$ decomposes as

$$\Omega_u(\tilde{v}, \tilde{v}')=\Omega_u^0(\tilde{v}, \tilde{v}')\oplus\Omega_u^1(\tilde{v}, \tilde{v}'),$$

where $\Omega_u^i(\tilde{v}, \tilde{v}')\in\mathfrak{g}_i$ *for* $i=0,1$, and the form $\Omega^i\in\mathcal{A}^2(P_0)\otimes\mathfrak{g}_i$ is the curvature of the connection above. Furthermore

3.29. $\Omega_u^1(\tilde{v}, \tilde{v}')$ commutes with $\mathrm{Ad}\,\gamma_1$.

Consider the G_0-invariant function on $\mathfrak{g}_0$ (1.7).

$$a\to j_{V_0}^{1/2}(a)=\left(\det_{V_0}\frac{\exp(a/2)-\exp(-a/2)}{a}\right)^{1/2}.$$

Extend it as in 1.15 to a function on $\Lambda^+T_uP\otimes\mathfrak{g}_0$ with values in Λ^+T_uP. Define the Chern-Weil form:

$$\hat{A}(M_0)_u=j_{V_0}^{-1/2}\left(-\frac{\Omega^0}{2i\pi}\right).$$

Then $\hat{A}(M_0)$ is basic i.e. the pull back of a form on M_0 which we denote by $\hat{A}(M_\gamma)$.

3.30. Let $g_1 \in G_1$, the function on $\mathfrak{g}_1$, $a \to (-1)^{l_1} \det_{V_1}(1-g_1 e^a)$ has a square root on $\mathfrak{g}_1$ denoted by $D^{1/2}(g_1, a)$ such that:

$$D^{1/2}(g_1, 0)=(-1)^{l_1} \operatorname{St}^1(g_1).$$

We have

$$D^{1/2}(g_1, a)=|\det_{V_1}(1-e^a g_1)|^{1/2}(\operatorname{St}^1(g_1))^{-1}|\det_{V_1}(1-g_1)|^{1/2}.$$

It is easy to see that the form $D^{1/2}(\gamma_1(u), -\Omega_u^1/2i\pi)$ on P_0 is basic and defines an invertible form on M_0. We denote its inverse by $D^{-1/2}(\gamma, \mathcal{N} M_0)$.

3.31. We denote by $\operatorname{ch}(\gamma, \mathcal{E})$ the form on M_0 given at the point x by

$$\operatorname{tr}_{E_x}\left(l(\gamma)_x \exp -\frac{\Omega_x^{\mathcal{E}}}{2i\pi}\right).$$

3.32 THEOREM (DONNELLY-PATODI [10], GILKEY [15], BISMUT [9]):

$\lim_{t \to 0} p_*(\Psi(x) A(t, \gamma, x) dx)=(\operatorname{ch}(\gamma, \mathcal{E}) D^{-1/2}(\gamma, \mathcal{N} M_\gamma) \hat{A}(M_\gamma))^{[\max]}$.

The proof will follow from several observations.

Let $x_0 \in M_0$. Let us fix $u \in P_0'$ above x_0. We write the image of u in P_0 as (u_0, u_1). The fiber $p^{-1}(x_0)$ of the fibration $p : U \to M_0$ is parametrized by a neighborhood of 0 in V_1, via the map

$$v \to x(v)=\exp_{x_0}(u_1(v)) \; (v \in V_1).$$

Denote by dv the volume form on V_1, and by dx_0 the Riemannian volume form on M_0. At the point $x(v)$ we write

$$dx=j(x_0, v)\, dx_0 \wedge dv.$$

The function $j(x_0, v)$ is C^∞ on a neighborhood of 0 in V_1 and

$$j(x_0, 0)=1.$$

Let us write

$$\varphi(v)=j(x_0, v)\Psi(x(v)).$$

Then φ is C^∞ and compactly supported and

$$\varphi(0)=1.$$

At the point x_0, we have:

$$p_*(\Psi(x)A(t, \gamma, x)\,dx)=A(t, \gamma, x_0)\,dx_0,$$

with

$$\bar{A}(t, \gamma, x_0)=\int_{V_1} A(t, \gamma, x(v))\,\Phi(v)\,dv.$$

The point $\exp_u \tilde{v}\in P'$ projects on $x(v)$, for $v\in V_1$. Thus we have:

$$A(t, \gamma, x(v))=\int_G \mathrm{st}\,(k_{\mathscr{E}}(t, \exp_u \tilde{v}, \gamma\,(\exp_u \tilde{v})g^{-1})(\rho(g)\otimes l(\gamma)_{x(v)})\,dg.$$

We have

$$\gamma \exp_u \tilde{v}=\exp_{\gamma u}\tilde{v}=\exp_{u\gamma_1}(\tilde{v})=\exp_u(\gamma_1 v)^{\sim}\gamma_1.$$

Thus:

3.33 $A(t, \gamma, x(v))$

$$=\int_G \mathrm{st}(k_{\mathscr{E}}(t, \exp_u \tilde{v}, \exp_u(\gamma_1 v)^{\sim}\gamma_1 g^{-1})(\rho(g)\otimes l(\gamma)_{x(v)})\,dg$$

$$=\int_G \mathrm{st}(k_{\mathscr{E}}(t, \exp_u \tilde{v}, \exp_u(\gamma_1 v)^{\sim}g^{-1})(\rho(g\gamma_1)\otimes l(\gamma)_{x(v)})\,dg.$$

Let us trivialize the bundle $\mathscr{E}\,|\,p^{-1}(x_0)$ by parallel transport along the geodesics $s\to x(sv)$. Thus all the fibers of $\mathscr{E}\,|\,p^{-1}(x_0)$ are identified with $E_0=\mathscr{E}_{x_0}$.

3.34 The linear map:

$$l(\gamma)_{x(v)}:\quad \mathscr{E}_{x(v)}\to\mathscr{E}_{x(\gamma_1 v)},$$

is identified with an endomorphism of E_0 which we denote by $l(\gamma, v)$. We write

$$l_0=l(\gamma, 0)=l(\gamma)_{x_0}.$$

Similarly the linear map

$$k_{\mathscr{E}}(t, \exp_u \tilde{v}, \exp_u(\gamma_1 v)^{\sim} g^{-1}): \quad S \otimes \mathscr{E}_{x(\gamma_1 v)} \to S \otimes \mathscr{E}_{x(v)},$$

is considered as an endomorphism of $S \otimes E_0$.

The function

$$t \to k_{\mathscr{E}}(t, \exp_u \tilde{v}, \exp_u(\gamma_1 v)^{\sim} g^{-1}) \varphi(v),$$

is rapidly decreasing as $t \to 0$ if g is far from the origin in G. Thus we can use exponential coordinates on G. *Let* $\Psi(a)$ be a cut-off function on $\mathfrak{g}$, equal to 1 on a small neighborhood of 0. Then $d(\exp a) = j_{\mathfrak{g}}(a)\, da$ (1.7) and

3.35 $I(\gamma, x_0) = \lim_{t \to 0} \bar{A}(t, \gamma, x_0)$

$$\lim_{t \to 0} \int_{\mathfrak{g} \times V_1} \operatorname{st}(k_{\mathscr{E}}(t, \exp_u \tilde{v}, \exp_u(\gamma_1 v)^{\sim} \exp(-a))$$
$$\times (\rho((\exp a)\gamma_1) \otimes l(\gamma, v))) \varphi(v) \Psi(a) j_{\mathfrak{g}}(a)\, da\, dv.$$

We will replace the kernel $k_{\mathscr{E}}$ by its expansion (3.4), with $N > \dim P/2$. This will not change the limit in 3.35.

Recall the notations 3.3. In the present situation, the sections $u \to U_i(u_0, u)$ of the bundle

$$\operatorname{End} S \otimes E_0 \otimes \pi^* \mathscr{E},$$

are determined by the system of differential equations

3.36 $$\nabla_{\mathscr{R}}(U_i r^i j_P^{1/2}) = r^i j_P^{1/2} (\Delta_{\mathscr{E}} - V_{\mathscr{E}}) U_{i-1},$$
$$U_0(u_0, u_0) = 1,$$

where $\Delta_{\mathscr{E}}$ denotes the Laplace operator on the bundle $\pi^* \mathscr{E}$. (We use the fact that $V_{\mathscr{E}}(x) = V_{\mathscr{E}}(x)^*$.)

We identify End S with the Clifford algebra $C(V)$. Then

$$V_{\mathscr{E}}(u) \in C^2(V) \otimes \operatorname{End} \mathscr{E}_x.$$

The differential equation implies immediately:

3.37. LEMMA:

$$U_i(u_0, u) \in C^{2i}(V) \otimes \operatorname{Hom}(\mathscr{E}_x, \mathscr{E}_{x_0}) \qquad \textit{for} \quad i = 0, 1, \ldots$$

Put

$$\varphi_i(a, v) = U_i(\exp_u \tilde{v}, \exp_u(\gamma_1 v)\tilde{\ } \exp(-a))$$

and

$$h(a, v) = q(\exp_u \tilde{v}, \exp_u(\gamma_1 v)\tilde{\ } \exp(-a)).$$

Let $Q_1(a)$ be the Hessian of the function $v \to h(a, v)$ at the crititcal point $0 \in V_1$. Since $\det_{V_1}(1-\gamma_1) \neq 0$ we see from 2.23 that $Q_1(a)$, considered as a symmetric endomorphism of V_1, is positive and invertible for small a. Thus 0 is a non degenerate critical point. By Morse lemma, there exists a local diffeomorphism of V_1:

$$v \to v' = F_a(v),$$

such that

$$F_a(0) = 0, \qquad dF_a(0) = Q_1(a)^{1/2},$$

and

$$h(a, v) = \|a\|^2 + \|v'\|^2.$$

Let us write:

$$dv = m(a, v')\,dv'; \qquad \text{then} \quad m(a, 0) = \det Q_1(a)^{-1/2},$$

$$\Phi_i'(a, v') = \Phi_i(a, v); \qquad \text{then} \quad \Phi_i'(a, 0) = \Phi_i(a, 0),$$

$$l'(\gamma, a, v') = l(\gamma, v); \qquad \text{then} \quad l'(\gamma, a, 0) = l_0,$$

$$\alpha(a, v') = \varphi(v)\Psi(a); \qquad \text{then} \quad \alpha(a, v') \text{ is compactly supported}$$

and $\alpha(a, 0)$ is identically equal to 1 in a neighborhood of 0 in $\mathfrak{g}$.

In 3.35 we replace k_ε by its expansion (3.4) with $N > 1/2 \dim P$ and we perform the change of variables $v \to v'$. We obtain

3.38 $$I(\gamma, x_0) = \lim_{t \to 0} (4\pi t)^{-l-\dim \mathfrak{g}/2}$$

$$\times \int_{\mathfrak{g} \times V_1} e^{-(\|a\|^2 + \|v\|^2)/4t}\, \mathrm{st}\big(\textstyle\sum_{i=0}^N t^i \Phi_i'(a, v)$$

$$\times (\rho(\exp a\gamma_1) \otimes l'(\gamma, a, v))\big) j_{\mathfrak{g}}(a)\, \alpha(a, v)\, m(a, v)\, da\, dv$$

$$= \lim_{t \to 0} (4\pi t)^{-l-\dim \mathfrak{g}/2} \int_{\mathfrak{g} \times V_1} e^{-(\|a\|^2 + \|v\|^2)/4t}$$

$$\textstyle\sum_{i=0}^N t^i\, \mathrm{st}((\rho(\exp a\gamma_1 \otimes 1) G_i(a, v))\, da\, dv,$$

with

$$G_i(a, v) = j_{\mathfrak{g}}(a)\,\alpha(a, v)\,m(a, v)\,(1 \otimes l'(\gamma, a, v))\,\Phi_i'(a, v).$$

Since $\Phi_i'(a, v) \in C^{2i}(V) \otimes \operatorname{End} E_0$, the limit in 3.38 exists by proposition 1.23 and is given by:

3.39 $$I(\gamma, x_0) = (i\pi)^{-l}\,\delta^{\mathfrak{g}_0}_{-1}(g),$$

where g is the function on $\mathfrak{g}_0$ defined by:

3.40 $$g(a) = \operatorname{st}^1(\gamma_1)\,\operatorname{tr}_{E_0}\left(\sigma\left(\sum_{i=0}^{N} \operatorname{gr}^{[2i]} G_i(a, 0)\right)\right]\left(\frac{a}{2}\right)\right),$$

where σ is a choice of a representatives in $S(\mathfrak{g})$ of elements of $\Lambda^+ V$ (1.19). We have:

3.41 $$G_i(a, 0) = j_{\mathfrak{g}}(a)\,(\det Q_1(a))^{-1/2}\,(1 \otimes l_0)\,\Phi_i(a, 0).$$

First, we compute $Q_1(a)$ for $a \in \mathfrak{g}_o$.
If

$$a = \sum_{1 \leqslant i < j \leqslant n_0} a_{ij}\, e_i \wedge e_j \in \mathfrak{g}_0 \qquad \text{and} \qquad u \in P_0,$$

then $(\Omega_u, a) = \sum_{1 \leqslant i < j \leqslant n_0} a_{ij}\,\Omega_u(\tilde{e}_i, \tilde{e}_j)$ decomposes by 3.29 as

$$(\Omega_u, a) = (\Omega_u^0, a) \oplus (\Omega_u^1, a),$$

where $(\Omega_u^i, a) \in \mathfrak{g}_i$ for $i = 0, 1$.

Since u is fixed, we omit the subscript u.

3.42. Lemma. — *For $a \in \mathfrak{g}_0$,*

$$Q_1(a) = (1 - \gamma_1^{-1} \exp((\Omega_1, a)/2)\left(\frac{(\Omega^1, a)/2}{\exp((\Omega^1, a)/2) - 1}\right)(1 - \gamma_1).$$

Proof. — We use 2.28. Note that a is replaced by $-a$. Since $(\Omega^1, a) \in \mathfrak{g}_1$ commutes with γ_1, it is enough by 2.28 to show that, for any $b \in \mathfrak{g}_1$ commuting with γ_1, the endomorphism

$$(I - \gamma_1^{-1} e^b)\left(\frac{b}{e^b - 1}\right)(1 - \gamma_1)$$

of V_1 is symmetric.

But this can be written as

$$\frac{b}{e^b-1}+\frac{b}{1-e^{-b}}-\gamma_1\frac{b}{e^b-1}-\frac{b}{1-e^{-b}}\gamma_1^{-1}.$$

It follows from 3.42 and 1.7 that for $a\in\mathfrak{g}_0$.

3.43 $$\det Q_1(a)=\det_{V_1}(1-\exp(-\Omega^1, a)/2)\gamma_1) \times\det_{V_1}(1-\gamma_1)\, j_{V_1}^{-1}((\Omega^1, a)/2).$$

We compute now $\mathrm{gr}^{[2i]}\Phi_i(a, 0)\in\Lambda^{2i}(V)\otimes\mathrm{End}\, E_0$.

Put:

$$R^{\mathscr{E}}(u)=\textstyle\sum_{1\leq i<j\leq n} e_i\wedge e_j\otimes\Omega_x^{\mathscr{E}}(ue_i, ue_j)\in\Lambda^2 V\otimes\mathrm{End}\, E_0.$$

Multiplication by $R^{\mathscr{E}}(u)$ defines an operator on the algebra $\Lambda^+ V\otimes\mathrm{End}\, E_0$. This operator is also denoted by $R^{\mathscr{E}}(u)$.

Let $a\to\tau(a)$ be the representation of $\mathfrak{g}$ on ΛV which extends the representation on V. We also denote by $\tau(a)$ the operator $\tau(a)\otimes 1$ on $\Lambda^+ V\otimes\mathrm{End}\, E_0$.

Put,

$$j_{P'}(a)=\det\theta(u, u\exp(-a))$$

Finally denote by 1 the unit element of $\Lambda^+ V\otimes\mathrm{End}\, E_0$. We have:

3.44. Proposition:

$$\textstyle\sum_{i\geq 0}\mathrm{gr}^{[2i]}\Phi_i(a, 0)=j_{P'}(a)^{-1/2}e^{\tau(a)}e^{-(\tau(a)+R^{\mathscr{E}}(u))}.\, 1.$$

Proof. — We consider the system of differential equations 3.36.

Put,

$$f_i(s)=j_{P'}(sa)^{1/2}\,\mathrm{gr}^{[2i]}\Phi_i(sa, 0),$$
$$F(s)=\Sigma s^i f_i(s)$$
$$R^{\mathscr{E}}(s)=R^{\mathscr{E}}(u\exp(-sa)).$$

Remark that $R^{\mathscr{E}}(s)=e^{s\tau(a)}R^{\mathscr{E}}(u)$.

The operator $\nabla_{\mathscr{R}}$ restricts to $u\exp(\mathfrak{g})=\exp_u(\mathfrak{g})$ as the Euler vector field. Thus, taking the gradation in account, 3.36 implies

$$\frac{d}{ds}F(s)=-R^{\mathscr{E}}(s)F(s)=-(e^{s\tau(a)}R^{\mathscr{E}}(u))F(s).$$

It follows that $B(s)=e^{-s\tau(a)}F(s)$ is a solution of

$$\frac{d}{ds}B(s)=-(\tau(a)+R^{\mathscr{E}}(u))\,B(s), \qquad B(0)=1,$$

and this proves the proposition.

Let us denote by $\mathscr{K}_0$ the kernel of the map

$$A_0\otimes 1:\quad \hat{S}(\mathfrak{g}_0)\otimes \operatorname{End} E_0 \to \Lambda^+(V_0)\otimes \operatorname{End} E_0$$

Using 3.44 we compute mod $\mathscr{K}_0$ the function

3.45 $$a\to\sigma(\Sigma\ \mathrm{gr}^{[2i]}\Phi_i(a,0))\left(\frac{a}{2}\right).$$

Denote by $\operatorname{ad} a$ the derivation of $\hat{S}(\mathfrak{g})$ associated to $a\in\mathfrak{g}$.

Then

$$\tau(a)\circ A=A\circ \operatorname{ad} a$$

and

$$(\operatorname{ad} a.\varphi)(a)=0 \qquad \text{for} \quad \varphi\in\hat{S}(\mathfrak{g}).$$

Therefore 3.45 is equal, mod $\mathscr{K}_0$, to

3.46 $$a\to j_{P'}(a)^{-1/2}\exp\left(-\frac{\langle \mathrm{R}^{\mathscr{E}}(u),a\rangle}{2}\right).$$

By 2.20, 2.21, we have

$$j_{P'}(a)=j_{\mathfrak{g}}(a)\,j_V\left(\frac{\langle\Omega,a\rangle}{2}\right).$$

For $a\in\mathfrak{g}_0$, we define

3.47 $$j_{P'}(a)=j_{\mathfrak{g}}(a)\,j_{V_0}\left(\frac{\langle\Omega^0,a\rangle}{2}\right)j_{V_1}\left(\frac{\langle\Omega^1,a\rangle}{2}\right).$$

By 3.43, 3.46 and 3.47 we have now identified all the ingredients necessary to the computation of the left hand side in 3.39. Remark that

$j_{g_0} = 1 \bmod \mathcal{K}_0$. Recalling the definition of $D^{1/2}(\gamma_1, a)$, (3.32), we obtain that the function g in 3.39 is equal mod $\mathcal{K}_0$ to

$$a \to D^{-1/2}\left(\gamma_1, -\frac{\langle \Omega^1, a \rangle}{2}\right) j_{V_0}^{-1/2}\left(\frac{\langle \Omega^0, a \rangle}{2}\right) \operatorname{tr}_{E_0}\left(l_0 \exp - \frac{\langle R^{\mathscr{E}}(u), a \rangle}{2}\right).$$

By 1.16, this implies the theorem.

BIBLIOGRAPHY

[1] ALVAREZ-GAUMÉ (L.), Supersymmetry and the Atiyah-Singer Index theorem, *Commun. Math. Phys.* Vol. 90, No. 101, 1983, pp. 161-173.

[2] ATIYAH (M. F.), Circular symmetry and stationary phase approximation. In *Proceedings of the Conference in honor of L. Schwartz, Asterisque*, Paris, Vol. 131, 1985, pp. 43-60.

[3] ATIYAH (M. F.) and BOTT (R.), A Lefschetz fixed point formula for elliptic complexes, *I. Ann. Math.*, Vol. 86, 1967, pp. 374-407; II, 88, 1968, pp. 451-481.

[4] ATIYAH (M. F.) and SEGAL (G. B.), The index of elliptic operators II, *Ann. of Math.*, Vol. 87, 1968, pp. 531-545.

[5] ATIYAH (M. F.) and SINGER (I. M.), The index of elliptic operators, I *Ann. of Math.*, Vol. 87, 1968, pp. 484-530; III *Ann. of Math.*, Vol. 87, 1968, pp. 546-604.

[6] BERGER (M.), GAUDUCHON (P.) and MAZET (E.), Le spectre d'une variété riemannienne, *Lecture notes in Mathematics*, 194, Springer-Verlag, Berlin-Heidelberg-New York.

[7] BERLINE (N.) and VERGNE (M.), The equivariant index and Kirillov's character formula, Amer. Math. Soc, *Amer. J. of Math.*, Vol. 107, n° 5, 1985, pp. 1159-1190.

[8] BERLINE (N.) and VERGNE (M.), Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante, *Comptes rendus Acad. Sc.*, Vol. 295, 1982, pp. 539-541.

[9] BISMUT (J. M.), The Atiyah-Singer theorems. A Probabilistic approach. I The index theorem, *J. Funct. Anal.*, Vol. 57, 1984, pp. 56-99. II The Lefschetz fixed point formulas, *J. Funct. Anal.*, Vol. 57, 1984, pp. 329-348.

[10] DONNELLY (H.) and PATODI (V. K.), Spectrum and the fixed point set of Isometries II, *Topology*, Vol. 16, 1977, pp. 1-11.

[11] FRIEDAN (D.) and WINDEY (H.), Supersymmetric Derivation of the Atiyah-Singer Index and the Chiral Anomaly, *Nuclear Physics B.*, Vol. B., No. 235 (FS 11), 1984, pp. 395-416.

[12] GETZLER (E.), Pseudo differential operators on super manifolds and the Atiyah-Singer index theorem, *Commun. Math. Phys.*, Vol. 92, 1983, pp. 163-178.

[13] GETZLER (E.), *A short proof of the Atiyah-Singer Index Theorem* (To appear).

[14] GILKEY (P.), Curvature and eigenvalues of the Laplacian for elliptic complexes. *Advances in Math.*, Vol. 10, 1973, pp. 344-382.

[15] GILKEY (P.), Lefschetz fixed point formulas and the heat equation, in *Partial Differential Equations and Geometry*, Proceedings, Park-City Conference, 1977, *Lecture Notes in Pure Appl. Math.*, No. 48, pp. 91-147, M. Dekker, New York, 1979.

[16] KIRILLOV (A. A.), Characters of unitary representations of Lie groups, *Funct. Anal. App.*, Vol. 2-2, 1967, pp. 40-55.

[17] LICHNEROWICZ (A.), Spineurs Harmoniques, *C.R. Acad. Sc.*, Vol. 257, 1963, pp. 7-9.

[18] MCKEAN (H. P), SINGER (I. M.), Curvature and the eigenvalues of the Laplacian, *J. Differential Geometry*, Vol. 1, 1967, pp. 43-69.

[19] MINAKSHISUNDARAM (S.) and PLEIJEL (A.), Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds, *Canad. J. Math.*, Vol. 1, 1949, pp. 242-256.

[20] PATODI (V. K.), Curvature and the Eigenforms of the Laplace Operators, *J. Diff. Geometry*, Vol. 5, 1971, pp. 233-249.

[21] PATODI (V. K.), Analytic proof of the Riemann-Roch theorem for Kaehler manifolds, *J. Diff. Geometry*, Vol. 5, 1971, pp. 251-283.

[22] VERGNE (M.), Formule de Kirillov et indice de l'opérateur de Dirac, *Proceedings of the International Congress of Mathematicians*, 1983, Warszawa.