BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 113 (1985), p. 231-240
<http://www.numdam.org/item?id=BSMF_1985__113__231_0>

Bull. Soc. math. France,
113, 1985, p.231-240

EXTREMAL PLURISUBHARMONIC FUNCTIONS AND INVARIANT PSEUDODISTANCES

BY

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RÉSUMÉ. — Nous introduisons et étudions une fonction extrémale plurisousharmonique qui généralise la fonction de Green d'une variable complexe. En utilisant cette fonction, nous définissons une distance invariante par rapport aux applications biholomorphes.

ABSTRACT. — An extremal plurisubharmonic function that generalizes Green's function of one complex variable is introduced and studied. It is used to define a biholomorphically invariant pseudodistance.

Introduction

In this article we define for any open connected set $\Omega \subset \mathbb{C}^n$ a plurisubharmonic function u_Ω that can be viewed as a counterpart of the generalized Green's function with pole at a given point. The organization of the paper is as follows. In the next section we introduce u_Ω and we prove that it decreases under holomorphic mappings. This generalizes the classical Lindelöf property of Green's functions of one complex variable (for other generalizations of the property, *see* [5]). Then we establish estimates for u_Ω in terms of the Caratheodory and Kobayashi pseudodistances on Ω and we prove that u_Ω satisfies the generalized Monge-Ampère equation. As regards the terminology related to the Monge-Ampère operator and used in this section the reader may consult [1] and [2]. In the last section we define an invariant pseudodistance on Ω using the function u_Ω and we study basic properties of the pseudodistance.

(*) Texte reçu le 27 septembre 1984, révisé le 31 janvier 1985.
(*) Research supported in part by the Irish Department of Education.

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1981, *Mathematics Subject Classification*, Primary 32 F 05, 32 H 15; Secondary 32 H 99.

It should be noted that plurisubharmonic counterparts of generalized Green's functions with pole at infinity have been known for more than two decades. They have been very useful in multidimensional complex analysis, especially in the theory of polynominal approximation (*see* [13], [11], [7] for references).

1. Plurisubharmonic counterparts of generalized Green's functions

Throughout this section we will assume that Ω is an open and connected subset of $\mathbb{C}^n$.

Fix $w \in \Omega$ and define:

$$u_\Omega(z, w) = \sup\{u(z)\},$$

where the supremum is taken over all non positive functions $u \in \mathrm{PSH}(\Omega)$ (including $-\infty$) such that the function $t \to u(t) - \log|t-w|$ is bounded from above in a neighbourhood of w. By ρ we will denote the Poincaré distance in the unit disc U. That is if a and b belong to U.

$$\rho(a, b) = \operatorname{arc\,tanh} |\varphi_b(a)|$$

where

$$\varphi_b(z) = \frac{b-z}{1-\bar{b}z} \qquad \text{for} \quad z \in U.$$

The Caratheodory pseudodistance for Ω is given by the formula

$$c_\Omega(z, w) = \sup\{\rho(f(z), f(w)) : f : \Omega \to U \text{ is holomorphic}\}.$$

Since the Poincaré metric is invariant with respect to automorphisms of U

$$c_\Omega(z, w) = \sup\{\rho(f(z), 0) : f : \Omega \to U \text{ is holomorphic, } f(w) = 0\}.$$

Now, assuming that $\inf \varnothing = +\infty$, put

$$\delta_\Omega(z, w) = \inf\{\rho(\xi, \eta)\}$$

where the infimum is being taken over all $\xi, \eta \in U$ for which there is a holomorphic mapping $f : U \to \Omega$ such that $f(\xi) = z$ and $f(\eta) = w$. The Kobayashi pseudodistance in Ω is defined as the largest pseudometric on Ω smaller than δ_Ω. In other words

$$k_\Omega(z, w) = \inf \textstyle\sum_{j=1}^m \delta_\Omega(a_j, a_{j+1}),$$

where the inf is taken over all chains of points $a_1, \ldots, a_{m+1}$ in Ω such that $a_1 = z$, $a_{m+1} = w$.

The function u_Ω has the following important property:

THEOREM 1.1. — *If $f : \Omega \to \Omega'$ is a holomorphic mapping between two open sets Ω and Ω' in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively, then*

$$u_{\Omega'}(f(z), f(w)) \leqslant u_\Omega(z, w),$$

for any $z \in \Omega - \{w\}$.

Proof. — If u is a function from the defining family for $u_{\Omega'}$ then $u \circ f \in \mathrm{PSH}(\Omega)$, $u \circ f$ is non positive and

$$u(f(z)) - \log|z-w| = u(f(z)) - \log|f(z) - f(w)| + \log\left|\frac{f(z)-f(w)}{z-w}\right|.$$

Thus the function $z \to u(f(z)) - \log|z-w|$ is upper bounded in a neighbourhood of w. Therefore $u \circ f \leqslant u_\Omega(., w)$. Since u was arbitrary, this proves the estimate in the theorem.

COROLLARY 1.2.

$$\log\tanh c_\Omega(z, w) \leqslant u_\Omega(z, w) \leqslant \log\tanh \delta_\Omega(z, w).$$

Proof. — It is easy to see that

$$\log\tanh c_\Omega(z, w) = \sup\{\log|f(z)|\},$$

where the supremum is taken over all holomorphic functions $f : \Omega \to U$ such that $f(w) = 0$. So the first estimate follows from the definition of u_Ω. The second estimate is a direct consequence of the above theorem and the fact thet u_U coincides with $\log\tanh\rho$.

COROLLARY 1.3. — *The function $u_\Omega(z, w)$ is plurisubharmonic in Ω with respect to z and $u_\Omega(z, w) - \log|z-w|$ is bounded from above in a neighbourhood of w.*

Proof. — Let us take $r > 0$ such that the ball

$$B = \{z \in \mathbf{C}^n : |z-w| < r\} \subset \Omega.$$

Then $u_\Omega(z, w) \leqslant u_B(z, w) \leqslant \log\tanh \delta_B(z, w)$ according to the above theorem.

But

$$\log\tanh\delta_B(z, w)=\log\tanh\rho\left(0, \frac{|z-w|}{r}\right)=\log|z-w|-\log r.$$

If v is the upper semicontinuous regularization of $u_\Omega(z, w)$ with respect to z, $v(z)\leqslant\log|z-w|-\log r$ near w and v is a nonpositive plurisubharmonic function on Ω. Thus $v\equiv u_\Omega(., w)$.

As a consequence we are able to prove the following.

COROLLARY 1.4. – *If* $\log\tanh\delta_\Omega(z, w)$ *is plurisubharmonic in* Ω *with respect to* z *when* w *is fixed then* $u_\Omega(z, w)=\log\tanh\delta_\Omega(z, w)$ *for all* $z\in\Omega$.

Proof. – As in the last proof we can prove that the function $z\to\log\tanh\delta_\Omega(z, w)$ belongs to the competing family in the definition of u_Ω. Combining this with the above theorem we get the corollary.

The simplest example of the situation when the right-hand side inequality in Corollary 1.2 is strict, is furnished by $\Omega=\mathbb{C}-\{0, 1\}$. In this case, $u_\Omega\equiv-\infty$ and k_Ω is non-trivial (*see* [8]). It is not difficult to prove that if Ω is an open annulus in the complex plane the left-hand side inequality in Corollary 1.2 is strict. In order to see this let us recall the formula for the Caratheodory distance in an annulus. Fix $q\in(0, 1)$ and put $A=\{z\in\mathbb{C}: q<|z|<1\}$.

For $z\neq0$ define

$$H(z)=\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}).$$

It is easy to see that H is holomorphic in $\mathbb{C}-\{0\}$ and $H\neq0$ except at the simple zeros $-q^{2n-1}$ for $n=0, \pm1, \pm2, \ldots$ It can be proved that H is equal to Jacobi's theta function up to a constant factor. Take $p\in(q, 1)$ and set

$$F_p(z)=H(p^{-1}qz)(H(q^{-1}pz))^{-1}.$$

Then F_p is a meromorphic function on $\mathbb{C}-\{0\}$ which is holomorphic in a neighbourhood of $\bar{A}$ and has a simple zero at the point $-p$. Moreover, F_p does not vanish at other points of $\bar{A}$. It follows from the definition of F_p that $|F_p(z)|=1$ when $|z|=1$ and $|F_p(z)|=p/q$ when $|z|=q$.

We have the following formula (*see* [14])

$$(*)\qquad c_A(a, b)=\operatorname{arc\,tanh}|f_b(a)|,$$

where

$$a,\ b\in A,$$
$$f_b(z)=z\,F_\alpha(e^{is}z)\,F_\beta(e^{it}z),$$
$$\alpha=q\,|b|^{-1},$$
$$\beta=|b|,$$
$$e^{it}=-|b|\,b^{-1},$$
$$e^{is}=|a|\,a^{-1}.$$

The formula (*) can be also easily derived from a result of R. M. ROBINSON [12].

Since $-u_A(.,\ w)$ is the Green's function for A with pole at w, we have

$$u_A(z,\ w)=-\gamma(w)\log|z|+\log|F(z)|,$$

where $\gamma(w)\log q=\log|w|-\log q$ and $F(z)=F_{|w|}(-w^{-1}|w|z)$. If $w_1=b$ and $w_2=-aq|ab|^{-1}$ *then* $\gamma(w_1)+\gamma(w_2)=-1$ and hence

$$\log|f_b(z)|=u_A(z,\ b)+u_A(z,\ w_2).$$

The above equality combined with (*) implies that for any two distinct points a and b of A

$$\log\tanh c_A(a,\ b)<u_A(a,\ b)$$

which proves our claim (*see* also [12]).

Using Corollary 1.3 we shall prove the following.

THEOREM 1.5. – *If $u=u_\Omega(.,\ w)\in L^\infty_{\mathrm{loc}}(\Omega-\{w\})$ then u satisfies the homogeneous Monge-Ampère equation $(dd^c u)^n=0$ in $\Omega-\{w\}$.*

Proof. – Take a ball B contained together with its closure in $\Omega-\{w\}$. Let (u_m) be a decreasing sequence of functions which are plurisubharmonic and smooth in a neighbourhood of $\bar B$ and such that $\lim_{m\to\infty}u_m=u$. In view of Theorem D in [1] for every m there is a function v_m that is continuous on $\bar B$, coincides with u_m on ∂B, is plurisubharmonic in B and satisfies the equation $(dd^c v_m)^n=0$ in B. Put $v=\lim_{m\to\infty}v_m$. By Theorem 2.1 in [2] $(dd^c v)^n=0$ in B. Sinve $v\geqslant u$ in B and $v=u$ on ∂B the function u_1 equal to u outside B and equal to v in B is plurisubharmonic in Ω. Moreover, Corollary 1.3 implies that u_1 belongs to the competing family in the definition of u. Thus $u=u_1$

in Ω. In particular, $(dd^c u)^n = 0$ in B because $u = v$ there. Since B was arbitrarily chosen, this completes the proof.

In one complex variable, if $\partial\Omega$ has positive logarithmic capacity, $-u_\Omega(z, w)$ coincides with the generalized Green's function of Ω with pole at w, and the Monge-Ampère equation reduces to the Laplace equation. In addition, if $\partial\Omega$ is regular with respect to the Dirichlet problem for the Laplace equation, Ω has the classical Green's function. Thus in this case, $u = u_\Omega(., w)$ has the following properties

$$\text{(I)}\qquad \begin{gathered} u \text{ is harmonic on } \Omega - \{w\}; \\ u(z) \to 0 \quad \text{if} \quad z \text{ approaches a boundary point of } \Omega; \\ u(z) - \log|z-w| = O(1) \text{ when } z \to w. \end{gathered}$$

Since the Monge-Ampère operator plays in several complex variables a similar role to that played by the Laplacian in the plane, it is natural to consider the following generalized Dirichlet problem (*see* also [9]).

$$\text{(II)}\qquad \begin{gathered} u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega - \{w\}); \\ (dd^c u)^n = 0 \quad \text{in } \Omega - \{w\}; \\ u(z) \to 0 \quad \text{if} \quad z \text{ approaches a boundary point of } \Omega; \\ u(z) - \log|z-w| = O(1) \quad \text{when } z \to w. \end{gathered}$$

It turns out that under certain assumptions about Ω the function $u(z) = u_\Omega(z, w)$ is a solution of (II). Namely we have:

PROPOSITION 1.6. — *Let Ω be a bounded open subset of $\mathbb{C}^n$. If Ω is either convex or pseudoconvex with $\mathscr{C}^1$-boundary then $u = u_\Omega(., w)$ solves the generalized Dirichlet problem* (II).

Proof. — Since Ω is bounded there is a positive constant M such that $\log|z-w| - M \leqslant u_\Omega(z, w)$. Thus $u \in L^\infty_{\mathrm{loc}}(\Omega - \{w\})$ and Theorem 1.5 implies that $(dd^c u)^n = 0$ in $\Omega - \{w\}$.

If Ω is convex and $p \in \partial\Omega$ then by the Hahn-Banach theorem there exists an $\mathbb{R}$-linear functional $h : \mathbb{C}^n \to \mathbb{R}$ such that $h < h(p)$ in Ω. Hence $f(z) = \exp(h(z) - ih(iz) - h(p))$ is an entire function such that $|f| < f(p) = 1$ on Ω. Therefore the function $v(z) = \log|(f(w)) - f(z))/(1 - \overline{f(w)}\, f(z))|$ satisfies the conditions in the definition of u so that $v \leqslant u$ in Ω. Consequently $\lim_{z \to p} u(z) = 0$.

If Ω is pseudoconvex with $\mathscr{C}^1$-boundary, a result of Kerzman and Rosay [6] implies that there is a $\mathscr{C}^\infty$-strictly plurisubharmonic negative function r on Ω such that $r(z)\to 0$ as $z\to\partial\Omega$. Let s be a real valued $\mathscr{C}^\infty$-function with compact support in Ω, such that $s=1$ in a neighbourhood of w. Define

$$v(z)=s(z)\log|z-w|+\mathrm{A}r(z) \qquad \text{where} \quad A>0.$$

If A is large enough, $v\in \mathrm{PSH}(\Omega)$ and hence $v\leqslant u$. This implies that $u(z)\to 0$ as $z\to\partial\Omega$.

Lempert has proved in [10] that if Ω is a bounded convex open set in $\mathbb{C}^n$ then $c_\Omega=\delta_\Omega=k_\Omega$. Combining this with Proposition 1.6, we obtain:

COROLLARY 1.7. — *If Ω is bounded and convex, $w\in\Omega$ and:*

$$u(z)=\log\tanh c_\Omega(z, w)=\log\tanh k_\Omega(z, w),$$

then u solves (II).

This can be viewed as a slight generalization of Theorem 4 in [9].

We would like to conclude this section with a few remarks about other properties of u_Ω.

Remark 1.8. — Suppose that the set $\mathbb{C}^n-\Omega$ is pluripolar (i.e. there exists a function $v\in\mathrm{PSH}(\mathbb{C}^n)$, $v\not\equiv-\infty$ and $\mathbb{C}^n-\Omega\subset\{v=-\infty\}$.) Then $u_\Omega\equiv-\infty$. Indeed, in this case $u_\Omega(.,w)$ can be extended to a plurisubharmonic function on $\mathbb{C}^n$ which is bounded from above by zero. So it must be equal to $-\infty$.

Remark 1.9. — If $\Omega\subset\Omega'$ and $\Omega'-\Omega$ is pluripolar then $u_\Omega=u_{\Omega'}$ on $\Omega\times\Omega$. It follows again from the fact that pluripolar sets are removable singularities for upperbounded plurisubharmonic functions.

Remark 1.10. — If $\Omega\subset\mathbb{C}$, then u_Ω is symmetric, i.e. $u_\Omega(z, w)=u_\Omega(w, z)$ for any z and w, $z\neq w$. Indeed, in the one-dimensional case, either-u_Ω is identically ∞ or it is equal to the generalized Green's function of Ω with pole at w and the conclusion follows from the symmetry of Green's functions (*see* e.g. [4]).

Remark 1.11. — If $\Omega\subset\mathbb{C}^n$ and for any two points z and w in Ω there is an automorphism f of Ω such that $f(z)=w$ and $f(w)=z$ then u is symmetric. It follows immediately from Theorem 1.1.

2. An invariant distance

The extremal function u_Ω can be used to define an invariant pseudo-distance.

Let Ω be an open connected subset of $\mathbb{C}^n$. Put

$$\sigma_\Omega^0(z, w) = \max\{\rho(\exp u_\Omega(z, w), 0), \rho(\exp u_\Omega(w, z), 0)\}.$$

Of course, if Ω is such that u_Ω is symmetric, $\sigma_\Omega^0 = \rho(\exp u_\Omega, 0)$ (Comp. Remarks 1.10, 1.11). In fact we could take any other symmetrization of $\rho(\exp u_\Omega(z, w), 0)$ as σ_Ω^0. Now define

$$\sigma_\Omega(z, w) = \inf \textstyle\sum_{j=1}^m \sigma_\Omega^0(a_{j-1}, a_j),$$

where the infimum is taken over all systems of points $a_0, \ldots, a_m$ in Ω such that $a_0 = z$ and $a_m = w$. Obviously, σ_Ω is a pseudodistance. Moreover, it is the largest pseudodistance on Ω smaller than σ_Ω^0.

As an immediate consequence of Theorem 1.1 and the fact that the function $\rho(t, 0)$ increases in $[0, 1)$ we obtain:

THEOREM 2.1. — *If $f : \Omega \to \Omega'$ is a holomorphic mapping:*

$$\sigma_{\Omega'}(f(z), f(w)) \leqslant \sigma_\Omega(z, w).$$

In particular, every automorphism of Ω is a σ_Ω-isometry.

It is well known that k_Ω is the largest pseudodistance on Ω for which every holomorphic mapping $f : U \to \Omega$ is a contraction. The Caratheodory pseudodistance c_Ω is the smallest pseudodistance on Ω for which every holomorphic mapping $f : \Omega \to U$ is a contraction. Combining these properties with Theorem 2.1 we get:

COROLLARY 2.2. — $c_\Omega \leqslant \sigma_\Omega \leqslant k_\Omega$.

This corollary yields the following:

COROLLARY 2.3. — *$\sigma_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is continuous.*

Proof. — By the triangle inequality

$$|\sigma_\Omega(z_0, w_0) - \sigma_\Omega(z, w)| \leqslant \sigma_\Omega(z_0, z) + \sigma_\Omega(w_0, w).$$

So the result follows from Corollary 2.2 and the continuity of k_Ω (*see* e. g. [3]).

COROLLARY 2.4. — *If Ω is bounded σ_Ω is a distance.*

Proof. — If $\sigma_\Omega(z, w)=0$, then $c_\Omega(z, w)=0$ and the corollary is a consequence of the corresponding property of c_Ω.

It is easy to find examples of open sets Ω such that $\sigma_\Omega \neq k_\Omega$. For instance, if $\Omega=\mathbb{C}\{0, 1\}$, $\sigma_\Omega \equiv 0$ and $k_\Omega \not\equiv 0$. In general, σ_Ω is different from the Bergmann metric b_Ω, because the latter does not have the property described in Theorem 2.1. The simplest example is furnished by

$$\Omega = U \times U \qquad \text{and} \qquad f(z_1, z_2)=(z_1, z_1).$$

Then $b_\Omega(0, (z_1, z_2))=[2(\text{arc tanh}^2|z_1|+\text{arc tanh}^2|z_2|)]^{1/2}$. Thus if $z \neq 0$, $b_\Omega(0, f(z, 0)) > b_\Omega(0, (z, 0))$. It is not clear, however, what is the relation of σ_Ω and c_Ω. We have already proved that if Ω is an annulus in the plane then:

$$c_\Omega < \sigma_\Omega^0.$$

It remains an open question whether this is true for σ_Ω instead of σ_Ω^0.

Finally, notice that if Ω is a bounded convex subset of $\mathbb{C}^n$ then in view of a result of Lempert [10] and Corollary 2.2, $c_\Omega=\sigma_\Omega=k_\Omega$.

Acknowledgement

The author is grateful to S. Dineen, R. M. Timoney and the referee for valuable comments.

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