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SOME INTRINSIC AND EXTRINSIC CHARACTERIZATIONS OF THE PROJECTIVE SPACE

BY

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RÉSUMÉ. — On démontre le théorème suivant : Le n-ième plongement de Veronese de $\mathbf{P}^r$ est la seule immersion $f: X \to \mathbf{P}^N$ où X est lisse, irréductible, de dimension r et $N=\binom{n+r}{r}-1$, telle que le n-ième espace osculateur en tout point x de X soit l'espace $\mathbf{P}^N$ tout entier.

Ensuite on conjecture — et démontre dans quelques cas particuliers — qu'une variété X lisse, irréductible, de dimension r, est isomorphe à $\mathbf{P}^r$ si le diviseur anti-canonique $-K$ est ample, et si, ou bien $-K$ est numériquement équivalent à $(r+1)H$ (H étant un diviseur positif quelconque), ou bien $\int c_1(O_X(-K))^r=(r+1)^r$.

ABSTRACT. — The following theorem is proved: The n-th Veronese embedding of $\mathbf{P}^r$ is the one and only immersion $f: X \to \mathbf{P}^N$ where X is a smooth, irreducible r-fold and $N=\binom{n+r}{r}-1$, such that the n-th osculating space at every point x of X is all of $\mathbf{P}^N$.

It is also conjectured — and verified in some cases — that a smooth, irreducible r-fold X is isomorphic to $\mathbf{P}^r$ if the anticanonical divisor $-K$ is ample and if either $-K$ is numerically equivalent to $(r+1)H$ for some divisor H, or if $\int c_1(O_X(-K))^r=(r+1)^r$.

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It is surprising in view of the work of the Italian school of algebraic geometry 70 years ago that the following theorem is new even in the special case characterizing the Veronese surface in the complex projective 5-space.

THEOREM. — *Over an algebraically closed field of any characteristic, the n-th Veronese embedding of* $\mathbb{P}^r$ *is the one and only one immersion* $f: X \to \mathbb{P}^N$ *where X is a smooth, irreducible r-fold and* $N=\binom{n+r}{r}-1$, *such that the n-th osculating space at every point x of X is all of* $\mathbb{P}^N$.

Proof. — Recall that the n-th osculating space at x is, by definition, the linear subspace of $\mathbb{P}^N$ determined by the first $N+1$ partial derivatives of f taken with respect to a system of local parameters for X at x and evaluated at x.

Suppose that $X=\mathbb{P}^r$. The n-th Veronese embedding is given in affine coordinates centered at x by

$$f(x_1, \ldots, x_r)=(x_1, \ldots, x_r, x_1^2, x_1 x_2, \ldots, x_r^2, x_1^3, \ldots, x_r^n).$$

Hence the first $N+1$ derivatives evaluated at x form the standard frame for $\mathbb{P}^N$. Thus the n-th osculating space is all of $\mathbb{P}^N$.

Conversely, for any smooth, irreducible r-fold X and for any immersion $f: X \to \mathbb{P}^N$ the n-th osculating space at x is, in other words, the space determined by the fiber at x of the following natural map of sheaves on X:

$$a: \quad O_X^{N+1} \to P_X^n(L),$$

where $L=f^* O(1)$ and where the target is the twisted sheaf of principal parts (*see* PIENE [15], § 2 and 6). Hence the osculating space is all of $\mathbb{P}^N$ if and only if a is injective at x. Assume $N=\binom{n+r}{r}-1$. Then a is injective at x if and only if a is an isomorphism at x.

Recall [EGA IV, 16.10.1] that there is a natural exact sequence:

$$(1) \qquad 0 \to (S^n \Omega_X^1) \otimes L \xrightarrow{b} P_X^n(L) \to P_X^{n-1}(L) \to 0.$$

It yields via a straightforward calculation the following relation among the first Chern classes:

$$(2) \qquad c_1(P_X^n(L))=\binom{n+r}{r+1} c_1(\Omega_X^1)+\binom{n+r}{r} c_1(L).$$

Assume now that a is an isomorphism at every point x. Form the composition $a^{-1}b$, dualize it and tensor with L. The result is a surjection,

$$L^{N+1} \twoheadrightarrow S^n T_X.$$

Since L is ample, therefore $S^n T_X$ is ample and hence T_X is ample, by Propositions 2.2 and 2.4 of HARTSHORNE [5]. Consequently, by Theorem 8 of MORI [13], $X=\mathbb{P}^r$.

Finally, since a is an isomorphism, (2) yields the relation:

$$(3) \qquad nc_1(T_X)=(r+1)c_1(L).$$

Since $X=\mathbb{P}^r$, therefore $L=O_X(n)$. Hence f is the n-th Veronese embedding, possibly followed by a projection and then an inclusion. However, each n-th osculating space is all of $\mathbb{P}^N$. So f is simply the Veronese embedding.

There is another proof that $X=\mathbb{P}^r$ when a is an isomorphism, which works in the following three cases:

(i) $r=1, 2$;

(ii) $r=3$ and the characteristic is 0;

(iii) n and $r+1$ are relatively prime and the characteristic is 0.

More generally, if $r=1, 2$, or if the characteristic is 0, or if n and $r+1$ are relatively prime, then this proof reduces the problem to establishing the following conjecture, which may be of interest in its own right (for other, related, conjectures, *see* FUJITA [2]).

CONJECTURE. — *Over an algebraically closed field of any characteristic, a smooth, irreducible r-fold X is isomorphic to* $\mathbb{P}^r$ *if an anticanonical divisor -K is ample and if either one of the following hypotheses is satisfied:*

(*a*) *there exists a divisor H such that* $(r+1)H$ *is numerically equivalent to -K;*

(*b*) $\int c_1(O_X(\text{-}K))^r=(r+1)^r$.

Second proof. — First, (3) says that $-K$ is ample. So, if $r=1$, then $X=\mathbb{P}^1$, and the proof is complete in this case. Secondly, suppose that n and $r+1$ are relatively prime, say $un+v(r+1)=1$. Say $L=O_X(D)$ and set $H=v(-K)+uD$. Then (3) implies that $(r+1)H$ is numerically equivalent to $-K$ (in fact, linearly equivalent, if the Chern classes are taken modulo linear equivalence). Thus (*a*) holds. Thirdly, the following argument shows that, if $r=2$ or if the characteristic is 0, then (b) holds; in fact, it shows that then all the Chern numbers of X are the same as those of $\mathbb{P}^r$. To complete the alternative proof, it will then suffice to establish the conjecture in the asserted cases, and this is done in the proposition below.

Set $c_i=c_i(T_X)$. Since a is an isomorphism,

$$\int c_1^{i_1}\ldots c_j^{i_j}.c_j(P_X^n(L))=0 \quad \text{whenever } i_1+2i_2+\ldots+ji_j+j=r.$$

Now, (1) and (3) imply that, for a certain nonzero integer q and a certain polynomial Q with rational coefficients,

$$c_j(P_X^n(L))=qc_j+Q(c_{j-1}, \ldots, c_1).$$

Combined, the two equations yield a relation among the Chern numbers

$$c(k_1, \ldots, k_r)=\int c_1^{k_1}\ldots c_r^{k_r} \qquad \text{where } k_1+2k_2+\ldots+rk_r=r.$$

Index the relation by the r-tuple $(i_1, \ldots, i_{j-1}, i_j+1, 0, \ldots, 0)$. Then, obviously, there is exactly one relation for each r-tuple $(m_1, \ldots, m_r)$ such that $m_1+2m_2+\ldots+rm_r=r$. Only one relation is trivial, the one with index $(r, 0, \ldots, 0)$. The remaining relations are linearly independent; indeed, if the r-tuples are put in reverse lexicographical order (that is, compare them starting with m_r), then the relation with index $(m_1, \ldots, m_r)$ involves a Chern number that does not appear in any of the successive relations, namely, $c(m_1, \ldots, m_r)$.

The Riemann-Roch theorem gives another relation, which expresses the Euler characteristic $\chi(O_X)$ in terms of the Chern numbers. It is independent of the other relations if $\chi(O_X)\neq 0$, because then it is the only relation with a nonzero constant term. Hence the relations determine the Chern numbers. So the numbers are those of $\mathbb{P}^r$, provided $\chi(O_X)=1$.

Suppose $r=2$. Then (1) yields the formula

$$c_2(P_X^n(L))=\binom{n+3}{4}(c_2+[(n^2-1)/3]c_1^2-2nc_1.c_1(L)+3c_1(L)^2).$$

Hence, in this case, the relations are as follows:

$$3c_2=c_1^2,$$
$$c_2+c_1^2=12\chi(O_X).$$

Since $-K$ is ample, $c_1^2 \geqslant 1$. Hence, $\chi(O_X) \geqslant 1$. On the other hand,

$$h^2(O_X) = h^0(K) \quad \text{and} \quad h^0(K) = 0,$$

because of duality and because $-K$ is ample. Since $h^0(O_X) = 1$, it follows that $\chi(O_X) = 1$.

For any r in characteristic 0, Kodaira's vanishing theorem yields $h^i(O_X) = 0$ for $i \geqslant 1$. Since $h^0(O_X) = 1$, therefore $\chi(O_X) = 1$.

It now remains to prove the following proposition.

PROPOSITION. — (1) (FUJITA [1]). *Under hypothesis (a), the conjecture holds in characteristic* 0.

(2) *Under hypothesis (b), the conjecture holds for $r = 1$, 2 in any characteristic, and for $r = 3$ in characteristic* 0.

Indeed, if $r = 1$, then $X = \mathbb{P}^1$, just because $-K$ is ample; in particular, (2) holds. If $r = 2$ or 3, then (2) holds by the classification of Del Pezzo surfaces (e.g., Theorem 24.4 (i) of MANIN [12]) and by the classification of Fano 3-folds (ISKOVSKIH [7], [8], MORI-MUKAI [14], L'VOVSKIĬ [11]). As to (1), $h^i(K + jH) = 0$ for $i, j \geqslant 1$ by Kodaira's vanishing theorem and for $i = 0$ and $j \leqslant r$ because $-K$ is ample and

$$(K + jH) . (-K)^{r-1} < 0.$$

Furthermore, $\chi(O_X) = 1$. Since $\chi(mH)$ is of degree r, therefore it must be $\binom{m+r}{r}$. Therefore $H^r = 1$ and $h^0(H) = r + 1$. By Theorem 1 of GOREN [3], $X = \mathbb{P}^r$. (While Goren does not assume X to be Cohen-Macaulay, he uses this hypothesis implicitly in the last line of the proof of Lemma 2. This interesting characterization of $\mathbb{P}^n$ has been rediscovered at least twice after Goren, by KOBAYASHI and OCHIAI [9] and by FUJITA [1]. Earlier HIRZEBRUCH and KODAIRA [6], Theorem 6, gave a weaker form of Goren's result, which is insufficient for our purposes.) FUJITA [1] proved (1), although he used linear equivalence in place of numerical equivalence. He determined the polynomial $\chi(mH)$ much as above, and he reproved Goren's theorem. KOLLAR [10] gave a similar proof, using the stronger form of Mori's theorem instead of Goren's theorem.

Remark. — The Hilbert polynomial of the n-th Veronese embedding of $\mathbb{P}^r$ does not always suffice to characterize the embedding. For example, the surface $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$ embedded by $O(2,2)$ has the same Hilbert polynomial (in fact the same Hilbert function) as the Veronese surface in

$\mathbb{P}^5$. Conceivably, the case $n=2$, $r=2$ is the only case in which the embedding is not characterized by its Hilbert polynomial.

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