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ON THE DIVISION OF FUNCTIONS OF CLASS C^r BY REAL ANALYTIC FUNCTIONS

BY

ZBIGNIEW SZAFRANIEC (*)

RÉSUMÉ. — Soit $(X, 0)$ un germe d'ensemble analytique cohérent. Supposons que les fonctions analytiques $g_1, \ldots, g_p$ engendrent un ideal $I(X)_0$. Il existe une fonction croissante $e: N \to N$ telle que, si une fonction f de classe $C^{e(r)}$ s'annule sur X, on a $f = \varphi_1 . g_1 + \ldots + \varphi_p . g_p$, ($\varphi_i$ étant des fonctions de classe C^r). Dans cet article nous démontrons une estimation de $e(r)$ dans des cas spéciaux.

ABSTRACT. — Let $(X, 0)$ be a germ of an analytic coherent set in R^n. Assume that analytic functions $g_1, \ldots, g_p$ generate ideal $I(X)_0$. There exists an increasing function $e: N \to N$ such that, for any function f of class $C^{e(r)}$ vanishing on X, there exist C^r-functions $\varphi_1, \ldots, \varphi_p$ such that $f = \varphi_1 . g_1 + \ldots + \varphi_p . g_p$. In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

Let $(X, 0)$ be a germ of an analytic coherent set in R^n. Assume that analytic functions $g_1, \ldots, g_p$ generate the ideal

$$I(X)_0 = \{ g \in \mathcal{O}_{n, 0} \mid g_{\mid X} \equiv 0 \}.$$

J. Cl. TOUGERON in [7] showed that there exists an increasing function $e: N \to N$ such that, for any $C^{e(r)}$-function f vanishing on X, there exist C^r-functions $\varphi_1, \ldots, \varphi_p$ such that

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p.$$

J. J. RISLER in [5] estimated precisely the function $e(r)$ in the case of plane curves.

In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

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1. Strongly irreducible polynomials

For any $x \in R^n \subset \mathbf{C}^n$, let us denote by $\mathcal{O}_{n,x}(\tilde{\mathcal{O}}_{n,x})$ the ring of germs of real analytic (holomorphic) functions at x. We denote by $\mathbf{m}_{n,x}(\tilde{\mathbf{m}}_{n,x})$ the maximal ideal of $\mathcal{O}_{n,x}(\tilde{\mathcal{O}}_{n,x})$.

DEFINITION 1. — Let:

$$P(X', X) = X^p + a_1(X')X^{p-1} + \ldots + a_p(X') \in \tilde{\mathcal{O}}_{n,0}[X]$$

be a distinguished polynomial. Let $\delta \in \tilde{\mathcal{O}}_{n,0}$ be the discriminant of the polynomial P. Assume that $\delta \not\equiv 0$. Denote by ω the initial form of δ at 0.

We say that P is *strongly irreducible* if there exist a constant $\varepsilon > 0$ and a set W such that the following conditions are satisfied:

(1.0) $W \subset \{(X', X) \in \mathbf{C}^{n+1} \mid 0 < \|X'\| < \varepsilon,$
$$P(X', X) = 0,\ \delta(X') \neq 0,\ \omega(X') \neq 0\},$$

(1.1) W is a nonempty, connected and open subset

of

$$\tilde{V}(P) = \{(X', X) \in \mathbf{C}^{n+1} \mid P(X', X) = 0\},$$

(1.2) If $w \in W$, $t \in \mathbf{C}$ and $0 < |t| \leqslant 1$ then

$$\pi^{-1}(t.\pi(w)) \cap \tilde{V}(P) \subset W,$$

where $\pi : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^n$ is the projection.

LEMMA. — 1 *Let* $P \in \tilde{\mathcal{O}}_{n,0}[X]$ *be a distinguished polynomial. Let* δ *be the discriminant of* P. *Assume that* $\delta \not\equiv 0$. *Denote by* ω *the initial form of* δ *at the origin.*

We require $H \subset \mathbf{C}^n$ *to be a complex hyperplane such that:*

(i) $\dim_{\mathbf{C}} H \geqslant 1$, $0 \in H$,

(ii) $P_{|H \times \mathbf{C}}$ *is irreductible in* $\tilde{\mathcal{O}}_{H \times \mathbf{C},0}$,

(iii) $\omega_{|H}$ *has no critical points, except possibly for the origin itself.*

Then the polynomial P is strongly irreducible.

The sketch of the proof.

Let $h = \dim_{\mathbf{C}} H$. We may assume that

$$H = \{X \in \mathbf{C}^n \mid X_{h+1} = \ldots = X_n = 0\}.$$

Denote by M the linear space of all complex $(n-h)\times h$-matrices. Let

$$\gamma=\{(L, v)\in \mathbb{C}P(h-1)\times \mathbb{C}^h \mid v\in L\},$$

be the canonical line bundle of $\mathbb{C}P(h-1)$.

We define a holomorphic map $\theta: M\times\gamma\to\mathbb{C}^n$ by $\theta(A, (L, v))=(v, A(v))$.

Of course : $\theta(0\times\gamma)=H$.

We use the notation:

$$G_1=\{(A,(L, v))\in M\times\gamma \mid A=0,\ v=0\},$$
$$G_2=\{(A, (L, v))\in M\times\gamma \mid A=0\},$$
$$S=\{(A, (L, v))\in M\times\gamma \mid v=0\}.$$

The homogeneous form $\omega_{|H}$ has an isolated singular point at the origin. Then there exist an open set $U_1\subset S$ and a closed complex manifold $N_1\subset U_1$ such that:

(1) $$G_1\subset U_1,$$

(2) N_1 is transverse to G_1 in S,

(3) $$\{(A, (L, v))\in M\times\gamma \mid \omega\circ\theta(A, (L, v))=0,\ (A, (L, 0))\in U_1\}$$
$$=\{(A, (L, v))\in M\times\gamma \mid (A, (L, 0))\in N_1\}\cup U_1.$$

The form $\omega_{|H}$ has an isolated singular point at the origin, so $\delta_{|H}$ has an isolated singular point at the origin.

It follows that there exists an open set $U_2\subset M\times\gamma$ and a closed complex manifold $N_2\subset U_2$ such that:

(4) $$G_1\subset U_2,$$

(5) $$U_2\cap S\subset U_1,$$

(6) N_2 is transverse to G_1 in U_2,

(7) $$\{(A, (L, v))\in M\times\gamma \mid \delta\circ\theta(A, (L, v))=0\}\cap U_2=N_2\cup(S\cap U_2),$$

(8) $$N_2\cap S=U_2\cap N_1.$$

Then there exist open sets $V_1\subset S$, $V_2\subset G_1$ and a constant $\varepsilon>0$ such that:

(9) $$N_1\cap G_1\subset V_2,$$

(10) $$G_1\subset V_1,$$

(11) $\{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \quad (0, (L, 0)) \notin V_2, 0 < \|v\| < \varepsilon\},$

is a deformation retract of

$\{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \quad (A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}.$

Denote

$$Z = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0,$$
$$(A, (L, 0)) \in V_1, (A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}.$$

By (7) the projection

$$\pi : \quad Z \to \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1,$$
$$(A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}$$

is a covering map.

Set

$$Z_1 = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0,$$
$$(A, (L, 0)) \in V_1, (0, (L, 0)) \notin V_2, 0 < \|v\| < \varepsilon\}.$$

By (11) Z_1 is a deformation retract of Z. Set

$$Z'_1 = \{(0, (L, v), X) \in (G_2 \setminus (N_2 \cup S)) \times \mathbb{C} \mid$$
$$P(\theta(0, (L, v)), X) = 0, 0 < \|v\| < \varepsilon\}.$$

The germ of $P_{|H \times \mathbb{C}}$ at 0 is irreducible, so, by ([4], Proposition 11, p. 55), we may assume that Z'_1 is connected. Then, if ε is sufficiently small, the sets Z and Z_1 are connected.

Denote $W = (\theta \times \mathrm{id}_{\mathbb{C}})(Z_1) \subset \mathbb{C}^n \times \mathbb{C}$. Then W is an open, connected subset of $\tilde{V}(P)$.

If V_1 is a sufficiently small neighbourhood of G_1 in S then, by (8) and (9), we have:

$$\pi(W) \subset \{(X', X) \in \mathbb{C}^n \times \mathbb{C} \mid 0 < \|X'\| < \varepsilon, P(X', X) = 0,$$
$$\delta(X') \neq 0, \omega(X') \neq 0\}.$$

By definition of W, if $w \in W$, $t \in \mathbb{C}$ and $0 < |t| \leqslant 1$ then

$$\pi^{-1}(t \cdot \pi(w)) \cap \tilde{V}(P) \subset W.$$

This completes the proof. ■

Example 1. — Let $P \in \tilde{\mathcal{O}}_{1,0}[X]$ be a distinguished irreducible polynomial. Then P is strongly irreducible.

Example 1. – Let $P \in \widetilde{\mathcal{O}}_{1,0}[X]$ be a distinguished irreducible polynomial. Then P is strongly irreducible.

Example 2. – Let $P(X', X) = X^2 + X_1^2 + X_2^2 + f(X_3, \ldots, X_n)$, where $f \in \tilde{\mathfrak{m}}_{n-2,0}$, $df(0) = 0$.

Set $H = \{ X' \in \mathbb{C}^n \mid X_3 = \ldots = X_n = 0 \}$. Then $P_{|H \times \mathbb{C}} = X^2 + X_1^2 + X_2^2$ is irreducible in $\widetilde{\mathcal{O}}_{H \times \mathbb{C}, 0}$ and $\omega_{|H} = -4(X_1^2 + X_2^2)$ has an isolated singular point at the origin.

Hence P is strongly irreducible.

COROLLARY 1. – *Let $P \in \widetilde{\mathcal{O}}_{n,0}[X]$ be a distinguished polynomial. Let δ be the discriminant of P. Assume that there exists a function $\Delta \in \widetilde{\mathcal{O}}_{n,0}$ such that $\Delta \not\equiv 0$ and $\widetilde{V}(\Delta) = \widetilde{V}(\delta)$. Denote by ω' the initial form of Δ at 0.*

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

(i) $\dim_{\mathbb{C}} H \geqslant 1$, $0 \in H$,

(ii) *the germ of $P_{|H \times \mathbb{C}}$ is irreducible in $\widetilde{\mathcal{O}}_{H \times \mathbb{C}, 0}$,*

(iii) *$\omega'_{|H}$ has an isolated singular point at 0.*

Then P is strongly irreducible.

Example 3. – Let $P(X', X) = X^3 + X_1^2 + X_2^2 + f(X_3, \ldots, X_n)$, where $f \in \tilde{\mathfrak{m}}_{n-2,0}$ and $df(0) = 0$.

Then $\delta(X') = -27(X_1^2 + X_2^2 + f(X_3, \ldots, X_n))^2$.

Set $\Delta(X') = X_1^2 + X_2^2 + f(X_3, \ldots, X_n)$. Of course $\widetilde{V}(\Delta) = \widetilde{V}(\delta)$.

Set

$$H = \{ X' \in \mathbb{C}^n \mid X_3 = \ldots X_n = 0 \}.$$

The germ of $P_{|H \times \mathbb{C}} = X^3 + X_1^2 + X_2^2$ at 0 is irreducible and $\omega'_{|H} = X_1^2 + X_2^2$ has an isolated singular point at the origin. Hence P is strongly irreducible.

LEMMA 2. – *Let $P(X', X) = X^p + a_1(X') X^{p-1} + \ldots + a_p(X') \in \widetilde{\mathcal{C}}_{n,0}[X]$ be a distinguished strongly irreducible polynomial.*

Denote by d the degree of the form ω.

Let $f_1, f_2 \in \widetilde{\mathcal{C}}_{n+1,0}$ be germs such that $f_1 \cdot f_2 \in \widetilde{\mathcal{C}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}^{e(r)}$, where $r \in \mathbb{N}$, $e(r) = 2p(r + [d/2] + 1)$, $[d/2]$ is the integer part of $d/2$.

Then $f_1 \in \widetilde{\mathcal{C}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}^r$ or $f_2 \in \widetilde{\mathcal{O}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}^r$.

This lemma is analoguous to Lemma 1.7 in [6].

Proof. – We define a map $h_1 : W \times \mathbb{C} \to \mathbb{C}^n$ by $h_1(w, t) = t^{p!} . \pi(w)$. By (1.2), if $0 < |t| \leqslant 1$ then $\pi^{-1}(h_1(w, t)) \cap \widetilde{V}(P) \subset W$. Denote $D = \{ t \in \mathbb{C} \mid |t| \leqslant 1 \}$. Since $\pi(W) \subset \mathbb{C}^n \setminus \widetilde{V}(\delta)$, so there exists a holomorphic function $h_2 : W \times (D \setminus \{0\}) \to \mathbb{C}$ such that:

$$P(h_1(w, t), h_2(w, t)) \equiv 0, \tag{1}$$

$$(h_1(w, 1), h_2(w, 1)) \equiv w. \tag{2}$$

The polynomial $P(X', X)$ is distinguished, so h_2 is bounded. Then there exists a holomorphic extension $h_2 : W \times D \to \mathbb{C}$. Hence $h = h_1 \times h_2 : W \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$ is holomorphic. Then, by Proposition 2.2 ([2], p. 55), there exists a constant $C_1 > 0$ such that:

$$|h_2(w, t)| \leqslant C_1 . \| h_1(w, t) \|^{1/p} \quad \text{for} \quad (w, t) \in W \times \mathbb{C}. \tag{3}$$

Then

$$|h_2(w, t)| \leqslant C_1 . |t|^{(p-1)!} \| \pi(w) \|^{1/p}. \tag{4}$$

By (1) and (4) there exist constants C_2, $C_3 > 0$ such that, for any $(w, t) \in W \times D$,

$$|(f_1 . f_2) \circ h(w, t)| \leqslant C_2 . \| h(w, t) \|^{e(r)} \leqslant C_3 . |t|^{2(r+[d/2]+1)p!}.$$

The set W is connected, so $W \times \{0\}$ is a connected complex submanifold of $W \times D$.

It follows that, for example,

$$f_1 \circ h(w, 0) \equiv \ldots \equiv \frac{\partial^{((r+[d/2]+1).p!-1)}}{\partial t^{((r+[d/2]+1).p!-1)}} (f_1 \circ h)(w, 0) \equiv 0.$$

Then there exists a continuous function $k : W \to R_+$ such that:

$$\begin{aligned} |f_1 \circ h(w, t)| &\leqslant k(w) . |t|^{(r+[d/2]+1).p!} \\ &= k(w) . \| \pi(w) \|^{-1} . \| t^{p!} . \pi(w) \|^{(r+[d/2]+1)} \\ &= k(w) . \| \pi(w) \|^{-1} . \| h_1(w, t) \|^{(r+[d/2]+1)}. \end{aligned} \tag{5}$$

From the preparation theorem we have:

$$f_1 = Q . P + \sum_{j=1}^{p} b_j(X') . X^{p-j}, \qquad \text{where} \quad b_j \in \tilde{\mathfrak{m}}_{n,0}.$$

Let $w_0 \in W$, $t \in D$. Denote by $\xi_1(t), \ldots, \xi_p(t)$ the roots of the polynomial $P(t^{p!} . \pi(w_0), X)$.

Then

$$f_1(t^{p!}.\pi(w_0),\ \xi_i(t))=\sum_{j=1}^{n} b_j(t^{p!}.\pi(w_0)).\xi_i^{p-j}(t).$$

By Cramer's rule

$$b_j(t^{p!}.\pi(w_0))=(\det[s_{kl}(t)])/(\prod_{1\leqslant n<m\leqslant p}(\xi_n(t)-\xi_m(t))),$$

where if $l\neq j$ then

$$s_{kl}(t)=\xi_k^{p-l}(t),\qquad s_{kj}(t)=f_1(t^{p!}.\pi(w_0),\ \xi_k(t)).$$

Of course

$$\left|\prod_{1\leqslant n<m\leqslant p}(\xi_n(t)-\xi_m(t))\right| = \left|\delta(t^{p!}.\pi(w_0))\right|^{1/2}.$$

By (1.0) $\pi(W)\subset\mathbb{C}^n\setminus\tilde{V}(\omega)$. By (5) there exist constants C_4, $C_5>0$ such that:

$$\left|\delta(t^{p!}.\pi(w_0))\right|^{1/2}>C_4.\left|t^{p!}\right|^{d/2},\ \left|\det[s_{kl}(t)]\right|<C_5.\left|t^{p!}\right|^{(r+[d/2]+1)}.$$

Then

$$\left|b_j(t^{p!}.\pi(w_0))\right|<(C_5/C_4).\left|t^{p!}\right|^r.$$

The set $\pi(W)$ is open in $\mathbb{C}^n$, so $b_j\in\tilde{\mathfrak{m}}^r{}_{n,\,0}$.

Then $f_1-Q.P\in\tilde{\mathfrak{m}}^r_{n+1,0}$. ■

COROLLARY 2. — *If $P\in\tilde{\mathcal{O}}_{n,\,0}[X]$ is strongly irreducible then P is irreducible in $\tilde{\mathcal{O}}_{n+1,0}$.*

2. Functions vanishing on an analytic set

DEFINITION 2. — Let $I\subset\mathcal{O}_{n,\,0}$ be an ideal. We denote by $\sqrt[R]{I}$ the ideal of germs vanishing on $V(I)_0$.

We say that I is *real* if $I=\sqrt[R]{I}$.

Let $\mathfrak{p}\subset\mathcal{O}_{n,\,0}$ be a prime ideal, $\{0\}\neq\mathfrak{p}\neq\mathcal{O}_{n,\,0}$. By [4] there exists, after a linear change of coordinates in $\mathbb{R}^n$, an interger k, $0<k\leqslant n$, such that $\mathcal{O}_{k,\,0}\to A=\mathcal{O}_{n,\,0}/\mathfrak{p}$ is an injection which makes A a finite $\mathcal{O}_{k,\,0}$-module.

Further, if K is the quotient field of $\mathcal{O}_{k,\,0}$, L that of A, we have $L=K(X_{k+1} \bmod \mathfrak{p})$, and for any $i\in[k+1,\ n]$, the minimal polynomial P_i

of X_i over K is in $\mathcal{O}_{k,0}[X]$ and is distinguished, so that there is a distinguished polynomial

(2.0) $P_i(X', X_i)=X_i^{p_i}+\sum_{j=1}^{p_i} a_{ij}(X')X_i^{p_i-j}, \qquad X'=(X_1, \ldots, X_k),$

with $P_i(X', X_i)\in\mathfrak{p}$.

Let $\delta(X')\in\mathcal{O}_{k,0}$ be the discriminant of the polynomial P_{k+1}. Then $\delta\notin\mathfrak{p}$.

Let $p=p_{k+1}$. There are polynomials Q_i of degree $<p$ in $\mathcal{O}_{k,0}[X]$ such that, for $i\in[k+2, n]$ we have $\delta.X_i-Q_i(X_{k+1})\in\mathfrak{p}$.

Let $\pi: R^n=R^k\times R^{n-k}\to R^k$ be the natural projection. There exists a fundamental system of neighbourhoods $\Omega=\Omega'\times\Omega''$ of 0 in $R^n=R^k\times R^{n-k}$ such that

(2.1) $$\pi_{|V(\mathfrak{p})\cap\Omega}\to\Omega' \text{ is proper.}$$

Lemma 3 (*see* [4]). — *There exists a constant $N\leqslant p^{n-k}$ such that for any point $x\in V(\mathfrak{p})\cap\Omega$ and any $f\in\mathcal{O}_{n,x}$:*

$$\delta^N.f\equiv g \qquad (\text{mod } P_{k+1}, \delta.X_{k+2}-Q_{k+2}, \ldots, \delta.X_n-Q_n),$$

where g is an element in $\mathcal{O}_{k,\pi(x)}[X_{k+1}]$.

Lemma 4 (*see* [7]). — *There exists a constant $\alpha\in N$, $\alpha\geqslant 1$, such that for any point $x'\in V(\delta)\cap\Omega'$ and any connected component U of $\Omega'\setminus V(\delta)$, if $x'\in\bar{U}$, then there exists a sequence (y^i) of points of U such that*

$$\lim y^i=x' \qquad \textit{and} \qquad \{y\in\Omega' \mid \|y-y^i\|<\|x'-y^i\|^\alpha\}\subset U.$$

Lemma 5 (*see* [7]). — *There exists a constant $\nu\in N$ such that for any $x\in V(\mathfrak{p})\cap\Omega$ and any germs $f_0, \ldots, f_{n-k}\in\mathcal{O}_{n,x}$, if*

$$h=f_0.\delta^N+f_1.P_{k+1}+\sum_{i=k+2}^n f_{i-k}.(\delta.X_i-Q_i)\in\mathfrak{m}_{n,x}^{r+\nu}, \qquad r\in N,$$

then there exist germs $g_0, \ldots, g_{n-k}\in\mathfrak{m}_{n,x}^r$ such that:

$$h=g_0.\delta^N+g_1.P_{k+1}+\sum_{i=k+2}^n g_{i-k}.(\delta.X_i-Q_i).$$

From now on we make the assumptions:

(2.2) $V(\mathfrak{p})$ is coherent in a neighbourhood of 0,

(2.3) the set $V(\mathfrak{p})\cap\Omega\setminus V(\delta)\times R^{n-k}$ is dense in $V(\mathfrak{p})\cap\Omega$, (2.4) If $(x', x_{k+1}) \in V(P_{k+1}) \cap (\Omega' \times R)$, then there exist polynomials

$$R_1, \ldots, R_{s(x', x_{k+1})}, \qquad Q \in \mathcal{O}_{k, x'}[X_{k+1} - x_{k+1}]$$

such that

$$P_{k+1} = R_1 \cdot \ldots \cdot R_{s(x', x_{k+1})} \cdot Q \quad \text{in } \mathcal{O}_{k, x'}[X_{k+1} - x_{k+1}],$$

(2.5) polynomials R_i are distinguished and strongly irreducible in $\tilde{\mathcal{O}}_{k, x'}[X_{k+1} - x_{k+1}]$,

(2.6) $$Q(x', x_{k+1}) \neq 0,$$

(2.7) for any $i \in [1, s(x', x_{k+1})]$

$$(x', x_{k+1}) \in \overline{V(R_i) \setminus (\bigcup_{j \neq i} V(R_j) \cup (V(\delta) \times R))}.$$

Example 4. — Assume that $f \in \mathfrak{m}_{n-3,0}$, $df(0) = 0$ and

(1) $$\{x \in R^{n-3} \mid f(x) \leqslant 0\} = \overline{\{x \in R^{n-3} \mid f(x) < 0\}}.$$

Define

$$P(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2 + X_n^2 \in \mathcal{O}_{n-1,0}[X_n].$$

The germ of P at 0 is irreducible, so $\mathfrak{p} = \mathcal{O}_{n,0} \cdot P$ is prime in $\mathcal{O}_{n,0}$.

If $x \in V(\mathfrak{p}) \setminus R^{n-3} \times \{0\}$, then $dP(x) \neq 0$, so the germ of P at x generates $I(V(\mathfrak{p}))_x$.

If $x = (x', x'') \in V(\mathfrak{p}) \cap R^{n-3} \times \{0\}$, where $x' \in R^{n-3}$ and $x'' \in R^3$ then $f(x') = 0$. Hence the germ of P at x is irreducible. By (1) the germ of $V(\mathfrak{p})$ at x contains regular points. From Lemma 2.5 ([3], p. 14), the germ of P at x generates $I(V(\mathfrak{p}))_x$. So $V(\mathfrak{p})$ is coherent.

Let $\delta = -4(f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2)$ be the discriminant of P.

By (1), $V(\mathfrak{p}) \setminus (V(\delta) \times R)$ is dense in $V(\mathfrak{p})$ in some neighbourhood of the origin.

If $x = (x', x'') \in V(\mathfrak{p}) \cap (R^{n-3} \times \{0\})$ then, by Example 2, the germ of P at x is strongly irreducible.

If $x \in V(\mathfrak{p}) \setminus (R^{n-3} \times \{0\})$, then $\delta(\pi(x)) \neq 0$ or $d\delta(\pi(x)) \neq 0$. Hence, by Definition 1 or lemma 1, the polynomial P is strongly irreducible.

So the conditions (2.2)-(2.7) are satisfied.

Let d = degree ω, where ω is the initial form of δ at 0. By induction we can define functions $e_i : N \to N$.

Set

$e_0(r)=p.(r+\nu+[d/2]+1)$,

$e_1(r)=p.\alpha.(e_0(r)-1)$,

$\vdots$

$e_i(r)=p.\alpha.(2p.(e_{i-1}(r)+[d/2]+1)-1)$,

Set $e'(r)=e_p(r)$.

THEOREM 1. — *Assume that* $\mathfrak{p}\subset\mathcal{O}_{n,0}$ *is a prime ideal satisfying the conditions* (2.2)-(2.7).

Let f be a function of class $C^{e'(r)}$ *vanishing on* $V(\mathfrak{p})$ *in a sufficiently small neighbourhood* Ω *of* 0.

Then, for any $x\in\Omega$, *the Taylor expansion* $T_x^{e'(r)} f\in I(V(\mathfrak{p}))_x+\mathfrak{m}^r_{n,x}$.

(*where* $T_x^{e'(r)} f=\sum_{|\beta|\leqslant e'(r)}(1/\beta!).(\partial^\beta f/\partial X^\beta)(x)(X-x)^\beta\in\mathcal{O}_{n,x}$).

Proof. — Let $x=(x', x_{k+1}, x'')\in V(\mathfrak{p})\cap\Omega$. Let $Y_{k+1}=X_{k+1}-x_{k+1}$. By (2.4) $P_{k+1}=R_1.\ldots.R_s.Q$ in $\mathcal{O}_{k,x'}[Y_{k+1}]$.

Denote by $\delta_i(\omega_i)$ the discriminant of the polynomial R_i (the initial form of δ_i at x').

From Lemma 3, there exists $g\in\mathcal{O}_{k,x'}[Y_{k+1}]$ such that

$$\delta^N.(T_x^{e'(r)} f)\equiv g \bmod I(V(\mathfrak{p}))_x.$$

The function f vanishes on $V(\mathfrak{p})_x$, so $T_x^{e'(r)} f$ and g are $e'(r)$-flat on $V(\mathfrak{p})_x$ (*see* [7]).

Every polynomial R_i has degree $\leqslant p$ and, by Corollary 2 and (2.5), is irreducible.

From Proposition 5.6 ([2], p. 50) there exist $a_{i1},\ldots,a_{ip}\in\mathcal{O}_{k,x'}$ such that

$$g^p+a_{i1}.g^{p-1}+\ldots+a_{ip}\in\mathcal{O}_{k+1,(x',x_{k+1})}.R_i.$$

Then a_{ip} is $e'(r)$-flat on $V(\mathfrak{p})\cap(V(R_i)\times R^{n-k-1})$.

By (2.0), (2.1) and Proposition 2.2 ([2], p. 55), a_{ip} is $(e'(r)/p)$-flat on $\pi(V(R_i))_{x'}$.

By (2.7) there exists a connected component U of

$$\{y'\in R^k \mid \|x'-y'\|<\varepsilon,\ \delta(y')\neq 0\}$$

such that $x'\in\bar{U}$ and $U\subset\pi(V(R_i))_{x'}$.

Then, from Lemma 5.11, [7] and Lemma 4

$$a_{ip} \in \mathfrak{m}_{k,x}^{2p(e_{p-1}(r)+[d/2]+1)}.$$

We have

$$g(g^{p-1}+a_{i1}.g^{p-2}+\ldots+a_{i,p-1}) \in \mathcal{O}_{k+1,x}.R_i+\mathfrak{m}_{k,x}^{2p(e_{p-1}(r)+[d/2]+1)}.$$

If x is sufficiently close to 0 then degree $\omega_i \leq d$.

Then, from Lemma 2,

$$g \in \mathcal{O}_{k+1,x}.R_i+\mathfrak{m}_{k+1,x}^{e_{p-1}(r)}$$

or

$$g^{p-1}+\ldots+a_{i,p-1} \in \mathcal{O}_{k+1,x}.R_i+\mathfrak{m}_{k+1,x}^{e_{p-1}(r)}.$$

In the second case, repeating this process $p-1$ times, we can prove that g is $e_0(r)$-flat on

$$\tilde{V}(R_i)=\{(y', y_{k+1}) \in \mathbb{C}^k \times \mathbb{C} \mid R_i(y', y_{k+1})=0\} \quad \text{at } (x', x_{k+1}).$$

Then g is $e_0(r)$-flat on $\tilde{V}(R_1.\ldots.R_s)=\tilde{V}(R_1) \cup \ldots \cup \tilde{V}(R_s)$ at (x', x_{k+1}).

From the preparation theorem we have

$$g=S.R_1.\ldots.R_s+\sum_{j=1}^{p} b_j.Y_{k+1}^{p-j}, \qquad \text{where } b_j \in \mathcal{O}_{k,x'}.$$

By Cramer's rule $b_j \in \mathfrak{m}_{k,x}^{r+\nu}$. The arguments are the same as in the proof of Lemma 2.

By (2.4)

$$\delta^N.(T_x^{e'(r)} f) \in (P_{k+1}, \delta.X_{k+2}-Q_{k+2}, \ldots, \delta.X_n-Q_n)\mathcal{O}_{n,x}+\mathfrak{m}_{n,x}^{r+\nu}.$$

From Lemma 5, there exists $h \in \mathfrak{m}_{n,x}^r$ such that

$$\delta^N.(T_x^{e'(r)} f-h) \in I(V(\mathfrak{p}))_x.$$

By (2.3), $T_x^{e'(r)} f-h \in I(V(\mathfrak{p}))_x$.

Then $T_x^{e'(r)} f \in I(V(\mathfrak{p}))_x+\mathfrak{m}_{n,x}^r$. ∎

THEOREM 2 (*see* [1]). — *Let* $g_1, \ldots, g_m \in \mathcal{O}_{n,0}$.

There exist a linear function $N \ni r \mapsto e''(r)=a''.r+b'' \in N$ *and an open neighbourhood* Ω *of* 0 *such that:*

if $f:\Omega\to R$ *is a function of class* $C^{e''(r)}$ *and, for any* $x\in\Omega$, $T_x^{e''(r)} f\in(g_1,\ldots,g_m).\mathcal{O}_{n,x}+\mathbf{m}_{n,x}^{e''(r)}$, *then there exist functions* $\varphi_1,\ldots,\varphi_m:\Omega\to R$ *of class* C^r *such that:*

$$f=\varphi_1\cdot g_1+\ldots+\varphi_m\cdot g_m.$$

THEOREM 3. – *Let* $(X,0)\subset(R^n,0)$ *be a germ of an analytic coherent set. Then* $I(X)_0=\mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_k$, *where* $\mathfrak{p}_1,\ldots,\mathfrak{p}_k$ *are prime ideals in* $\mathcal{O}_{n,0}$.

Suppose that every ideal $\mathfrak{p}_i$ *satisfies assumptions* (2.2)-(2.7). *Then there exists a linear function* $N\ni r\mapsto e(r)=a.r+b\in N$ *such that for any function* f *of class* $C^{e(r)}$ *vanishing on* X:

$$f=\varphi_1\cdot g_1+\ldots+\varphi_m\cdot g_m,$$

where $g_1,\ldots,g_m\in I(X)_0$ *and* $\varphi_1,\ldots,\varphi_m$ *are germs of function of class* C^r.

This theorem is a sharpened version of the result of J.-Cl. TOUGERON (*see* Theorem 5.12, [7]).

Proof. – We have $I(X)_0\subset\sqrt{I(X)_0}=\mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_k$, where

$$\mathfrak{p}_1,\ldots,\mathfrak{p}_k\subset\mathcal{O}_{n,0}$$

are prime ideals. The germ of X at 0 is coherent, so the ideal $I(X)_0$ is real. Then $\sqrt{I(X)_0}\subset\sqrt[R]{I(X)_0}=I(X)_0$. Hence $I(X)_0=\mathfrak{p}_1\cap\ldots\cap\mathfrak{p}_k$.

From Theorem 1 there exists a linear function $N\ni r\mapsto e'(r)=a'.r+b'\in N$ such that, for any function f of class $C^{e'(r)}$ vanishing on X and any $x\in X$ we have

$$T_x^{e'(r)} f\in\bigcap_{i=1}^k(I(V(\mathfrak{p}_i))_x+\mathbf{m}_{n,x}^r).$$

By (2.3) and ([7], Theorem 3.8), there exists a constant $\nu'\in N$ such that, for any $x\in X$ we have

$$\bigcap_{i=1}^k(I(V(\mathfrak{p}_i))_x+\mathbf{m}_{n,x}^{r+\nu'})\subset\bigcap_{i=1}^k I(V(\mathfrak{p}_i))_x+\mathbf{m}_{n,x}^r\subset I(X)_x+\mathbf{m}_{n,x}^r.$$

Let $g_1,\ldots,g_m$ be generators of $I(X)_0$. Let $e''(r)$ be a function as in Theorem 2.

Define $e(r)=e'(e''(r)+\nu')=a.r+b$.

Let f be a germ of class $C^{e(r)}$ vanishing on X.

Then, for any $x \in X$ in some neighbourhood of 0 we have

$$T_x^{e(r)} f \in \bigcap_{i=1}^k (I(V(\mathfrak{p}_i))_x + \mathbf{m}_{n,x}^{e''(r)+v'}) \subset I(X)_x + \mathbf{m}_{n,x}^{e''(r)}.$$

From Theorem 2 there exist functions $\varphi_1, \ldots, \varphi_m$ of class C^r such that

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m.$$

This completes the proof. ■

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