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On the division of functions of class $C^r$ by real analytic functions


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ON THE DIVISION OF FUNCTIONS OF CLASS $C^r$
BY REAL ANALYTIC FUNCTIONS

BY

ZBIGNIEW SZAFRANIEC (*)

Résumé. — Soit $(X, 0)$ un germe d'ensemble analytique cohérent. Supposons que les fonctions analytiques $g_1, \ldots, g_p$ engendrent un idéal $I(X)_0$. Il existe une fonction croissante $e : N \rightarrow N$ telle que, si une fonction $f$ de classe $C^{(r)}$ s'annule sur $X$, on a $f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$ ($\varphi$ étant des fonctions de classe $C^r$). Dans cet article, nous démontrons une estimation de $e(r)$ dans des cas spéciaux.

Abstract. — Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate ideal $I(X)_0$. There exists an increasing function $e : N \rightarrow N$ such that, for any function $f$ of class $C^{(r)}$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that $f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$. In this paper, we investigate the problem of the estimation of $e(r)$ in some special cases.

Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate the ideal

$I(X)_0 = \{ g \in \mathcal{O}_{\mathbb{R}^n, 0} | g \mid_X \equiv 0 \}$.

J. Cl. Tougeron in [7] showed that there exists an increasing function $e : N \rightarrow N$ such that, for any $C^{(r)}$-function $f$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that

$f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$.

J. J. Risler in [5] estimated precisely the function $e(r)$ in the case of plane curves.

In this paper, we investigate the problem of the estimation of $e(r)$ in some special cases.

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1. Strongly irreducible polynomials

For any \( x \in \mathbb{R}^n \subset \mathbb{C}^n \), let us denote by \( \mathcal{O}_{n, x}(\mathcal{D}_{n, x}) \) the ring of germs of real analytic (holomorphic) functions at \( x \). We denote by \( \mathfrak{m}_{n, x}(\mathfrak{m}_{n, x}) \) the maximal ideal of \( \mathcal{O}_{n, x}(\mathcal{D}_{n, x}) \).

**Definition 1.** — Let:

\[
P(X', X) = X^n + a_1(X')X^{n-1} + \ldots + a_p(X') \in \mathcal{D}_{n, 0}[X]
\]

be a distinguished polynomial. Let \( \delta \in \mathcal{D}_{n, 0} \) be the discriminant of the polynomial \( P \). Assume that \( \delta \neq 0 \). Denote by \( \omega \) the initial form of \( \delta \) at \( 0 \).

We say that \( P \) is **strongly irreducible** if there exist a constant \( \varepsilon > 0 \) and a set \( W \) such that the following conditions are satisfied:

1. \( (X', X) \in \mathbb{C}^{n+1} \mid 0 < \|X'\| < \varepsilon, \quad P(X', X) = 0, \quad \delta(X') \neq 0, \quad \omega(X') \neq 0 \),

2. \( W \) is a nonempty, connected and open subset of \( \mathcal{V}(P) = \{ (X', X) \in \mathbb{C}^{n+1} \mid P(X', X) = 0 \} \),

3. If \( w \in W, \ t \in \mathbb{C} \) and \( 0 < |t| < 1 \) then

\[
\pi^{-1}(t, \pi(w)) \cap \mathcal{V}(P) \subset W,
\]

where \( \pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \) is the projection.

**Lemma.** — 1 Let \( P \in \mathcal{D}_{n, 0}[X] \) be a distinguished polynomial. Let \( \delta \) be the discriminant of \( P \). Assume that \( \delta \neq 0 \). Denote by \( \omega \) the initial form of \( \delta \) at the origin.

We require \( H \subset \mathbb{C}^n \) to be a complex hyperplane such that:

(i) \( \dim_{\mathbb{C}} H \geq 1, \ 0 \in H \),

(ii) \( P \mid _H \times \mathbb{C} \) is irreducible in \( \mathcal{D}_H \times \mathbb{C}, 0 \).

(iii) \( \omega \mid _H \) has no critical points, except possibly for the origin itself.

Then the polynomial \( P \) is strongly irreducible.

The sketch of the proof.

Let \( h = \dim_{\mathbb{C}} H \). We may assume that

\[
H = \{ X \in \mathbb{C}^n \mid X_{h+1} = \ldots = X_n = 0 \}.
\]
Denote by $M$ the linear space of all complex $(n-h) \times h$-matrices. Let 
$$\gamma = \{(L, v) \in \mathbb{C} P(h-1) \times \mathbb{C}^* \mid v \in L \},$$
be the canonical line bundle of $\mathbb{C} P(h-1)$.

We define a holomorphic map $\theta : M \times \gamma \to \mathbb{C}^*$ by $\theta(A, (L, v)) = (v, A(v))$. Of course: $\theta(0 \times \gamma) = H$.

We use the notation:

- $G_1 = \{(A, (L, v)) \in M \times \gamma \mid A = 0, v = 0 \}$,
- $G_2 = \{(A, (L, v)) \in M \times \gamma \mid A = 0 \}$,
- $S = \{(A, (L, v)) \in M \times \gamma \mid v = 0 \}$.

The homogeneous form $\omega_H | H$ has an isolated singular point at the origin. Then there exist an open set $U_1 \subset S$ and a closed complex manifold $N_1 \subset U_1$ such that:

1. $G_1 \subset U_1$,
2. $N_1$ is transverse to $G_1$ in $S$,
3. $$\{(A, (L, v)) \in M \times \gamma \mid \omega \circ \theta(A, (L, v)) = 0, (A, (L, 0)) \in U_1 \} = \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in N_1 \} \cup U_1.$$

The form $\omega_H | H$ has an isolated singular point at the origin, so $\delta_H | H$ has an isolated singular point at the origin.

It follows that there exists an open set $U_2 \subset M \times \gamma$ and a closed complex manifold $N_2 \subset U_2$ such that:

4. $G_1 \subset U_2$,
5. $U_2 \cap S \subset U_1$,
6. $N_2$ is transverse to $G_1$ in $U_2$,
7. $$\{(A, (L, v)) \in M \times \gamma \mid \delta \circ \theta(A, (L, v)) = 0 \} \cap U_2 = N_2 \cup (S \cap U_2).$$
8. $N_2 \cap S = U_2 \cap N_1$.

Then there exist open sets $V_1 \subset S$, $V_2 \subset G_1$ and a constant $\varepsilon > 0$ such that:

9. $N_1 \cap G_1 \subset V_2$,
10. $G_1 \subset V_1$. 

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(11) \{(A, (L, v)) \in M \times \gamma | (A, (L, 0)) \in V_1, \ (0, (L, 0)) \notin V_2, \ 0 < \|v\| < \varepsilon\},
is a deformation retract of
\{(A, (L, v)) \in M \times \gamma | (A, (L, 0)) \in V_1, \ (A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon\}.

Denote
\[Z = \{(A, (L, v), X) \in M \times \gamma \times C \mid P(\theta(A, (L, v)), X) = 0, \]
\[(A, (L, 0)) \in V_1, \ (A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon\}.

By (7) the projection
\[\pi : Z \to \{(A, (L, v)) \in M \times \gamma | (A, (L, 0)) \in V_1, \]
\[(A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon\}
is a covering map.

Set
\[Z_1 = \{(A, (L, v), X) \in M \times \gamma \times C \mid P(\theta(A, (L, v)), X) = 0, \]
\[(A, (L, 0)) \in V_1, (0, (L, 0)) \notin V_2, \ 0 < \|v\| < \varepsilon\}.

By (11) \(Z_1\) is a deformation retract of \(Z\). Set
\[Z_1' = \{(0, (L, v), X) \in (G_2 \setminus (N_2 \cup S)) \times C \mid \]
P\((\theta(0, (L, v)), X) = 0, \ 0 < \|v\| < \varepsilon\}.

The germ of \(P_{|H \times C}\) at 0 is irreducible, so, by ([4], Proposition 11, p. 55), we may assume that \(Z_1'\) is connected. Then, if \(\varepsilon\) is sufficiently small, the sets \(Z\) and \(Z_1\) are connected.

Denote \(W = (\theta \times \text{id}_C)(Z_1) \subset C^* \times C\). Then \(W\) is an open, connected subset of \(\bar{V}(P)\).

If \(V_1\) is a sufficiently small neighbourhood of \(G_1\) in \(S\) then, by (8) and (9), we have:
\[\pi(W) \subset \{(X', X) \in C^* \times C | 0 < \|X'\| < \varepsilon, \ P(X', X) = 0, \]
\[\delta(X') \neq 0, \ \omega(X') \neq 0\}.

By definition of \(W\), if \(w \in W, \ t \in C\) and \(0 < |t| \leq 1\) then
\[\pi^{-1}(t \cdot \pi(w)) \cap \bar{V}(P) \subset W.

This completes the proof. \(\blacksquare\)

Example 1. — Let \(P \in \mathcal{O}_{1,0}[X]\) be a distinguished irreducible polynomial. Then \(P\) is strongly irreducible.
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Example 2. — Let \( P(X', X) = X^2 + X_1^2 + X_2^2 + \cdots + X_n \) where \( f \in \mathcal{O}_{-2,0}, \) \( df(0) = 0. \)

Set \( H = \{ X' \in \mathbb{C}^n | X_3 = \ldots = X_n = 0 \} \). Then \( P_{|H \times \mathbb{C}, 0} = X^2 + X_1^2 + X_2^2 \) is irreducible in \( \mathcal{O}_{H \times \mathbb{C}, 0} \) and \( \omega_{|H} = -4(X_1^2 + X_2^2) \) has an isolated singular point at the origin.

Hence \( P \) is strongly irreducible.

Corollary 1. — Let \( P \in \mathcal{O}_{n,0}[X] \) be a distinguished polynomial. Let \( \delta \) be the discriminant of \( P \). Assume that there exists a function \( \Delta \in \mathcal{O}_{n,0} \) such that \( \Delta \neq 0 \) and \( \bar{\Delta}(\Delta) = \bar{\Delta}(\delta) \). Denote by \( \omega' \) the initial form of \( \Delta \) at 0.

We require \( H \subset \mathbb{C}^n \) to be a complex hyperplane such that:

1. \( \dim_{\mathbb{C}} H = 1, 0 \in H \),
2. the germ of \( P_{|H \times \mathbb{C}} \) is irreducible in \( \mathcal{O}_{H \times \mathbb{C}, 0} \),
3. \( \omega_{|H} \) has an isolated singular point at 0.

Then \( P \) is strongly irreducible.

Example 3. — Let \( P(X', X) = X^3 + X_1^2 + X_2^2 + \cdots + X_n \) where \( f \in \mathcal{O}_{-2,0}, df(0) = 0. \)

Then \( \Delta(X') = -27(X_1^2 + X_2^2 + \cdots + X_n) \).

Set \( \Delta(X') = X_1^2 + X_2^2 + \cdots + X_n \). Of course \( \bar{\Delta}(\Delta) = \bar{\Delta}(\delta) \).

Set \( H = \{ X' \in \mathbb{C}^n | X_3 = \ldots X_n = 0 \} \).

The germ of \( P_{|H \times \mathbb{C}} = X^2 + X_1^2 + X_2^2 \) at 0 is irreducible and \( \omega_{|H} = X_1^2 + X_2^2 \) has an isolated singular point at the origin. Hence \( P \) is strongly irreducible.

Lemma 2. — Let \( P(X', X) = \sum a_\rho(X') X'^\rho + \cdots + a_0(X') \in \mathcal{O}_{n,0}[X] \) be a distinguished strongly irreducible polynomial.

Denote by \( \rho \) the degree of the form \( \omega \).

Let \( f_1, f_2 \in \mathcal{O}_{n+1,0} \) be germs such that \( f_1 \cdot f_2 \in \mathcal{O}_{n+1,0} \cdot P + \mathcal{O}_{n+1,0} \cdot f_2 \).

Then \( f_1 \in \mathcal{O}_{n+1,0} \cdot P + \mathcal{O}_{n+1,0} \cdot f_2 \) or \( f_2 \in \mathcal{O}_{n+1,0} \cdot P + \mathcal{O}_{n+1,0} \).

This lemma is analogous to Lemma 1.7 in [6].

Proof. — We define a map \( h_1 : W \times \mathbb{C} \to \mathbb{C}^n \) by \( h_1(w, t) = t^{p.} \cdot \pi(w) \).

By (1.2), if \( 0 < |t| \leq 1 \) then \( \pi^{-1}(h_1(w, t)) \cap \bar{\Delta}(P) \subset W \).

Denote \( D = \{ t \in \mathbb{C} | |t| \leq 1 \} \).

Since \( \pi(W) \subset \mathbb{C}^n \setminus \bar{\Delta}(\delta) \), so there exists a holomor-
The polynomial $P(X', X)$ is distinguished, so $h_2$ is bounded. Then there exists a holomorphic extension $h_2 : W \times D \to \mathbb{C}$. Hence $h = h_1 \times h_2 : W \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$ is holomorphic. Then, by Proposition 2.2 ([2], p. 55), there exists a constant $C_1 > 0$ such that:

$$(3) \quad |h_2(w, t)| \leq C_1 \cdot \|h_1(w, t)\|^{1/p} \quad \text{for} \quad (w, t) \in W \times \mathbb{C}.$$ 

Then

$$(4) \quad |h_2(w, t)| \leq C_1 \cdot |t|^{(p-1)\cdot \|\pi(w)\|^{1/p}}.$$ 

By (1) and (4) there exist constants $C_2, C_3 > 0$ such that, for any $(w, t) \in W \times D$,

$$|(f_1 \cdot f_2) \circ h(w, t)| \leq C_2 \cdot \|h(w, t)\|^{(p) \cdot \|\pi(w)\|^{1/p}} \cdot |t|^{2 \cdot (r+\lfloor d/2 \rfloor + 1)}.$$ 

The set $W$ is connected, so $W \times \{0\}$ is a connected complex submanifold of $W \times D$.

It follows that, for example,

$$f_1 \circ h(w, 0) = \ldots = \frac{\partial^{(r+\lfloor d/2 \rfloor + 1) \cdot p-1}}{\partial t^{(r+\lfloor d/2 \rfloor + 1) \cdot p-1}} (f_1 \circ h)(w, 0) = 0.$$ 

Then there exists a continuous function $k : W \to \mathbb{R}_+$ such that:

$$(5) \quad |f_1 \circ h(w, t)| \leq k(w) \cdot |t|^{(r+\lfloor d/2 \rfloor + 1) \cdot p-1} = k(w) \cdot \|\pi(w)\|^{1/p} \cdot \|\pi(w)\|^{(r+\lfloor d/2 \rfloor + 1)} = k(w) \cdot \|\pi(w)\|^{1/p} \cdot \|h_1(w, t)\|^{(r+\lfloor d/2 \rfloor + 1)}.$$ 

From the preparation theorem we have:

$$f_1 = Q \cdot P + \sum_{j=1}^p b_j(X') \cdot X^{p-1}, \quad \text{where} \quad b_j \in \tilde{m}_{n_0}.$$ 

Let $w_0 \in W, t \in D$. Denote by $\xi_1(t), \ldots, \xi_p(t)$ the roots of the polynomial $P(t^{p-1} \cdot \pi(w_0), X)$.
Then
\[ f_1(t^{p^j} \cdot \pi(w_0), \xi(t)) = \sum_{j=1}^n b_j(t^{p^j} \cdot \pi(w_0)) \cdot \xi_{j}^{p^j} \cdot (t). \]

By Cramer's rule
\[ b_j(t^{p^j} \cdot \pi(w_0)) = \frac{(\text{det } [s_{kl}(t)])(\prod_{1 \leq n < m < p} (\xi_m(t) - \xi_n(t)))}{\prod_{1 \leq n < m < p} (\xi_m(t) - \xi_n(t))}, \]

where if \( l \neq j \) then
\[ s_{kl}(t) = \xi_k^{p^j}(t), \quad s_{kj}(t) = f_1(t^{p^j} \cdot \pi(w_0), \xi_k(t)). \]

Of course
\[ \prod_{1 \leq n < m < p} (\xi_m(t) - \xi_n(t)) = |\delta(t^{p^j} \cdot \pi(w_0))|^{1/2}. \]

By (1.0) \( \pi(W) \subset \mathbb{C}^n \setminus \bar{V}(\omega) \). By (5) there exist constants \( C_4, C_5 > 0 \) such that:
\[ |\delta(t^{p^j} \cdot \pi(w_0))|^{1/2} > C_4 \cdot |t^{p^j}|^{d/2}, \quad |\text{det } [s_{kl}(t)]| < C_5 \cdot |t^{p^j}|^{(r+1/d^2+1)}. \]

Then
\[ |b_j(t^{p^j} \cdot \pi(w_0))| < (C_5/C_4) \cdot |t^{p^j}|^r. \]

The set \( \pi(W) \) is open in \( \mathbb{C}^n \), so \( b_j \in \mathbb{C}^{n+1,0} \).

Then \( f_1 - Q \cdot P \in \mathbb{C}^{n+1,0} \).

**COROLLARY 2.** — If \( P \in \mathcal{O}_{n,0}[X] \) is strongly irreducible then \( P \) is irreducible in \( \mathcal{O}_{n+1,0} \).

### 2. Functions vanishing on an analytic set

**DEFINITION 2.** — Let \( I \subset \mathcal{O}_{n,0} \) be an ideal. We denote by \( \sqrt{I} \) the ideal of germs vanishing on \( V(I)_0 \).

We say that \( I \) is **real** if \( I = \sqrt{I} \).

Let \( p \subset \mathcal{O}_{n,0} \) be a prime ideal, \( \{0\} \neq p \neq \mathcal{O}_{n,0} \). By [4] there exists, after a linear change of coordinates in \( \mathbb{R}^n \), an integer \( k, 0 < k \leq n \), such that \( \mathcal{O}_{k,0} \rightarrow A = \mathcal{O}_{n,0}/p \) is an injection which makes \( A \) a finite \( \mathcal{O}_{k,0} \)-module.

Further, if \( K \) is the quotient field of \( \mathcal{O}_{k,0} \), \( L \) that of \( A \), we have \( L = K(X_{k+1} \text{ mod } p) \), and for any \( i \in [k+1, n] \), the minimal polynomial \( P_i \)
of $X_i$ over $K$ is in $\mathcal{O}_{k,0}[X]$ and is distinguished, so that there is a distinguished polynomial

\begin{equation}
(2.0) \quad P_i(X', X_i) = X'_i + \sum_{j=1}^{p_i} a_{ij}(X') X'^{i-1}, \quad X' = (X_1, \ldots, X_k),
\end{equation}

with $P_i(X', X_i) \in \mathfrak{p}$.

Let $\delta(X') \in \mathcal{O}_{k,0}$ be the discriminant of the polynomial $P_{k+1}$. Then $\delta \notin \mathfrak{p}$.

Let $p = p_{k+1}$. There are polynomials $Q_i$ of degree $< p$ in $\mathcal{O}_{k,0}[X]$ such that, for $i \in [k+2, n]$ we have $\delta \cdot X_i - Q_i(X_{k+1}) \in \mathfrak{p}$.

Let $\pi : R^n = R^k \times R^{n-k} \to R^k$ be the natural projection. There exists a fundamental system of neighbourhoods $\Omega = \Omega' \times \Omega^*$ of 0 in $R^n = R^k \times R^{n-k}$ such that

\begin{equation}
(2.1) \quad \pi |_{V(p) \cap \Omega} \to \Omega^* \text{ is proper.}
\end{equation}

**Lemma 3 (see [4]).** — There exists a constant $N \leq p^{n-k}$ such that for any point $x \in V(p) \cap \Omega$ and any $f \in \mathcal{O}_{n,x}$:

\begin{equation}
\delta^N \cdot f \equiv g \pmod{P_{k+1}, \delta \cdot X_{k+2} - Q_{k+2}, \ldots, \delta \cdot X_n - Q_n},
\end{equation}

where $g$ is an element in $\mathcal{O}_{k+1}(X_{k+1})$.

**Lemma 4 (see [7]).** — There exists a constant $\alpha \in N$, $\alpha \geq 1$, such that for any point $x' \in V(\delta) \cap \Omega'$ and any connected component $U$ of $\Omega \setminus V(\delta)$, if $x' \in \bar{U}$, then there exists a sequence $(y^i)$ of points of $U$ such that

\begin{equation}
\lim_{i \to \infty} y^i = x' \quad \text{and} \quad \{ y \in \Omega' | \|y - y^i\| < \|x' - y^i\|^\alpha \} \subset U.
\end{equation}

**Lemma 5 (see [7]).** — There exists a constant $\nu \in N$ such that for any $x \in V(p) \cap \Omega$ and any germs $f_0, \ldots, f_{n-k} \in \mathcal{O}_{n,x}$ if

\begin{equation}
h = f_0 \cdot \delta^N + f_1 \cdot P_{k+1} + \sum_{i=k+2}^{n} f_{i-k} \cdot (\delta \cdot X_i - Q_i) \in \mathfrak{m}_{n,x}^\nu,
\end{equation}

then there exist germs $g_0, \ldots, g_{n-k} \in \mathfrak{m}'_{n,x}$ such that:

\begin{equation}
h = g_0 \cdot \delta^N + g_1 \cdot P_{k+1} + \sum_{i=k+2}^{n} g_{i-k} \cdot (\delta \cdot X_i - Q_i).
\end{equation}

From now on we make the assumptions:

(2.2) $V(p)$ is coherent in a neighbourhood of 0,

(2.3) the set $V(p) \cap \Omega \setminus V(\delta) \times R^{n-k}$ is dense in $V(p) \cap \Omega$,
(2.4) If \((x', x_{k+1}) \in V(P_{k+1}) \cap (\Omega' \times R)\), then there exist polynomials

\[ R_1, \ldots, R_s(x', x_{k+1}), \quad Q \in \mathcal{O}_{k, x} [X_{k+1} - x_{k+1}] \]

such that

\[ P_{k+1} = R_1 \ldots R_s(x', x_{k+1})Q \text{ in } \mathcal{O}_{k, x} [X_{k+1} - x_{k+1}], \]

(2.5) polynomials \(R_i\) are distinguished and strongly irreducible in \(\mathcal{O}_{k, x} [X_{k+1} - x_{k+1}]\),

(2.6) \(Q(x', x_{k+1}) \neq 0\),

(2.7) for any \(i \in \{1, s(x', x_{k+1})\}\)

\[(x', x_{k+1}) \in \overline{V(R_i) \setminus (\bigcup_{j \neq i} V(R_j) \cup (V(\delta) \times R))}. \]

**Example 4.** Assume that \(f \in m_{n-3,0}, df(0) = 0\) and

\[(1) \quad \{ x \in R^{n-3} \mid f(x) \leq 0 \} = \{ x \in R^{n-3} \mid f(x) < 0 \}. \]

Define

\[ P(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2 + X_n^2 \in \mathcal{O}_{n-1,0} [X_n]. \]

The germ of \(P\) at 0 is irreducible, so \(p = \mathcal{O}_{n,0}. P\) is prime in \(\mathcal{O}_{n,0}\).

If \(x \in V(p) \setminus R^{n-3} \times \{0\}\), then \(dP(x) \neq 0\), so the germ of \(P\) at \(x\) generates \(I(V(p))_x\).

If \(x = (x', x^n) \in V(p) \cap R^{n-3} \times \{0\}\), where \(x' \in R^{n-3}\) and \(x^n \in R^3\) then \(f(x') = 0\). Hence the germ of \(P\) at \(x\) is irreducible. By (1) the germ of \(V(p)\) at \(x\) contains regular points. From Lemma 2.5 ([3], p. 14), the germ of \(P\) at \(x\) generates \(I(V(p))_x\). So \(V(p)\) is coherent.

Let \(\delta = -4(f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2)\) be the discriminant of \(P\).

By (1), \(V(p) \setminus (V(\delta) \times R)\) is dense in \(V(p)\) in some neighbourhood of the origin.

If \(x = (x', x^n) \in V(p) \cap (R^{n-3} \times \{0\})\) then, by Example 2, the germ of \(P\) at \(x\) is strongly irreducible.

If \(x \in V(p) \setminus (R^{n-3} \times \{0\})\), then \(\delta(\pi(x)) \neq 0\) or \(d\delta(\pi(x)) \neq 0\). Hence, by Definition 1 or lemma 1, the polynomial \(P\) is strongly irreducible.

So the conditions (2.2)-(2.7) are satisfied.

Let \(d = \text{degree } \omega\), where \(\omega\) is the initial form of \(\delta\) at 0. By induction we can define functions \(e_i : N \rightarrow N\).

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Set
\[ e_0(r) = p \cdot (r + \nu + [d/2] + 1), \]
\[ e_1(r) = p \cdot \alpha \cdot (e_0(r) - 1), \]
\[ \vdots \]
\[ e_i(r) = p \cdot \alpha \cdot (2p \cdot (e_{i-1}(r) + [d/2] + 1) - 1), \]
\[ \text{Set } e'(r) = e_p(r). \]

Theorem 1. — Assume that \( p \subset \mathfrak{O}_{n,0} \) is a prime ideal satisfying the conditions (2.2)-(2.7).

Let \( f \) be a function of class \( C^{r(\alpha)} \) vanishing on \( V(p) \) in a sufficiently small neighbourhood \( \Omega \) of 0.

Then, for any \( x \in \Omega \), the Taylor expansion \( T^r_x(r) f \in I(V(p))_x + \mathfrak{m}_{x}^r \)
\[ \text{(where } T^r_x(r) f = \sum_{|\beta| \leq e'(r)(1/\beta !)} \partial^\beta f / \partial X^\beta (x)(X - x)^\beta \in \mathfrak{O}_{n, x}. \)

Proof. — Let \( x = (x', x_{k+1}, x') \in V(p) \cap \Omega \). Let \( Y_{k+1} = x_{k+1} - x_{k+1} \). By \( (2.4) \) \( P_{k+1} = R_1, \ldots, R_x, Q \) in \( \mathfrak{O}_{k, x} [Y_{k+1}] \).

Denote by \( \delta_i(\omega_i) \) the discriminant of the polynomial \( R_i \) (the initial form of \( \delta_i \) at \( x' \)).

From Lemma 3, there exists \( g \in \mathfrak{O}_{k, x} [Y_{k+1}] \) such that
\[ \delta^N \cdot (T^r_x(r) f) \equiv g \mod I(V(p))_x. \]

The function \( f \) vanishes on \( V(p)_x \), so \( T^r_x(r) f \) and \( g \) are \( e'(r) \)-flat on \( V(p)_x \)
\( \text{(see [7]).} \)

Every polynomial \( R_i \) has degree \( \leq p \) and, by Corollary 2 and (2.5), is irreducible.

From Proposition 5.6 ([2], p. 50) there exist \( a_{i_1}, \ldots, a_{i_p} \in \mathfrak{O}_{k, x} \) such that
\[ g + a_{i_1} + \ldots, a_{i_p} \in \mathfrak{O}_{k+1, x} [x', x_{k+1}], R_i. \]

Then \( a_{i_p} \) is \( e'(r) \)-flat on \( V(p) \cap (V(R_i) \times R^{n-k-1}) \).

By (2.0), (2.1) and Proposition 2.2 ([2], p. 55), \( a_{i_p} \) is \( e'(r)/p \)-flat on \( \pi(V(R_i))_x \).

By (2.7) there exists a connected component \( U \) of
\[ \{ y' \in R^k \mid \| x' - y' \| < \varepsilon \text{, } \delta(y') \neq 0 \} \]
such that \( x' \in U \) and \( U \subset \pi(V(R_i))_x \).

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Then, from Lemma 5.11, [7] and Lemma 4

\[ a_{ip} \in m_{k,x}^{2p(\nu_{p-1}(r) + [d/2] + 1)}. \]

We have

\[ g(g^{p-1} + a_{i1} \cdot g^{p-2} + \ldots + a_{i,p-1}) \in \mathcal{O}_{k+1,x} \cdot R_t + m_{k,x}^{2p(\nu_{p-1}(r) + [d/2] + 1)}. \]

If \( x \) is sufficiently close to 0 then degree \( \omega_i \leq d. \)

Then, from Lemma 2,

\[ g \in \mathcal{O}_{k+1,x} \cdot R_t + m_{k,x}^{2p-1(\nu_{p-1}(r))} \]

or

\[ g^{p-1} + \ldots + a_{i,p-1} \in \mathcal{O}_{k+1,x} \cdot R_t + m_{k,x}^{2p-1(\nu_{p-1}(r))}. \]

In the second case, repeating this process \( p-1 \) times, we can prove that \( g \) is \( e_0(r) \)-flat on

\[ \mathcal{V}(R_t) = \{ (y', y_{k+1}) \in \mathbb{C}^k \times \mathbb{C} | R_t(y', y_{k+1}) = 0 \} \text{ at } (x', x_{k+1}). \]

Then \( g \) is \( e_0(r) \)-flat on \( \mathcal{V}(R_1, \ldots, R_s) = \mathcal{V}(R_1) \cup \ldots \cup \mathcal{V}(R_s) \) at \( (x', x_{k+1}) \).

From the preparation theorem we have

\[ g = S \cdot R_1 \ldots R_s + \sum_{j=1}^{p} b_j \cdot Y_{s+1}^j, \text{ where } b_j \in \mathcal{O}_{k,x}. \]

By Cramer's rule \( b_j \in m_{k,x}^{* \nu_{s+1}} \). The arguments are the same as in the proof of Lemma 2.

By (2.4)

\[ \delta^N(T_{x}^r) f \in (P_{k+1}, \delta, X_{k+2} - Q_{k+2}, \ldots, \delta, X_n - Q_n) \mathcal{O}_{n,x} + m_{n,x}^{* \nu_{s+1}}. \]

From Lemma 5, there exists \( h \in m_{n,x}^{*} \) such that

\[ \delta^N(T_{x}^r) f - h \in I(V(p)). \]

By (2.3), \( T_{x}^r f - h \in I(V(p)). \)

Then \( T_{x}^r f \in I(V(p)) + m_{n,x}^{* \nu_{s+1}} \). \[ \blacksquare \]

**Theorem 2 (see [1]).** Let \( g_1, \ldots, g_m \in \mathcal{O}_{n,0}. \)

There exist a linear function \( \nu \exists r \mapsto e^* (r) = a^* . r + b^* \in N \) and an open neighbourhood \( \Omega \) of 0 such that:
If $f: \Omega \to \mathbb{R}$ is a function of class $C^{r}(\Omega)$ and, for any $x \in \Omega$, $T_{x}^{r}(f) \in (g_{1}, \ldots, g_{m})_{\mathbb{C}_{n, x} + m_{n, x}^{r}}$, then there exist functions $\varphi_{1}, \ldots, \varphi_{m}: \Omega \to \mathbb{R}$ of class $C^{r} \cap \mathbb{R}$ such that:

$$f = \varphi_{1} \cdot g_{1} + \cdots + \varphi_{m} \cdot g_{m}.$$ 

**Theorem 3.** — Let $(X, 0) \subset (\mathbb{R}^{n}, 0)$ be a germ of an analytic coherent set. Then $I(X)_{0} = p_{1} \cap \ldots \cap p_{k}$, where $p_{1}, \ldots, p_{k}$ are prime ideals in $\mathbb{C}_{n, 0}$.

Suppose that every ideal $p_{i}$ satisfies assumptions (2.2)-(2.7). Then there exists a linear function $N \exists r \mapsto e(r) = a \cdot r + b \in \mathbb{N}$ such that for any function $f$ of class $C^{r}(\Omega)$ vanishing on $X$:

$$f = \varphi_{1} \cdot g_{1} + \cdots + \varphi_{m} \cdot g_{m},$$

where $g_{1}, \ldots, g_{m} \in I(X)_{0}$ and $\varphi_{1}, \ldots, \varphi_{m}$ are germs of function of class $C^{r}$.

This theorem is a sharpened version of the result of J.-C. Tougeron (see Theorem 5.12, [7]).

**Proof.** — We have $I(X)_{0} \subset \sqrt{I(X)}_{0} = p_{1} \cap \ldots \cap p_{k}$, where $p_{1}, \ldots, p_{k} \subset \mathbb{C}_{n, 0}$ are prime ideals. The germ of $X$ at $0$ is coherent, so the ideal $I(X)_{0}$ is real. Then $\sqrt{I(X)}_{0} \subset \sqrt{\sqrt{I(X)}}_{0} = I(X)_{0}$. Hence $I(X)_{0} = p_{1} \cap \ldots \cap p_{k}$.

From Theorem 1 there exists a linear function $N \exists r \mapsto e'(r) = a' \cdot r + b' \in \mathbb{N}$ such that, for any function $f$ of class $C^{r}(\Omega)$ vanishing on $X$ and any $x \in X$ we have

$$T_{x}^{r}(f) \in \bigcap_{i=1}^{k} (I(V(p_{i}))_{x} + m_{n, x}^{r}).$$

By (2.3) and ([7], Theorem 3.8), there exists a constant $v' \in \mathbb{N}$ such that, for any $x \in X$ we have

$$\bigcap_{i=1}^{k} (I(V(p_{i}))_{x} + m_{n, x}^{r} \cdot v') \subset \bigcap_{i=1}^{k} I(V(p_{i}))_{x} + m_{n, x}^{r} \subset I(X)_{x} + m_{n, x}^{r}.$$

Let $g_{1}, \ldots, g_{m}$ be generators of $I(X)_{0}$. Let $e''(r)$ be a function as in Theorem 2.

Define $e'(r) = e''(e''(r) + v') = a' \cdot r + b'$.

Let $f$ be a germ of class $C^{r}(\Omega)$ vanishing on $X$. 

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Then, for any $x \in X$ in some neighbourhood of 0 we have

$$T_{x}^e f \in \bigcap_{i=1}^{k} (I(V(p_i))_{x} + m_{n_{x}}^{e_{x}}(r^{e} + r^{n})) \subseteq I(X)_{x} + m_{n_{x}}^{e_{x}}(r^{e}).$$

From Theorem 2 there exist functions $\varphi_{1}, \ldots, \varphi_{m}$ of class $C^{e}$ such that

$$f = \varphi_{1} \cdot g_{1} + \ldots + \varphi_{m} \cdot g_{m}.$$

This completes the proof. ■

REFERENCES

[1] JEDDARI (L.), Sur la divisibilité des fonctions de classe $C^{e}$ par les fonctions analytiques réelles.