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On the division of functions of class $C^r$ by real analytic functions


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ON THE DIVISION OF FUNCTIONS OF CLASS $C^r$
BY REAL ANALYTIC FUNCTIONS

BY

ZBIGNIEW SZAFAFRANIEC (*)

Résumé. — Soit $(X, 0)$ un germe d'ensemble analytique cohérent. Supposons que les fonctions analytiques $g_1, \ldots, g_p$ engendrent un idéal $I(X)_0$. Il existe une fonction croissante $e : N \to N$ telle que, si une fonction $f$ de classe $C^{(r)}$ s'annule sur $X$, on a $f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$ ($\varphi_i$ étant des fonctions de classe $C^r$). Dans cet article nous démontrons une estimation de $e(r)$ dans des cas spéciaux.

Abstract. — Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate ideal $I(X)_0$. There exists an increasing function $e : N \to N$ such that, for any function $f$ of class $C^{(r)}$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that $f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$. In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate the ideal $I(X)_0 = \{ g \in \mathcal{O}_{\mathbb{R}^n, 0} | g \mid_X = 0 \}$.

J. Cl. Tougeron in [7] showed that there exists an increasing function $e : N \to N$ such that, for any $C^{(r)}$-function $f$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that $f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p$.

J. J. Risler in [5] estimated precisely the function $e(r)$ in the case of plane curves.

In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

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1. Strongly irreducible polynomials

For any $x \in \mathbb{R}^n \subset \mathbb{C}^n$, let us denote by $\mathcal{O}_{n, x}(\partial_{n, x})$ the ring of germs of real analytic (holomorphic) functions at $x$. We denote by $m_{n, x}(\bar{m}_{n, x})$ the maximal ideal of $\mathcal{O}_{n, x}(\partial_{n, x})$.

**Definition 1.** — Let:

$$P(X', X) = X^n + a_1(X')X^{n-1} + \ldots + a_p(X') \in \bar{m}_{n, 0}[X]$$

be a distinguished polynomial. Let $\delta \in \bar{m}_{n, 0}$ be the discriminant of the polynomial $P$. Assume that $\delta \neq 0$. Denote by $\omega$ the initial form of $\delta$ at $0$.

We say that $P$ is **strongly irreducible** if there exist a constant $\varepsilon > 0$ and a set $W$ such that the following conditions are satisfied:

1. $W \subset \{(X',X) \in \mathbb{C}^{n+1} \mid 0 < \|X'\| < \varepsilon, P(X', X) = 0, \delta(X') \neq 0, \omega(X') \neq 0\}$,
2. $W$ is a nonempty, connected and open subset of $\tilde{P}(P) = \{(X', X) \in \mathbb{C}^{n+1} \mid P(X', X) = 0\}$,
3. If $w \in W$, $t \in \mathbb{C}$ and $0 < |t| \leq 1$ then
   $$\pi^{-1}(t, \pi(w)) \cap \tilde{P}(P) \subset W,$$

where $\pi: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is the projection.

**Lemma.** — 1 Let $P \in \bar{m}_{n, 0}[X]$ be a distinguished polynomial. Let $\delta$ be the discriminant of $P$. Assume that $\delta \neq 0$. Denote by $\omega$ the initial form of $\delta$ at the origin.

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

1. $\dim_{\mathbb{C}} H \geq 1, 0 \in H$,
2. $P|_{H \times \mathbb{C}}$ is irreducible in $\bar{m}_{H \times \mathbb{C}, 0}$,
3. $\omega|_{H}$ has no critical points, except possibly for the origin itself.

Then the polynomial $P$ is strongly irreducible.

The sketch of the proof.

Let $h = \dim_{\mathbb{C}} H$. We may assume that

$$H = \{X \in \mathbb{C}^n \mid X_{h+1} = \ldots = X_n = 0\}.$$
Denote by $M$ the linear space of all complex $(n-h) \times h$-matrices. Let

$$\gamma = \{(L, v) \in CP(h-1) \times \mathbb{C}^* \mid v \in L\},$$

be the canonical line bundle of $CP(h-1)$.

We define a holomorphic map $\theta : M \times \gamma \to \mathbb{C}^*$ by $\theta(A, (L, v)) = (v, A(v))$. Of course, $\theta(0 \times \gamma) = H$.

We use the notation:

$$G_1 = \{(A, (L, v)) \in M \times \gamma \mid A = 0, v = 0\},$$

$$G_2 = \{(A, (L, v)) \in M \times \gamma \mid A = 0\},$$

$$S = \{(A, (L, v)) \in M \times \gamma \mid v = 0\}.$$

The homogeneous form $\omega_{|H}$ has an isolated singular point at the origin. Then there exist an open set $U_1 \subset S$ and a closed complex manifold $N_1 \subset U_1$ such that:

1. $G_1 \subset U_1$,
2. $N_1$ is transverse to $G_1$ in $S$,
3. $\{(A, (L, v)) \in M \times \gamma \mid \omega \circ \theta(A, (L, v)) = 0, (A, (L, 0)) \in U_1\}
   = \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in N_1\} \cup U_1$.

The form $\omega_{|H}$ has an isolated singular point at the origin, so $\delta_{|H}$ has an isolated singular point at the origin.

It follows that there exists an open set $U_2 \subset M \times \gamma$ and a closed complex manifold $N_2 \subset U_2$ such that:

4. $G_1 \subset U_2$,
5. $U_2 \cap S \subset U_1$,
6. $N_2$ is transverse to $G_1$ in $U_2$,
7. $\{(A, (L, v)) \in M \times \gamma \mid \delta \circ \theta(A, (L, v)) = 0\} \cap U_2 = N_2 \cup (S \cap U_2),$
8. $N_2 \cap S = U_2 \cap N_1$.

Then there exist open sets $V_1 \subset S$, $V_2 \subset G_1$ and a constant $\varepsilon > 0$ such that:

9. $N_1 \cap G_1 \subset V_2$,
10. $G_1 \subset V_1$. 

BULLETIN DE LA SOCIÉTÉ MATHEMATIQUE DE FRANCE
(11) \( \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \ \ (0, (L, 0)) \notin V_2, \ 0 < \|v\| < \varepsilon \} \),
is a deformation retract of
\( \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \ (A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon \} \).

Denote
\[ Z = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0, \]
\[ (A, (L, 0)) \in V_1, (A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon \}. \]

By (7) the projection
\[ \pi : Z \to \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \]
\[ (A, (L, v)) \notin N_2 \cup S, \ 0 < \|v\| < \varepsilon \} \]
is a covering map.

Set
\[ Z_1 = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0, \]
\[ (A, (L, 0)) \in V_1, (0, (L, 0)) \notin V_2, \ 0 < \|v\| < \varepsilon \}. \]

By (11) \( Z_1 \) is a deformation retract of \( Z \). Set
\[ Z'_1 = \{(0, (L, v), X) \in (G_2 \setminus (N_2 \cup S)) \times \mathbb{C} \mid P(\theta(0, (L, v)), X) = 0, \ 0 < \|v\| < \varepsilon \}. \]

The germ of \( P_{|H \times \mathbb{C}} \) at 0 is irreducible, so, by ([4], Proposition 11, p. 55), we may assume that \( Z'_1 \) is connected. Then, if \( \varepsilon \) is sufficiently small, the sets \( Z \) and \( Z_1 \) are connected.

Denote \( W = (\theta \times \text{id}_\mathbb{C})(Z_1) \subset \mathbb{C}^* \times \mathbb{C} \). Then \( W \) is an open, connected subset of \( \overline{V(P)} \).

If \( V_1 \) is a sufficiently small neighbourhood of \( G_1 \) in \( S \) then, by (8) and (9), we have:
\[ \pi(W) \subset \{(X', X) \in \mathbb{C}^* \times \mathbb{C} \mid 0 < \|X'\| < \varepsilon, \ P(X', X) = 0, \ \delta(X') \neq 0, \ \omega(X') \neq 0 \}. \]

By definition of \( W \), if \( w \in W, \ t \in \mathbb{C} \) and \( 0 < |t| \leq 1 \) then
\[ \pi^{-1}(t, \pi(w)) \cap \overline{V(P)} \subset W. \]

This completes the proof. \( \blacksquare \)

Example 1. – Let \( P \in \partial_{1, \omega}[X] \) be a distinguished irreducible polynomial. Then \( P \) is strongly irreducible.
Example 1. — Let $P \in \mathcal{O}_{1,0}[X]$ be a distinguished irreducible polynomial. Then $P$ is strongly irreducible.

Example 2. — Let $P(X', X) = X^2 + X_1^2 + X_2 + f(X_3, \ldots, X_n)$, where $f \in \mathfrak{m}_{-2,0}$, $df(0) = 0$.

Set $H = \{X' \in \mathbb{C}^n \mid X_3 = \ldots = X_n = 0\}$. Then $P_{|_H \times \mathbb{C}} = X^2 + X_1^2 + X_2^2$ is irreducible in $\mathcal{O}_{H \times \mathbb{C}, 0}$ and $\omega'_H = -4(X_1^2 + X_2^2)$ has an isolated singular point at the origin.

Hence $P$ is strongly irreducible.

Corollary 1. — Let $P \in \mathcal{O}_{1,0}[X]$ be a distinguished polynomial. Let $\delta$ be the discriminant of $P$. Assume that there exists a function $\Delta \in \mathcal{O}_{1,0}$ such that $\Delta \neq 0$ and $\tilde{V}(\Delta) = \tilde{V}(\delta)$. Denote by $\omega'$ the initial form of $\Delta$ at 0.

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

(i) $\dim_C H \geq 1, \ 0 \in H$,

(ii) the germ of $P_{|_H \times \mathbb{C}}$ is irreducible in $\mathcal{O}_{H \times \mathbb{C}, 0}$.

(iii) $\omega'_H$ has an isolated singular point at 0.

Then $P$ is strongly irreducible.

Example 3. — Let $P(X', X) = X^3 + X_1^2 + X_2 + f(X_3, \ldots, X_n)$, where $f \in \mathfrak{m}_{-2,0}$ and $df(0) = 0$.

Then $\delta(X') = -27X_1^2 + X_2 + f(X_3, \ldots, X_n))^2$.

Set $\Delta(X') = X_1^2 + X_2 + f(X_3, \ldots, X_n)$. Of course $\tilde{V}(\Delta) = \tilde{V}(\delta)$.

Set

$$H = \{X' \in \mathbb{C}^n \mid X_3 = \ldots X_n = 0\}.$$

The germ of $P_{|_H \times \mathbb{C}} = X^3 + X_1^2 + X_2^2$ at 0 is irreducible and $\omega'_H = X_1^2 + X_2^2$ has an isolated singular point at the origin. Hence $P$ is strongly irreducible.

Lemma 2. — Let $P(X', X) = X^p + a_1(X')X^{p-1} + \ldots + a_p(X') \in \mathcal{O}_{n, 0}[X]$ be a distinguished strongly irreducible polynomial.

Denote by $d$ the degree of the form $\omega$.

Let $f_1, f_2 \in \mathcal{O}_{n+1,0}$ be germs such that $f_1 \cdot f_2 \in \mathcal{O}_{n+1,0}$, $P + \mathfrak{m}_{n+1,0}$, where $r \in \mathbb{N}, e(r) = 2p(r + [d/2] + 1)$, $[d/2]$ is the integer part of $d/2$.

Then $f_1 \in \mathcal{O}_{n+1,0}$, $P + \mathfrak{m}_{n+1,0}$ or $f_2 \in \mathcal{O}_{n+1,0}$, $P + \mathfrak{m}_{n+1,0}$.

This lemma is analogous to Lemma 1.7 in [6].

Proof. — We define a map $h_1 : W \times \mathbb{C} \to \mathbb{C}^n$ by $h_1(w, t) = t^{p-1}\pi(w)$. By (1.2), if $0 < |t| \leq 1$ then $\pi^{-1}(h_1(w, t)) \cap \tilde{V}(P) \subset W$. Denote $D = \{t \in \mathbb{C} \mid |t| \leq 1\}$. Since $\pi(W) \subset \mathbb{C}^n \backslash \tilde{V}(\delta)$, so there exists a holomor-
Phic function $h_2 : W \times (D \setminus \{0\}) \to \mathbb{C}$ such that:

(1) $P(h_1(w, t), h_2(w, t)) = 0,$

(2) $(h_1(w, 1), h_2(w, 1)) = w.$

The polynomial $P(X', X)$ is distinguished, so $h_2$ is bounded. Then there exists a holomorphic extension $h_2 : W \times D \to \mathbb{C}.$ Hence $h = h_1 \times h_2 : W \times C \to \mathbb{C}^n \times \mathbb{C}$ is holomorphic. Then, by Proposition 2.2 ([2], p. 55), there exists a constant $C_1 > 0$ such that:

(3) $|h_2(w, t)| \leq C_1 \cdot \|h_1(w, t)\|^{1/p}$ for $(w, t) \in W \times C.$

Then

(4) $|h_2(w, t)| \leq C_1 \cdot |t|^{(p-1)/p} \cdot \|\pi(w)\|^{1/p}.$

By (1) and (4) there exist constants $C_2, C_3 > 0$ such that, for any $(w, t) \in W \times D,$

$|(f_1 \circ f_2) \circ h(w, t)| \leq C_2 \cdot \|h(w, t)\|^{e(r)} \leq C_3 \cdot |t|^{2 (r + [d/2] + 1)/p}.$

The set $W$ is connected, so $W \times \{0\}$ is a connected complex submanifold of $W \times D.$

It follows that, for example,

$f_1 \circ h(w, 0) = \ldots = \frac{\partial^{(r + [d/2] + 1)/p - 1}}{\partial t^{(r + [d/2] + 1)/p - 1}} (f_1 \circ h)(w, 0) = 0.$

Then there exists a continuous function $k : W \to \mathbb{R}_+$ such that:

(5) $|f_1 \circ h(w, t)| \leq k(w) \cdot |t|^{(r + [d/2] + 1)/p},$

$k(w) \cdot \|\pi(w)\|^{-1} \cdot \|X^{p-1} \cdot \pi(w)\|^{(r + [d/2] + 1)}$

$k(w) \cdot \|\pi(w)\|^{-1} \cdot \|h_1(w, t)\|^{(r + [d/2] + 1)}.$

From the preparation theorem we have:

$f_1 = Q \cdot P + \sum_{j=1}^{p} b_j(X') \cdot X^{p-j},$ where $b_j \in \mathfrak{m}_n.0.$

Let $w_0 \in W, t \in D.$ Denote by $\xi_1(t), \ldots, \xi_p(t)$ the roots of the polynomial $P(t^{p-1} \cdot \pi(w_0), X).$
Then
\[ f_1(t^{p-1}, \pi(w_0), \xi_i(t)) = \sum_{j=1}^{n} b_j(t^{p-1}, \pi(w_0)) \cdot \xi_i^{p-1}(t). \]

By Cramer's rule
\[ b_j(t^{p-1}, \pi(w_0)) = \frac{\det [s_{kl}(t)]/(\prod_{1 \leq n \leq m \leq p} (\xi_n(t) - \xi_m(t)))}{J(0)}. \]

where if \( l \neq j \) then
\[ s_{kl}(t) = \xi_k^{p-1}(t), \quad s_{kj}(t) = f_1(t^{p-1}, \pi(w_0), \xi_k(t)). \]

Of course
\[ \left| \prod_{1 \leq n \leq m \leq p} (\xi_n(t) - \xi_m(t)) \right| = \left| \delta(t^{p-1}, \pi(w_0)) \right|^{1/2}. \]

By (1.0) \( \pi(W) \subset \mathbb{C}^n \setminus \mathbb{P}(\omega) \). By (5) there exist constants \( C_4, C_5 > 0 \) such that:
\[ \left| \delta(t^{p-1}, \pi(w_0)) \right|^{1/2} > C_4, \left| t^{p-1} \right|^{d/2}, \left| \det [s_{kl}(t)] \right| < C_5, \left| t^{p-1} \right|^{(r+[d/2]+1)}. \]

Then
\[ |b_j(t^{p-1}, \pi(w_0))| < (C_5/C_4) \cdot \left| t^{p-1} \right|^r. \]

The set \( \pi(W) \) is open in \( \mathbb{C}^n \), so \( b_j \in \mathbb{m}_{n+1,0} \).

Then \( f_1 - Q \cdot P \in \mathbb{m}_{n+1,0} \).

**Corollary 2.** — If \( P \in \mathfrak{O}_{n,0}[X] \) is strongly irreducible then \( P \) is irreducible in \( \mathfrak{O}_{n+1,0} \).

**2. Functions vanishing on an analytic set**

**Definition 2.** — Let \( I \subset \mathfrak{O}_{n,0} \) be an ideal. We denote by \( \sqrt{I} \) the ideal of germs vanishing on \( V(I)_0 \).

We say that \( I \) is real if \( I = \sqrt{I} \).

Let \( p \subset \mathfrak{O}_{n,0} \) be a prime ideal, \( \{0\} \neq p \neq \mathfrak{O}_{n,0} \). By [4] there exists, after a linear change of coordinates in \( \mathbb{R}^n \), an integer \( k, 0 < k \leq n \), such that \( \mathfrak{O}_{k,0} \rightarrow A = \mathfrak{O}_{n,0}/p \) is an injection which makes \( A \) a finite \( \mathfrak{O}_{k,0} \)-module.

Further, if \( K \) is the quotient field of \( \mathfrak{O}_{k,0} \), \( L \) that of \( A \), we have \( L = K(X_{k+1} \text{ mod } p) \), and for any \( i \in [k+1, n] \), the minimal polynomial \( P_i \)
of $X_i$ over $K$ is in $\mathcal{O}_{k,0}[X]$ and is distinguished, so that there is a
distinguished polynomial

$$ (2.0) \quad P_i(X', X_i) = X_i^{p_i} + \sum_{j=1}^{p_i} a_{ij}(X') X_i^{p_i-j}, \quad X' = (X_1, \ldots, X_k), $$

with $P_i(X', X_i) \in \mathfrak{p}$.

Let $\delta(X') \in \mathcal{O}_{k,0}$ be the discriminant of the polynomial $P_{k+1}$. Then
$\delta \notin \mathfrak{p}$.

Let $p = p_{k+1}$. There are polynomials $Q_i$ of degree $<p$ in $\mathcal{O}_{k,0}[X]$ such
that, for $i \in [k+2, n]$ we have $\delta \cdot X_i - Q_i(X_{i+1}) \in \mathfrak{p}$.

Let $\pi: R^n = R^k \times R^{n-k} \to R^k$ be the natural projection. There exists a
fundamental system of neighbourhoods $\Omega = \Omega' \times \Omega^*$ of 0 in $R^n = R^k \times R^{n-k}$
such that

$$ (2.1) \quad \pi|_{V(p) \cap \Omega} : \Omega' \to \Omega^* \text{ is proper.} $$

**Lemma 3** (see [4]). — There exists a constant $N = p^{n-k}$ such that for any
point $x \in V(p) \cap \Omega$ and any $f \in \mathcal{O}_{n,x}$ :

$$ \delta^N \cdot f \equiv g \pmod{P_{k+1}, \delta \cdot X_{k+2} - Q_{k+2}, \ldots, \delta \cdot X_n - Q_n}, $$

where $g$ is an element in $\mathcal{O}_{k,0}[X_{k+1}]$.

**Lemma 4** (see [7]). — There exists a constant $\alpha \in N$, $\alpha \geq 1$, such that for
any point $x' \in V(\delta) \cap \Omega'$ and any connected component $U$ of $\Omega \setminus V(\delta)$, if
$x' \in U$, then there exists a sequence $(y^i)$ of points of $U$ such that

$$ \lim_{i \to \infty} y^i = x' \quad \text{and} \quad \{ y \in \Omega' \mid \|y - y^i\| < \|x' - y^i\|^{\alpha} \} \subset U. $$

**Lemma 5** (see [7]). — There exists a constant $\nu \in N$ such that for any
$x \in V(p) \cap \Omega$ and any germs $f_0, \ldots, f_{n-k} \in \mathcal{O}_{n, x}$, if

$$ h = f_0 \cdot \delta^N + f_1 \cdot P_{k+1} + \sum_{i=k+2}^{n-k} f_{i-k} \cdot (\delta \cdot X_i - Q_i) \in \mathfrak{m}_{n,x}^\nu, \quad r \in N, $$

then there exist germs $g_0, \ldots, g_{n-k} \in \mathfrak{m}_{n,x}^\nu$ such that:

$$ h = g_0 \cdot \delta^N + g_1 \cdot P_{k+1} + \sum_{i=k+2}^{n-k} g_{i-k} \cdot (\delta \cdot X_i - Q_i). $$

From now on we make the assumptions:

(2.2) $V(p)$ is coherent in a neighbourhood of $0$,

(2.3) the set $V(p) \cap \Omega \setminus V(\delta) \times R^{n-k}$ is dense in $V(p) \cap \Omega$. 

**TOME 113 - 1985 - N 2**
(2.4) If \((x', x_{k+1}) \in V(P_{k+1}) \cap (\Omega' \times R)\), then there exist polynomials
\[ R_1, \ldots, R_s(x', x_{k+1}), \quad Q \in \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \]
such that
\[ P_{k+1} = R_1 \cdots R_s(x', x_{k+1}) \cdot Q \quad \text{in } \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \]

(2.5) polynomials \( R_i \) are distinguished and strongly irreducible in
\[ \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \]

(2.6) \( Q(x', x_{k+1}) \neq 0 \),

(2.7) for any \( i \in \{1, s(x', x_{k+1})\} \)
\[ (x', x_{k+1}) \in V(R_i) \setminus \left( \bigcup_{j \neq i} V(R_j) \cup (V(\delta) \times R) \right) \]

Example 4. – Assume that \( f \in m_{-3,0} \), \( df(0) = 0 \) and
\[ f(x') = x \quad x' \in R_3 \]

Define
\[ P(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-3}) + X_{n-2} + X_{n-1}^2 + X_n \in \mathcal{O}_{n-1,0}[X_n]. \]

The germ of \( P \) at 0 is irreducible, so \( p = \mathcal{O}_{n,0} \). \( P \) is prime in \( \mathcal{O}_{n,0} \).

If \( x \in V(p) \setminus R^{n-3} \times \{0\} \), then \( dP(x) \neq 0 \), so the germ of \( P \) at \( x \) generates
\( I(V(p)) \).

If \( x' = (x', x') \in V(p) \cap R^{n-3} \times \{0\} \), where \( x' \in R^{n-3} \) and \( x' \in R^3 \) then \( f(x') = 0 \). Hence the germ of \( P \) at \( x \) is irreducible. By (1) the germ of \( V(p) \) at \( x \) contains regular points. From Lemma 2.5 ([3], p. 14), the germ of \( P \) at \( x \) generates \( I(V(p))_x \). So \( V(p) \) is coherent.

Let \( \delta = -4(f(X_1, \ldots, X_{n-3}) + X_{n-2} + X_{n-1}^2) \) be the discriminant of \( P \).

By (1), \( V(p) \setminus (V(\delta) \times R) \) is dense in \( V(p) \) in some neighbourhood of the origin.

If \( x' = (x', x') \in V(p) \cap (R^{n-3} \times \{0\}) \) then, by Example 2, the germ of \( P \) at \( x \) is strongly irreducible.

If \( x \in V(p) \setminus (R^{n-3} \times \{0\}) \), then \( \delta(\pi(x)) \neq 0 \) or \( d\delta(\pi(x)) \neq 0 \). Hence, by Definition 1 or Lemma 1, the polynomial \( P \) is strongly irreducible.

So the conditions (2.2)-(2.7) are satisfied.

Let \( d = \text{degree } \omega \), where \( \omega \) is the initial form of \( \delta \) at 0. By induction we can define functions \( e_i : N \to N \).
Set
\[ e_0(r) = p \cdot (r + \nu + \lfloor d/2 \rfloor + 1), \]
\[ e_1(r) = p \cdot \alpha \cdot (e_0(r) - 1), \]
\[ \vdots \]
\[ e_i(r) = p \cdot \alpha \cdot (2p \cdot (e_{i-1}(r) + \lfloor d/2 \rfloor + 1) - 1), \]

Set \( e'(r) = e_p(r). \)

**Theorem 1.** — Assume that \( p \in \mathcal{O}_{n,0} \) is a prime ideal satisfying the conditions (2.2)-(2.7).

Let \( f \) be a function of class \( C^{(r)} \) vanishing on \( V(p) \) in a sufficiently small neighbourhood \( \Omega \) of 0.

Then, for any \( x \in \Omega \), the Taylor expansion \( T_x^{(r)} f \in I(V(p))_x + \mathfrak{m}^r x \).

(Where \( T_x^{(r)} f = \sum_{|\beta| \leq e'(r)} (1/\beta !) \cdot (\partial^\beta f/(\partial X^\beta))(x)(X-x)^\beta \in \mathcal{O}_{n,x} \).

**Proof.** — Let \( x = (x', x_{k+1}, x') \in V(p) \cap \Omega \). Let \( Y_{k+1} = X_{k+1} - x_{k+1} \). By (2.4) \( P_{k+1} = R_1, \ldots, X_0, Q \in \mathcal{O}_{k,x}[Y_{k+1}] \).

Denote by \( \delta_i(\omega_i) \) the discriminant of the polynomial \( R_i \) (the initial form of \( \delta_i \) at \( x' \)).

From Lemma 3, there exists \( g \in \mathcal{O}_{k,x}[Y_{k+1}] \) such that
\[ \delta^N \cdot (T_x^{(r)} f) \equiv g \mod I(V(p))_x. \]

The function \( f \) vanishes on \( V(p)_x \), so \( T_x^{(r)} f \) and \( g \) are \( e'(r) \)-flat on \( V(p)_x \) (see [7]).

Every polynomial \( R_i \) has degree \( \leq p \) and, by Corollary 2 and (2.5), is irreducible.

From Proposition 5.6 ([2], p. 50) there exist \( a_{i_1}, \ldots, a_{i_p} \in \mathcal{O}_{k,x} \) such that
\[ g^p + a_{i_1} \cdot g^{p-1} + \ldots + a_{i_p} \in \mathcal{O}_{k+1, X'(x', x_{k+1})} R_i. \]

Then \( a_{i_p} \) is \( e'(r) \)-flat on \( V(p) \cap (V(R_i) \times R^{n-k-1}) \).

By (2.0), (2.1) and Proposition 2.2 ([2], p. 55), \( a_{i_p} \) is \( (e'(r)/p) \)-flat on \( \pi(V(R_i))_x \).

By (2.7) there exists a connected component \( U \) of
\[ \{ y' \in R^k \mid || x' - y' || < \varepsilon, \delta(y') \neq 0 \} \]
such that \( x' \in \overline{U} \) and \( U \subset \pi(V(R_i))_x \).
Then, from Lemma 5.11, [7] and Lemma 4

\[ a_{ip} \in m_{k,x}^{2p(e_p-1) + [d/2] + 1}. \]

We have

\[ g(g^{p-1} + a_{i_1}g^{p-2} + \ldots + a_{i_p-1}) \in \mathcal{O}_{k+1,x}R_i + m_{k,x}^{2p(e_p-1) + [d/2] + 1}. \]

If \( x \) is sufficiently close to 0 then degree \( \omega_i \leq d. \)

Then, from Lemma 2,

\[ g \in \mathcal{O}_{k+1,x}R_i + m_{k,x}^{e_{p-1}(e_p-1) + [d/2] + 1}. \]

or

\[ g^{p-1} + \ldots + a_{i_p-1} \in \mathcal{O}_{k+1,x}R_i + m_{k,x}^{e_{p-1}(e_p-1) + [d/2] + 1}. \]

In the second case, repeating this process \( p - 1 \) times, we can prove that \( g \) is \( e_0(r) \)-flat on

\[ \tilde{V}(R_i) = \{(y', y_{k+1}) \in C^k \times C | R_i(y', y_{k+1}) = 0 \} \text{ at } (x', x_{k+1}). \]

Then \( g \) is \( e_0(r) \)-flat on \( \tilde{V}(R_1 \ldots R_x) = \tilde{V}(R_1) \cup \ldots \cup \tilde{V}(R_x) \) at \( (x', x_{k+1}). \)

From the preparation theorem we have

\[ g = \sum_{j=1}^p b_j Y_{k+1}^j, \quad \text{where } b_j \in \mathcal{O}_{k,x}. \]

By Cramer's rule \( b_j \in m_{k,x}^{e_{p-1}} \). The arguments are the same as in the proof of Lemma 2.

By (2.4)

\[ \delta^N(T_x^{e(r)} f) \in (P_{k+1}, \delta, X_{k+2} - Q_{k+2}, \ldots, \delta, X_n - Q_n) \mathcal{O}_{k,x} + m_{k,x}^{e_{p-1}}. \]

From Lemma 5, there exists \( h \in m_{k,x}^{e_{p-1}} \) such that

\[ \delta^N(T_x^{e(r)} f - h) \in I(V(p)), \]

By (2.3), \( T_x^{e(r)} f - h \in I(V(p)), \)

Then \( T_x^{e(r)} f \in I(V(p)) + m_{k,x}^{e_{p-1}}. \]

**Theorem 2 (see [1]).** - Let \( g_1, \ldots, g_m \in \mathcal{O}_{k,0}. \)

There exist a linear function \( N \ni r \mapsto e^*(r) = a^* \cdot r + b^* \in N \) and an open neighbourhood \( \Omega \) of 0 such that:
if \( f : \Omega \to \mathbb{R} \) is a function of class \( C^{\infty} \) and, for any \( x \in \Omega \), 
\[ T^x f \in (g_1, \ldots, g_m) \cdot \mathcal{O}_{n, x} + \mathfrak{m}^x \] 
then there exist functions \( \varphi_1, \ldots, \varphi_m : \Omega \to R \) of class \( C^r \) such that:

\[ f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m. \]

**Theorem 3.** — Let \( (X, 0) \subset (\mathbb{R}^n, 0) \) be a germ of an analytic coherent set. Then \( I(X)_0 = p_1 \cap \ldots \cap p_k \), where \( p_1, \ldots, p_k \) are prime ideals in \( \mathcal{O}_{n, 0} \).

Suppose that every ideal \( p_i \) satisfies assumptions (2.2)-(2.7). Then there exists a linear function \( N : r \mapsto e(r) = a \cdot r + b \in \mathbb{N} \) such that for any function \( f \) of class \( C^{\infty} \) vanishing on \( X \):

\[ f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m, \]

where \( g_1, \ldots, g_m \in I(X)_0 \) and \( \varphi_1, \ldots, \varphi_m \) are germs of function of class \( C^r \).

This theorem is a sharpened version of the result of J.-Cl. Tougeron (see Theorem 5.12, [7]).

**Proof.** — We have \( I(X)_0 \subset \sqrt{I(X)_0} = p_1 \cap \ldots \cap p_k \), where

\[ p_1, \ldots, p_k \subset \mathcal{O}_{n, 0} \]

are prime ideals. The germ of \( X \) at 0 is coherent, so the ideal \( I(X)_0 \) is real. Then \( \sqrt{I(X)_0} \subset \sqrt{I(X)_0} = I(X)_0 \). Hence \( I(X)_0 = p_1 \cap \ldots \cap p_k \).

From Theorem 1 there exists a linear function \( N : r \mapsto e'(r) = a' \cdot r + b' \in \mathbb{N} \) such that, for any function \( f \) of class \( C^{\infty} \) vanishing on \( X \) and any \( x \in X \) we have

\[ T^x f \in \bigcap_{i=1}^k \left( I(V(p_i))_x + \mathfrak{m}^x \right). \]

By (2.3) and ([7], Theorem 3.8), there exists a constant \( v' \in \mathbb{N} \) such that, for any \( x \in X \) we have

\[ \bigcap_{i=1}^k \left( I(V(p_i))_x + \mathfrak{m}^x \right) = \bigcap_{i=1}^k \left( I(V(p_i))_x + \mathfrak{m}^x \right). \]

Let \( g_1, \ldots, g_m \) be generators of \( I(X)_0 \). Let \( e'(r) \) be a function as in Theorem 2.

Define \( e(r) = e'(e'(r) + v') = a \cdot r + b \).

Let \( f \) be a germ of class \( C^{\infty} \) vanishing on \( X \).
Then, for any $x \in X$ in some neighbourhood of $0$ we have

$$T^x \in \bigcap_{i=1}^k (I(V(p_i))_x + \mathfrak{m}_x^{s_i} (r) + \mathfrak{m}_x^{s''_i (r)}) \subseteq I(X)_x + \mathfrak{m}_x^{s''_i (r)}.$$

From Theorem 2 there exist functions $\varphi_1, \ldots, \varphi_m$ of class $C^r$ such that

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m.$$

This completes the proof. \(\blacksquare\)

REFERENCES

[1] JEDDARI (L.), Sur la divisibilité des fonctions de classe $C^r$ par les fonctions analytiques réelles.