ZBIGNIEW SZAFRANIEC

On the division of functions of class $C^r$ by real analytic functions


<http://www.numdam.org/item?id=BSMF_1985__113__143_0>
ON THE DIVISION OF FUNCTIONS OF CLASS $C^r$
BY REAL ANALYTIC FUNCTIONS

BY

ZBIGNIEW SZAFRANIEC (*)

RéSUMÉ. — Soit $(X, 0)$ un germe d'ensemble analytique cohérent. Supposons que les fonctions analytiques $g_1, \ldots, g_p$ engendrent un idéal $I(X)_0$. Il existe une fonction croissante $e : N \rightarrow N$ telle que, si une fonction $f$ de classe $C^{(r)}$ s'annule sur $X$, on a
\[ f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p \] (\(\varphi_i\) étant des fonctions de classe $C^r$). Dans cet article nous démontrons une estimation de $e(r)$ dans des cas spéciaux.

ABSTRACT. — Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate ideal $I(X)_0$. There exists an increasing function $e : N \rightarrow N$ such that, for any function $f$ of class $C^{(r)}$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that
\[ f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p. \] In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

Let $(X, 0)$ be a germ of an analytic coherent set in $\mathbb{R}^n$. Assume that analytic functions $g_1, \ldots, g_p$ generate the ideal
\[ I(X)_0 = \{ g \in \mathcal{O}_{n, 0} \mid g \mid_x \equiv 0 \}. \]

J. Cl. TOUGERON in [7] showed that there exists an increasing function $e : N \rightarrow N$ such that, for any $C^{(r)}$-function $f$ vanishing on $X$, there exist $C^r$-functions $\varphi_1, \ldots, \varphi_p$ such that
\[ f = \varphi_1 \cdot g_1 + \ldots + \varphi_p \cdot g_p. \]

J. J. RISLER in [5] estimated precisely the function $e(r)$ in the case of plane curves.

In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

(*) Texte reçu le 5 mars 1984.
Z. SZAFRANIEC, Institute of Mathematics, 80-952 Gdansk, Wita Stwosza 57, Poland.

BULLETIN DE LA SOCIÉTÉ MATHEMATIQUE DE FRANCE — 0037-9484/1985/02 143 13/$ 3.30
© Gauthier-Villars
1. Strongly irreducible polynomials

For any \( x \in \mathbb{R}^n \subseteq \mathbb{C}^n \), let us denote by \( \mathcal{O}_{n,x} \) the ring of germs of real analytic (holomorphic) functions at \( x \). We denote by \( m_{n,x} \) the maximal ideal of \( \mathcal{O}_{n,x} \).

**Definition 1.** Let:

\[
P(X', X) = X^p + a_1(X')X^{p-1} + \ldots + a_p(X') \in \mathcal{O}_{n,0}[X]
\]

be a distinguished polynomial. Let \( \delta \in \mathcal{O}_{n,0} \) be the discriminant of the polynomial \( P \). Assume that \( \delta \neq 0 \). Denote by \( \omega \) the initial form of \( \delta \) at 0.

We say that \( P \) is *strongly irreducible* if there exist a constant \( \varepsilon > 0 \) and a set \( W \) such that the following conditions are satisfied:

1. \( W \subset \{(X', X) \in \mathbb{C}^{n+1} | 0 < \|X'\| < \varepsilon, P(X', X) = 0, \delta(X') \neq 0, \omega(X') \neq 0 \} \),
2. \( W \) is a nonempty, connected and open subset of \( \mathbb{C}^{n+1} \),
3. If \( w \in W, t \in \mathbb{C} \) and \( 0 < |t| < 1 \) then \( \pi^{-1}(t, \pi(w)) \cap \mathcal{V}(P) \subset W \),

where \( \pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \) is the projection.

**Lemma.** Let \( P \in \mathcal{O}_{n,0}[X] \) be a distinguished polynomial. Let \( \delta \) be the discriminant of \( P \). Assume that \( \delta \neq 0 \). Denote by \( \omega \) the initial form of \( \delta \) at the origin.

We require \( H \subset \mathbb{C}^n \) to be a complex hyperplane such that:

1. \( \dim H \geq 1, 0 \in H \),
2. \( P \mid_{H \times \mathbb{C}} \) is irreducible in \( \mathcal{O}_{H \times \mathbb{C},0} \).
3. \( \omega \mid_H \) has no critical points, except possibly for the origin itself.

Then the polynomial \( P \) is strongly irreducible.

The sketch of the proof.

Let \( h = \dim H \). We may assume that

\[
H = \{ X \in \mathbb{C}^n \mid X_{n+1} = \ldots = X_n = 0 \}.
\]
Denote by $M$ the linear space of all complex $(n-h) \times h$-matrices. Let
\[ \gamma = \{(L, v) \in \mathbb{C}P(h-1) \times \mathbb{C}^* | v \in L\}, \]
be the canonical line bundle of $\mathbb{C}P(h-1)$.

We define a holomorphic map $\theta : M \times \gamma \to \mathbb{C}^*$ by $\theta(A, (L, v)) = (v, A(v))$.

Of course : $\theta(0 \times \gamma) = H$.

We use the notation:
\[ G_1 = \{(A, (L, v)) \in M \times \gamma | A = 0, v = 0\}, \]
\[ G_2 = \{(A, (L, v)) \in M \times \gamma | A = 0\}, \]
\[ S = \{(A, (L, v)) \in M \times \gamma | v = 0\}. \]

The homogeneous form $\omega_1 H$ has an isolated singular point at the origin. Then there exist an open set $U_1 \subset S$ and a closed complex manifold $N_1 \subset \mathbb{C}^*$ such that:

1. $G_1 \subset U_1$,
2. $N_1$ is transverse to $G_1$ in $S$,
3. $\{(A, (L, v)) \in M \times \gamma | \omega \circ \theta(A, (L, v)) = 0, (A, (L, 0)) \in U_1\}
   = \{(A, (L, v)) \in M \times \gamma | (A, (L, 0)) \in N_1\} \cup U_1$.

The form $\omega_1 H$ has an isolated singular point at the origin, so $\delta_1 H$ has
an isolated singular point at the origin.

It follows that there exists an open set $U_2 \subset M \times \gamma$ and a closed complex
manifold $N_2 \subset U_2$ such that:

4. $G_1 \subset U_2$,
5. $U_2 \cap S \subset U_1$,
6. $N_2$ is transverse to $G_1$ in $U_2$,
7. $\{(A, (L, v)) \in M \times \gamma | \delta \circ \theta(A, (L, v)) = 0\} \cap U_2 = N_2 \cup (S \cap U_2)$,
8. $N_2 \cap S = U_2 \cap N_1$.

Then there exist open sets $V_1 \subset S$, $V_2 \subset G_1$ and a constant $\varepsilon > 0$ such
that:

9. $N_1 \cap G_1 \subset V_2$,
10. $G_1 \subset V_1$.
(11) \( \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \quad (0, (L, 0)) \notin V_2, \quad 0 < \|v\| < \varepsilon \} \),
is a deformation retract of\n\( \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \quad (A, (L, v)) \notin N_2 \cup S, \quad 0 < \|v\| < \varepsilon \} \).

Denote\n\[ Z = \{(A, (L, v), X) \in M \times \gamma \times C \mid P\left(\theta (A, (L, v)), X\right) = 0, \]
\[ (A, (L, 0)) \in V_1, \quad (A, (L, v)) \notin N_2 \cup S, \quad 0 < \|v\| < \varepsilon \} \].

By (7) the projection\n\[ \pi : \quad Z \to \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \]
\[ (A, (L, v)) \notin N_2 \cup S, \quad 0 < \|v\| < \varepsilon \} \]
is a covering map.

Set\n\[ Z_1 = \{(A, (L, u), X) \in M \times \gamma \times C \mid P\left(\theta (A, (L, v)), X\right) = 0, \]
\[ (A, (L, 0)) \in V_1, \quad (0, (L, 0)) \notin V_2, \quad 0 < \|v\| < \varepsilon \} \].

By (11) \( Z_1 \) is a deformation retract of \( Z \). Set\n\[ Z'_1 = \{(0, (L, v), X) \in (G_2 \setminus (N_2 \cup S)) \times C \mid \]
\[ P\left(\theta (0, (L, v)), X\right) = 0, \quad 0 < \|v\| < \varepsilon \} \].

The germ of \( P_{1_H \times C} \) at 0 is irreducible, so, by ([4], Proposition 11, p. 55), we may assume that \( Z'_1 \) is connected. Then, if \( \varepsilon \) is sufficiently small, the sets \( Z \) and \( Z_1 \) are connected.

Denote \( W = (\theta \times \text{id}_C)(Z_1) \subset C^* \times C \). Then \( W \) is an open, connected subset of \( \overline{V}(P) \).

If \( V_1 \) is a sufficiently small neighbourhood of \( G_1 \) in \( S \) then, by (8) and (9), we have:
\[ \pi(W) \subset \{(X', X) \in C^* \times C \mid 0 < \|X'\| < \varepsilon, \quad P(X', X) = 0, \quad \delta(X') \neq 0, \quad \omega(X') \neq 0 \} \].

By definition of \( W \), if \( w \in W, \ t \in C \) and \( 0 < |t| \leq 1 \) then\n\[ \pi^{-1}(t \cdot \pi(w)) \cap \overline{V}(P) \subset W. \]

This completes the proof. \( \blacksquare \)

Example 1. — Let \( P \in \partial L_0[X] \) be a distinguished irreducible polynomial. Then \( P \) is strongly irreducible.
Example 1. — Let $P \in \mathcal{O}_{1,0}[X]$ be a distinguished irreducible polynomial. Then $P$ is strongly irreducible.

Example 2. — Let $P(X', X) = X^2 + X_1^2 + X_2^2 + f(X_3, \ldots, X_n)$, where $f \in \mathcal{m}_{-2,0}$. Then $P|_{H \times \mathbb{C}} = X^2 + X_1^2 + X_2^2$ is irreducible in $\mathcal{O}_{H \times \mathbb{C}, 0}$ and $\omega|_H = -4(X_1^2 + X_2^2)$ has an isolated singular point at the origin.

Hence $P$ is strongly irreducible.

Corollary 1. — Let $P \in \mathcal{O}_{n,0}[X]$ be a distinguished polynomial. Let $\delta$ be the discriminant of $P$. Assume that there exists a function $\Delta \in \mathcal{O}_{n,0}$ such that $\Delta \neq 0$ and $\hat{\mathcal{V}}(\Delta) = \hat{\mathcal{V}}(\delta)$. Denote by $\omega'$ the initial form of $\Delta$ at 0.

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

(i) $\dim_H H \geq 1, 0 \in H$,
(ii) the germ of $P|_{H \times \mathbb{C}}$ is irreducible in $\mathcal{O}_{H \times \mathbb{C}, 0}$,
(iii) $\omega'|_H$ has an isolated singular point at 0.

Then $P$ is strongly irreducible.

Example 3. — Let $P(X', X) = X^3 + X_1^3 + X_2^3 + f(X_3, \ldots, X_n)$, where $f \in \mathcal{m}_{-2,0}$ and $df(0) = 0$.

Then $\Delta(X') = -27(X_1^3 + X_2^3 + f(X_3, \ldots, X_n))^2$.

Set $\Delta(X') = X_1^2 + X_2^2 + f(X_3, \ldots, X_n)$. Of course $\hat{\mathcal{V}}(\Delta) = \hat{\mathcal{V}}(\delta)$.

Set $H = \{X' \in \mathbb{C}^n | X_3 = \ldots X_n = 0\}$.

The germ of $P|_{H \times \mathbb{C}} = X^3 + X_1^3 + X_2^3$ at 0 is irreducible and $\omega'|_H = X_1^2 + X_2^2$ has an isolated singular point at the origin. Hence $P$ is strongly irreducible.

Lemma 2. — Let $P(X', X) = X^p + a_1(X')X^{p-1} + \ldots + a_p(X') \in \mathcal{O}_{n,0}[X]$ be a distinguished strongly irreducible polynomial.

Denote by $d$ the degree of the form $\omega$.

Let $f_1, f_2 \in \mathcal{O}_{n+1,0}$ be germs such that $f_1 \cdot f_2 \in \mathcal{O}_{n+1,0} \cdot P + \tilde{m}^{(r)}_{n+1,0}$. where $r \in \mathbb{N}$, $e(r) = 2p(r + [d/2] + 1)$, $[d/2]$ is the integer part of $d/2$.

Then $f_1 \in \mathcal{O}_{n+1,0} \cdot P + \tilde{m}^{(r)}_{n+1,0}$ or $f_2 \in \mathcal{O}_{n+1,0} \cdot P + \tilde{m}^{(r)}_{n+1,0}$.

This lemma is analogous to Lemma 1.7 in [6].

Proof. — We define a map $h_1 : W \times \mathbb{C} \to \mathbb{C}^n$ by $h_1(w, t) = t^p \cdot \pi(w)$. By (1.2), if $0 < |t| \leq 1$ then $\pi^{-1}(h_1(w, t)) \cap \hat{\mathcal{V}}(P) \subset W$. Denote $D = \{t \in \mathbb{C} | |t| \leq 1\}$. Since $\pi(W) \subset \mathbb{C}^n \setminus \hat{\mathcal{V}}(\delta)$, so there exists a holomor-
phic function $h_2 : W \times (D \setminus \{0\}) \to \mathbb{C}$ such that:

(1) \[ P(h_1(w, t), h_2(w, t)) = 0, \]

(2) \[ (h_1(w, 1), h_2(w, 1)) \equiv w. \]

The polynomial $P(X', X)$ is distinguished, so $h_2$ is bounded. Then there exists a holomorphic extension $h_2 : W \times D \to \mathbb{C}$. Hence $h = h_1 \times h_2 : W \times C \to C^n \times C$ is holomorphic. Then, by Proposition 2.2 ([2], p. 55), there exists a constant $C_1 > 0$ such that:

(3) \[ |h_2(w, t)| \leq C_1 \cdot \|h_1(w, t)\|^{1/p} \quad \text{for} \quad (w, t) \in W \times C. \]

Then

(4) \[ |h_2(w, t)| \leq C_1 \cdot |t|^{(p-1)/p} \cdot \|\pi(w)\|^{1/p}. \]

By (1) and (4) there exist constants $C_2, C_3 > 0$ such that, for any $(w, t) \in W \times D$,

\[ |f_1 \circ h(w, t)| \leq C_2 \cdot \|h(w, t)\|^{p-1} \cdot \|\pi(w)\| \leq C_3 \cdot |t|^{2(v+\lfloor d/2 \rfloor+1)p}. \]

The set $W$ is connected, so $W \times \{0\}$ is a connected complex submanifold of $W \times D$.

It follows that, for example,

\[ f_1 \circ h(w, 0) = \ldots = \frac{d^{(v+\lfloor d/2 \rfloor+1)p-1}}{d^{(v+\lfloor d/2 \rfloor+1)p-1}} (f_1 \circ h)(w, 0) \equiv 0. \]

Then there exists a continuous function $k : W \to R_+$ such that:

(5) \[ |f_1 \circ h(w, t)| \leq k(w) \cdot |t|^{(v+\lfloor d/2 \rfloor+1)p}. \]

\[
= k(w) \cdot \|\pi(w)\|^{-1} \cdot \|\pi(w)\|^{(v+\lfloor d/2 \rfloor+1)} \cdot \|h_1(w, t)\|^{v+\lfloor d/2 \rfloor+1}.
\]

From the preparation theorem we have:

\[ f_1 = Q \cdot P + \sum_{j=1}^{p} b_j(X') \cdot X^{p-1}, \quad \text{where} \quad b_j \in \mathbb{R}. \]

Let $w_0 \in W, t \in D$. Denote by $\xi_1(t), \ldots, \xi_p(t)$ the roots of the polynomial $P(t^{p-1} \cdot \pi(w_0), X)$. 


TOME 113 - 1985 - N 2
Then
\[ f_1(t^{p^1}. \pi(w_0), \xi_i(t)) = \sum_{j=1}^{n} b_j(t^{p^1}. \pi(w_0)) \cdot \xi_j^{p-1}(t). \]

By Cramer's rule
\[ b_j(t^{p^1}. \pi(w_0)) = (\det [s_{kj}(t)]/\prod_{1 \leq m < p} (\xi_m(t) - \xi_m(0))), \]
where if \( l \neq j \) then
\[ s_{kl}(t) = \xi_k^{p-1}(t), \quad s_{kj}(t) = f_1(t^{p^1}. \pi(w_0), \xi_k(t)). \]

Of course
\[ |\prod_{1 \leq m < p} (\xi_m(t) - \xi_m(0))| = |\delta(t^{p^1}. \pi(w_0))|^{1/2}. \]

By (1.0) \( \pi(W) \subset C^* \setminus \mathcal{P}(\omega) \). By (5) there exist constants \( C_4, C_5 > 0 \) such that:
\[ |\delta(t^{p^1}. \pi(w_0))|^{1/2} > C_4 \cdot |t^{p^1}|^{d/2}, \quad |\det [s_{kj}(t)]| < C_5 \cdot |t^{p^1}|^{(r+\lfloor d/2 \rfloor + 1)}. \]

Then
\[ |b_j(t^{p^1}. \pi(w_0))| < (C_5/C_4) \cdot |t^{p^1}|^r. \]

The set \( \pi(W) \) is open in \( C^* \), so \( b_j \in m_{+1,0}. \)

Then \( f_1 - Q \cdot p \in m_{+,1,0}. \)

**Corollary 2.** — If \( P \in \mathcal{O}_{n,0}[X] \) is strongly irreducible then \( P \) is irreducible in \( \mathcal{O}_{n+1,0}. \)

2. Functions vanishing on an analytic set

**Definition 2.** — Let \( I \subset \mathcal{O}_{n,0} \) be an ideal. We denote by \( \sqrt[\mathcal{O}_{n,0}]{I} \) the ideal of germs vanishing on \( V(I)_0 \).

We say that \( I \) is real if \( I = \sqrt[\mathcal{O}_{n,0}]{I} \).

Let \( p \subset \mathcal{O}_{n,0} \) be a prime ideal, \( \{0\} \neq p \neq \mathcal{O}_{n,0} \). By (4) there exists, after a linear change of coordinates in \( R^n \), an interger \( k, 0 < k \leq n \), such that \( \mathcal{O}_{k,0} \rightarrow A = \mathcal{O}_{n,0}/p \) is an injection which makes \( A \) a finite \( \mathcal{O}_{k,0} \)-module.

Further, if \( K \) is the quotient field of \( \mathcal{O}_{k,0} \), \( L \) that of \( A \), we have \( L = K(X_{k+1} \mod p) \), and for any \( i \in [k+1, n] \), the minimal polynomial \( P_i \).
of $X_i$ over $K$ is in $\mathcal{O}_{k,0}[X]$ and is distinguished, so that there is a distinguished polynomial

\begin{equation}
P_i(X', X_i) = X_i^{p_i} + \sum_{j=1}^{p_i} a_{ij}(X') X_i^{p_i-j}, \quad X' = (X_1, \ldots, X_k),
\end{equation}

with $P_i(X', X_i) \in \mathfrak{p}$.

Let $\delta(X') \in \mathcal{O}_{k,0}$ be the discriminant of the polynomial $P_{k+1}$. Then $\delta \notin \mathfrak{p}$.

Let $p = p_{k+1}$. There are polynomials $Q_i$ of degree $< p$ in $\mathcal{O}_{k,0}[X]$ such that, for $i \in [k+2, n]$ we have $\delta \cdot X_i - Q_i(X_{k+1}) \in \mathfrak{p}$.

Let $\pi : R^n = R^k \times R^{n-k} \to R^k$ be the natural projection. There exists a fundamental system of neighbourhoods $\Omega = \Omega' \times \Omega^*$ of 0 in $R^n = R^k \times R^{n-k}$ such that

\begin{equation}
\pi|_{V(p) \cap \Omega} \to \Omega' \text{ is proper.}
\end{equation}

**Lemma 3 (see [4]).** — There exists a constant $N \leq p^{n-k}$ such that for any point $x \in V(p) \cap \Omega$ and any $f \in \mathcal{O}_{n,x}$:

\[ \delta^N \cdot f \equiv g \pmod{P_{k+1}, \delta \cdot X_{k+2} - Q_{k+2}, \ldots, \delta \cdot X_n - Q_n}, \]

where $g$ is an element in $\mathcal{O}_{k,n(x)}[X_{k+1}]$.

**Lemma 4 (see [7]).** — There exists a constant $\alpha \in N$, $\alpha \geq 1$, such that for any point $x' \in V(\delta) \cap \Omega'$ and any connected component $U$ of $\Omega' \setminus V(\delta)$, if $x' \in U$, then there exists a sequence $(y')$ of points of $U$ such that

\[ \lim_{y'} y' = x' \quad \text{and} \quad \{ y \in \Omega' \mid \| y - y' \| < \| x' - y' \|^\alpha \} \subset U. \]

**Lemma 5 (see [7]).** — There exists a constant $\nu \in N$ such that for any $x \in V(p) \cap \Omega$ and any germs $f_0, \ldots, f_{n-k} \in \mathcal{O}_{n,x}$, if

\[ h = f_0 \cdot \delta^N + f_1 \cdot P_{k+1} + \sum_{i=k+2}^{n} f_{i-k} \cdot (\delta \cdot X_i - Q_i) \in \mathfrak{m}_{n,x}^\nu, \quad r \in N, \]

then there exist germs $g_0, \ldots, g_{n-k} \in \mathfrak{m}_{n,x}$ such that:

\[ h = g_0 \cdot \delta^N + g_1 \cdot P_{k+1} + \sum_{i=k+2}^{n} g_{i-k} \cdot (\delta \cdot X_i - Q_i). \]

From now on we make the assumptions:

\begin{enumerate}
\item[(2.2)] $V(p)$ is coherent in a neighbourhood of 0,
\item[(2.3)] the set $V(p) \cap \Omega' \setminus V(\delta) \times R^{n-k}$ is dense in $V(p) \cap \Omega$.
\end{enumerate}
(2.4) If \((x', x_{k+1}) \in V(P_{k+1}) \cap (\Omega' \times R)\), then there exist polynomials
\[ R_1, \ldots, R_s(x', x_{k+1}), \quad Q \in \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \]
such that
\[ P_{k+1} = R_1 \cdots R_s(x', x_{k+1}) \cdot Q \] in \( \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \).

(2.5) Polynomials \( R_i \) are distinguished and strongly irreducible in
\( \mathcal{O}_{k, x}[X_{k+1} - x_{k+1}] \).

(2.6) \( Q(x', x_{k+1}) \neq 0 \).

(2.7) For any \( i \in \{1, \ldots, s(x', x_{k+1})\} \),
\[ (x', x_{k+1}) \in \overline{V(R_i)} \setminus (\bigcup_{j \neq i} V(R_j) \cup (V(\delta) \times R)). \]

Example 4. Assume that \( f \in m_{-3,0} \), \( df(0) = 0 \) and
\[ \{ x \in R^{n-3} \mid f(x) \leq 0 \} = \{ x \in R^{n-3} \mid f(x) < 0 \}. \]

Define
\[ P(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2 + X_n^2 \in \mathcal{O}_{n-1,0}[X_n]. \]

The germ of \( P \) at \( 0 \) is irreducible, so \( p = \mathcal{O}_{n,0} \). \( P \) is prime in \( \mathcal{O}_{n,0} \).

If \( x \in V(p) \setminus R^{n-3} \times \{0\} \), then \( \partial P(x) \neq 0 \), so the germ of \( P \) at \( x \) generates \( I(V(p))_x \).

If \( x = (x', x^*) \in V(p) \cap R^{n-3} \times \{0\} \), where \( x' \in R^{n-3} \) and \( x^* \in R^3 \) then \( f(x^*) = 0 \). Hence the germ of \( P \) at \( x \) is irreducible. By (1) the germ of \( V(p) \) at \( x \) contains regular points. From Lemma 2.5 ([3], p. 14), the germ of \( P \) at \( x \) generates \( I(V(p))_x \). So \( V(p) \) is coherent.

Let \( \delta = -4(f(X_1, \ldots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2) \) be the discriminant of \( P \).

By (1), \( V(p) \setminus (V(\delta) \times R) \) is dense in \( V(p) \) in some neighbourhood of the origin.

If \( x = (x', x^*) \in V(p) \cap (R^{n-3} \times \{0\}) \) then, by Example 2, the germ of \( P \) at \( x \) is strongly irreducible.

If \( x \in V(p) \setminus (R^{n-3} \times \{0\}) \), then \( \delta(\pi(x)) \neq 0 \) or \( d\delta(\pi(x)) \neq 0 \). Hence, by Definition 1 or Lemma 1, the polynomial \( P \) is strongly irreducible.

So the conditions (2.2)-(2.7) are satisfied.

Let \( d = \deg \omega \), where \( \omega \) is the initial form of \( \delta \) at \( 0 \). By induction we can define functions \( e_i : N \to N \).
Set
\[ e_0(r) = p \cdot (r + v + [d/2] + 1), \]
\[ e_1(r) = p \cdot \alpha \cdot (e_0(r) - 1), \]
\[ \vdots \]
\[ e_i(r) = p \cdot \alpha \cdot (2p \cdot (e_{i-1}(r) + [d/2] + 1) - 1), \]

Set \( e'(r) = e_p(r). \)

**Theorem 1.** — Assume that \( p \subseteq \mathcal{O}_{n,0} \) is a prime ideal satisfying the conditions (2.2)-(2.7).

Let \( f \) be a function of class \( C^e(r) \) vanishing on \( V(p) \) in a sufficiently small neighbourhood \( \Omega \) of \( 0. \)

Then, for any \( x \in \Omega, \) the Taylor expansion \( T_x^{e(r)} f \in I(V(p))_x + m^n_{x, x}. \)

(Where \( T_x^{e(r)} f = \sum_{1 \leq j \leq e'(r)} (1/j!) \cdot (\partial^j f/\partial X^j)(x)(X-x)^j \in \mathcal{O}_{n, x}. \)

**Proof.** — Let \( x = (x', x_{k+1}, x^*) \in V(p) \cap \Omega. \) Let \( Y_{k+1} = X_{k+1} - x_{k+1}. \) By (2.4) \( P_{k+1} = R_1, \ldots, R_k, Q \in \mathcal{O}_{k, x}[Y_{k+1}]. \)

Denote by \( \delta_i(\omega_i) \) the discriminant of the polynomial \( R_i \) (the initial form of \( \delta_i \) at \( x^* \)).

From Lemma 3, there exists \( g \in \mathcal{O}_{k, x}[Y_{k+1}] \) such that

\[ \delta^N \cdot (T_x^{e(r)} f) \equiv g \bmod I(V(p))_x. \]

The function \( f \) vanishes on \( V(p)_x, \) so \( T_x^{e(r)} f \) and \( g \) are \( e'(r) \)-flat on \( V(p)_x \)

Every polynomial \( R_i \) has degree \( \leq p \) and, by Corollary 2 and (2.5), is irreducible.

From Proposition 5.6 ([2], p. 50) there exist \( a_{i_1}, \ldots, a_{i_p} \in \mathcal{O}_{k, x} \) such that

\[ g^p + a_{i_1} \cdot g^{p-1} + \ldots + a_{i_p} \in \mathcal{O}_{k+1, (x', x_{k+1})} \cdot R_i. \]

Then \( a_{i_p} \) is \( e'(r) \)-flat on \( V(p) \cap (V(R_i) \times R^{-k-1}). \)

By (2.0), (2.1) and Proposition 2.2 ([2], p. 55), \( a_{i_p} \) is \( (e'(r)/p) \)-flat on \( \pi(V(R_i))_x. \)

By (2.7) there exists a connected component \( U \) of

\[ \{ y' \in R^k \mid \| x' - y' \| < \varepsilon, \delta(y') \neq 0 \} \]

such that \( x' \in U \) and \( U \subseteq \pi(V(R_i))_x. \)
Then, from Lemma 5.11, [7] and Lemma 4

\[ a_i \in m_{x}^{2p(e_p - 1) + [d/2] + 1} \]

We have

\[ g(g^{p-1} + a_1 \cdot g^{p-2} + \ldots + a_{i, p-1}) \in \mathcal{O}_{k+1, x} \cdot R_i + m_{x}^{2p(e_p - 1) + [d/2] + 1} \]

If \( x \) is sufficiently close to 0 then degree \( \omega_i \leq d \).

Then, from Lemma 2,

\[ g \in \mathcal{O}_{k+1, x} \cdot R_i + m_{x}^{2p(e_p - 1) + (r)} \]

or

\[ g^{p-1} + \ldots + a_{i, p-1} \in \mathcal{O}_{k+1, x} \cdot R_i + m_{x}^{2p-1(r)} \]

In the second case, repeating this process \( p - 1 \) times, we can prove that \( g \) is \( e_0(r) \)-flat on

\[ \mathcal{V}(R_i) = \{ (y', y_{k+1}) \in C^k \times C | R_i(y', y_{k+1}) = 0 \} \] at \( (x', x_{k+1}) \).

Then \( g \) is \( e_0(r) \)-flat on \( \mathcal{V}(R_1 \ldots R_s) = \mathcal{V}(R_1) \cup \ldots \cup \mathcal{V}(R_s) \) at \( (x', x_{k+1}) \).

From the preparation theorem we have

\[ g = s \cdot R_1 \ldots R_s + \sum_{j=1}^{p} b_j \cdot Y_{k+1}^{p-j}, \quad \text{where} \quad b_j \in \mathcal{O}_{k, x}. \]

By Cramer's rule \( b_j \in m_{x}^{r \cdot } \). The arguments are the same as in the proof of Lemma 2.

By (2.4)

\[ \delta N. (T_{x}^{*} f) \in (P_{k+1}, 8 \cdot X_{k+2} - Q_{k+2}, \ldots, 8 \cdot X_n - Q_n) \mathcal{O}_{n, x} + m_{n, x}. \]

From Lemma 5, there exists \( h \in m_{n, x} \) such that

\[ \delta N. (T_{x}^{*} f - h) \in I(V(p))_x. \]

By (2.3), \( T_{x}^{*} f - h \in I(V(p))_x. \)

Then \( T_{x}^{*} f \in I(V(p))_x + m_{n, x} \). ~

**Theorem 2** (see [1]). — Let \( g_1, \ldots, g_m \in \mathcal{O}_{n, 0}. \)

There exist a linear function \( N \ni r \mapsto e^*(r) = a^* \cdot r + b^* \in N \) and an open neighbourhood \( \Omega \) of 0 such that:

**BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE**
if $f: \Omega \to R$ is a function of class $C^{e^{r}(\tau)}$ and, for any $x \in \Omega$, $T_x^{e^{r}(\tau)} f \in (g_1, \ldots, g_m, \mathcal{O}_{n,x} + m^{e^{r}(\tau)}_{n,x})$ then there exist functions $\varphi_1, \ldots, \varphi_m: \Omega \to R$ of class $C^r$ such that:

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m.$$  

**Theorem 3.** Let $(X, 0) \subset (R^n, 0)$ be a germ of an analytic coherent set. Then $I(X)_0 = p_1 \cap \ldots \cap p_k$, where $p_1, \ldots, p_k$ are prime ideals in $\mathcal{O}_{n,0}$.

Suppose that every ideal $p_i$ satisfies assumptions (2.2)-(2.7). Then there exists a linear function $N \ni r \mapsto e(r) = a \cdot r + b \in N$ such that for any function $f$ of class $C^{e^{r}(\tau)}$ vanishing on $X$:

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m,$$

where $g_1, \ldots, g_m \in I(X)_0$ and $\varphi_1, \ldots, \varphi_m$ are germs of function of class $C^r$.

This theorem is a sharpened version of the result of J.-C. Tougeron (see Theorem 5.12, [7]).

**Proof.** We have $I(X)_0 \subset \sqrt{I(X)_0} = p_1 \cap \ldots \cap p_k$, where $p_1, \ldots, p_k \subset \mathcal{O}_{n,0}$ are prime ideals. The germ of $X$ at 0 is coherent, so the ideal $I(X)_0$ is real. Then $\sqrt{I(X)_0} \subset \sqrt{I(X)_0} = I(X)_0$. Hence $I(X)_0 = p_1 \cap \ldots \cap p_k$.

From Theorem 1 there exists a linear function $N \ni r \mapsto e'(r) = a' \cdot r + b' \in N$ such that, for any function $f$ of class $C^{e^{r}(\tau)}$ vanishing on $X$ and any $x \in X$ we have

$$T_x^{e^{r}(\tau)} f \in \cap_{i=1}^k (I(V(p_i))_x + m^{e^{r}(\tau)}_{n,x}).$$

By (2.3) and ([7], Theorem 3.8), there exists a constant $v' \in N$ such that, for any $x \in X$ we have

$$\cap_{i=1}^k (I(V(p_i))_x + m^{e^{r}(\tau)}_{n,x}) \subset \cap_{i=1}^k I(V(p_i))_x + m^{e^{r}(\tau)}_{n,x} \subset I(X)_x + m^{e^{r}(\tau)}_{n,x}.$$

Let $g_1, \ldots, g_m$ be generators of $I(X)_0$. Let $e^{e^{r}(\tau)}(r)$ be a function as in Theorem 2.

Define $e(r) = e'(e^{e^{r}(\tau)} + v') = a \cdot r + b$.

Let $f$ be a germ of class $C^{e^{r}(\tau)}$ vanishing on $X$. 

_Tome 113 - 1985 - s 2_
FUNCTIONS OF CLASS $C^r$

Then, for any $x \in X$ in some neighbourhood of 0 we have

$$T_x^{(r)} f \in \bigcap_{l=1}^k \left( I(V(p_l))_x + \mathfrak{m}_n^e (x)^{r+l} \right) \subset I(X)_x + \mathfrak{m}_n^e (x)^r.$$

From Theorem 2 there exist functions $\varphi_1, \ldots, \varphi_m$ of class $C^r$ such that

$$f = \varphi_1 \cdot g_1 + \ldots + \varphi_m \cdot g_m.$$

This completes the proof. ■

REFERENCES

[1] JEDDARI (L.), *Sur la divisibilité des fonctions de classe $C^r$ par les fonctions analytiques réelles.*


